1. Let X be a random variable with density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Compute the density function of $Y = X^3$.

$$F_Y(y) = P(X^3 \le y) = P(X \le y^{1/3}) = F_X(y^{1/3})$$

$$f_Y(y) = f_X(y^{1/3}) \frac{1}{3} y^{-2/3}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y^{2/3}/2} \frac{1}{3} y^{-2/3}$$

$$= \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}y^{2/3}} y^{-2/3}$$

2. Let X_1 and X_2 be independent identically distributed random variables with density function

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Compute the density function of (Y_1, Y_2), where $Y_1 = X_1/X_2$ and $Y_2 = X_1^2 X_2$.

$$x_1 = (y_1 y_2)^{1/3} = y_1^{1/3} y_2^{1/3}$$

$$x_2 = y_2^{1/3} y_1^{-2/3}$$

$$J = \begin{pmatrix} \frac{1}{3} y_1^{-2/3} y_2^{1/3} & \frac{1}{3} y_2^{-2/3} y_1^{1/3} \\ -\frac{2}{3} y_1^{-5/3} y_2^{1/3} & \frac{1}{3} y_2^{-2/3} y_1^{-2/3} \end{pmatrix}$$

$$|\det(J)| = \left| \frac{1}{9} y_1^{-4/3} y_2^{-1/3} + \frac{2}{9} y_1^{-4/3} y_2^{-1/3} \right|$$

$$= \frac{1}{3} y_1^{-4/3} y_2^{-1/3}$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1(\underline{y}), x_2(\underline{y})) \, |\det(J)|$$

$$= \begin{cases} \exp\left(-y_1^{1/3} y_2^{1/3} - y_2^{1/3} y_1^{-2/3}\right) \frac{1}{3} y_1^{-4/3} y_2^{-1/3} & \text{for } y_1, y_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

3. Let X be a random variable with density function

$$f(x) = \frac{1}{2}e^{-|x|}.$$

Compute the moment generating function of X and use it to compute $E(X^2)$.

Assume $t \in (-1,1)$.

$$M_X(t) = E\left(e^{tX}\right) = \int_{\mathbb{R}} e^{tx}\,\tfrac{1}{2}e^{-|x|}\,dx$$

$$= \int_0^{\infty} \tfrac{1}{2} e^{x(t-1)}\,dx + \int_{-\infty}^{0} \tfrac{1}{2} e^{x(t+1)}\,dx$$

$$= \frac{1}{2(t-1)} e^{x(t-1)} \Big|_{x=0}^{x=\infty} + \frac{1}{2(t+1)} e^{x(t+1)} \Big|_{x=-\infty}^{x=0}$$

$$= 0 - \frac{1}{2(t-1)} + \frac{1}{2(t+1)} - 0$$

$$= \frac{1}{2(t+1)} - \frac{1}{2(t-1)} = \tfrac{1}{2}\left((t+1)^{-1} - (t-1)^{-1}\right)$$

$$M'(t) = \tfrac{1}{2}\left(-1\,(t+1)^{-2} + 1\,(t-1)^{-2}\right)$$

$$E(X^2) = M''(t)\Big|_{t=0} = \tfrac{1}{2}\left(2(t+1)^{-3} - 2(t-1)^{-3}\right)\Big|_{t=0}$$

$$= \tfrac{1}{2}(2+2) = \boxed{2}$$

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Find the cumulative distribution function of $X_{n:n}$.

$$P(X_{n:n} \leq y) = (F(y))^n = \begin{cases} (1 - e^{-y})^n & y \geq 0 \\ 0 & y < 0 \end{cases}$$

5. Let X_1 and X_2 be independent random variables with density functions

$$f(x) = \frac{1}{2}e^{-|x|}$$

and

$$g(x) = \begin{cases} 1/2 & \text{if } x \in (-1,1) \\ 0 & \text{if } x \notin (-1,1), \end{cases}$$

respectively. Compute the density function of $X_1 + X_2$.

$$f_{X_1+X_2}(y) = \int_{\mathbb{R}} f(y-s)\,g(s)\,ds = \int_{-1}^{1} \frac{1}{4} e^{-|y-s|}\,ds$$

$$= \begin{cases} \int_{-1}^{1} \frac{1}{4} e^{-y+s}\,ds & y > 1 \\ \int_{-1}^{y} \frac{1}{4} e^{-y+s}\,ds + \int_{y}^{1} \frac{1}{4} e^{y-s}\,ds & \\ \int_{-1}^{1} \frac{1}{4} e^{y-s}\,ds & y < -1 \end{cases}$$

$$= \begin{cases} \frac{1}{4} e^{-y}\left(e - \frac{1}{e}\right) & y > 1 \\ \frac{1}{2} - \frac{1}{4} e^{-y-1} - \frac{1}{4} e^{y-1} & y \in [-1,1] \\ \frac{1}{4} e^{y}\left(e - \frac{1}{e}\right) & y < -1 \end{cases}$$

6. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with cumulative distribution function

$$F(x) = \frac{1}{1+e^{-x}}.$$

Find the limiting distribution of $X_{1:n} + \log(n)$.

$$P(X_{1:n} + \log(n) \leq y) = P(X_{1:n} \leq y - \log(n))$$

$$= 1 - P(X_{1:n} > y - \log(n))$$

$$= 1 - (1 - F(y - \log(n)))^n$$

$$= 1 - \left(1 - \frac{1}{1+e^{-y+\log(n)}}\right)^n$$

$$= 1 - \left(1 - \frac{1}{1+ne^{-y}}\right)^n$$

$$= 1 - \left(\frac{1+ne^{-y}-1}{1+ne^{-y}}\right)^n = 1 - \left(\frac{ne^{-y}}{1+ne^{-y}}\right)^n$$

$$= 1 - \left(\frac{1+ne^{-y}}{ne^{-y}}\right)^{-n} = 1 - \left(1 + \frac{e^{+y}}{n}\right)^{-n}$$

$$\longrightarrow 1 - e^{(-e^y)}$$

7. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

Approximate $P\left(\sum_{i=1}^{20} X_i \le 12\right)$ in terms of $\Phi(\cdot)$, the cdf of a standard normal.

$$= P\left(\frac{\sum X_i - 10}{\sqrt{\frac{20}{12}}} \le \frac{2}{\sqrt{20/12}}\right) \approx \Phi\left(\frac{2}{\sqrt{5/3}}\right)$$