**Chapter #8 – Limit Theorems**

**Question #5:** Fifty numbers are rounded off to the nearest integer and then summed. If the individual round off errors are uniformly distributed over $(-0.5,0.5)$, approximate the probability that the resultant sum differs from the exact sum by more than 3.

* If we let $X\_{i}$ denote the ith round off error, then $X\_{i}\~UNIF(-0.5,0.5)$ so that the density is $f\_{X\_{i}}\left(x\_{i}\right)=1$ whenever $x\in (-0.5,0.5)$ for $i=1,…,50$. We can then calculate that $E\left(X\_{i}\right)=\frac{0.5+(-0.5)}{2}=0$ and $Var\left(X\_{i}\right)=\frac{\left(0.5-\left(-0.5\right)\right)^{2}}{12}=\frac{1}{12}$. We then note that $X\_{1},…,X\_{50}$ are independent and let $X=\sum\_{i=1}^{50}X\_{i}$. The central limit theorem guarantees that $X$ will be approximately normally distributed with $μ=50\*0=0$ and $σ^{2}=50\*\frac{1}{12}=\frac{25}{6}$, so that $P\left(\left|X\right|>3\right)=2P\left(X>3\right)=2P\left(\frac{X-0}{\sqrt{25/6}}>\frac{3-0}{\sqrt{25/6}}\right)≈2P\left(Z>1.47\right)=0.1416$.

**Question #13:** Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64. (a) Approximate the probability that the average test score in the class of size 25 exceeds 80. (b) Repeat part (a) for the class of size 64. (c) Approximate the probability that the average test score in the larger class exceeds that of the other class by over 2.2 points. (d) Approximate the probability that the average test score in the smaller class exceeds that of the other class by over 2.2 points.

1. $P\left(\overbar{X}>80\right)=P\left(\frac{\overbar{X}-74}{\frac{14}{\sqrt{25}}}>\frac{80-74}{\frac{14}{\sqrt{25}}}\right)≈P\left(Z>2.14\right)≈0.0162$.
2. $P\left(\overbar{Y}>80\right)=P\left(\frac{\overbar{Y}-74}{\frac{14}{\sqrt{64}}}>\frac{80-74}{\frac{14}{\sqrt{64}}}\right)≈P\left(Z>3.43\right)≈0.0003$.
3. We have that the variance is given by $Var\left(\overbar{Y}-\overbar{X}\right)=Var\left(\overbar{Y}\right)+Var\left(\overbar{X}\right)=Var\left(\frac{1}{64}\sum\_{i=1}^{64}y\_{i}\right)+Var\left(\frac{1}{25}\sum\_{i=1}^{25}x\_{i}\right)=\frac{1}{64^{2}}Var\left(\sum\_{i=1}^{64}y\_{i}\right)+\frac{1}{25^{2}}Var\left(\sum\_{i=1}^{25}x\_{i}\right)=\frac{1}{64^{2}}\sum\_{i=1}^{64}Var(y\_{i})+\frac{1}{25^{2}}\sum\_{i=1}^{25}Var(x\_{i})=\frac{1}{64^{2}}\left(64\*14^{2}\right)+\frac{1}{25^{2}}\left(25\*14^{2}\right)=\frac{196}{64}+\frac{196}{25}$, so the standard deviation is therefore $sd\left(\overbar{Y}-\overbar{X}\right)=\sqrt{\frac{196}{64}+\frac{196}{25}}≈3.3$. This allows us to calculate $P\left(\overbar{Y}-\overbar{X}>2.2\right)=P\left(\frac{\overbar{Y}-\overbar{X}-0}{3.3}>\frac{2.2-0}{3.3}\right)≈P\left(Z>0.67\right)≈0.2514$.
4. $P\left(\overbar{X}-\overbar{Y}>2.2\right)=P\left(\frac{\overbar{X}-\overbar{Y}-0}{3.3}>\frac{2.2-0}{3.3}\right)≈P\left(Z>0.67\right)≈0.2514$.

**Question #15:** An insurance company has 10,000 automobile policyholders. The expected yearly claim per policyholder is $240, with a standard deviation of $800. Approximate the probability that the total yearly claim exceeds $2.7 million.

* $P\left(\sum\_{i=1}^{10,000}X\_{i}>2,700,000\right)≈P\left(Z>\frac{2,700,000-10,000\left(240\right)}{\sqrt{10,000\left(800^{2}\right)}}\right)=P\left(Z>3.75\right)≈0.0000$.

**Question #1:** The number of automobiles sold weekly at a certain dealership is a random variable with expected value 16. Give an upper bound to the probability that (a) next week’s sales exceed 18; (b) next week’s sales exceed 25.

1. Markov’s inequality states that if $X$ is a random variable that takes only nonnegative values, then for any value $a>0$ we have $P\left(X\geq a\right)\leq \frac{E\left(X\right)}{a}$. Here, we wish to find an upper bound for $P\left(X>18\right)=P\left(X\geq 19\right)\leq \frac{E\left(X\right)}{19}=\frac{16}{19}=0.84$.
2. Similarly, $P\left(X>25\right)=P\left(X\geq 26\right)\leq \frac{E\left(X\right)}{26}=\frac{16}{26}=0.62$.

**Question #2:** Suppose in Problem 1 that the variance of the number of automobiles sold weekly is 9. (a) Give a lower bound to the probability that next week’s sales are between 10 and 22, inclusively. (b) Give upper bound to the probability that next week’s sales exceed 18.

1. Chebyshev’s inequality states that if $X$ is a random variable with finite mean$ μ$ and variance $σ^{2}$, then for any value $k>0$ we have $P\left(\left|X-μ\right|\geq k\right)\leq \frac{σ^{2}}{k^{2}}$. We wish to find an upper bound for $P\left(10\leq X\leq 22\right)=P\left(\left|X-16\right|\leq 6\right)=1-P\left(\left|X-16\right|>6\right)\geq 1-\frac{9}{6^{2}}=1-\frac{9}{36}=\frac{27}{36}=\frac{3}{4}=0.75$.

1. Chebyshev’s one-sided inequality states that if $X$ is a random variable with mean 0 and finite variance $σ^{2}$, then for any $a>0$ we have $P\left(X\geq a\right)\leq \frac{σ^{2}}{σ^{2}+a^{2}}$. Here we wish to find $P\left(X>18\right)=P\left(X\geq 19\right)=P\left(X-16\geq 3\right)\leq \frac{9}{9+3^{2}}=\frac{9}{9+9}=\frac{9}{18}=\frac{1}{2}=0.5$.

**Question #3:** If $E[X]=75$, $E[Y]=75$, $Var[X]=10$, $Var[Y]=12$, $Cov(X,Y)=-3$, then give an upper bound to (a) $P\left(\left|X-Y\right|>15\right)$, (b) $P\left(X>Y+15\right)$ and (c) $P\left(Y>X+15\right)$.

1. First note that the expected value $E\left(X-Y\right)=E\left(X\right)-E\left(Y\right)=75-75=0$ while the variance $Var\left(X-Y\right)=Var\left(X\right)+Var\left(Y\right)-2Cov\left(X,Y\right)=10+12-2\left(-3\right)=28$. Using Chebyshev’s inequality, $P\left(\left|X-Y\right|>15\right)\leq \frac{σ^{2}}{15^{2}}=\frac{28}{225}=0.12$.
2. Using Chebyshev’s one-sided inequality, we have $P\left(X>Y+15\right)=P\left(X-Y>15\right)=\frac{σ^{2}}{σ^{2}+15^{2}}=\frac{28}{28+225}=\frac{28}{253}=0.11$.
3. $P\left(Y>X+15\right)=P\left(Y-X>15\right)=\frac{σ^{2}}{σ^{2}+15^{2}}=\frac{28}{28+225}=\frac{28}{253}=0.11$

**Question #4:** Suppose that the number of units produced daily at factory A is a random variable with mean 20 and standard deviation 3 and the number produced at factory B is a random variable with mean 18 and standard deviation 6. Assuming independence, derive an upper bound for the probability that more units are produced today at factory B than A.

* If $X$ is the number produced at factory A and $Y$ the number produced at factory B, then we have $E\left(Y-X\right)=E\left(Y\right)-E\left(X\right)=-2$ and $Var\left(Y-X\right)=Var\left(Y\right)+Var\left(X\right)=9+36=45$. Thus, $P\left(Y-X>0\right)=P\left(Y-X\geq 1\right)=P\left(Y-X+2\geq 3\right)\leq \frac{45}{45+9}=\frac{45}{54}$.

**Question #7:** The servicing of a machine requires two separate steps, with the time needed for the first step being an exponential random variable with mean 0.2 hour and the time for the second step being an independent exponential random variable with mean 0.3 hour. If a repair person has 20 machines to service, approximate the probability that all the work can be completed in 8 hours.

* If $X$ is the time required to service a machine, then $E\left(X\right)=0.2+0.3=0.5$. Also, since the variance of an exponential random variable is equal to the square of its mean, we have that $Var\left(X\right)=\left(0.2\right)^{2}+\left(0.3\right)^{2}=0.13$. If we then let $X\_{i}$ denote the time required to service job $i$ for $i=1,2,…,20$, then we have that $E\left(\sum\_{i=1}^{20}X\_{i}\right)=0.5\left(20\right)=10$ while $Var\left(\sum\_{i=1}^{20}X\_{i}\right)=0.13\left(20\right)=2.6$. Therefore, $P\left(\sum\_{i=1}^{20}X\_{i}>8\right)=P\left(\frac{\sum\_{i=1}^{20}X\_{i}-10}{\sqrt{2.6}}>\frac{8-10}{\sqrt{2.6}}\right)≈P\left(Z>-1.24\right)=1-Φ\left(-1.24\right)=0.1074$.

**Question #8:** On each bet, a gambler loses 1 with probability 0.7, loses 2 with probability 0.2, or wins 10 with probability 0.1. Approximate the probability that the gambler will be losing after his first 100 bets.

* If we let $X$ denote the gambler’s winnings on each bet, then we can calculate that $E\left(X\right)=-1\left(0.7\right)-2\left(0.2\right)+10\left(0.1\right)=-0.1$ and $Var\left(X\right)=E\left(X^{2}\right)-\left[E\left(X\right)\right]^{2}=\left(-1\right)^{2}\left(0.7\right)+\left(-2\right)^{2}\left(0.2\right)+\left(10\right)^{2}\left(0.1\right)-\left(-0.1\right)^{2}=11.5-0.01=11.49$. We then have $E\left(\sum\_{i=1}^{100}X\_{i}\right)=100\left(-0.1\right)=-10$ and $Var\left(\sum\_{i=1}^{100}X\_{i}\right)=100\left(11.49\right)=1149$, so $P\left(\sum\_{i=1}^{100}X\_{i}\leq -0.5\right)=P\left(\frac{\sum\_{i=1}^{100}X\_{i}-(-10)}{\sqrt{1149}}\leq \frac{-0.5-(-10)}{\sqrt{1149}}\right)≈P\left(Z\leq 0.28\right)=Φ\left(0.28\right)=0.61$.

**Question #10:** A tobacco company claims that the amount of nicotine in one of its cigarettes is a random variable with mean 2.2 mg and standard deviation 0.3 mg. However, the average nicotine content of 100 randomly chosen cigarettes was 3.1 mg. Approximate the probability that the average would have been as high as or higher than 3.1 if the claims were true.

* If the claim were true, then by the central limit theorem, the average nicotine content (call it $X$) would approximately have a normal distribution with mean 2.2 and standard deviation 0.03. Thus, the probability that it would be as high as 3.1 is given by $P\left(X>3.1\right)=P\left(\frac{X-2.2}{\sqrt{0.03}}>\frac{3.1-2.2}{\sqrt{0.03}}\right)≈P\left(Z>5.2\right)=1-Φ\left(5.2\right)≈0$.