**Chapter #6 – Jointly Distributed Random Variables**

**Question #1:** Two fair dice are rolled. Find the joint probability mass function of and when (a) is the largest value obtained on any die and is the sum of the values; (b) is the value on the first die and is the larger of the two values; (c) is the smallest and is the largest value obtained on the dice. (The answers are based on the following table. Also note that only the solution to part (a) is presented as parts (b) and (c) are solved similarly.)

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Sum of two dice** | | | | | | | |
| **Second Die** | **First Die** | | | | | | |
|  | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

1. If is the max on either die and is the sum of the dice, then the joint probability mass function is given in the following table.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **X** | | | | | | | | |
| **Y** |  | 1 | 2 | 3 | 4 | 5 | 6 |  |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1/36 | 0 | 0 | 0 | 0 | 0 | 1/36 |
| 3 | 0 | 2/36 | 0 | 0 | 0 | 0 | 2/36 |
| 4 | 0 | 1/36 | 2/36 | 0 | 0 | 0 | 3/36 |
| 5 | 0 | 0 | 2/36 | 2/36 | 0 | 0 | 4/36 |
| 6 | 0 | 0 | 1/36 | 2/36 | 2/36 | 0 | 5/36 |
| 7 | 0 | 0 | 0 | 2/36 | 2/36 | 2/36 | 6/36 |
| 8 | 0 | 0 | 0 | 1/36 | 2/36 | 2/36 | 5/36 |
| 9 | 0 | 0 | 0 | 0 | 2/36 | 2/36 | 4/36 |
| 10 | 0 | 0 | 0 | 0 | 1/36 | 2/36 | 3/36 |
| 11 | 0 | 0 | 0 | 0 | 0 | 2/36 | 2/36 |
| 12 | 0 | 0 | 0 | 0 | 0 | 1/36 | 1/36 |
|  | 1/36 | 3/36 | 5/36 | 7/36 | 9/36 | 11/36 |  |

Note that the marginal probability mass function for is given by summing the columns while the marginal mass function for is given by summing the rows.

**Question #2:** Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls (13 total). Let equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of (a),; (b) ,,.

1. We have that and . We then calculate , , and .
2. We have that , and . We can therefore calculate , , , , , , , .

**Question #9:** The joint probability density function of X and Y is given by the following where and . (a) Verify that this is indeed a joint density function. (b) Compute the probability density function of and . (c) Compute the . (d) Find the probability . (e) Find . (f) Find .

1. We must verify that the sum is .
2. We must find the following integral . Thus, where . Similarly, , .
3. We calculate that .
4. We have .
5. Since where , we have .
6. Since where , we have that .

**Question #13:** A man and a woman agree to meet at a certain location at about 12:30 PM. (a) If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1 PM, find the probability that the first to arrive waits no longer than 5 minutes. (b) What is the probability that the man arrives first?

1. If we let denote the arrival time of the man, then while the density is when and zero otherwise. Similarly, if we let denote the arrival time of the women, then and is when and zero otherwise. Since and are independent, we have that the joint density function , which implies that whenever we have that and zero otherwise. Therefore, the probability that the first to arrive waits no longer than 5 minutes is given by .
2. The probability that the man arrives first is .

**Question #17:** Three points , and are selected at random on a line L. What is the probability that lies between and ?

* The probability is since each of the points is equally likely to be the middle one.

**Question #21:** Let the joint density function where and while we let otherwise. (a) Show that is a joint probability density function. (b) Find the . (c) Find the .

1. We must verify that the sum .
2. We first find the marginal density for . We thus have . Then we have .
3. By the same reasoning as above, we find that .

**Question #29:** The gross weekly sales at a certain restaurant is a normal random variable with mean $2200 and standard deviation $230. What is the probability that (a) the total gross sales over the next 2 weeks exceeds $5000; (b) weekly sales exceed $2000 in at least 2 of the next 3 weeks? What independence assumptions have you made (independence)?

1. If we let be the random variable equal to the restaurant’s sales in week , then we see that . We can then define the random variable and by proposition 3.2, we have that or that . We therefore have that , where .
2. In any given week, we calculate that . The probability that weekly sales will exceed this amount in at least 2 of the next 3 weeks can be calculated with the binomial distribution where is the random variable equal to the number of weeks that sales exceed $2,000. We therefore have .

**Question #34:** Jay has two jobs to do, one after the other. Each attempt at job takes one hour and is successful with probability . If and , what is the probability that it will take Jay more than 12 hours to be successful on both jobs?

* Assume that Jay’s strategy is to attempt job 1 until he succeeds, and then move on to job 2 after. The probability that Jay never gets job 1 done in 12 attempts is given by . The probability that Jay fails job 1 in his first attempts, succeeding in the attempts and then never getting job 2 done is given by . Thus, the total probability is given by .

**Question #38:** Choose a number at random from the set of numbers {1, 2, 3, 4, 5}. Now choose a number at random from the subset no larger than , that is, from {1, . . . ,X}. Call this second number Y. (a) Find the joint mass function of and . (b) Find the conditional mass function of given that . Do it for i = 1, 2, 3, 4, 5. (c) Are and independent? Why?

1. We can represent the joint probability mass function of and either with a table or with an analytic formula. For the formula, we may write the joint PMF of these two random variables as . The table below presents this joint PMF of and also, where the row sums give , the probability mass function of , while the column sums give , the probability mass function of .

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **X** | | | | | | | |
| **Y** |  | 1 | 2 | 3 | 4 | 5 |  |
| 1 | 1/5 | 1/10 | 1/15 | 1/20 | 1/25 | 0.46 |
| 2 | 0 | 1/10 | 1/15 | 1/20 | 1/25 | 0.26 |
| 3 | 0 | 0 | 1/15 | 1/20 | 1/25 | 0.16 |
| 4 | 0 | 0 | 0 | 1/20 | 1/25 | 0.09 |
| 5 | 0 | 0 | 0 | 0 | 1/25 | 0.04 |
|  | 1/5 | 1/5 | 1/5 | 1/5 | 1/5 |  |

1. We have that . The construction of this PMF follows from the definition of conditional distributions.
2. No, because the distribution of depends on the value of .

**Question #41:** The joint density function of X and Y is given by whenever and otherwise. (a) Find the conditional density of , given , and that of , given . (b) Find the density function of .

1. We have that for and that for .
2. We calculate the CDF of as , and then differentiate to obtain , which implies that the density is given by for . (Or use the CDF technique.)

**Question #52:** Let and denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is whenever we have and otherwise. Find the joint density function of the polar coordinates and .

* We have and wish to find , where the random variables and have been transformed into and by the function and the function . From the change of variables formula, we know that and so we are able to uniquely solve for and in terms of and . The next step is to compute the Jacobian for this transformation, which is . We then use the formula to find that whenever , . Note that we could also compute the Jacobian as and just add .

**Question #55:** X and Y have joint density function when and (i.e. whenever ) and otherwise. (a) Compute the joint density function of and . (b) What are the marginal densities?

1. We have and wish to find , where the random variables and have been transformed into and by the function and the function . We can solve for and uniquely in terms of and and find that we have and . The next step is to compute the Jacobian, which is given by . We can then find that joint density is whenever and which reduces to and .
2. We have that whenever and that for and similarly for we have .

**Question #56:** If and are independent and identically distributed uniform random variables on (0, 1), compute the joint density functions of each of the following random variables (a) and ; (b) and ; (c) and .

1. We have and if and zero otherwise. The independence of and implies that their joint probability density function is if and . The transformations are then and , so implying that and that . This allows to find , so we can calculate . To find the bounds, we have so and so .
2. We have so and so , which allows us to calculate the Jacobian as . We then have that if and . Combining the bounds gives us that whenever .
3. We have and so and , which implies that the Jacobian is . Thus, whenever we have that the bounds are and .