**Final – Practice Exam**

**Question #1:** If and denotes a random sample of size from the population with density function , then a) find the joint density of and , and b) find the marginal probability density function of .

1. By the independence of the two random variables and , we have that their joint density is given by whenever and . Then and implies that and , so we can compute the Jacobian as . This implies that the joint density of and is whenever and . This also shows and are independent.
2. We have if .

**Question #2:** Find the Maximum Likelihood Estimators (MLE) of and based on a random sample from where and .

* We first note that the likelihood function is given by . Then by inspection, we see that . To find the MLE of , we compute so that the derivative with respect to is . Since the second derivative is zero, we have found that .

**Question #3:** If is a random sample from , then a) find a UMVUE for when is known, and b) find a UMVUE for when is known.

1. We have previously shown that the Normal distribution is a member of the Regular Exponential Class (REC), where and are jointly complete sufficient statistics for the unknown parameters and . Since we have that , the statistic will be unbiased for the unknown parameter . The Lehmann-Scheffe Theorem then guarantees that is a UMVUE for when is known.
2. Since , the statistic will be unbiased for the unknown parameter when is known. The Lehmann-Scheffe Theorem states that is a UMVUE for .

**Question #4:** Consider a random sample from , where is unknown but is known. Find the constants depending on the values of and such that is a 90% confidence interval for .

* We first compute the CDF of the population whenever . Then we use this to compute the CDF of the largest order statistic . In order for to be a 90% confidence interval for , it must be the case that and . We solve each of these two equations for the unknown constants, so . Similarly, we find that .

**Question #5:** If and are independent and identically distributed from such that their density is , then find the joint density of and .

* We have whenever we have that and .

**Question #6:** Let be a random sample from with known such that their common density is . Find the MLE of .

* We have so that . Then we can calculate that the . Therefore, we have found .

**Question #7:** Use the Cramer-Rao Lower Bound to show that is a UMVUE for based on a random sample of size from an distribution where .

* We first find the Cramer-Rao Lower Bound; since , we have . Then we have so that and . Finally, implies that . To verify that is a UMVUE for , we first show that . Then we check that the variance achieves the Cramer-Rao Lower Bound from above, so . Thus, is a UMVUE for .

**Question #8:** Use the Lehmann-Scheffe Theorem to show that is a UMVUE for based on a random sample of size from an distribution where .

* We know that is a member of the Regular Exponential Class (REC) with , so the statistic is complete sufficient for the parameter . Then implies that the estimator is unbiased for , so the Lehmann-Scheffe Theorem guarantees that it will be the Uniform Minimum Variance Unbiased Estimator of .

**Question #9:** Find a confidence interval for based on a random sample of size from an distribution where . Use the facts that , that is a scale parameter, and that .

* Since is a scale parameter, we know that will be a pivotal quantity. Note that we used the fact that , which was derived in a previous exercise. We begin by noting that since each , we can conclude that . In order to obtain a chi-square distributed pivotal quantity, we use the modified pivot . To verify its distribution, we use the CDF technique so implies that . This proves that so is the desired confidence interval for the unknown parameter .

**Question #10:** Let be independent and identically distributed from the population with CDF given by . Find the limiting distribution of .

* We have that . Then is the limiting distribution since for all real numbers and .

**Question #11:** Let be independent and identically distributed from the population with a PDF given by . Approximate using .

* From the given density, we know that so and . Then if we define , we have that and . These facts allow us to compute .

**Question #12:** Suppose that is a random variable with density for . Compute the Moment Generating Function (MGF) of and use it to find .

* By definition, we have . After integrating and collecting like terms, we obtain . We then compute and so that .

**Question #13:** Let be a random variable with density . Compute the probability density function of the transformed random variable .

* We have so that if .

**Question #14:** Let be a random sample from the density whenever and zero otherwise. Find a complete sufficient statistic for the parameter .

* We begin by verifying that the density is a member of the Regular Exponential Class (REC) by showing that , where and . Then we know that the statistic is complete sufficient for the parameter .

**Question #15:** Let be a random sample from the density whenever and zero otherwise. Find the Maximum Likelihood Estimator (MLE) for .

* We have so that and . Thus, we have found that the Maximum Likelihood Estimator is .

**Question #16:** Let be a random sample from the density whenever and zero otherwise. Find the Method of Moments Estimator (MME) for .

* We have , so that so the desired estimator is .