**Chapter #2 – Axioms of Probability**

**Question #1:** A box contains 3 marbles (1 red, 1 green and 1 blue). Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. (a) Describe the sample space of this experiment. (b) Repeat when the second marble is drawn without replacing the first marble.

1. $S=\{(R,R),(R,G),(R,B),(G,G),(G,R),(G,B),(B,B),(B,R),(B,G)\}$
2. $S=\{(R,G),(R,B),(G,R),(G,B),(B,R),(B,G)\}$

**Question #5:** A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x\_{1}, x\_{2},x\_{3},x\_{4}, x\_{5})$, where $x\_{i}$ is equal to 1 if component $i$ is working and is equal to 0 if component $i$ is failed. (a) How many outcomes are in the sample space of this experiment? (b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3 and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W. (c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A? (d) Write all the outcomes in the event $A∩W$.

1. $2\*2\*2\*2\*2=2^{5}=32$ (we have two choices, 1 or 0, for each five components)
2. $W=\{ \left(11000\right),\left(11001\right),\left(11010\right),\left(11100\right),\left(11011\right),\left(11110\right),\left(11101\right),\left(11111\right),$

 $\left(00110\right),\left(00111\right),\left(10110\right),\left(01110\right),\left(10111\right),\left(01111\right),\left(10101\right) \}$

 (8 vectors = 1,2 working, 6 vectors = 3,4 working and 1 vector = 1,3,5 working)

1. $2\*2\*2=2^{3}=8$ (we have two choices, 1 or 0, for each of the three remaining components unrelated to 4 and 5 which are both failed in event A)
2. $A=\{\left(00000\right),\left(10000\right),\left(01000\right),\left(00100\right),\left(11000\right),\left(01100\right),\left(10100\right),\left(11100\right)\}$

$$A∩W=\{\left(11000\right),(11100)\}$$

**Question #9:** A store accepts either American Express or VISA credit card. A total of 24% of its customers carry an American Express card, 61% carry a VISA card and 11% carry both. What percentage of its customers carry a credit card that the establishment will accept?

* Let $V$ be the event that the customer carries VISA and $A$ be the event that they carry American Express. We want to find $P(A∪V)$. The details of the problem reveal that $P\left(A\right)=0.24$, $P\left(V\right)=0.61$ and $P\left(A∩V\right)=0.11$. We can plug these into the formula $P\left(A∪V\right)=P\left(A\right)+P\left(V\right)-P\left(A∩V\right)=0.24+0.61-0.11=0.74$. Thus, it is 74%.

**Question #13:** A town with a population of 100,000 has three newspapers: I, II and III. The proportions of townspeople who read these papers are the following: (I – 10%) (II – 30%) (III – 5%) (I and II – 8%) (I and III – 2%) (II and III – 4%) (I and II and III – 1%). (a) Find the number of people who read only one newspaper. (b) How many people read at least two newspapers? (c) If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper? (d) How many people in the town do not read any newspapers at all? (e) How many people in the town read only one morning paper and one evening paper? Note: see physical copy for Venn diagram of this question.

1. $\left(0.01+0.19+0.00\right) \* 100,000= 20,000$
2. $\left(0.07+0.01+0.03+0.01\right) \* 100,000= 12,000$
3. $\left(0.07+0.03+0.01\right) \* 100,000= 11,000$
4. $\left(1-0.32\right) \* 100,000= 68,000$
5. $\left(0.07+0.03\right) \* 100,000= 10,000$

**Question #17:** If 8 rooks (castles) are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others. That is, compute the probability that no row or file contains more than one rook.

* ${(64\*49\*36\*25\*16\*9\*4\*1)}/{(64\*63\*62\*61\*60\*59\*58\*57)}=0.37$ (the denominator the total ordered ways to arrange the rooks; the numerator is a sequential choice for each rook, beginning with 64 which gets reduced to the next lowest square after placing each rook; imagine placing each rook in the bottom-left corner of the board each time, so that after each rook is placed a new, smaller board is then available for the following rook to be placed)
* $\frac{8!}{C\left(64,8\right)}=0.37$ (alternative solution where order doesn’t matter so use combinations)

**Question #21:** A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have four children, and 1 has five children. (a) If one of these families is chosen at random, what is the probability it has $i $children, where $i=1, 2, 3, 4, 5$? (b) If one of the children is randomly chosen, what is the probability that child comes from a family having $i$ children, where $i=1, 2, 3, 4, 5$? Note: there are 48 children.

1. $P\left(1\right)=\frac{4}{20}, P\left(2\right)=\frac{8}{20}, P\left(3\right)=\frac{5}{20}, P\left(4\right)=\frac{2}{20}, P\left(5\right)=\frac{1}{20}$
2. $P\left(1\right)=\frac{4}{48}, P\left(2\right)=\frac{16}{48}, P\left(3\right)=\frac{15}{48}, P\left(4\right)=\frac{8}{48}, P\left(5\right)=\frac{5}{48}$

**Question #25:** A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first. Hint: Let $E\_{n}$ denote the event that a 5 occurs on the $n^{th}$ roll and no 5 or 7 occurs on the first $n-1$ rolls. Compute $P(E\_{n})$ and argue that $\sum\_{i=1}^{\infty }P(E\_{n})$ is the probability.

* $P\left(E\_{n}\right)=\left(\frac{26}{36}\right)^{n-1}\left(\frac{4}{36}\right)=\left(\frac{13}{18}\right)^{n-1}\left(\frac{1}{9}\right)$ (after drawing a 6 by 6 grid of possible outcomes, we see that there are six outcomes where the sum is 7 and four outcomes where the sum is 5; thus there are ten outcomes where the sum is 5 or 7; the 26/36 represents the probability that the sum is something other than 5 or 7; we raise to the $(n-1)$ power since the event $E\_{n}$ specifies no 5 or 7 in the first $(n-1)$ rolls; the 4/36 represents the probability that the sum is 5 on the $n^{th}$ roll)
* $\sum\_{n=1}^{\infty }P\left(E\_{n}\right)=\sum\_{n=1}^{\infty }\left(\frac{13}{18}\right)^{n-1}\left(\frac{1}{9}\right)=\frac{1}{9}\sum\_{n=1}^{\infty }\left(\frac{13}{18}\right)^{n-1}=\frac{1}{9}\*\left(\frac{1}{1-\frac{13}{18}}\right)=\frac{2}{5}$ (this is a geometric series with $r=\frac{13}{18}<1$, so we use the formula $\left(\frac{1}{1-r}\right)$ to calculate the infinite sum to obtain the probability that a 5 will occur first before a 7 occurs)

**Question #29:** An urn contains $n$ white and $m$ black balls, where $n$ and $m$ are positive numbers. (a) If two balls are randomly withdrawn, what is the probability that they are the same color? (b) If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color? (c) Show that the probability in part (b) is always larger than the one in part (a).

1. $\frac{\left(\genfrac{}{}{0pt}{}{n}{2}\right)+\left(\genfrac{}{}{0pt}{}{m}{2}\right)}{\left(\genfrac{}{}{0pt}{}{n+m}{2}\right)}=\frac{n^{2}-n+m^{2}-m}{\left(m+n\right)^{2}-(m+n)}$ (the work for the simplification has been omitted here)
2. $\frac{n^{2}+m^{2}}{\left(n+m\right)^{2}}$ (we use the basic principle of counting and recall that there is replacement)
3. We must prove $\frac{n^{2}+m^{2}}{\left(m+n\right)^{2}}>\frac{n^{2}-n+m^{2}-m}{\left(m+n\right)^{2}-(m+n)}$ (get least common denominator and simplify)

**Question #33:** A forest contains 20 elk, of which 5 are captured, tagged, and then released. A certain time later, 4 of the 20 elk are captured. What is the probability that 2 of these 4 have been tagged? What assumptions are you making?

* $\frac{C\left(5,2\right)\*C(15,2)}{C(20,4)}$ (the total number of ways to capture 4 of the 20 elk is $C(20,4)$, the number of ways to get 2 of the 5 tagged elk is $C(5,2)$ and the remaining 2 elk in the group of four must be selected from the 15 that aren’t tagged, so this is $C(15,2)$ ways)

**Question #37:** An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly (a) all 5 problems? (b) at least 4 of the problems?

1. $\frac{C\left(7,5\right)}{C(10,5)}=1/12$ (the $C\left(7,5\right)$ is the number of ways to choose 5 problems from the 7 that the student knows while the $C\left(10,5\right)$ the number of ways to choose 5 problems)
2. $\frac{C\left(7,4\right)\*C\left(3,1\right)+C\left(7,5\right)}{C(10,5)}=1/2$ (the $C\left(7,4\right)\*C\left(3,1\right)$ is the number of ways to get 4 questions correct and 1 incorrect while the $C\left(7,5\right)$ is the number of ways to get all five questions correct; we add these in the numerator because the probability of getting at least four correct is the probability of getting 4 correct or the probability of getting 5 correct; recall that with “or” we add while with “and” we multiply)

**Question #41:** If a die is rolled 4 times, what is the probability that 6 comes up at least once?

* $1-(5^{4}/6^{4})$ (the $5^{4}$ is the number of ways to not roll a 6 in four rolls of the die)

**Question #45:** A woman has $n$ keys, of which one will open her door. (a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her $k^{th}$ try? (b) What if she does not discard previously tried keys?

1. $\frac{\left(n-1\right)\left(n-2\right)…(n-k+1)}{n\left(n-1\right)\left(n-2\right)…(n-k+1)}=\frac{1}{n}$ (in the numerator, we have $(n-1)$ choices of wrong keys to choose from and after each one doesn’t work, there is one less wrong key to choose so we keep doing this until the key right before the $k^{th}$ key; in the denominator, we simply list all the possible choices at each step included the right key)
2. $\frac{(n-1)^{k-1}}{n^{k}}$ (in the numerator, we have $(n-1)$ choices of the wrong key each time (until the kth time when there is only one option to choose the right key); in the denominator, we have $n$ choices each time).

**Question #49:** A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

* $\frac{C\left(6,3\right)\*C(6,3)}{C(12,6)}$ (the $C(12,6)$ is the total number of ways to choose six people from the group of 12 people while the $C(6,3)$ is the number of ways to choose 3 men from the group of 6; we do it twice because it’s also how we choose 3 women from group of 6)