MATH 5010 – Exam 2

Name:

Date:

You will notice that the questions on this exam resemble questions that you have seen before. However, many of them are NOT quite the same. Read each question carefully. Each question is worth 3 points for a total of 27 points. Although partial credit is not guaranteed, some partial credit may be awarded. Therefore it may be in your best interest to write your steps down neatly and carefully.

1. I have a p-coin. Each toss of the coin is independent of other tosses, and each toss results in HEADS with probability p. I will toss this coin until it lands on heads 5 times. Let X be the number of times it will land on TAILS. Find the mass function of X.

$$p(x) = \begin{cases} \binom{x+4}{4} p^5 (1-p)^x & x \in \{0, 1, 2, \ldots\}, \\ \\ 0 & o/w. \end{cases}$$

2. Find the fourth moment, $E(X^4)$, of an exponential random variable with parameter 1 (i.e. EXP(1)).

$$E(X^4) = \int_0^\infty x^4 e^{-x} \, dx$$

$$= -e^{-x}\left[x^4 + 4x^3 + 12x^2 + 24x + 24\right]\Big|_{x=0}^{x=\infty}$$

$$= 24$$

3. Suppose X and Y have joint density, which, for some value of C, is given by

$$f(x,y) = \begin{cases} Ce^{-x-3y} & x, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the variance of Y.

$$E(Y) = \int_0^\infty y \, 3e^{-3y} \, dy$$

$$= -e^{-3y}\left[y + \frac{1}{3}\right]\Big|_{y=0}^{y=\infty}$$

$$= \frac{1}{3}$$

$$E(Y^2) = \int y^2 \, 3e^{-3y} \, dy$$

$$= -e^{-3y}\left[y^2 + \frac{2y}{3} + \frac{2}{9}\right]\Big|_{y=0}^{y=\infty} = \frac{2}{9}$$

$$\text{var}(Y) = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$
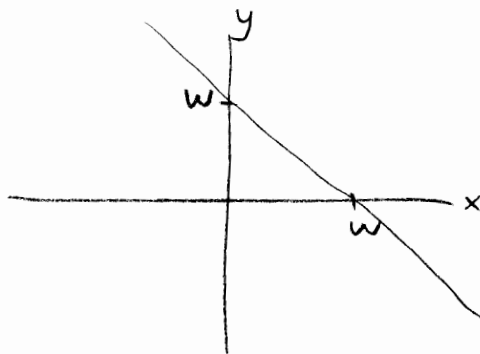
4. Suppose X and Y are i.i.d UNIF(-1,1). Find the density of $W = X + Y$.

$$F_W(w) = P(X + Y \leq w)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f(x,y) \, dy \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{w-x} f_Y(y) \, dy \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \, F_Y(w-x) \, dx$$

$$\Rightarrow \quad f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x) \, dx$$

$$= \begin{cases} \int_{\max(-1, w-1)}^{\min(1, w+1)} \frac{1}{4} \, dx & w \in (-2,2), \\ 0 & o/w. \end{cases}$$

$$= \begin{cases} \frac{1}{4}\left(\min(1, w+1) - \max(-1, w-1)\right) & w \in (-2,2), \\[2em] 0 & o/w. \end{cases}$$

5. Suppose X and Y are independent. Suppose also that X is uniform on $(0,1)$ and Y is exponentially distributed with parameter 1. Let $U = 2X + 4Y$ and $V = -X/3 + Y$. Find the joint density of U and V.

$$\frac{1}{6}U + V = \frac{5}{3}Y \quad \Rightarrow \quad Y = \frac{3}{5}\left(\frac{1}{6}U + V\right) = \frac{1}{10}U + \frac{3}{5}V$$

$$U - 4V = \frac{10}{3}X \quad \Rightarrow \quad X = \frac{3}{10}\left(U - 4V\right) = \frac{3}{10}U - \frac{6}{5}V$$

$$f_{XY}(x,y) = \begin{cases} e^{-y} & x \in (0,1), \ y > 0, \\ 0 & o/w. \end{cases}$$

$$f_{UV}(u,v) = f_{XY}\left(\frac{3}{10}u - \frac{6}{5}v, \ \frac{1}{10}u + \frac{3}{5}v\right) \cdot \frac{3}{10}$$

$$= \begin{cases} \frac{3}{10} e^{-\left(\frac{1}{10}u + \frac{3}{5}v\right)} & \frac{3}{10}u - \frac{6}{5}v \in (0,1) \\ & \text{and } \frac{1}{10}u + \frac{3}{5}v > 0, \\ 0 & o/w. \end{cases}$$

$$\boxed{.0004}$$

6. The width of a slot of a duralumin forging is (in inches) normally distributed with a mean of .7 and a variance of .004 The specification limits were given as $.8 \pm .005$. What percentage of forgings will be within specification? Express your answer in terms of $\Phi(x)$, the cdf of a standard normal, and then use the table at the back of this exam to evaluate it.

$$\mathbb{P}\left(.8 - .005 < X < .8 + .005\right)$$

$$= \mathbb{P}\left(\frac{.8 - .005 - .7}{.02} < \frac{X - .7}{.02} < \frac{.8 + .005 - .7}{.02}\right)$$

$$= \Phi(5.25) - \Phi(4.75)$$

$$= \Phi(-4.75) - \Phi(-5.25)$$

$$< .0002$$

7. Jay has two jobs to do, one after the other. Each attempt at job i takes one hour and is successful with probability p_i. If $p_1 = .3$ and $p_2 = .5$, what is the probability that it will take Jay more than 5 hours to be successful on both jobs?

Let X = # of attempts needed to do job 1.

Let Y = # of attempts needed to do job 2.

$$p(x,y) = \begin{cases} (.3)(.5)(.7)^{x-1}(.5)^{y-1} & x, y \in \{1,2,\ldots\}, \\[2em] 0 & o/w. \end{cases}$$

$\mathbb{P}\left(\text{Jay takes more than 5 hours}\right)$

$$= 1 - \sum_{x=1}^{4} \sum_{y=1}^{5-x} .3(.5)^{y}(.7)^{x-1}$$