**Midterm #2 – Practice Exam**

**Question #1:** Let be independent normal random variables such that for . Assuming that and , find the value of such that .

* Since all of these random variables have zero mean and for any random variable , we have that and . We can then transform each of these random variables to standard normal ones, so we have that and . Next, squaring these standard normal random variables transforms them into chi-square distributed random variables, so that and . Since the sum of independent chi-square random variables is also distributed chi-square with the parameter being the sum of the individual parameters, we see that and . Finally, we know that the ratio of two chi-square random variables divided by their respective degrees of freedom follows the F distribution, so that . Using these facts above, we can compute that where . From the F table of values, we see that , so we equate and solve for the value of , so that .

**Question #2:** If are three independent normal random variables such that , , and , then compute the probability .

* From the known distributions of the three random variables, we can conclude that and . Also, we can see that which implies that . From the definition of the t distribution, we know that , which allows us to find , where the random variable and we used the t table of values.

**Question #3:** Let be independent and identically distributed random variables from the density function . Find the Method of Moments Estimator (MME) for the unknown parameter and show that it is unbiased.

* We first compute . Using integration by parts with , and , we can compute that . We then equate this population moment to the corresponding sample moment and solve for the unknown parameter in . Therefore, we have that , so the estimator is . To show this estimator is unbiased for , we compute .

**Question #4:** Let be independent and identically distributed random variables from the density function . Find the Maximum Likelihood Estimator (MLE) for the unknown parameter .

* We first find the likelihood function so . We then differentiate the log likelihood function with respect to the parameter to obtain . We equate this to zero and solve for to obtain . Since the second derivative will be negative, verifying that the likelihood function is maximized, we have found that .

**Question #5:** If is a random variable from the density function whenever , then compute the Cramer-Rao Lower Bound (CRLB).

* We know that , so we compute each of these components individually. Since , we have . We then compute . We then calculate Fisher’s Information since , so we have that and . We must therefore compute . This integral can be solved using integration by parts with , and , , so . We can then find Fisher’s information since Combining these results, we have that . This means that for any estimator which satisfies .

**Question #6:** Let be a random sample from (in particular, the mean and variance are both equal to ). Find the Maximum Likelihood Estimator (MLE) of .

* Since , its density is . This allows us to find and the log of the likelihood function . We then differentiate this to obtain . Finally, we set this equal to zero and solve, so that . The solutions to this quadratic equation are , but we choose the positive solution since . Therefore, we have found that .

**Question #7:** Let be a random sample from such that whenever . a) Find the Cramer-Rao Lower Bound (CRLB) on the variance of unbiased estimators of , and b) is the estimator a UMVUE of ?

1. Since , then and . Then since , we have ,, and . Since this is a constant, we have so that .
2. We first verify unbiasedness by calculating . Then we compute the variance to check if it equals the Cramer-Rao Lower Bound. Thus, we have . This then verifies that the estimator is the uniformly minimum variance unbiased estimator.

**Question #8:** Let be a random sample from such that whenever . a) Find a single sufficient statistic and then assume that it is complete, and b) State the Lehmann-Scheffe Theorem and use it to show that is a UMVUE of .

1. We verify that is a member of the REC and use that to find a complete sufficient statistic. This is clear since , where . Thus, we know is a complete sufficient statistic.
2. The Lehmann-Scheffe Theorem states that if have joint density function , is a complete sufficient statistic for the parameter , is a statistic that is unbiased for the parameter and is a function of , then is a UMVUE for . Since and , it is a UMVUE for .

**Question #9:** Let be a random sample from the population with CDF given by , where and . Find the corresponding density function and the Maximum Likelihood Estimators (MLE) for and .

* We have . Then the likelihood function is given by . By inspection, we can see that is the MLE for the unknown parameter . To find the MLE for , we first construct the log likelihood function and then find . Since the second derivative is negative, we have that .

**Question #10:** If is a random sample from the distribution where , then a) find the Cramer-Rao Lower Bound for the variance of unbiased estimators of the unknown parameter , and b) use this information to find a UMVUE for .

1. We have , so . Then we find , so that . We then compute , which means . Thus, .
2. By inspection, we see that the estimator is both unbiased and achieved the lower found. To verify unbiasedness, note that . Then we can compute , so is a UMVUE for .

**Question #11:** If is a random sample from where , then a) find a pair of jointly sufficient statistics, and b) find a single sufficient statistic based on those two.

1. Since , we have . We then note that this can be written as , so the factorization criterion guarantees that and are jointly sufficient for , where depends on the only through the sufficient statistics and does not depend on .
2. We can also write the density as by multiplying the first indicator function by a negative 1. But then , so by the factorization criterion the single statistic is sufficient.

**Question #12:** If is a random sample from the distribution where , then a) find a complete sufficient statistic for using the Regular Exponential Class (REC), and b) state the Lehmann-Scheffe Theorem and use it to find a UMVUE for .

1. For , we have , which verifies that the density is a member of the REC. Then we know that the statistic is complete sufficient for the parameter .
2. The Lehmann-Scheffe Theorem states that if have joint density function , is a complete sufficient statistic for the parameter , is a statistic that is unbiased for the parameter and is a function of , then is a UMVUE for . We thus need to find a statistic which is both unbiased and a function of . We do this by first noting that , so that we have . This means that will be unbiased, so the Lehmann-Scheffe Theorem guarantees that it will be a UMVUE for .

**Question #13:** Let be a random sample from such that for and let the prior distribution of be from such that the density is whenever . Note that a gamma random variable has mean and variance . a) Find the posterior distribution, and b) find the Bayes Estimator (BE) of the parameter under a squared error loss function .

1. Since the posterior distribution is , we can write that it is simply proportional to since is a constant term. Therefore, we have . Except for the constant term, this clearly resembles a gamma distribution, with first parameter and second parameter . Thus, we have found that the posterior distribution of the parameter is .
2. We know that under the squared error loss function , the Bayes estimator of is given by the conditional mean of the posterior distribution. However, the mean of a gamma random variable is simply the product of its parameters, so we have that . Under the absolute error loss function , the Bayes estimator is simply given by the median of the posterior distribution. However, there is no closed form expression for the median of a gamma random variable, so the Bayes estimator in this case would have to be computed numerically.

**Question #14:** Let be a random sample from the distribution where we have and the mass function is . a) If it is assumed that has prior density , then find the posterior density given the sample, and b) find the Bayes Estimator (BE) assuming squared error loss.

1. Note that the prior distribution of is , where the general density function of a random variable is . Then we have that . After combining like terms, we see that which clearly resembles a beta random distribution except for the constant term. Thus, we have found that the posterior distribution is
2. Since the Bayes Estimator under a squared error loss function is the mean of the posterior distribution and the mean of a is , we can conclude that . Under the absolute error function, the Bayes Estimator is the median which is approximately in this case.