1. Let X_1, X_2, ..., X_n be independent identically distributed random variables with cumulative distribution function
$$F(x) = \frac{1}{1+e^{-x}}.$$
Find the limiting distribution of $X_{1:n} + \log(n)$.

$$P(X_{1:n} + \log(n) < y) = \mathbb{P}(X_{1:n} < y - \log(n))$$

$$= 1 - \mathbb{P}(X_{1:n} \geq y - \log(n))$$

$$= 1 - \left(1 - \frac{1}{1+e^{-y+\log(n)}}\right)^n$$

$$= 1 - \left(\frac{1+e^{-y}n - 1}{1+e^{-y}n}\right)^n$$

$$= 1 - \left(\frac{e^{-y}n}{1+e^{-y}n}\right)^n$$

$$= 1 - \left(\frac{1}{e^{-y}n} + 1\right)^{-n}$$

$$= 1 - e^{-e^{y}}$$

Note: $F(x) = \begin{cases} x^2 & x \in (0,1), \\ 1 & x \geq 1, \\ 0 & x \leq 0. \end{cases}$

2. Let X_1, X_2, ..., X_n be independent identically distributed random variables with density function

$$f(x) = \begin{cases} 2x & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

Find the limiting distribution of $n(1 - X_{n:n})$.

$$\mathbb{P}\left(n(1-X_{n:n}) \leq y\right) = \mathbb{P}\left(1 - X_{n:n} \leq \frac{y}{n}\right) 1\{y \in [0,n]\}$$

$$= \mathbb{P}\left(X_{n:n} \geq 1 - \frac{y}{n}\right) 1\{y \in [0,n]\}$$

$$= \left(1 - \mathbb{P}\left(X_{n:n} < 1 - \frac{y}{n}\right)\right) 1\{y \in [0,n]\}$$

$$= \begin{cases} 1 - \left(\left(1 - \frac{y}{n}\right)^2\right)^n & y \in [0,n] \\ 0 & \text{o/w} \end{cases}$$

$$\rightarrow \begin{cases} 1 - e^{-2y} & y \geq 0, \\ 0 & y < 0. \end{cases}$$

Note: $\mathrm{Var}\left(\sum X_i\right) = n\sigma^2 = n\left(\frac{2}{3} - 0^2\right) = \frac{2n}{3}$

$E(X) = 0$

$E(X^2) = \int_{-1}^{1} x^2 \tfrac{1}{2} dx \quad \frac{1}{2}\frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}\frac{1}{2} = \frac{1}{3}$

3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(x) = \begin{cases} 1/2 & \text{if } x \in (-1,1) \\ 0 & \text{if } x \notin (-1,1). \end{cases}$$

Approximate $P\left(\sum_{i=1}^{100} X_i \le 0\right)$ in terms of $\Phi(\cdot)$, the cdf of a standard normal.

$$\mathbb{P}\left(\sum_{i=1}^{100} X_i \le 0\right) = \mathbb{P}\left(\frac{\sum X_i}{\sqrt{\frac{200}{3}}} \le 0\right) \approx \Phi(0) = \frac{1}{2}$$

4. Let X be a random variable with density function

$$f(x) = \begin{cases} 1/2 & \text{if } x \in (-1,1) \\ 0 & \text{if } x \notin (-1,1). \end{cases}$$

Compute the moment generating function of X.

$$M_X(t) = E(e^{tX}) = \int_{-1}^{1} e^{tx} \frac{1}{2}\, dx = \frac{1}{2t} e^{tx} \Big|_{x=-1}^{x=1}$$

$$= \begin{cases} \frac{1}{2t}\left[e^{t} - e^{-t}\right]. & t \neq 0 \\ 1 & t = 0 \end{cases}$$

5. Let X_1 and X_2 be independent random variables with density functions

$$X_1 \qquad f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

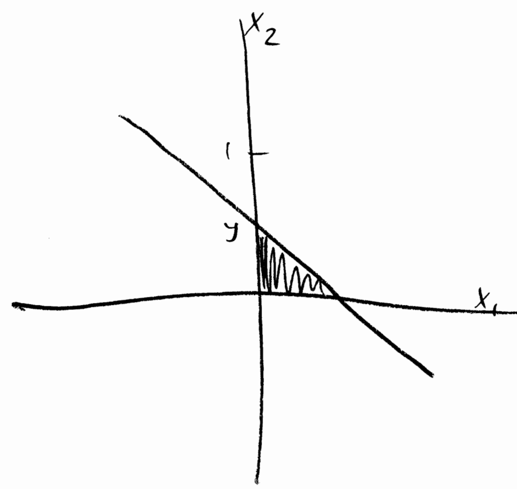
and

$$X_2 \qquad g(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1), \end{cases}$$

respectively. Compute the density function of $X_1 + X_2$.

$$F(y) = \mathbb{P}(X_1 + X_2 \leq y) = \begin{cases} \int_0^{\min(1,y)} \int_0^{y-x_2} e^{-x_1} \, dx_1 \, dx_2 & y \geq 1, \\ & \text{o/w.} \end{cases}$$

$$= \begin{cases} \int_0^{\min(1,y)} \left(-e^{-x_1}\right) \Big|_{x_1=0}^{x_1=y-x_2} dx_2 \\ \end{cases}$$



$$= \begin{cases} \int_0^{\min(1,y)} \left(1 - e^{x_2 - y}\right) dx_2 \\ \end{cases}$$

$$= \begin{cases} 1 - e^{-y}(e-1) & y \geq 1, \\ y - e^{-y}(e^y - 1) = y - 1 + e^{-y} & y \in (0,1), \\ 0 & y \leq 0. \end{cases}$$

$$\Rightarrow \quad f_{X_1+X_2}(y) = \begin{cases} e^{-y}(e-1) & y \geq 1, \\ 1 - e^{-y} & y \in (0,1), \\ 0 & y \leq 0. \end{cases}$$

6. Let X_1 and X_2 be independent identically distributed random variables with density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Compute the density function of (Y_1, Y_2), where $Y_1 = X_1$ and $Y_2 = X_1 + 2X_2$.

$$X_1 = y_1 \qquad\qquad J = \begin{bmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

$$X_2 = \frac{y_2 - y_1}{2}$$

$$|\det(J)| = \frac{1}{2}$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(y_1, \frac{y_2 - y_1}{2}\right) \cdot \frac{1}{2} \cdot 1\{y_2 > y_1 > 0\}$$

$$= \begin{cases} e^{-y_1} e^{-\frac{y_2}{2}} e^{\frac{y_1}{2}} \cdot \frac{1}{2} & y_2 > y_1 > 0 \\ & \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-\frac{1}{2}(y_1 + y_2)} & y_2 > y_1 > 0, \\ 0 & \text{o/w}. \end{cases}$$

7. Let X be a random variable with density function

$$f(x) = \begin{cases} 1/4 & \text{if } x \in (-2,2) \\ 0 & \text{if } x \notin (-2,2). \end{cases}$$

Compute the density function of $Y = X^2$.

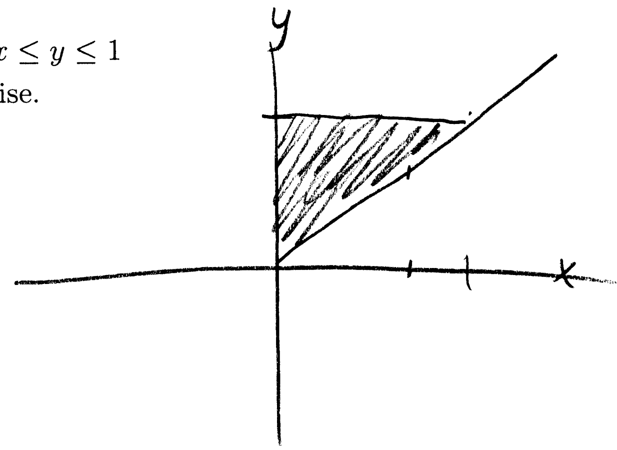
$$F_Y(y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}} = \begin{cases} \frac{1}{4\sqrt{y}} & y \in (0,4), \\ 0 & \text{o/w}. \end{cases}$$

8. Let X and Y be ~~a~~ random variables with joint density function

$$f(x,y) = \begin{cases} k(x+y) & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of the constant, k.



$$k \int_0^1 \int_x^1 (x+y) \, dy \, dx$$

$$= k \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big|_x^1 \, dx$$

$$= k \int_0^1 \left(x + \frac{1}{2}\right) - \left(x^2 + \frac{x^2}{2}\right) dx$$

$$= k \int_0^1 \left(-\frac{3}{2} x^2 + x + \frac{1}{2} \right) dx$$

$$= k \left[\frac{-3}{2} \frac{1}{3} + \frac{1}{2} + \frac{1}{2} \right]$$

$$= k \left[\frac{-1}{2} + \frac{1}{2} + \frac{1}{2} \right]$$

$$\Rightarrow k = 2$$

$f(x, y)$

9. Let X and Y be a random variables with joint density function

$$f(x) = \begin{cases} \frac{2}{3}(x+1) & \text{if } x \in (0,1) \text{ and } y \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginals.

$$f_X(x) = \begin{cases} \int_0^1 \frac{2}{3}(x+1)\,dy & x \in (0,1) \\ 0 & \text{o/w} \end{cases}$$

$$= \begin{cases} \frac{2}{3}(x+1) & x \in (0,1), \\ 0 & \text{o/w}. \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^1 \frac{2}{3}(x+1)\,dx & y \in (0,1), \\ 0 & \text{o/w}. \end{cases}$$

$$= \begin{cases} 1 & y \in (0,1), \\ 0 & \text{o/w}. \end{cases}$$