2. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with $P\{X_1 = 1\} = \theta$ and $P\{X_1 = 0\} = 1-\theta$ for some $\theta \in (0,1)$. Find the uniformly minimum variance unbiased estimator for $\theta(1-\theta)$.

$X_i \sim iid \; BER(\theta)$

$$f(\underline{x};\theta) = \prod p^{x_i}(1-p)^{1-x_i} 1\{x_i \in \{0,1\}\}$$

$$= p^{\sum x_i}(1-p)^{n-\sum x_i} 1\{x_i \in \{0,1\} \text{ for } i=1,2,\ldots,n\}.$$

Thus $\sum X_i$ is sufficient for θ.

Since $BER(\theta)$ is a member of the REC, $\sum X_i$ is also complete.

$E(\bar{X}) = \theta$, so $\bar{X}$ is the UMVUE for θ

$$E(\bar{X}^2) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n} E(X_i X_k) = \frac{1}{n^2}\left(n E(X_1^2) + (n^2-n)(EX_i)^2\right)$$

$$= \frac{1}{n^2}\left(n\theta + n^2\theta^2 - n\theta^2\right) = \frac{\theta}{n} + \theta^2 - \frac{\theta^2}{n}$$

$$= \theta^2\left(1-\frac{1}{n}\right) + \frac{\theta}{n}$$

Thus $\bar{X} - \left[\dfrac{\left(\bar{X}^2 - \frac{\bar{X}}{n}\right)}{1-\frac{1}{n}}\right]$ is the UMVUE for $\theta - \theta^2$.