**Midterm #1 – Practice Exam**

**Question #1:** Let and be independent and identically distributed random variables with probability density function such that they come from the distribution . Compute the joint density function of and .

* Since and are independent, their joint density function is given by whenever , . We have the transformation and so and , so we can compute the Jacobian as . This implies that the joint density whenever and .

**Question #2:** Let be a standard normal random variable. Compute the Moment Generating Function (MGF) of using the standard normal distribution function .

* For some continuous random variable with density , its moment generating function is given by . Since , we know that its density function is given by . We can therefore compute . Separating out the variable and making the substitution and then yields . Thus, the Moment Generating Function of is given by .

**Question #3:** Let and be independent random variables with respective probability density functions and . Calculate the probability density function of the transformed random variable .

* Since and are independent random variables, their joint density is if and . We have and generate so and , so we can compute the Jacobian as . This implies that the joint density whenever and or . We then integrate out to find the PDF . Thus, we find that whenever and zero otherwise.

**Question #4:** Let be independent and identically distributed random variables from a cumulative distribution function . Find the limiting distribution of the transformed first order statistic given by .

* We can calculate that when . Since there is no dependence on , we may simply compute the desired limiting distribution as .

**Question #5:** Let be independent and identically distributed random variables with density function . That is, they are sampled from a random variable distributed as . Approximate using .

* Since , we know that and . Then if we let , we have and so that . Thus where .

**Question #6:** If such that , what transformation of will yield a random variable ? What about a random variable ?

* Let , so we can compute that and then by differentiating we obtain so that .
* Since we need to end up with to conclude that , it is clear that we must make the transformation to obtain this result. We can therefore compute so after differentiating we have so that we conclude .

**Question #7:** Does convergence in probability imply convergence in distribution? Does convergence in distribution imply convergence in probability? Give a counterexample if not.

* We have that , but that . As a counterexample, consider where . Then so . However, we have . This expression won’t converge to zero as since there is no dependence on in the probability expression. This implies that there is no convergence in probability, that is .

**Question #8:** State the law of large numbers (LLN), including its assumptions. Then state the central limit theorem (CLT), including its assumptions.

* *Law of Large Numbers:* If are independent and identically distributed with finite mean and variance , then the sample mean .
* *Central Limit Theorem:* If are independent and identically distributed with finite mean and variance , then and .

**Question #10:** Let and be independent random variables with Cumulative Distribution Function (CDF) given by for . Find the joint probability density function of the random variables and .

* We first find the density for . Then since and are independent random variables, their joint density function is for . We then have that and so that and . These computations allow us to compute the Jacobian as . This implies that the joint density function is whenever . This work also reveals that and are not independent since their joint probability density function cannot be factored into two functions of and .

**Question #11:** Let be independent random variables and suppose that we know , , and . What is the distribution of ?

* We use the Moment Generating Function (MGF) technique to note that since the three random variables are independent, we have . This reveals that the random variable since when .

**Question #12:** Let be independent random variables. Give examples of random variables defined in terms of the random variables such that we have , , , and .

* We have that , , , and . These random variables are constructed since if some , then . This implies that , so that . Finally, we note that where and that .

**Question #13:** If and are independent random variables such that their densities are , then find the joint density of and .

* By independence, if and . We have and so , which implies that so . Then . This allows us to compute the Jacobian as . Thus, the joint density is whenever we have that and zero otherwise.

**Question #14:** If are a random sample of size from the distribution, then a) find the joint density of the order statistics , b) find the marginal densities of and , and c) find the mean of the sample range .

1. We have if .
2. We have if and that whenever .
3. We have , so we compute and . Then, we have that . Alternatively, we could have found and computed as a double integral.