1. Let X_1, X_2 and X_3 be three independent random variables. X_1 is $\chi^2(3)$ and X_2 is $\chi^2(2)$ and X_3 is $\chi^2(2)$. Compute

$$P\{X_1/(X_2+X_3) \leq 1\}$$

using one of the enclosed tables.

$$\mathbb{P}\left(\frac{X_1}{X_2+X_3} \leq 1\right) = \mathbb{P}\left(\frac{X_1/3}{\frac{X_2+X_3}{4}} \leq \frac{4}{3}\right)$$

$$= \mathbb{P}\left(Y \leq 4/3\right) \quad \text{where } Y \sim F(3,4)$$

2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} 0, & \text{if } t \notin [0,1] \\ \theta t^{\theta-1}, & \text{if } t \in [0,1]. \end{cases}$$

Find a sufficient statistic for θ.

$$f(\underline{x};\theta) = \prod \theta x_i^{\theta-1} \, 1\{x_i \in [0,1]\}$$

$$= \theta^n e^{(\theta-1)\sum \log(x_i)} \, 1\{x_i \in [0,1]\}.$$

Therefore $\sum \log(X_i)$ is sufficient for θ.

$\prod X_i$ is also sufficient for θ.

3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} 0, & \text{if } t \notin [0,\theta] \\ \frac{2t}{\theta^2}, & \text{if } t \in [0,\theta]. \end{cases}$$

Find a sufficient statistic for θ.

$$f(\underline{x};\theta) = \prod \frac{2x_i}{\theta^2} \, 1\{x_i \in [0,\theta]\}$$

$$= \frac{2^n}{\theta^{2n}} \left(\prod x_i\right) 1\{x_{n:n} \leq \theta\} \, 1\{x_{1:n} \geq 0\}.$$

Thus $X_{n:n}$ is sufficient for θ.

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \frac{1}{2\theta} e^{-|t|/\theta}, \quad -\infty < t < \infty.$$

Find the maximum likelihood estimator for θ and compute the asymptotic variance of the maximum likelihood estimator.

$L(\theta) = \frac{1}{2^n \theta^n} e^{-\frac{1}{\theta}\sum |x_i|}$, which is maximized

at $\hat{\theta} = \frac{\sum |x_i|}{n}$.

The asymptotic variance of the unique MLE is the CRLB, $\frac{\theta^2}{n}$.

5. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} 0, & \text{if } t \notin [0,\theta] \\ \dfrac{2t}{\theta^2}, & \text{if } t \in [0,\theta]. \end{cases}$$

Find the maximum likelihood estimator for θ and compute the bias of the maximum likelihood estimator for θ.

$$L(\theta) = 2^n \theta^{-2n} \prod x_i \, \mathbb{1}\{x_i \in [0,\theta]\}$$

$$= 2^n \theta^{-2n} \left(\prod x_i\right) \mathbb{1}\{x_{n:n} \le \theta\},$$

which is maximized at $\hat{\theta} = x_{n:n}$.

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta = \frac{-\theta}{2n+1}$$

6. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$h(t;\theta) = \begin{cases} 0, & \text{if } -\infty < t < 1 \\ \theta t^{-(\theta+1)}, & \text{if } 1 \leq t < \infty \end{cases}$$

Find a moment estimator for θ.

$$\bar{X} = \int_1^\infty x\,\theta\,x^{-(\theta+1)}\,dx = \frac{-\theta}{1-\theta} \qquad \text{for } \theta > 1$$

Thus $\hat{\theta} = \dfrac{\bar{X}}{\bar{X}-1}$

7. Let X_1, X_2 and X_3 be three independent normal random variables. X_1 is normal $N(0,4)$, X_2 is normal $N(0,12)$ and X_3 is normal $N(0,9)$. Compute

$$P\left\{\frac{X_1+X_2}{|X_3|} \leq 3\right\}$$

using one of the enclosed tables.

$$\mathbb{P}\left(\frac{X_1+X_2}{|X_3|} \leq 3\right) = \mathbb{P}\left(\frac{\frac{X_1+X_2}{4}}{\sqrt{\frac{X_3^2}{3^2}}} \leq \frac{3}{4}\cdot 3\right)$$

$$= \mathbb{P}\left(Y \leq \frac{9}{4}\right) \quad \text{where} \quad Y \sim T(1).$$

8. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with probaility mass function

$$f(t;\theta) = \frac{2!}{t!(2-t)!}\theta^t(1-\theta)^{2-t}, \text{ if } t = 0, 1, 2.$$

Find the uniformly minimum variance estimator for θ.

$$f(\underline{x};\theta) = \prod \binom{2}{x_i} \theta^{x_i} (1-\theta)^{2-x_i} \, 1\{x_i \in \{0,1,2\}\}$$

$$= \left(\prod \binom{2}{x_i} 1\{x_i \in \{0,1,2\}\}\right) \theta^{\sum x_i} (1-\theta)^{2n - \sum x_i}.$$

Thus $\sum X_i$ is sufficient for θ.

$\sum X_i \sim BIN(2n, \theta)$, which is complete since it is a member of the REC.

Thus $\dfrac{\sum X_i}{2n}$ is a UMVUE for θ.