**Chapter #1 – Combinatorial Analysis**

**Question #1:** (a) How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers? (b) Repeat this question under the assumption that no letter or number can be repeated in a single license plate.

1. $26\*26\*10\*10\*10\*10\*10=67,600,000 $
2. $26\*25\*10\*9\*8\*7\*6=19,656,000$

**Question #4:** John, Jim, Jay and Jack have formed a band with 4 instruments. (a) If each of the boys can play all 4 instruments, how many different arrangements are possible? (b) What if John and Jim can play all 4 instruments but Jay and Jack can each play only piano, drums?

1. $4\*3\*2\*1=24$ (order of boys doesn’t matter)
2. $2\*1\*2\*1=4$ (order is Jay, Jack, John and Jim)

**Question #7:** (a) In how many ways can 3 boys and 3 girls sit in a row? (b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are to sit together? (c) In how many ways if only the boys must sit together? (d) In how many ways if no two people of the same sex are allowed to sit together?

1. $6!=6\*5\*4\*3\*2\*1=720$ (since there are six people in total)
2. $\left(2!\right)\*\left(3!\right)\*\left(3!\right)=2\*6\*6=72$ (the 2! is for whether the boys or girls are first)
3. $\left(4!\right)\*\left(3!\right)=24\*6=144$ (the 4! is because we now have four objects to arrange, one set of boys and three girls, while the 3! is the arrangements of the boys within the set)
4. $\left(2!\right)\*\left(3!\right)\*\left(3!\right)=72$ (the 2! is for whether we start with a boy or a girl)

**Question #10:** In how many ways can 8 people be seated in a row if (a) there are no restrictions on the seating arrangement? (b) the persons A and B must sit next to each other? (c) there are 4 men and 4 women and no 2 men or 2 women can sit together? (d) there are 5 men and they must sit next to each other? (e) there are 4 married couples and each couple must sit together?

1. $8!=40,320$ (since there are eight people in total)
2. $\left(2!\right)\*\left(7!\right)=10,080$ (the 2! is the arrangements for persons A and B, while the 7! is because we now have seven objects to arrange since A, B are now one object)
3. $\left(2!\right)\*\left(4!\right)\*\left(4!\right)=1,152$ (the 2! is for whether we start with a man or woman)
4. $\left(5!\right)\*\left(4!\right)=2,880$ (the 5! is the arrangements for the five men, while the 4! is because we now have four objects to arrange, the set of men and three remaining women)
5. $\left(2!\right)\*\left(2!\right)\*\left(2!\right)\*\left(2!\right)\*\left(4!\right)=(2!)^{4}\*\left(4!\right)=384$ (the 2!s are the arrangements of each of the couples, while the 4! Is because we now have four different objects to arrange, one for each couple that must sit together)

**Question #13:** Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

* The question is in how many ways we can choose two people from the twenty available to shake hands (without double-counting). This is a combination problem, so the answer is $C\left(n,k\right)=C\left(20,2\right)=(20!)/(2!)(18!)=(20\*19)/2=190$.

**Question #16:** A student has to sell 2 books from a collection of 6 math, 7 science and 4 economics books. How many choices are possible if (a) both books are to be on the same subject? (b) the books are to be on different subjects?

1. $C\left(6,2\right)+C\left(7,2\right)+C\left(4,2\right)=15+21+6=42$ (we use combinations within each subject because Book #1 plus Book #2 is the same as Book #2 plus Book #1 and add them up to get the total number of choices which can be sold; in the answer, the subjects are ordered as math first, science second and economics third)
2. $\left(6\*7\right)+\left(6\*4\right)+\left(7\*4\right)=42+24+28=94$ (in the answer, the ordering is math/science, math/economics and science/economics; we use the principle of counting for each of the first two choices of the textbooks to sold and add them up)

**Question #19:** From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if (a) there are no restrictions on the committees? (b) 2 of the men refuse to serve together? (c) 2 of the women refuse to serve together? (d) 1 man and 1 woman refuse to serve together?

1. $C\left(8,3\right)\*C\left(6,3\right)=56\*20=1,120$ (use the generalized principle of counting for the two procedures of choosing 3 members from 8 women and 3 members from 6 men)
2. $C\left(8,3\right)\*\left[C\left(6,3\right)-C\left(4,1\right)\right]=56\*\left(20-4\right)=896$ (the choice of 3 members from 8 women remains the same while the choice of 3 members from 6 men is reduced by $C\left(4,1\right)$, which is the number of ways that both men are on the committee; the reason this is so is because we force the two men to be on the committee and must choose one more man from the remaining four to round out the group of 3 male members)
3. $C\left(6,3\right)\*\left[C\left(8,3\right)-C\left(6,1\right)\right]=20\*\left(56-6\right)=1,000$ (the choice of 3 members from 6 men remains the same while the choice of 3 members from 8 women is reduced by $C\left(6,1\right)$, which is the number of ways both women are on the committee; the reasoning for this reduction is the same as for the two men who refused to work together)
4. $\left[C\left(8,3\right)\*C\left(6,3\right)\right]-\left[C\left(7,2\right)\*C\left(5,2\right)\right]=1,120-210=910$ (the first part is without restrictions while the second part is all the groups with the man and woman chosen)

**Question #21:** Consider a 4 by 5 grid of points (see book). Suppose that, starting at the point labeled A (bottom left), you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B (top right) is reached. How many different paths from A to B are there? To reach B from A, you must take 4 steps right and 3 steps up.

* $C\left(4+3,3\right)=C\left(7,3\right)=C\left(7,4\right)=(7!)/(3!)(4!)=35$ (we choose a path of 3 steps up from a total of 7 that must be completed to get from A to B, which is the same as choosing a path of 4 steps right from the total of 7 that must be completed)
* An alternative solution is to imagine that you have a bag with four “right” arrows and three “up” arrows and seven open slots to fill. Thus, you can begin with the right arrows and obtain $C\left(7,4\right)\*C\left(3,3\right)=C\left(7,4\right)\*1=C(7,4)$ or you can begin with the up arrows and obtain $C\left(7,3\right)\*C\left(4,4\right)=C\left(7,3\right)\*1=C(7,3$).

**Question #22:** In Problem 21, how many different paths are there from A to B that go through point X which is circled in the following lattice (2 steps right and 2 steps up)?

* $C\left(4,2\right)\*C\left(3,2\right)=6\*3=18$ (use the basic principle of counting by breaking the journey up into two parts; the first is from A to X which can be done in six ways while the second is from X to B which can be done in 3 ways)
* As above, we have $\left[C\left(4,2\right)\*C\left(2,2\right)\right]\*\left[C\left(3,2\right)\*C\left(1,1\right)\right]=C\left(4,2\right)\*C\left(3,2\right)=18$.

**Question #25:** The game of bridge is played by 4 players, each of whom is dealt 13 cards from a deck of 52 total cards. How many bridge deals are possible?

* $C\left(52,13\right)\*C\left(39,13\right)\*C\left(26,13\right)\*C\left(13,13\right)=52!/(13!)^{4}$ (we must choose 13 cards from the total of 52 for the first player, 13 cards from the remaining 39 for the second player and so on; by the generalized counting principle, we multiply these together)
* $C\left(52, \left(13,13,13,13\right)\right)=52!/(13!)^{4}$ (this is a special formula from section five in this chapter which gives an equivalent solution to the one above)

**Question #28:** (a) If 8 new teachers are to be divided among 4 schools, how many divisions are possible? (b) What if each school must receive 2 teachers?

1. $4\*4\*4\*4\*4\*4\*4\*4=4^{8}=65,536$ (since each of the eight teachers can be assigned to any of the four schools, there are 4 choices for the first teacher, 4 choices for the second teacher and so on; we multiply by the generalized counting principle)
2. $C\left(8,2\right)\*C\left(6,2\right)\*C\left(4,2\right)\*C\left(2,2\right)=8!/\left(2!\right)^{4}=2,520$ (like the bridge example, we must choose 2 teachers from the total of eight for the first school, 2 teachers from the remaining 6 for the second school and so on; we multiply these choices together)