**Chapter #7 – Properties of Expectation**

**Question #4:** Let when and and otherwise. (a) Verify that is a density function, (b) find , (c) find , (d) find , and (e) compute the covariance between and given by .

1. We must show that .
2. From Proposition 2.1, we know that if the random variables and have the joint density function , then . Here we are given , so we have that .
3. We are now given that , so we can calculate .
4. We have , so .
5. To derive a computational formula, we have . We therefore see that for these random variables .

**Question #5:** The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are to the point is . If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.

* We first let , so we would like to compute . Since the sides of the square have length 3, we have that and , which implies that and . Finally, note that . Therefore, we have that .

**Question #6:** A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.

* We have a sum of 10 independent random variables . We therefore have that . We could also have found this from noting that for all , so that .

**Question #9:** A total of balls, numbered 1 through , are put into urns, also numbered 1 through in such a way that ball is equally likely to go into any of the urns . Find (a) the expected number of urns that are empty; (b) probability none of the urns is empty.

1. Let be the number of empty urns and define an indicator variable if urn is empty and otherwise. We must find the expected value of this indicator random variable, which is simply the probability that it is one: . Since a ball can land in any of the urns with equal probability, the probability that the ith ball will not land in urn is . Similarly, the probability that the i + 1st ball will not land in urn is . Using this reasoning, we calculate that . We can then use this to calculate the .
2. For all of the urns to have at least one ball in them, the nth ball must be dropped into the nth urn, which occurs with probability . Similarly, the n – 1st ball must be dropped into the n – 1st urn, which has a probability of , and so on. We can therefore calculate that .

**Question #11:** Consider independent flips of a coin having probability of landing on heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if and the outcome is , then there are 3 changeovers. Find the expected number of changeovers. Hint: Express the number of changeovers as the sum of a total of Bernoulli random variables.

* Let if a changeover occurs on the ith flip and otherwise. Thus, whenever . If we let denote the number of changeovers, then we have that .

**Question #20:** In an urn containing balls, the ith ball has weight , . The balls are removed without replacement, one at a time, according to: At each selection, the probability that a given ball in the urn is chosen is equal to its weight divided by the sum of the weights remaining in the urn. For instance, if at some time is the set of balls remaining in the urn, then the next selection will be with probability , . Compute the expected number of balls that are withdrawn before ball number 1 is removed.

* Let if ball is removed before ball 1 and otherwise. Then we see that .

**Question #33:** If and , find (a) , and (b) .

1. We calculate that the , which can calculate using the identity that .
2. .

**Question #38:** If the joint density when and and otherwise, compute .

* Since , we must compute the following:
  + .
  + Since , we have by doing integration by parts with , so that , .
  + with and .
* Thus, .

**Question #41:** A pond contains 100 fish, of which 30 are carp. If 20 fish are caught, what are the mean and variance of the number of carp among the 20?

* Let denote the number of carp caught of the 20 fish caught. We assume that each of the ways to catch 20 fish from 100 are equally likely, so it is clear that . This implies that , while .

**Question #22:** Suppose that , and are independent Poisson random variables with respective parameters , and . Let and . Calculate (a) and , and (b) .

1. and since whenever a random variable , we have that .
2. since the three random variables are independent (implying that their covariance is zero) and .