**Chapter #3 – Conditional Probability**

**Question #1:** Two fair dice are rolled. What is the conditional probability that at least one lands on 6 (event A) given that the dice land on different numbers (event B)?

* $P\left(B\right)=\frac{P(A∩B)}{P(B)}=\frac{P\left\{first six, second not\right\}+P\{first not,second six\}}{P\{both different\}}=\frac{(5/36)+(5/36)}{1-(6/36)}=1/3$

**Question #13:** Suppose that an ordinary deck of 52 cards (which contains 4 aces) is randomly divided into 4 hands of 13 cards each. We are interested in determining $p$, the probability that each hand has an ace. Let $E\_{i}$ be the event that the $i^{th}$ hand has exactly one ace. Determine $p= P\left(E\_{1}∩E\_{2}∩E\_{3}∩E\_{4}\right)$ by using the multiplication rule.

* $P\left(E\_{1}\right)\*P\left(E\_{1}\right)\*P\left(E\_{1}∩E\_{2}\right)\*P\left(E\_{1}∩E\_{2}∩E\_{3}\right)=\frac{\left(\genfrac{}{}{0pt}{}{4}{1}\right)\left(\genfrac{}{}{0pt}{}{48}{12}\right)}{\left(\genfrac{}{}{0pt}{}{52}{13}\right)}\*\frac{\left(\genfrac{}{}{0pt}{}{3}{1}\right)\left(\genfrac{}{}{0pt}{}{36}{12}\right)}{\left(\genfrac{}{}{0pt}{}{39}{13}\right)}\*\frac{\left(\genfrac{}{}{0pt}{}{2}{1}\right)\left(\genfrac{}{}{0pt}{}{24}{12}\right)}{\left(\genfrac{}{}{0pt}{}{26}{13}\right)}\*1$

**Question #19:** A total of 48 percent of the women and 37 percent of the men that took a certain “quit smoking” class remained nonsmokers for at least one year after completing the class. These people then attended a success party at the end of a year. If 62 percent of the original class was male, (a) what percentage of those attending the party were women? (b) what percentage of the original class attended the party?

1. $P\left(A\right)=\frac{P(W∩A)}{P(A)}=\frac{P(A|W)\*P(W)}{P(A|W)\*P(W)+P(A|W^{c})\*P(W^{c})}=\frac{\left(0.48\right)\*\left(0.38\right)}{\left(0.48\right)\*\left(0.38\right)+\left(0.37\right)\*(0.62)}=0.44$
2. $P\left(A\right)=P\left(W\right)P\left(W\right)+P\left(W^{c}\right)P\left(W^{c}\right)=\left(0.48\right)\*\left(0.38\right)+\left(0.37\right)\*\left(0.62\right)=0.41$

**Question #25:** The following method was proposed to estimate the number of people over the age of 50 who reside in a town of known population 100,000: “As you walk along the streets, keep a running count of the percentage of people you encounter who are over 50. Do this for a few days; then multiply the percentage you obtain by 100,000 to obtain the estimate.” Comment on this method. Hint: Let $p$ denote the proportion of people in the town who are over 50. Furthermore, let $α\_{1}$ denote the proportion of time that a person under the age of 50 spends in the streets, and let $α\_{2}$ be the corresponding value for those over 50. What quantity does the method estimate? When is the estimate approximately equal to $p$?

* Let F denote the event that a person is over fifty and denote this probability by p which is also the number we desire to estimate. Let $α\_{1}$ denote the proportion of the time a person under fifty spends on the streets and $α\_{2}$ the same proportion for people over fifty. Let S denote the event that a person (of any age) is found in the streets. Then this event S can be decomposed into the sets where the person on the streets is less than or greater than fifty as $S=(S∩F)∪(S∩F^{c})$.
* $P\left(S\right)=\frac{P(S|F)\*P(F)}{P(S|F)\*P(F)+P(S|F^{c})\*P(F^{c})}=\frac{(α\_{2}\*p)}{(α\_{2}\*p)+(α\_{1}\*\left(1-p\right))}$
* When $α\_{1}=α\_{2}$, we have that $P\left(S\right)=p$.

**Question #31:** Ms. Aquina has just had a biopsy on a possibly cancerous tumor. Not wanting to spoil a weekend family event, she does not want to hear any bad news in the next few days. But if she tells the doctor to call only if the news is good, then if the doctor does not call, Ms. Aquina can conclude that the news is bad. So, being a student of probability, Ms. Aquina instructs the doctor to flip a coin. If it comes up heads, the doctor is to call if the news is good and not call if the news is bad. If the coin comes up tails, the doctor is not to call. In this way, even if the doctor doesn’t call, the news is not necessarily bad. Let $α$ be the probability that the tumor is cancerous; let $β$ be the conditional probability that the tumor is cancerous given that the doctor does not call. (a) Which should be larger, $α$ or $β$? (b) Find $β$ in terms of $α$, and prove your answer in part (a). Note: Let C be the event she has cancer and N be no call.

1. $β=P\left(N\right)=\frac{P(C∩N)}{P(N)}=\frac{P(N|C)\*P(C)}{P(N|C)\*P(C)+P(N|C^{c})\*P(C^{c})}=\frac{\left(1\right)\*(α)}{\left(1\right)\*\left(α\right)+\left(1/2\right)\*(1-α)}=\frac{α}{α/2+1/2}=\frac{2α}{α+1}$
2. We found that $β=\frac{2α}{α+1}\geq α \rightarrow β\geq α$.

**Question #37:** (a) A gambler has a fair coin and a two-headed coin in his pocket. He selects one of the coins at random; when he flips it, it shows heads. What is the probability that it is the fair coin? (b) Suppose that he flips the same coin a second time and, again, it shows heads. Now what is the probability that it is the fair coin? (c) Suppose that he flips the same coin a third time and it shows tails. Now what is the probability that it is the fair coin?

1. $P\left(H\right)=\frac{P(H|F)\*P(F)}{P(H|F)\*P(F)+P(H|F^{c})\*P(F^{c})}=\frac{\left(\frac{1}{2}\right)\*\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\*\left(\frac{1}{2}\right)+1\*(\frac{1}{2})}=\frac{1}{3}$
2. $P\left(H∩H\right)=\frac{P(H∩H|F)\*P(F)}{P(H∩H|F)\*P(F)+P(H∩H|F^{c})\*P(F^{c})}=\frac{\left(\frac{1}{2}\right)\*\left(\frac{1}{2}\right)\*\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\*\left(\frac{1}{2}\right)\*\left(\frac{1}{2}\right)+1\*(\frac{1}{2})}=\frac{1}{5}$
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**Question #55:** In a class, there are 4 freshman boys, 6 freshman girls, and 6 sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

* Let $n$ be the number of sophomore girls, F be the event freshman and B be the event boy. Then we know $P\left(F\right)=\frac{10}{16+n}$, $P\left(B\right)=\frac{10}{16+n}$ and $P\left(F∩B\right)=\frac{4}{16+n}$. Sex and class will be independent if and only if $P\left(F∩B\right)=P\left(F\right)\*P(B)$. Therefore, we must solve $\frac{4}{16+n}=\left(\frac{10}{16+n}\right)\*\left(\frac{10}{16+n}\right)$, which implies that $n=9$ is the correct answer.

**Question #61:** Genes relating to albinism are denoted by A and a. Only those people who receive the a gene from both parents will be albino. Persons having the gene pair A, a are normal in appearance and, because they can pass on the trait to their offspring, are called carriers. Suppose that a normal couple has two children, exactly one of whom is an albino. Suppose that the nonalbino child mates with a person who is known to be a carrier for albinism. (a) What is the probability that their first offspring is an albino? (b) What is the conditional probability that second offspring is an albino given that their firstborn is not?

1. Since the normal couple has one albino child, they are both carriers, having gene pair (A, a). Hence, the nonalbino child has 1/3 chance to have gene (A, A) and 2/3 chance to have gene (A, a). The (A, A) child has probability 0 to have an albino child and the (A, a) child will have 1/4 chance to get an albino child when he mates with a carrier. Therefore the first probability that the first offspring is an albino is given by the following: $\left(\frac{2}{3}\right)\*\left(\frac{1}{4}\right)=\frac{1}{6}.$
2. The probability of the first offspring is nonalbino is given by the following formula $P\left(first nonalbino\right)=1-P\left(first albino\right)=1-\frac{1}{6}=\frac{5}{6}$. The probability of the first one being nonalbino and the second one being albino is given by the following $P\left(first nonalbino and second albino\right)=P\left(first nonalbino\right)P\left(second albino\right)=\left(\frac{1}{2}\right)\*\left(\frac{1}{4}\right)=\frac{1}{8} . $Since we already know that they have one albino child, the gene of both parents are (A, a). Therefore, the probability of having a nonalbino child is 1/2 and the probability of having an albino child is 1/4. Thus, we can calculate that $P\left(second albino \right| first nonalbino)=\frac{P\left(first nonalbino and second albino\right)}{P\left(first nonalbino\right)}=\frac{1/8}{5/6}=\frac{3}{20}$.

**Question #73:** Suppose that each child born to a couple is equally likely to be a boy or a girl, independently of the sex distribution of the other children in the family. For a couple having 5 children, compute the probabilities of the following events: (a) All children are of the same sex. (b) The 3 eldest are boys and the others girls. (c) Exactly 3 are boys. (d) The 2 oldest are girls. (e) There is at least 1 girl.

1. $C\left(2,1\right)\*\left(\frac{1}{2}\right)^{5}= \frac{1}{16}$
2. $\left(\frac{1}{2}\right)^{2}\*\left(\frac{1}{2}\right)^{3}=\left(\frac{1}{2}\right)^{5}= \frac{1}{32}$
3. $C\left(5,3\right)\*\left(\frac{1}{2}\right)^{3}\*\left(\frac{1}{2}\right)^{2}=10\*\left(\frac{1}{32}\right)= \frac{5}{16}$
4. $\left(\frac{1}{2}\right)^{2}= \frac{1}{4}$
5. $1-\left(\frac{1}{2}\right)^{5}= \frac{31}{32}$

**Question #79:** In successive rolls of a pair of fair dice, what is the probability of getting 2 sevens before 6 even numbers?

* We know that $P\left(sum 7\right)=\frac{1}{6}$ and $P\left(sum even\right)=\frac{1}{2}$. We then say that each roll that is either a 7 or an even number will be 7 with probability $p=\frac{P(sum 7)}{P\left(sum 7\right)+P(sum even)}=\frac{1}{4}$. Therefore from Example 4i, $\sum\_{i=2}^{7}\left(\genfrac{}{}{0pt}{}{7}{i}\right)\left(\frac{1}{4}\right)^{i}\left(\frac{3}{4}\right)^{7-i}$ is the desired probabily.

**Question #84:** An urn contains 12 balls, of which 4 are white. Three players—A, B and C—successively draw from the urn, A first, then B, then C, then A, and so on. The winner is the first one to draw a white ball. Find the probability of winning for each player if (a) each ball is replaced after it is drawn; (b) the balls that are withdrawn are not replaced.