MATH 5010 – Final Exam

Name:

Date:

Each question is worth 5 points. Although partial credit is not guaranteed, some partial credit may be awarded. You may notice that the questions on this exam are identical to quiz or exam questions that you have seen before. Because these questions are identical to those you have seen before, you are required to neatly and carefully show all of your work as well as the final answer. No credit will be given unless you show your work. Five points will be subtracted if your work does not match the solution you write down. Thus, you may theoretically receive a score as low as -50 *or as high as* 50 *on this exam.*

1. Bad drivers crash in a given year with probability 0.4 while a safe driver will crash with probability 0.2. An insurance company receives an application from a person that was in an accident last year and wants to charge an appropriate amount depending on whether the applicant is a good driver. It is estimated that 40% of the applicants are good drivers. What is the probability that this particular applicant is a good driver?

2. Consider a 2-dimensional integer lattice. Suppose that a robot starting at (0,0) must pick up a load at (27,5) and deliver it to (45,17). In a single move, the robot can either go up by one (eg (0,0) to (0,1)) or go right by one (eg (0,0) to (1,0)). The robot chooses a path at random so that each possible path from (0,0) through (27,5) to (45,17) is equally likely. What is the probability that the robot will hit the landmine at (40,10)?

3. The Jones family and the Smith family each have two children. Assume that each child born is male with probability 0.5 independently of other births. The oldest child in the Jones family is a girl and at least one of the children in the Smith family is a girl. What is the probability that both children in both families are girls?

4. Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate-specific antigen (PSA) that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately .135, increasing to approximately .268 if the man does have cancer. If, on the basis of other factors, a physician is 70 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that the test indicated an elevated PSA level?

5. I have a p-coin. Each toss of the coin is independent of other tosses, and each toss results in HEADS with probability p. I will toss this coin until it lands on heads 5 times. Let X be the number of times it will land on TAILS. Find the mass function of X.

6. Suppose X has c.d.f. given by

$$F(x) = \begin{cases} x^2 & x \in (0,1), \\ 1 & x \geq 1, \\ 0 & x \leq 0. \end{cases}$$

Find the variance of X.

7. Suppose X and Y are i.i.d UNIF(-1,1). Find the density of $W = X + Y$.

8. Suppose X and Y are independent. Suppose also that X is uniform on $(0, 1)$ and Y is exponentially distributed with parameter 1. Let $U = 2X + 4Y$ and $V = -X/3 + Y$. Find the joint density of U and V.

9. Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 35 and the other to a class of size 64. Approximate the probability that the average score in the class of size 35 exceeds the other class's average by more than 2 points. Express your answer in terms of $\Phi(x)$.

10. An insurance company has 10,000 automobile policyholders. The expected yearly claim per policyholder is \$240, with a standard deviation of \$800. Approximate the probability that the total yearly claim exceeds \$2.5 million. Express your answer in terms of $\Phi(x)$.

11. Please indicate how many quizzes you missed due to absences that you have discussed with me already. Please also refresh my memory regarding the reason for the absence(s). If the number of homework assignments you missed is different than the number of quizzes, also indicate the number of assignments that you missed. If you have not discussed your absences with me, you do not need to complete this.

MATH 5010 – Exam 1

Name:

Date:

You may notice that some questions on this exam resemble questions that you have seen before. However, many of them are NOT quite the same. Read each question carefully. Each question is worth 3 points; your total exam score will be either 25 or the sum of your individual points, whichever is less. Although partial credit is not guaranteed, some partial credit may be awarded. Therefore it may be in your best interest to write your steps down neatly and carefully.

1. Bad drivers crash in a given year with probability 0.4 while a safe driver will crash with probability 0.2. An insurance company receives an application from a person that was in an accident last year and wants to charge an appropriate amount depending on whether the applicant is a good driver. It is estimated that 40% of the applicants are good drivers. What is the probability that this particular applicant is a good driver?

$A = \{\text{Safe driver}\}$

$B = \{\text{Crashed last year}\}$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{(0.2)(0.4)}{\mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c)}$$

$$= \frac{(0.2)(0.4)}{(0.2)(0.4) + (0.4)(0.6)} = \frac{1}{4}$$

2. Consider a 2-dimensional integer lattice. Suppose that a robot starting at (0,0) must pick up a load at (27,5) and deliver it to (45,17). In a single move, the robot can either go up by one (eg (0,0) to (0,1)) or go right by one (eg (0,0) to (1,0)). The robot chooses a path at random so that each possible path is equally likely. What is the probability that the robot will hit the landmine at (40,10)?

$$\frac{\binom{32}{27}\binom{18}{13}\binom{12}{5}}{\binom{32}{27}\binom{30}{18}}$$

5. The Jones family and the Smith family each have two children. Assume that each child born is male with probability 0.5 independently of other births. The oldest child in the Jones family is a girl and at least one of the children in the Smith family is a girl. What is the probability that both children in both families are girls?

$A = \{\text{Jones kids are both girls}\}$

$B = \{\text{Smith kids are both girls}\}$

$$\mathbb{P}(A) = \frac{N(\{GG\})}{N(\{GG, GB\})} = \frac{1}{2}$$

$$\mathbb{P}(B) = \frac{N(\{GG\})}{N(\{GG, GB, BG\})} = \frac{1}{3}$$

$$\mathbb{P}(A \cap B) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$$

6. Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate-specific antigen (PSA) that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately .135, increasing to approximately .268 if the man does have cancer. If, on the basis of other factors, a physician is 70 percent certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that the test indicated an elevated PSA level?

$C = \{\text{has cancer}\}$

$P = \{\text{elevated PSA}\}$

$\mathbb{P}(P \mid C^c) = 0.135$

$\mathbb{P}(P \mid C) = 0.268$

$\mathbb{P}(C) = 0.70$

$$\mathbb{P}(C \mid P) = \frac{\mathbb{P}(P \mid C)\,\mathbb{P}(C)}{\mathbb{P}(P)} = \frac{\mathbb{P}(P \mid C)\,\mathbb{P}(C)}{\mathbb{P}(P \mid C)\,\mathbb{P}(C) + \mathbb{P}(P \mid C^c)\,\mathbb{P}(C^c)}$$

$$= \frac{0.268\,(0.7)}{0.268(0.7) + 0.135(0.3)}$$

$$\approx 0.82$$

MATH 5010 – Exam 2

Name:

Date:

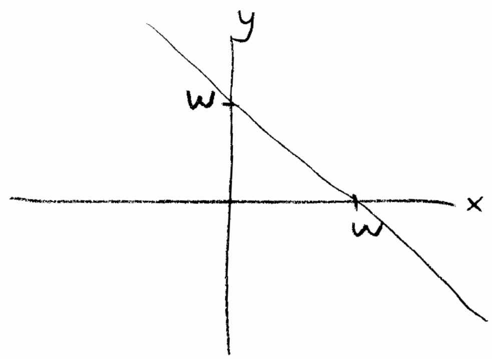
You will notice that the questions on this exam resemble questions that you have seen before. However, many of them are NOT quite the same. Read each question carefully. Each question is worth 3 points for a total of 27 points. Although partial credit is not guaranteed, some partial credit may be awarded. Therefore it may be in your best interest to write your steps down neatly and carefully.

1. I have a p-coin. Each toss of the coin is independent of other tosses, and each toss results in HEADS with probability p. I will toss this coin until it lands on heads 5 times. Let X be the number of times it will land on TAILS. Find the mass function of X.

$$p(x) = \begin{cases} \binom{x+4}{4} p^5 (1-p)^x & x \in \{0, 1, 2, \ldots\}, \\ 0 & \text{o/w.} \end{cases}$$

4. Suppose X and Y are i.i.d UNIF(-1,1). Find the density of $W = X + Y$.

$$F_W(w) = \mathbb{P}(X+Y \leq w)$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{w-x} f(x,y) \; dy\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{w-x} f_Y(y)\, dy \; dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, F_Y(w-x)\, dx$$

$$\Rightarrow \quad f_W(w) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx$$

$$= \begin{cases} \displaystyle\int_{\max(-1,w-1)}^{\min(1,w+1)} \frac{1}{4}\, dx & w \in (-2,2), \\ 0 & \text{o/w.} \end{cases}$$

$$= \begin{cases} \frac{1}{4}\left(\min(1,w+1) - \max(-1,w-1)\right) & w \in (-2,2), \\ 0 & \text{o/w.} \end{cases}$$

5. Suppose X and Y are independent. Suppose also that X is uniform on $(0,1)$ and Y is exponentially distributed with parameter 1. Let $U = 2X + 4Y$ and $V = -X/3 + Y$. Find the joint density of U and V.

$$\frac{1}{6}U + V = \frac{5}{3}Y \Rightarrow Y = \frac{3}{5}\left(\frac{1}{6}U + V\right) = \frac{1}{10}U + \frac{3}{5}V$$

$$U - 4V = \frac{10}{3}X \Rightarrow X = \frac{3}{10}(U - 4V) = \frac{3}{10}U - \frac{6}{5}V$$

$$f_{XY}(x,y) = \begin{cases} e^{-y} & x \in (0,1),\ y > 0, \\ 0 & \text{o/w.} \end{cases}$$

$$f_{UV}(u,v) = f_{XY}\left(\frac{3}{10}u - \frac{6}{5}v,\ \frac{1}{10}u + \frac{3}{5}v\right) \cdot \frac{3}{10}$$

$$= \begin{cases} \frac{3}{10} e^{-\left(\frac{1}{10}u + \frac{3}{5}v\right)} & \frac{3}{10}u - \frac{6}{5}v \in (0,1) \text{ and } \frac{1}{10}u + \frac{3}{5}v > 0, \\ 0 & \text{o/w.} \end{cases}$$

MATH 5010 – Quiz 17

Name:

Date:

8.15 An insurance company has 10,000 automobile policyholders. The expected yearly claim per policy-holder is \$240, with a standard deviation of \$800. Approximate the probability that the total yearly claim exceeds \$2.5 million. Express your answer in terms of $\Phi(x)$.

$$\mathbb{P}\left(\sum_{i=1}^{10^4} X_i > 2.5 * 10^6\right) = \mathbb{P}\left(\frac{\sum X_i - 10^4(240)}{\sqrt{10^4\, 800^2}} > \frac{2.5*10^6 - 10^4(240)}{\sqrt{10^4\, 800^2}}\right)$$

$$= 1 - \Phi\left(\frac{10^4(2.5 * 10^2 - 240)}{8 * 10^2\, 100}\right)$$

$$= 1 - \Phi\left(\frac{10^5}{8 * 10^4}\right) = 1 - \Phi\left(\frac{5}{4}\right).$$

MATH 5010 – **Quiz 16**

Name:

Date:

8.13 Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 35 and the other to a class of size 64. Approximate the probability that the average score in the class of size ~~25~~ 35 exceeds the other class's average by more than 2 points. Express your answer in terms of $\Phi(x)$.

$$\mathbb{P}(\bar{X}-\bar{Y} > 2) = \mathbb{P}\left(\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{14^2}{35}+\frac{14^2}{64}}} > \frac{2}{14\sqrt{\frac{1}{35}+\frac{1}{64}}}\right) = 1 - \Phi\left(\frac{2}{14\sqrt{\frac{1}{35}+\frac{1}{64}}}\right)$$