

1. Let X be a random variable with density function

$$f(x) = \begin{cases} 1/4 & \text{if } x \in (-2, 2) \\ 0 & \text{if } x \notin (-2, 2). \end{cases}$$

Compute the density function of $Y = (X - 1)^2$.

$$\begin{aligned} F_Y(y) &= P((X-1)^2 \leq y) = P(-\sqrt{y} \leq X-1 \leq \sqrt{y}) = P(1-\sqrt{y} \leq X \leq 1+\sqrt{y}) \\ &= F_X(1+\sqrt{y}) - F_X(1-\sqrt{y}) \end{aligned}$$

$$f_Y(y) = f_X(1+\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(1-\sqrt{y}) \frac{1}{2\sqrt{y}}$$

$$= \begin{cases} \frac{1}{4\sqrt{y}} & y \in (0, 1], \\ \frac{1}{8\sqrt{y}} & y \in (1, 9), \\ 0 & \text{o/w.} \end{cases}$$

2. Let X_1 and X_2 be independent identically distributed random variables with density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Compute the density function of (Y_1, Y_2) , where $Y_1 = X_1$ and $Y_2 = X_1 + 2X_2$.

$$\begin{aligned} f_{Y_1, Y_2}(y_1, y_2) &= f_{X_1, X_2}(x_1(\underline{y}), x_2(\underline{y})) |\det(J)| \\ &= e^{-y_1 - \left(\frac{y_2 - y_1}{2}\right)} \cdot \frac{1}{2} \cdot 1\{y_1 > 0, y_2 > y_1\} \end{aligned}$$

$$= \boxed{\frac{1}{2} e^{-\frac{(y_1 + y_2)}{2}} 1\{y_2 > y_1 > 0\}}$$

$$x_1 = y_1$$

$$x_2 = \frac{y_2 - x_1}{2} = \frac{y_2 - y_1}{2}$$

$$J = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

3. Let X_1 and X_2 be independent identically distributed random variables with density function

$$f(x) = \begin{cases} \frac{e^x}{e-1} & \text{if } x \in (0, 1) \\ 0 & \text{if } x \notin (0, 1). \end{cases}$$

Compute the moment generating function of $2X_1 - 3X_2 + 2$.

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \int_0^1 \frac{e^{tx}}{e-1} e^x dx = \frac{1}{e-1} \int_0^1 e^{x(t+1)} dx \\ &= \frac{1}{e-1} \left. \frac{e^{x(t+1)}}{t+1} \right|_0^1 = \frac{e^{t+1} - 1}{(e-1)(t+1)}. \end{aligned}$$

$$\begin{aligned} M_{2X_1 - 3X_2 + 2}(t) &= M_{2X_1}(t) M_{-3X_2}(t) M_2(t) \\ &= M_X(2t) M_X(-3t) e^{2t} \end{aligned}$$

$$= \left[\frac{e^{2t+1} - 1}{(e-1)(2t+1)} \right] \left[\frac{e^{-3t+1} - 1}{(e-1)(-3t+1)} \right] e^{2t}$$

4. Let X_1 and X_2 be independent random variables with density functions

$$X \quad f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$Y \quad g(x) = \begin{cases} 1 & \text{if } x \in (0, 1) \\ 0 & \text{if } x \notin (0, 1), \end{cases}$$

respectively. Compute the density function of $X_1 + X_2$.

$$\begin{aligned} f_{X+Y}(w) &= \int_{-\infty}^{\infty} f_X(s) f_Y(w-s) ds \\ &= \int_{\max\{0, w-1\}}^w e^{-s} ds = -e^{-s} \Big|_{\max\{0, w-1\}}^w \\ &= -e^{-w} + e^{-\max\{0, w-1\}} \end{aligned} \quad \left. \begin{array}{l} \text{Assuming} \\ w > 0 \end{array} \right\}$$

So, in general,

$$f_{X+Y}(w) = \begin{cases} 1 - e^{-w} & w \in (0, 1] \\ -e^{-w} + e^{1-w} & w > 1 \\ 0 & \text{o/w} \end{cases}$$

5. Let X_1, X_2, \dots, X_n be independent identically distributed random variables with cumulative distribution function

$$F(x) = \begin{cases} 1 - 1/x & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases} = \begin{cases} 1 - \frac{1}{x} & x > 1, \\ 0 & x \leq 1. \end{cases}$$

Find the limiting distribution of $X_{n:n}/n$.

$$P\left(\frac{X_{n:n}}{n} \leq x\right) = P(X_{n:n} \leq nx) = \left[P(X \leq nx)\right]^n = (F(nx))^n$$

$$= \begin{cases} \left(1 - \frac{1}{nx}\right)^n & nx \geq 1 \\ 0 & nx < 1 \end{cases} \rightarrow 1_{\{x > 0\}} e^{-1/x}$$

6. Let X_1, X_2, \dots, X_n be independent identically distributed random variables with cumulative distribution function

$$F(x) = \begin{cases} 1 - 1/x^4 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}$$

Find the limiting distribution of $\frac{X_{n:n}}{n^{1/4}}$.

$$\begin{aligned} P\left(\frac{X_{n:n}}{n^{1/4}} \leq x\right) &= P\left(X_{n:n} \leq x n^{1/4}\right) = \left(F(x n^{1/4})\right)^n \\ &= \begin{cases} \left(1 - \frac{1}{x^4 n}\right)^n & x n^{1/4} \geq 1 \\ 0 & x n^{1/4} < 1 \end{cases} \rightarrow 1_{\{x > 0\}} e^{-1/x^4} \end{aligned}$$

7. Let X_1, X_2, \dots, X_n be independent identically distributed random variables with density function

$$f(x) = \begin{cases} 1 & \text{if } x \in (0, 1) \\ 0 & \text{if } x \notin (0, 1). \end{cases}$$

Approximate $P(\sum_{i=1}^{90} X_i \leq 77)$ in terms of $\Phi(\cdot)$, the cdf of a standard normal.

$$P(\sum X_i \leq 77) = P\left(\frac{(\sum X_i) - 45}{\sqrt{\frac{90}{12}}} \leq \frac{32}{\sqrt{\frac{90}{12}}}\right) \\ \approx \boxed{\Phi\left(\frac{32}{\sqrt{90/12}}\right)}$$

$$\text{var}(X_i) = E(X^2) - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\text{Since } E(X^2) = \int_0^1 x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3}$$

$$\text{Therefore } \text{var}(\sum X_i) = 90/12.$$