THREE DIMENSIONAL FC ARTIN GROUPS ARE CAT(0)

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Abstract. Following earlier work of T. Brady, we construct locally CAT(0) classifying spaces for those Artin groups which are three dimensional and which satisfy the FC (flag complex) condition. The approach is to verify the "link condition" by applying gluing arguments for CAT(1) spaces and by using curvature testing techniques as suggested by the work of M. Elder and J. McCammond.

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1. Introduction

Geometric group theory is concerned with the study of groups via their actions on geometric or topological spaces. Of particular interest is the class of groups which act on "non-positively curved metric spaces".

A.D. Aleksandrov [A] formulated the following definition of a non-positively curved metric space: Let (X, d) be a *geodesic metric space*, i.e. a metric space with the property that any two points may be joined by a length minimizing path. Such a path is called a geodesic segment. By analogy with the Euclidean and hyperbolic planes, non-positive curvature is characterized by a thin triangle condition. A triangle in X is set of three geodesic segments joining three distinct points; a comparison triangle is a Euclidean triangle with the same edge lengths. If a triangle Δ in X is such that for every pair of points $p, q \in \Delta$ the following inequality holds:

$$d(p,q) \leq |\bar{p} - \bar{q}|, \tag{1}$$

where $\bar{p}$ and $\bar{q}$ denote the corresponding points of in a comparison triangle $\bar{\Delta} \subset \mathbb{E}^2$, then we say that Δ satisfies the *CAT(0) inequality*. (These letters honor the geometers Cartan, Aleksandrov, and Toponogov who first studied the curvature of Riemannian manifolds in terms of comparison triangles.) If every triangle in X satisfies the CAT(0) inequaltiy, then X is a *CAT(0) space*. A CAT(0) space should be thought of as a metric space with *global* non-positive curvature. If X is locally CAT(0), then X is said to have curvature ≤ 0.

The class of groups which act *geometrically* (properly discontinuously, cocompactly, and by isometries) on a CAT(0) space share a number of notable group theoretic properties. For instance, these groups are finitely presented, have solvable word and conjugacy problems, and every solvable subgroup is virtually abelian. We will call such a group a *CAT(0) group*. (The book by M. Bridson and A. Haefliger [BH] is an excellent reference.)

In spite of their well-understood properties, it remains a difficult problem to construct interesting examples of CAT(0) groups, especially if the group is expected to act on a CAT(0) space (typically a piecewise Euclidian polyhedral cell complex) of dimension ≥ 3. The purpose of the present article is to provide interesting examples of CAT(0) groups in dimension three and to demonstrate the effectiveness of the "curvature testing" techniques for polyhedral cell complexes proposed by

M. Elder and J. McCammond [EM]. As per the title, we will show that certain "three dimensional" Artin groups are CAT(0).

In general, it is unknown whether or not every Artin group acts geometrically on a CAT(0) space. The answer is not even known for the braid groups on more than four strings. An affirmative answer to the CAT(0) question would give a geometric proof of a number of group-theoretic properties which conjecturally hold for all Artin groups, including solvable word and conjugacy problems.

There are, however, some partial answers to the CAT(0) question. R. Charney and M. Davis [CD] have shown that each Artin group acts geometrically on its "Salvetti complex". This is a piecewise Euclidean cube complex, which is CAT(0) if and only if the Artin group is "right-angled".

More recently, T. Brady and J. McCammond [BM] approached the CAT(0) problem by finding new presentations for "two dimensional" Artin groups. They showed that many of the associated presentation 2-complexes admit locally CAT(0) metrics. It follows from the Cartan-Hadamard theorem for locally CAT(0) spaces that the universal cover of such a complex is (globally) CAT(0); so, the fundamental group is acting geometrically on a CAT(0) space via deck transformations. So, these two-dimensional Artin groups are CAT(0) groups.

T. Brady [Br1] continued this line of investigation for the finite type Artin groups with three generators. These are the "three dimensional" Artin groups whose associated Coxeter group is an essential finite reflection group on $\mathbb{R}^3$; there are precisely three such Coxeter groups which do not split as a direct product, namely the symmetry groups of the tetrahedron, the cube, and the dodecahedron. To each such Artin group, G, Brady associated a three dimensional, connected, piecewise Euclidean complex K with a single vertex v_0 such that $\pi_1(K, v_0) \cong G$. He then showed that K is a locally CAT(0) space by cleverly verifying the "link condition", i.e. that the geometric link of v_0 is a CAT(1) space. (A geodesic metric space is *CAT(1)* if every triangle of perimeter $< 2\pi$ satisfies the inequality in Equation 1 with respect to a spherical comparison triangle.) It follows that the universal covering space of K is CAT(0) and that G acts geometrically via deck transformations. As CAT(0) spaces are contractible, he has also shown that K is a non-positively curved $K(G, 1)$. Inspired by this last result, we will prove the following:

Main Theorem. *Every three dimensional FC Artin group is CAT(0).*

The complex we will consider is an amalgamation of the spaces considered by Brady. However, unlike Brady's complex, verifying the link condition does not reduce to 1-dimensional computation. In Brady's case, the link is the (spherical) suspension of a 1-complex. Thus, it sufficed to check that a certain 1-complex was CAT(1). (This is essentially a combinatorial condition.) However, in the complexes we will consider, the link is not a suspension. The difficulty, then, is to check that a given piecewise spherical 2-complex is CAT(1). With the exceptions of Gromov's "all-right" criterion for all-right piecewise spherical complexes [G] and Moussong's Lemma for piecewise spherical complexes with polyhedral cells of with edge lengths $\geq \pi/2$ [M], there are no known combinatorial characterizations of CAT(1) 2-complexes. We will overcome this difficulty by using gluing arguments for CAT(1) spaces and by using the curvature testing techniques of [EM]. When combined with some deep results of B. Bowditch on locally CAT(1) spaces [Bow], we will demonstate that curvature testing is an effective way to study piecewise spherical 2-complexes.

The construction of the complex we will consider is closely related to the structure of special subgroups in Coxeter groups. Thus, we begin with an overview of Artin groups and their associated Coxeter groups.

2. Artin groups and Coxeter groups

Definition. Let S be a finite set of cardinality n. A *Coxeter matrix* for S is an $n \times n$ symmetric matrix with entries $m_{ij} \in \{1, 2, \ldots, \infty\}$ such that $m_{ij} = 1 \iff i = j$. Fix a Coxeter matrix M, and let A be the group given by the following presentation:

$$A = \langle S \mid \langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}} \rangle,$$

where $\langle s_i, s_j \rangle^{m_{ij}}$ means the string $s_i s_j s_i \cdots$ having m_{ij} letters when $m_{ij} < \infty$. Such a relation will be called an *Artin relation* of length m_{ij}. If $m_{ij} = \infty$, the relation $\langle s_i, s_j \rangle^{m_{ij}} = \langle s_j, s_i \rangle^{m_{ij}}$ is omitted from the presentation. The pair (A, S) is called an *Artin system* and the group A is called an *Artin group.*

Similarly, we can define a group W by the following presentation:

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where, again, we omit the relation if $m_{ij} = \infty$. The pair (W, S) is called a *Coxeter system*, and the group W is called a *Coxeter group.* We will refer to (W, S) as the Coxeter system associated to (A, S).

If the associated Coxeter group W is finite, we say that the Artin system is *spherical.* Likewise, a *spherical* Coxeter system is one in which the Coxeter group is a finite group. If the associated Coxeter group is infinite, then the Artin or Coxeter system is of *infinite type.*

(Note that it is common in the literature to use the term "finite type Artin group" instead of "spherical Artin group".)

Example. Let M be the $n \times n$ Coxeter matrix with entries $m_{ij} = 2$ if $|i - j| > 2$, $m_{i,i+1} = 3$, and $m_{ii} = 1$. Then the presentation for A is exactly the usual presentation of the braid group on $n + 1$ strings. The generator s_i represents the braid which crosses the $(i+1)$-st string over the i-th string and which leaves the other $n-1$ strings fixed. The braid relations are exactly those appearing in the presentation for A, namely s_i and s_j commute if $|i - j| > 2$ and s_i and s_{i+1} satisfy the relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. The associated Coxeter group W is the symmetric group on $n + 1$ letters. As the symmetric group is a finite group, the braid group is an example of a spherical Artin group.

The associated Coxeter group W is naturally a quotient of its Artin group. There is a surjective homomorphism $\pi : A \to W$ sending each generator $s \in S \subset A$ to the generator denoted by the same letter in W. The kernel of this map is referred to as the *pure Artin group.* In the example above, the kernel of the map from Braid(n+1) to Sym(n+1) is the pure braid group— those braids whose strings begin and end at the same node.

Definition. Given a (possibly empty) subset $T \subseteq S$. Define A_T and W_T to be the subgroups of A and, respectively, W generated by T. These subgroups are called *special subgroups.* The spherical special subgroups will be called, simply, *spherical subgroups.* They are indexed by the *spherical subsets* of S:

$$\mathcal{S} = \{\, T \subseteq S \mid W_T \text{ is finite} \,\}.$$

For any subset $T \subseteq S$, one can define a new Artin system, $(A(T), T)$, or a new Coxeter system, $(W(T), T)$, by forming the Coxeter matrix M_T whose entries are those entries of M indexed by pairs $(i, j) \in T \times T$. There are obvious homomorphisms $A(T) \to A_T$ and $W(T) \to W_T$. In fact, these maps are isomorphisms; and, for $T, T' \in \mathcal{S}$, $A_T \cap A_{T'} = A_{T \cap T'}$ and $W_T \cap W_{T'} = W_{T \cap T'}$. The proofs for Coxeter groups can be found in Bourbaki [Bo], while the proofs for Artin groups appear in van der Lek's Ph.D. thesis [L].

Definition. There is another, more visual, way to specify an Artin system or Coxeter system: let S be a set of cardinality n and let M be a Coxeter matrix for S. Let $\mathcal{G}$ be the labeled graph with vertex set S having a single edge labeled m_{ij} joining s_i to s_j whenever $1 < m_{ij} < \infty$. The graph $\mathcal{G}$ is called a *Coxeter graph.* The Coxeter graph contains precisely the same information as a Coxeter matrix.

Remark. Note that what we call a Coxeter graph is not the same as the "Dynkin diagrams" encountered in the study of Lie algebras. In Lie theory, the graph has, again, S as its vertex set; but its edges join those vertices for which $m_{ij} > 2$. Such an edge is labeled by m_{ij} if $m_{ij} > 3$ and no label is given if $m_{ij} = 3$.

Definition. Let Γ have vertex set S. Say that a nonempty set of vertices $T \subseteq S$ spans a simplex in Γ whenever $T \in \mathcal{S}$. We will refer to Γ as the *link complex* of the Artin or Coxeter system. As the only two generator spherical Coxeter groups are the finite dihedral groups ($m_{ij} < \infty$), the graph $\mathcal{G}$ (without labels) is precisely the 1-skeleton of Γ.

Definition. The *dimension* of an Artin system (A, S) is $\max\{|T| : T \in \mathcal{S}\}$. It follows that $\dim\ (A, S) = \dim\ \Gamma + 1$, where Γ is the link complex of (A, S). When the context is clear, we say that $\dim(A, S)$ is the dimension of the Artin group.

Remark. It is conjectured and, in many cases, known that the dimension of an Artin group is the same as its cohomological dimension [CD]. To each Artin system, there is an associated complexified hyperplane complement Q. The Artin group acts freely on the universal cover $\widetilde{Q}$, and the quotient space is conjectured to be a $K(A, 1)$ space. Those familiar with this work will recognize the cone on the link complex as the fundamental chamber of the *Deligne complex*— a piecewise Euclidean cell complex which is homotopy equivalent to $\widetilde{Q}$. For many Artin groups, it is known that the Deligne complex is CAT(0). However, this does not answer the CAT(0) question for Artin groups— the groups do not act properly on this complex.

Definition. An Artin system is said to satisfy the *FC condition* if for each $T \subseteq S$ we have $T \in \mathcal{S} \iff m_{ij} < \infty$ for all $s_i, s_j \in T$. This is equivalent to the requirement that the link complex Γ be a flag complex (FC), i.e. a subset $T \subseteq S$ spans a simplex of Γ if and only if every distinct pair of vertices $s_i, s_j \in T$ spans an edge. When the context is clear, we say that the group is FC.

Example. Let (A, S) be the Artin system with $S = \{s_1, s_2, s_3\}$ and with $m_{ij} = 3$ for $i \neq j$. Geometrically, the associated Coxeter group, W, can be realized as a subgroup of isometries of the Euclidean plane generated by affine reflections across three lines which meet pairwise, forming an equilateral triangle. The product of two such reflections is a rotation by $2\pi/3$. So, each special subgroup with two generators is a dihedral group of order six. But, the group W is not finite— the W-orbit of any equilateral triangle covers the entire plane. So, in terms of the complex Γ, we have that each distinct pair $\{s_i, s_j\}$ spans a simplex; but $\{s_1, s_2, s_3\}$ does not— the subgroup, namely all of W, is not finite. Thus, (A, S) is not FC.

We briefly recall some of the theory of Coxeter groups. The reader is referred to the books by N. Bourbaki [Bo], K. Brown [Bro], and J. Humphreys [H].

Definition. Let (W, S) be a Coxeter system. The *reflections* of (W, S) are the elements of the set $R = \{\, wsw^{-1} \in W \mid w \in W, s \in S \,\}$. Given $1 \neq w \in W$, define its *reflection length* or simply its *length*, denoted by $\ell(w)$, to be the smallest $k \in \{1, 2, \dots\}$ such that $w = r_1 \dots r_k$, where each r_i is a reflection. By convention, $\ell(1) = 0$.

Similarly, given $T \subset S$, we define $R_T := \{\, wtw^{-1} \in W \mid w \in W_T, t \in T \,\}$. In fact, $R \cap W_T = R_T$. If $w \in W_T$, we denote its reflection length with respect to R_T by $\ell_T(w)$. Let $\mathcal{R}$ denote the union of all R_T such that $T \in \mathcal{S}$. We refer to these reflections as the *reflections of spherical type.*

Remark. In the standard references on Coxeter groups, $\ell(w)$ denotes the length of w with respect to the Coxeter generating set S. As we will have no need to use this length function, there should be no confusion.

The term "reflection" is justified by the following fundamental theorem on Coxeter groups.

Theorem 2.1. *(Geometric Representation) Let (W, S) be a Coxeter system. Let V be a vector space of dimension $|S|$. Then there is a canonical faithful linear representation $\sigma : W \to GL(V)$.*

A proof of this theorem can be found in any of the references on Coxeter groups. As in [Bo], the geometric representation is used to study the relationship between W and its special subgroups W_T. Passing to the contragredient representation of σ, W acts on the dual space V^*. There is a polyhedral cone $\bar{C}$ which is a strict fundamental domain for the action of W on a W-invariant subset $U \subset V^*$ called the Tits cone. $\bar{C}$ is the disjoint union of its open faces $\{C_T\}_{T \in \mathcal{S}}$, and $C_{T'} \subset \bar{C_T} \iff T \subset T'$. The stabilzer of any point in the open face C_T is precisely the special subgroup W_T. The maximal open face, $C = C_\emptyset$, is called the fundamental chamber.

If (W, S) is a spherical Coxeter system, then W can be faithfully represented as a discrete subgroup of $O(V)$, where $O(V)$ is the subgroup of $GL(V)$ preserving a positive definite bilinear form. Moreover, the matrix which represents the form with respect to the standard basis is $(-\cos(\pi/m_{ij}))$, where $M = (m_{ij})$ is the Coxeter matrix for (W, S).

The set of elements of W which act as orthogonal reflections with respect to this form is precisely the set of reflections defined above. In fact, the finite Coxeter groups are precisely the finite subgroups of $O(n)$ generated by reflections. For this reason, the finite Coxeter groups are often called finite reflection groups.

When (W, S) is an infinite type Coxeter system, this form is no longer positive definite. Still, each reflection $r \in R$ acts by a "reflection" in the sense that $\sigma(r)$ fixes a codimension 1 hyperplane in V, has a simple (-1) eigenvalue, and $\sigma(r)^2 = 1$.

Coxeter groups admit other interesting geometric interpretations too. Every Coxeter group W acts geometrically on its *Coxeter-Davis complex* X ([DM] is a good survey). It was shown by Moussong [M] that X admits a very natural CAT(0) metric wherein the elements of $\mathcal{R}$ are acting by reflections in the "walls" of X. Thus, Coxeter groups are examples of CAT(0) groups.

Definition. Let (W, S) be a Coxeter system, where S has cardinality n. An element $x \in W$ of the form $x = s_{i_1} \dots s_{i_n}$, where $\{ i_1, \dots, i_n \}$ is a permutation of $\{ 1, \dots, n \}$, is called a *Coxeter element.*

The construction of a non-positively curved $K(A, 1)$ complex for a three dimensional FC Artin group is closely related to a partial ordering of the associated Coxeter group with respect to a family of Coxeter elements indexed by the spherical subsets. We describe this partial ordering in the next section.

3. Allowable elements and allowable expressions

Definition. Let (W, S) be a spherical Coxeter system. Let R be the set of reflections. Then the reflection length ℓ defines a relation $\leq$ on W as follows:

$$w \leq w' \iff \ell(w) + \ell(w^{-1}w') = \ell(w').$$

Regarding R as a generating set for W, we say a word, $r_1 \dots r_k$, is *reduced* if $\ell(r_1 \dots r_k) = k$. A *prefix* of a reduced word $r_1 \dots r_k$ is a subword of the form $r_1 \dots r_i$ for some $i, 1 \leq i \leq k$. The empty word is also considered a prefix.

Proposition 3.1. *Let (W, S) be a spherical Coxeter system, let R be the set of reflections, and let $\leq$ be the relation as above. Suppose $w, w' \in W$. Then $w \leq w'$ if and only if w is prefix of an R-reduced word representing w'. Thus, the relation, $\leq$, defines a partial order on W.*

Proof. Exercise. □

Now, suppose (W, S) is an arbitrary Coxeter system. For each $T \in \mathcal{S}$, we have a partial order $\leq_T$ and a length function ℓ_T defined on W_T with respect to the reflections R_T. We will show that these partial orders and length functions agree on the intersection of any collection of spherical subgroups. The proof relies on a theorem of R. Carter (refer to Lemma 2.8 in [Ca] for a proof). The theorem in its stated form (below) can be found in [Be]. We also recommend reading Proposition 2.2 in [BW] for an independent proof.

Theorem 3.2. *(Carter's Lemma) Let (W, S) be a finite Coxeter system with reflections R and reflection length function ℓ. Suppose $\rho : W \to GL(V)$ is a faithful linear representation of W on a finite dimensional vector space $V \cong \mathbb{R}^n$ such that, for every $w \in W$, $codim(ker(\rho(w) - Id)) = 1 \iff w \in R$. Suppose $w \in W$. Then the reflection length of w is equal to the codimension of its fixed subspace: $\ell(w) = codim(ker(\rho(w) - Id))$.*

Remark. For any Coxeter group, the geometric representation $\sigma : W \to GL(V)$ has the stated property: a non-trivial $w \in W$ fixes a codimension one hyperplane in V if and only if w is a reflection. However, the conclusion of Carter's Lemma does not hold for arbitrary infinite Coxeter groups. For instance, the Coxeter group W with the Coxeter graph consisting of three disjoint vertices acts on $\mathbb{R}^3$ via its geometric representation. But the square of the product of the three generators has reflection length four. However, Carter's Lemma does hold for the $\tilde{A}_n$ Coxeter groups [BW], and likely holds for all Euclidean Coxeter groups.

The following theorem is due to R. Charney and the author. It is inspired by a similar result for spherical Coxeter groups in [CP].

Theorem 3.3. *Let (W, S) be a Coxeter system and let R be the set of reflections. Suppose that $w = r_1 \dots r_k$ is R-reduced. If $w \in W_T$ and $T \in \mathcal{S}$, then $r_i \in R_T$ for all i.*

Proof. Let $n = |S|$ and consider the geometric representation $\sigma : W \to GL(V)$. Pass to the contragredient representation, so that W acts on V^*. Each reflection of W acts by a reflection in a hyperplane of $V^* \cong \mathbb{R}^n$. Let $w \in W_T$ and write $w = r_1 \dots r_k$ as an R-reduced word. Assume that $T \in \mathcal{S}$ so that W_T is a finite Coxeter group. Write w as an R_T-reduced word: $w = q_1 \dots q_l$, where each $q_i \in R_T$. Necessarily, $k \leq l$. Let $F := \cap_{i=1}^{k} H_i$, where each H_i is the codimension one hyperplane fixed by r_i. Let $Fix(w) := \{ v \in V^* : w.v = v \}$. Carter's Lemma, applied to σ restricted to W_T, tells us that $l = \ell_T(w)$ is equal to the codimension of $Fix(w) \subset V^*$. On the other hand, w fixes the subspace F; so $F \subset Fix(w)$. As $\operatorname{codim}(F) \leq k \leq l$, we must have equality: $F = Fix(w)$. In particular, each r_i fixes every point in $Fix(w)$. Choose a point $x \in C_T$, the open face of the fundamental chamber C. The stabilizer of x is W_T. So, $x \in Fix(w) = F$, and, hence, each r_i fixes x. Thus, each r_i belongs to W_T; and so, each $r_i \in R \cap W_T = R_T$. □

The following is an easy corollary:

Corollary 3.4. *Let (W, S) be a Coxeter system, let R be the set of reflections, and let ℓ denote the reflection length. If $w \in W_T$ and $T \in \mathcal{S}$, then $\ell(w) = \ell_T(w)$. In particular, the reflection length functions, $\{\ell_T\}_{T\in\mathcal{S}}$, and the partial orders, $\{\leq_T\}_{T\in\mathcal{S}}$, agree on the intersection of spherical subgroups.*

Proof. Exercise. □

Proposition 3.5. *Let (W, S) be a Coxeter system and let ℓ denote the reflection length with respect to R. Then W is a poset via the relation $w \leq w' \iff w, w' \in W_T$ and $w \leq_T w'$ for some $T \in \mathcal{S}$. Moreover, $w \leq w' \iff w$ is a prefix of some R-reduced expression for w' and $w, w' \in W_T$ for some $T \in \mathcal{S}$.*

Proof. The only non-trivial verification is that the relation is transitive. Suppose $w \leq w'$ and $w' \leq w''$, then there exists $T, T' \in \mathcal{S}$ such that $w \leq_T w' \in W_T$ and $w' \leq_{T'} w'' in W_{T'}$. By Proposition 3.1, w is an R_T reduced prefix of w'. But $w' \in W_{T'}$, so, by Theorem 3.3, each reflection appearing in this reduced word also belongs to $R_{T'}$. And, as $\ell_T(w) = \ell_{T'}(w)$, the word is $R_{T'}$-reduced. Substituting this word for the prefix representing w' in an $R_{T'}$-reduced word for w'', we see that $w \leq_{T''} w''$. □

Definition. Let (W, S) be a Coxeter system together with a total ordering $\prec$ on S. For each $T \in \mathcal{S}$, let $x_T := t_1 \dots t_k$ where $T = \{t_1 \prec \cdots \prec t_k\}$. Thus, a total ordering chooses a Coxeter element x_T for each Coxeter system (W_T, T).

For each $T \in \mathcal{S}$, define the *x_T–allowable elements* thus:

$$Allow(x_T) := \{\, w \in W_T \mid 1 \neq w \leq_T x_T \,\}.$$

Define the allowable elements of W to be the set

$$Allow(W) := \bigcup_{T\in\mathcal{S}} Allow(x_T).$$

By Proposition 3.1, the x_T-allowable elements are precisely the non-trivial elements of W_T which can be represented as a prefix of an R_T-reduced expression of x_T. In particular, by repeated application of the move $x = r_1 r_2 \dots r_n = r_2(r_2^{-1} r_1 r_2) r_3 \dots r_n$, it is easy to deduce that $T \subset Allow(x_T)$. In fact, as we shall prove later (Corollary 5.2), every reflection in R_T is allowable, i.e. $R_T \subset Allow(x_T)$.

The following is an easy consequence of Proposition 3.5:

Proposition 3.6. *Let (W, S) be a Coxeter system together with a total ordering of S. Then*

$$Allow(W) = \{w \in W : 1 < w \leq x_T \text{ for some } T \in \mathcal{S}\}.$$

Proof. Exercise. □

The total ordering of S makes a consistent choice of Coxeter elements in the following sense:

Proposition 3.7. *Let (W, S) be a Coxeter system together with a total ordering of S. Suppose $T, T' \in \mathcal{S}$ are such that $T \subset T'$. Then $Allow(x_T) \subset Allow(x_{T'})$ is an inclusion of posets.*

Proof. This is immediate from Proposition 3.5 and the fact that $x_T \leq x_{T'}$. □

Remark. These partial orders have been studied by a number of other researchers in the case where (W, S) is a spherical Coxeter system, cf. D. Bessis [Be]; D. Bessis, F. Digne, J. Michelle [BDM]; J. Birman, K. Ko, & J. Lee [BKL]; T. Brady [Br2]; T. Brady & C. Watts [BW]; and M. Picantin [P]. Our notation is consistent with [Be]. In each of these articles, the object of study is a "dual braid monoid". D. Bessis, building on the other authors' partial results, proves that if the Coxeter element x_S is correctly chosen, then the group of fractions of the dual braid monoid is isomorphic to the associated Artin group. However, it is unclear how or if this result generalizes to an infinite type Artin group.

Definition. We say a sequence of allowable elements $(w_1, \dots, w_k)$ in $Allow(x_T)$ defines an *allowable expression* of length k, $k > 0$, if the product $w_1 \cdots w_k \in Allow(x_T)$ and $\sum_{i=1}^{k} \ell(w_i) = \ell(w_1 \dots w_k)$. Denote the allowable expressions of length k by $Expr(x_T; k)$ and all the allowable expressions by $Expr(x_T)$.

We define allowable expressions in W to be the set

$$Expr(W) = \bigcup_{T \in \mathcal{S}} Expr(x_T).$$

In particular, the allowable expressions of length $|T|$ in $Expr(x_T)$ correspond to all the R-reduced words which represent x_T. It also noteworthy that, equivalently, an allowable expression is a sequence

$(w_1, \ldots, w_k)$ of elements of $Allow(W)$ such that $\Sigma_{i=1}^k \ell(w_i) = \ell(w_1 \ldots w_k)$, and $w_1 \ldots w_k \in Allow(W)$. For, by definition $w_1 \ldots w_k \leq x_T$ for some $T \in \mathcal{S}$. It follows that each $w_i \leq x_T$, as well. Thus, $(w_1, \ldots, w_k) \in Expr(x_T; k)$.

Proposition 3.8. *If $T \subset T' \in \mathcal{S}$, then $Expr(x_T) \subset Expr(x_{T'})$.*

Proof. If $(w_1, \ldots, w_k) \in Expr(x_T; k)$, then, by Proposition 3.7 each w_i belongs to $Allow(x_{T'})$ and so does the product $w_1 \ldots w_k$. Now use the fact that $\ell_{T'}$ agrees with ℓ_T on W_T. □

The next two propositions address the question of whether there are allowable elements in W_T which are not x_T-allowable.

Proposition 3.9. *Let (W, S) be a Coxeter system of dimension ≤ 3 together with a total ordering of S. Let $T, T' \in \mathcal{S}$ and let $w \in W$. Suppose $w \leq x_T, x_{T'}$. If $\ell(w) = |T \cap T'|$, then $w = x_{T \cap T'}$. Moreover, for each nonempty $T \in \mathcal{S}$, there is a unique allowable element w of length $|T|$ belonging to W_T.*

Proof. First, observe that if $w \leq x_T, x_{T'}$, then $w \in W_T \cap W_{T'} = W_{T \cap T'}$. If $T \subset T'$, then $\ell(w) = |T \cap T'| = |T|$. But $\ell(w) + \ell(w^{-1} x_T) = \ell(x_T) = |T|$. So, $w = x_T = x_{T \cap T'}$. Everything so far is true regardless of the dimension.

Now suppose that neither T nor T' is a subset of the other. If the intersection has cardinality one, then w belongs to $W_{T \cap T'}$— a group with only one non-trivial element. So $w \in T \cap T'$ and the conclusion holds. The only remaining possibility is that w has length two and T and T' have have three elements. Write $w = r_1 r_2$, where $r_1, r_2 \in R_{T \cap T'}$. Suppose $T = \{t_1 \prec t_2 \prec t_3\}$ and $T \cap T' = \{t_i \prec t_j\}$. Then, by shifting, we can write $x_T = t_i t_j u$, reduced, for some $u \in R$. But u cannot belong to $R_{T \cap T'}$, because $x_T \notin W_{T \cap T'}$. As $w \leq x_T$ has length two, so there exists a $q \in R$ such that $r_1 r_2 q = x_T = t_i t_j u$. So, $uq = t_j t_i r_1 r_2 \in W_{T \cap T'}$. By Theorem 3.3, uq cannot be reduced— u would belong to $R_{T \cap T'}$. So, $uq = 1$. Thus, $w = r_1 r_2 = t_i t_j = x_{T \cap T'}$.

For the second statement, suppose $w \in Allow(W; |T|) \cap W_T$, where $T \in \mathcal{S}$ is nonempty. So, $w \leq x_{T'}$ for some $T' \in \mathcal{S}$; and we can find a reduced word for $w = r_1 \ldots r_k$ as a product of $k = |T|$ reflections. By Theorem 3.3, each r_i belongs to $R_{T \cap T'}$. Consequently, $|T \cap T'| = |T|$; and so, $T \subset T'$. If $T = T'$, then, as above, $w \leq x_T$ and $\ell(w) = |T|$ imply that $w = x_T$. So, we may assume that $T \subsetneq T'$. Now, consider cases: If $|T| = 1$, there is only one nontrivial element in W_T; so, $w = x_T$. If $|T| = 2$, then $|T'| = 3$. Again, there is an R-reduced expression $x_{T'} = wq = t_i t_j u$, where q and u are reflections and $T = \{t_i \prec t_j\} \subset T'$. We must have that $uq = 1$. So, $w = t_i t_j = x_T$. □

Proposition 3.10. *Let (W,S) be a Coxeter system of dimension ≤ 3 together with a total ordering of S. Suppose $T \in \mathcal{S}$. Then $Allow(W) \cap W_T = Allow(x_T)$ and $Expr(W;k) \cap (W_T)^k = Expr(W_T;k)$ for each k.*

Proof. Suppose $w \in Allow(W) \cap W_T$. If $\ell(w) = 1$, then w is a reflection in R_T. Corollary 5.2 (proof in the next section) implies that $w \in R_T \subset Allow(x_T)$. Suppose $\ell(w) = 2$ and $w \leq x_{T'}$ for some $T' \in \mathcal{S}$. By Proposition 3.7, we may assume T is minimal. So, $T \subset T'$. Either $|T| = 2$, forcing $y = x_T$, or $T = T'$. In either case, $w \in Allow(x_T)$. Finally if $\ell(w) = 3$, then $w = x_T$. Thus, $Allow(W) \cap W_T \subset Allow(x_T)$. The opposite inclusion is immediate. Similarly, there is only one nontrivial inclusion for the second statement. Suppose $(w_1, \ldots, w_k) \in Expr(W;k) \cap (W_T)^k$. Then, each $w_i \in Allow(W) \cap W_T = Allow(x_T)$. By subgroup closure, the product $w_1 \ldots w_k$ belongs to $Allow(W) \cap W_T$. It follows that $(w_1, \ldots, w_k) \in Expr(x_T;k)$. ☐

Remark. Propositions 3.9 and 3.10 probably admit generalizations to all dimensions. However, the brute force arguments given here provide little insight. The work of D. Bessis [Be] is particularly recommended to those who may want to generalize these statements.

We are now ready to define the proposed non-positively curved $K(A,1)$ space for a three dimensional FC Artin group A. This complex is defined purely in terms of a three dimensional Coxeter system together with a total ordering of its generating set.

4. Brady's complex

Let Γ be a link complex of dimension together with a total ordering of its vertices. Γ defines a Artin system (A,S) together with a total ordering of its generating set S. To emphasize its origin, we may denote the Artin group by A_Γ and its associated Coxeter group by W_Γ. We will assume that Γ is a simplicial complex of dimension ≤ 2. Let $K = K_\Gamma$ be the cell complex (*Brady's complex*) defined as follows:

- 0-skeleton $K^{(0)}$: a single vertex labeled v_0.

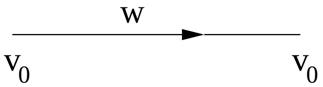

Figure 1. A labelled, oriented 1-cell.

- 1-skeleton $K^{(1)}$: a labelled, oriented edge for each allowable element, $w \in Allow(W)$, with both ends attached at v_0.

- 2-skeleton $K^{(2)}$: a two simplex for each allowable expression of length two: $(w_1, w_2) \in \mathit{Expr}(W; 2)$. The edges of the simplex are labeled by w_1, w_2, and w_1w_2, viewed as elements of W, not as words. The boundary of this 2-simplex is attached to the 1-skeleton according to its labeling and orientation. Refer to Figure 2.

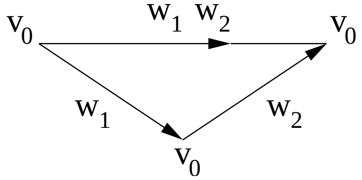

FIGURE 2. A 2-cell.

- 3-skeleton $K^{(3)}$: a three simplex for each allowable expression of length three: $(w_1, w_2, w_3) \in \mathit{Expr}(W; 3)$. It's six edges are labeled by the following elements of W:

$$w_1, w_1w_2, w_2, w_2w_3, w_3, w_1w_2w_3.$$

Their orientation and adjacency is shown in Figure 3. The boundary of this 3-simplex is attached to the 2-skeleton accordingly.

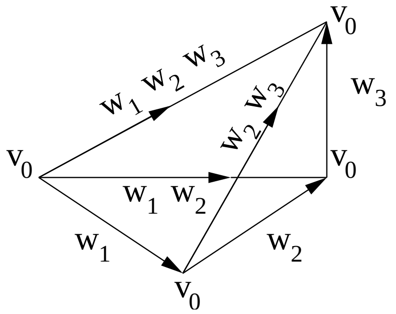

FIGURE 3. A typical 3-cell in K.

From the description of the the 2-skeleton, it follows that the fundamental group of K is presented thus:

$$\pi_1(K, v_0) = \langle \{[w] : w \in \mathit{Allow}(W)\} \mid \{[w_1][w_2] = [w_1w_2] : (w_1, w_2) \in \mathit{Expr}(W; 2)\}\rangle,$$

or, informally, $\pi_1(K) = \langle \mathit{Allow}(W) \mid \mathit{Expr}(W; 2)\rangle$. The generator $[w]$ is called a *lift* of the allowable element $w \in W$. The brackets are used to distinguish an element of the fundamental group from an element of the Coxeter group.

If A_Γ is a spherical Artin group with generators $S = \{b \prec a \prec c\}$ such that $m_{ac} = 2$, then K is exactly the complex considered by T. Brady [Br1]. If A_Γ is a spherical Artin group with two generators, this is the 2-complex considered by T. Brady and J. McCammond in [BM]. And if A_Γ is a sperical Artin group with one generator ($A \cong \mathbb{Z}$), then K_Γ is just a circle with a single vertex.

In the case of a three dimensional Artin group, each $T \in \mathcal{S}$ gives rise to one of these Brady complexes above. Define K_T to be the Brady complex associated to the Coxeter system (W_T, T) together with the total ordering of S restricted to T. Each K_T is a subcomplex of K_Γ. This follows from Proposition 3.10. Similarly, it follows from Propositions 3.7 and 3.8 that $K_{T \cap T'} = K_T \cap K'_T$, whenever $T, T' \in \mathcal{S}$.

The goal of the next two sections is to prove that $\pi_1(K_\Gamma, v_0) \cong A_\Gamma$.

5. The fundamental group of K_Γ

If A_Γ is a spherical Artin group of dimension ≤ 3 and W_Γ is its associated Coxeter group, then, regardless of the choice of total ordering, the cell complex K_Γ always has the same fundamental group:

Theorem 5.1. *Let (W, S) be a spherical Coxeter system. Let x and y be two Coxeter elements for (W, S). Then there is an element $w_0 \in W$ such that $w_0 y w_0^{-1} = x$ in W. The automorphism of W, ϕ, mapping $w \mapsto w_0 w w_0^{-1}$ restricts to a bijection $Allow(y) \to Allow(x)$. This, induces a bijection $Expr(y) \to Expr(x)$. Moreover, if W has three or fewer generators, ϕ induces an isomorphism $\Phi : \pi_1(K_y) \to \pi_1(K_x)$, by defining $\Phi([a]) := [\phi(a)]$, where K_y and K_x are the Brady complexes defined by their respective Coxeter elements.*

Proof. The first statement, that all Coxeter elements in a finite Coxeter group are conjugate, is classical [Bo]. That ϕ is a bijection follows from the fact that it permutes the reflections and preserves the reflection length of any element of W.

In the special case of length two expressions, we see that the relation $[\phi(a_1)][\phi(a_2)] = [\phi(a_1 a_2)]$ holds in $\pi_1(K_x)$. Hence,

$$\Phi([a_1][a_2]) = [\phi(a_1)][\phi(a_2)] = [\phi(a_1 a_2)] = \Phi([a_1 a_2]).$$

Thus, Φ takes relators to relators, and so, Φ is a well-defined group homomorphism. Moreover, Φ is clearly invertible and, hence, is an isomorphism. □

As a corollary, we deduce that all the reflections in a spherical Coxeter system are allowable, regardless of the choice of Coxeter element:

Corollary 5.2. *Let (W, S) be a spherical Coxeter system and let x be a Coxeter element. Then every reflection is x-allowable.*

Proof. The hard work has already been done by D. Bessis. He proves that every spherical Coxeter system admits a "chromatic" Coxeter element y for which the set of reflections are y-allowable (use Lemma 1.3.4 in [Be]). Now, apply the bijection between $Allow(x)$ and $Allow(y)$ from the theorem above. The bijection preserves the set of reflections. Hence, every reflection is x-allowable. $\square$

To establish an isomorphism between an (infinite type) Artin group A_Γ and the fundamendal group of its Brady complex K_Γ, we seek compatible isomophisms $A_T \to \pi_1(K_T, v_0)$ for each $T \in \mathcal{S}$. As A_Γ will be assumed to have dimension ≤ 3, the spherical subgroups will have at most three generators. Each case is treated separately. Recall that $\pi : A \to W$ denotes the natural map.

If $A = A_T$ is a spherical Artin group with one generator, s, and W is its associated Coxeter group, then:

(1) $A = \langle\, s \mid \quad \rangle \cong \mathbb{Z}$,
(2) $W = \langle\, s \mid s^2 = 1 \,\rangle \cong \mathbb{Z}/2\mathbb{Z}$, and
(3) $\pi_1(K) = \langle\, [s] \mid \quad \rangle \cong \mathbb{Z}$.

So, the map $\beta : s \to [\pi(s)]$ defines an isomorphism $A \to \pi_1(K)$.

If $A = A_T$ is a spherical Artin group with two generators, s_1 and s_2, and W is its associated Coxeter group, then there is an integer $m > 1$ such that

(1) $A = \langle\, s_1, s_2 \mid < s_1, s_2 >^m = < s_2, s_1 >^m \,\rangle$,
(2) $W = \langle\, a, b \mid (ab)^m = 1 \,\rangle$, a dihedral group of order $2m$, and
(3) $\pi_1(K) = \langle\, [x], [r_1], \dots, [r_m] \mid [x] = [r_1][r_2], \dots, [x] = [r_m][r_1] \,\rangle$.

The last statement appears in [BM]:

Proposition 5.3. *(T. Brady, J. McCammond* [BM]*) Let (A, S) be the two generator spherical Artin group with generating set $S = \{s_1, s_2\}$, and let (W, S) be the associated Coxeter system. Fix a Coxeter element: either $x = s_1 s_2$ or $x = s_2 s_1$. In either case, we have $\pi_1(K) = \langle\, [x], [r_1], \dots, [r_m] \mid [x] = [r_1][r_2], \dots, [x] = [r_m][r_1] \,\rangle$, where K is the Brady complex of (W, S) with respect to x.*

As in the one generator case, the fundamental group of the Brady complex is naturally isomorphic to the Artin group:

Proposition 5.4. *(T. Brady, J. McCammond* [BM]*) Let (A, S) be the two generator spherical Artin group with generating set $S = \{s_1, s_2\}$, and let (W, S) be the associated Coxter system. Then, for either choice* *of Coxeter element, the map β, which takes each generator $s_i \in S$ to the allowable element $[\pi(s_i)]$, defines an isomorphism of A onto $\pi_1(K, v_0)$.*

Refer to [BM] for a proof of Proposition 5.4. Bascially, consider the cases of when m is odd or even separately, and then directly verify that the desired relations hold in both presentations.

Theorem. *Let (A, S) be a spherical three generator Artin group and let W be its associated Coxeter group. Then, regardless of the choice of Coxeter element in W, the map $\beta : A \to \pi_1(K, v_0)$ sending each generator s to $[\pi(s)]$ is an isomorphism.*

We will prove this theorem in the next section (Theorem 6.2). Assuming this result, we now prove that the fundamental group of K_Γ is isomorphic to A_Γ.

Theorem 5.5. *Let Γ define an Artin group of dimension ≤ 3 together with a total ordering of the generating set. Then $\pi_1(K_\Gamma, v_0) \cong A_\Gamma$.*

Proof. Let (A, S) be the Artin system defined by Γ and let (W, S) be the associated Coxter system. Suppose $T \subsetneq T' \in \mathcal{S}$. Let $i : A_T \to A_{T'}$ be the map induced by the inclusion $T \subset T'$. Then i is injective. (This is result is due to van der Lek [L] in general, though inclusions of spherical Artin groups were known to be injective by the earlier work of Brieksorn & Saito [BS] and Deligne [De].) Let $j :\ \pi_1(K_T) \to \pi_1(K_{T'})$ be induced by the inclusion $Allow(x_T) \subset Allow(x_{T'}$. This is well-defined because $Expr(x_T; 2) \subset Expr(x_{T'}; 2)$. Now consider the following diagram:

$$\begin{tikzcd} A_T \arrow[r] \arrow[d] & A_{T'} \arrow[d] \\ \pi_1(K_T) \arrow[r] & \pi_1(K_{T'}) \end{tikzcd}$$

The horizontal maps are the maps i and j above; the vertical maps are the maps β_T and $\beta_{T'}$ which map $t \mapsto [\pi(t)]$.

We have shown that β_T and $\beta_{T'}$ are isomorphisms when $|T'| \leq 2$. Assuming Theorem 6.2, these maps are also isomorphisms when $|T'| = 3$. To check the diagram commutes, it suffices to chase the generating set T. By construction, these maps restrict to the identity on T. So, the diagram commutes. As the vertical maps are isomorphisms, it follows that the bottom horizontal map is also injective. Taking colimits over $T \in \mathcal{S}$, we get an isomorphism

$$\operatorname*{colim}_{T \in \mathcal{S}} A_T \cong \operatorname*{colim}_{T \in \mathcal{S}} \pi_1(K_T).$$

By examining the presentation for $\pi_1(K_\Gamma)$, we see that we have defined the desired isomorphism. □

6. Three generator spherical Artin groups

Let (A, S) be an Artin system. Let W be the associated Coxeter group, and let $R \subset W$ be the set of reflections. Let $\pi : A \to W$ be the natural map.

Definition. Given $S' \subset A$, we say that S' is *equivalent* to S if

- S' generates A
- there is a bijection $S \leftrightarrow S'$, and
- the corresponding Artin relations, $< s'_i, s'_j >^{m_{ij}} = < s'_j, s'_i >^{m_{ij}}$, are a set of defining relators for A.

We specialize to the case where (A, S) a three generator spherical Artin group. Let $S = \{s_1, s_2, s_3\}$. We can choose $m = m_{12}, n = m_{23}$, and $2 = m_{13}$. This group is finite if $m = 2$ and $2 \leq n < \infty$ or when $m = 3$, and $n \in \{3, 4, 5\}$.

Suppose $S' = \{s'_1, s'_2, s'_3\} \subset A$ generates A as an Artin group, where $s_i \leftrightarrow s'_i$. Let $x' = s'_1 s'_2 s'_3$. We emphasize that x' is the product of the elements of the Artin generating set in a particular order; specifically, the first two generators satisfy an Artin relation of length m and the last two generators commute. Denote the corresponding Coxeter element by $y' := \pi(x')$. We may then construct Brady's complex; we denote it by $K_{y'}$. In particular, we can describe the fundamental group as follows (refer to [Br1]:

$$\pi_1(K_{y'}) = \langle\, \{[y']\} \cup \{[r] : r \in R\} \mid \{[y'] = [r_1][r_2][r_3] : r_1 r_2 r_3 =_W y', r_i \in R\}\, \rangle.$$

T. Brady has shown that the map $\beta_{S',x'} : A \to \pi_1(K_{y'})$ taking $s' \mapsto [\pi(s')]$ is an isomorphism. The inverse map is obtained by noting that the set of reflections is the closure of $\pi(S') \subset W$ under the action of conjugation by y'. Given $r \in R$, we can thus write

$$r = (y')^{-k} \pi(s'_i)(y')^k = \pi((x')^{-k} s'_i (x')^k)$$

for some integer k. The inverse map sends $[r] \mapsto (x')^{-k} s'_i (x')^k \in A$. That this map is well-defined is the main content of Brady's proof. We wish to extend this result to the case where K_y is defined by an arbitrary Coxeter element $y \in W$.

Remark. Given an arbitrary Coxeter element y in a spherical Coxeter system (W, S), it need not be true that the reflections are the closure of S under the action of conjugation by y. D. Bessis [Be] has shown that this is the case provided y is a "chromatic" Coxeter element, i.e. 2-color the Coxeter *diagram* of W (the Coxeter diagram of a spherical Coxeter group is a forest with at most one vertex of valence three in any component) by say 'red' and 'blue'; then a Coxeter element which is a product of the red generators followed by the product of the blue generators is called *chromatic*. However, in the case of an infinite type Coxeter system, it is not always possible to make a consistent choice of chromatic Coxeter elements for each spherical subgroup. Nonetheless, as we argue below, given y, one can find an alternate Coxeter generating set for which y is chromatic. We only have a case by case proof for the three generator spherical Coxeter groups. It is reasonable to expect that this could generalized in a unified way by, perhaps, viewing alternate generating sets as different chambers in the Coxeter-Davis complex.

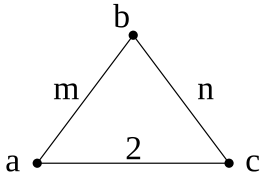

FIGURE 4. The Coxeter graph of (W, S).

Let $S = \{a, b, c\}$ so that the Coxeter graph is as shown in Figure 4. Let $x = abc \in A$, and let $y := \pi(x) \in W$ be the Coxeter element. Define $S' := aSa^{-1} = \{a' := a, b' := aba^{-1}, c' = c\} \subset A$. Then, as conjugation by a is an automormism of A and $c' = aca^{-1}$, we see that S' is equivalent to S. Let $x' := b'a'c'$ and $y' := \pi(x')$. Then (S', x', y') has the form of Brady's setup, namely x' is defined to be the product of the elements of an Artin generating set in the prefered order. So, the map $\beta_{S',x'} : A \to \pi_1(K_{y'})$ is an isomorphism. To simplify our notation, we use the following convention: if $g \in A$, then $[g] := [\pi(g)]$.

Lemma 6.1. *The map* $\beta_{S,x} : A \to \pi_1(K_y)$ *sending* $s \mapsto [s]$ *is an isomorphism. In fact,* $\beta_{S,x} = \beta_{S',x'}$*.*

Proof. The second statement makes sense because $\pi_1(K_{y'}) = \pi_1(K_y)$. The choice of x' is such that $x' = (aba^{-1})(a)(c) = abc = x$. So, $y' = y$ and the presentations of the fundamental groups are identical. It suffices to prove that $\beta_{S',x'}(s') = \beta_{S,x}(s')$ for every $s' \in S'$. If $s' = a'$ or c', there is nothing to prove. If $s' = b'$, then

$$\beta_{S,x}(b') = \beta_{S,x}(aba^{-1}) = [a][b][a]^{-1}.$$

On the other hand, as $y = \pi(abc) = \pi(aba^{-1})\pi(a)\pi(c)$, the relation

$$[a][b][c] = [b'][a][c]$$

holds in $\pi_1(K_y)$. Solving the equation in the group, we find that

$$[b'] = [a][b][a]^{-1}.$$

Hence, $\beta_{S',x'}(b') = \beta_{S,x}(b')$, as claimed. □

The cases where the Coxeter elements are defined by other permutations of the set $S = \{a, b, c\}$ are similar. Given $x =$ permutation of S, we define a corresponding element $g \in A$:

- if $x = bac$ or bca, let $g = 1$
- if $x = abc$, let $g = a$ (this is the case above)
- if $x = cba$, let $g = c$
- if $x = acb$ or cab, let $g = ac$

Let $S' := gSg^{-1}$. Then $a' := gag^{-1} = a, b' := gbg^{-1}$, $c' := gcg^{-1} = c$ and S' is equivalent to S. Let $x' = b'a'c'$. Then $x' = gbacg^{-1} = x$. In particular, x and x' define the same Coxeter element. Moreover, (S', x') takes the form of Brady's theorem.

The Brady complexes have identical fundamental groups. To show that $\beta_{S',x'} = \beta_{S,x}$, we check that this holds for each generator $s' \in S'$. As $a' = a$ and $c' = c$, we need only check that $\beta_{S',x'}(b') = \beta_{S,x}(b)$. So, we write b' in terms of the generating set S: $b' = gbg^{-1}$. So, $\beta_{S,x}(b') = (\beta_{S,x}(g))[b](\beta_{S,x}(g))^{-1}$. To see that this equals $\beta_{S',x'}(b') = [b']$, we only need to verify that that the relation $(\beta_{S,x}(g))[b] = [b'](\beta_{S,x}(g))$ holds in $\pi_1(K)$. We return to cases:

- if $g = 1$, then $\beta_{S,x}(g) = 1$ and the relation follows from $b' = b$.
- if $g = a$, this is the case above
- if $g = c$, then $\beta_{S,x}(g) = [c]$ and the relation follows from
$$[x] = [c][b][a] = [cbc^{-1}][c][a] = [b'][c][a]$$
- if $g = ac$, then $\beta_{S,x}(g) = [a][c]$ and the relation follows from
$$[x] = [a][c][b] = [acbc^{-1}a^{-1}][a][c] = [b'][a][c]$$

Thus, we have proven the desired theorem:

Theorem 6.2. *Given a three generator spherical Artin system (A, S) and given a Coxeter element $y \in W$ chosen by any total ordering of S, the map $\beta : A \to \pi_1(K_y)$ sending $s \mapsto [\pi(s)]$ is an isomorphism.*

7. The geometry of Brady's complex

Let Γ define an Artin group of dimension ≤ 3 together with a total ordering of the generating set. Define a piecewise Euclidean structure on K by assigning a length of $\sqrt{k}$ to each edge labelled by an allowable element of length k. The metric on each cell is then determined. The model polyhedral 3-cell is shown below.

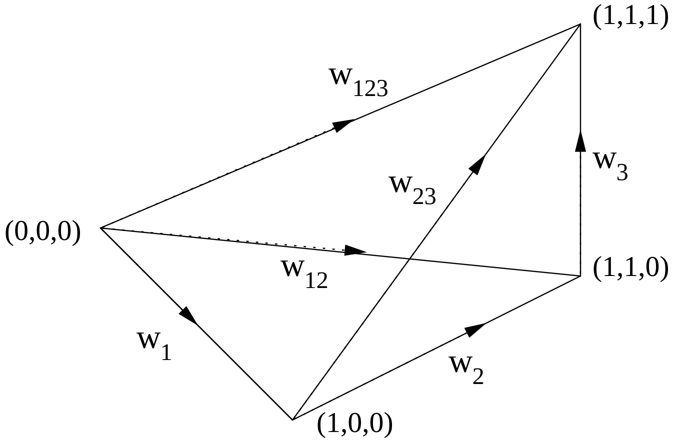

Figure 5. A model metric 3-cell in K.

Above is the 3-cell corresponding to the allowable expression (w_1, w_2, w_3). It is isometric to the tetrahedron $\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant y \leqslant x \leqslant 1\}$. In Figure 5, the allowable rotations (length two allowable elements) are denoted by $w_{12} := w_1w_2$ and $w_{23} := w_2w_3$; a Coxeter element is denoted by $w_{123} := w_1w_2w_3$.

We will study the geometry of K within the formal framework of M_κ- polyhedral and simplicial complexes. Let M^n_κ denote the complete simply connected Riemannian manifold of constant curvature κ and dimension n. Thus, M^n_0 is Euclidean n-space, M^n_1 is the unit n-sphere, and M^n_{-1} is the hyperbolic n-space.

Roughly speaking, an M_κ-complex is cell complexes constructed by gluing convex polyhedral cells (in M_κ) along isometric faces. If X is an M_κ complex, we will denote an n-dimensional cell by C^n_λ and its attatching map by $q_\lambda : C^n_\lambda \to X$. M. Bridson proved that every M_κ complex having only finitely many isometry classes of cells (referred to as *finite shapes*) is a geodesic metric space with respect to the intrisic length metric. The the reader is refered to [BH] for precise definitions and for a proof of this theorem. Using these definitions it is not hard to see that K_Γ, with its cells metrized as above, is an M_0-polyhedral complex.

An important class of these complexes are the M_κ simplicial complexes. In this case, cells are required to be simplices and the attatching maps are required to be injective. Note that the attatching maps of K_Γ are not injective. However, as we will see in the next section, the link of K_Γ is an M_1-simplicial complex.

8. The link of v_0 in K_Γ

Let $L = L_\Gamma := Lk(v_0, K_\Gamma)$ be the geometric link of v_0 in K. By definition, this is the cell complex defined by the unit tangent vectors based at vertices in each of the convex polyhedral cells which are attatched to the vertex v_0 in K_Γ. With this cell structure, L is an M_1-polyhedral complex.

Each 1-cell of K contributes exactly two vertices to L. Suppose that a 1-cell C^1_λ is oriented from a vertex v_1 to a vertex v_2 and that its edge is labelled by the allowable element w. The attaching map q_λ maps both vertices to v_0 in K. Thus, $Lk(v_0, C^1_\lambda)$ consists of two vertices—one for the initial tangent vector of the geodesic path from v_1 to v_2 and one for the initial tangent vector of the reverse path. We label the vertex in the link contributed by the path from v_1 to v_2 by $(w, 1)$; the other vertex is labelled $(w, -1)$. (Refer to Figure 6.) As every vertex of L arises in this way, we deduce that the vertices of L are in bijective correspondence with the set $Allow(W) \times \{\pm 1\}$. Given a vertex (w, ϵ) of L, we say it has *length* $\ell(w)$ and *sign* ϵ.

Remark. We regard $Allow(W) \times \{\pm 1\}$ as a poset via reverse lexicographic ordering: $(w_1, \epsilon_1) \leq (w_2, \epsilon_2) \iff \epsilon_1 < \epsilon_2$ or $\epsilon_1 = \epsilon_2$ and $w_1 \leq w_2$. We can use this discription to uniquely label the cells of L in terms of their vertices.

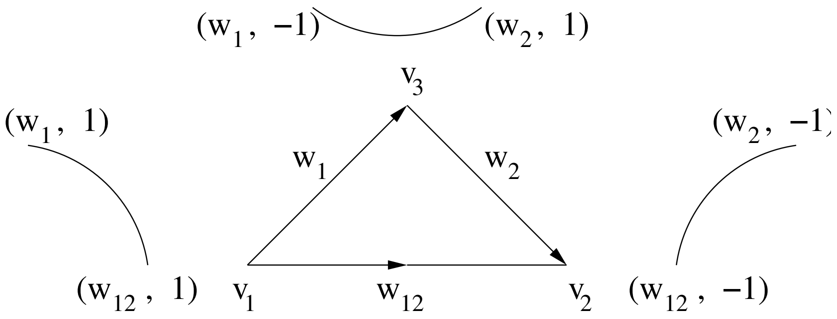

Figure 6. Each vertex contributes a 1-cell in L.

Suppose C^2_λ is a 2-cell in K. C^2_λ is isometric to a Euclidean triangle and indexed by an allowable expression $\lambda := (w_1, w_2)$ of length 2. Suppose the vertices of C^2_λ are v_1, v_2, and v_3, and suppose the directed edge from v_i to v_{i+1} be labelled by w_i for $i = 1, 2$. Label the directed edge from v_1 to v_3 by $w_{12} := w_1 w_2 \in W$. The attaching map q_λ maps all of the vertices to v_0 and maps each directed edge to the 1-cell of K with the same label and orientation. Thus, the link of of a 2-cell of K

consists three disjoint arcs (refer to Figure 6):

$$Lk(v_0, q_\lambda(C^2_\lambda)) = \bigsqcup_{i=1,2,3} Lk(v_i, C^2_\lambda).$$

The vertices of a 1-cell in L are related by the reverse lexicographic ordering on $Allow(W) \times \{\pm\epsilon\}$. Making the convention that vertices are listed in ascending order, we can list the 1-cells according to their vertex set as follows:

$$[(w_1, 1), (w_{12}, 1)],\ [(w_1, -1), (w_2, 1)],\ \text{and } [(w_2, -1), (w_{12}, -1)].$$

This is a complete list if we range over all ordered pairs $(w_1, w_2) \in Expr(W; 2)$.

Proposition 8.1. *Let* $w_1, w_2 \in Allow(W)$*.*

1. *The vertices* $\{(w_1, 1), (w_2, 1)\}$ *or* $\{(w_1, -1), (w_2, -1)\}$*, span a* 1*-cell in* $L \iff w_1, w_2 \in Allow(x_T)$*, for some* $T \in \mathcal{S}$ *and either* $w_1 < w_2$ *or* $w_2 < w_1$*.*
2. *The vertices* $\{(w_1, -1), (w_2, 1)\}$ *span a* 1*-cell in* $L \iff (w_1, w_2) \in Expr(W; 2)$*.*

Notation. Henceforth, we will use the convention that the reflections will be indicated by the letters p, q, r, s or t. Likewise, the *rotations* (i.e. elements in W of length two) will be indicated by the letters y or z. Lastly, we will reserve the letter x or x_T for elements of length three in W.

In Figure 7, we list the three different oriented, metric 2-cells of K. They correspond to expressions of the form $(r, s), (y, t)$, and (q, y) in $Expr(x; 2)$. Recall that the lengths of the edges are 1 for a reflection, $\sqrt{2}$ for a rotation, and $\sqrt{3}$ for and element of length three. The first triangle is an isoceles right triangle. The angles in the second two triangles are indicated, where $\alpha = \arctan(\sqrt{2})$ and $\beta = \arctan(1/\sqrt{2})$. So, $0 < \beta < \pi/4 < \alpha < \pi/2$.

The left 2-cell contributes the following 1-cells to the link:

- $[(r, 1), (y, 1)]$ of length $\pi/4$; (algebraically: $r < y$)
- $[(r, -1), (s, 1)]$ of length $\pi/2$; ($rs = y$ is a reduced expression)
- $[(s, -1), (y, -1)]$ of length $\pi/4$; ($s < y$)

The middle 2-cell contributes:

- $[(y, 1), (x, 1)]$ of length β; ($y < x$)
- $[(y, -1), (t, 1)]$ of length $\pi/2$; ($yt = x$ is reduced)
- $[(t, -1), (x, -1)]$ of length α; ($t < x$)

And the right 2-cell contributes:

- $[(q, 1), (x, 1)]$ of length α; ($q < x$)

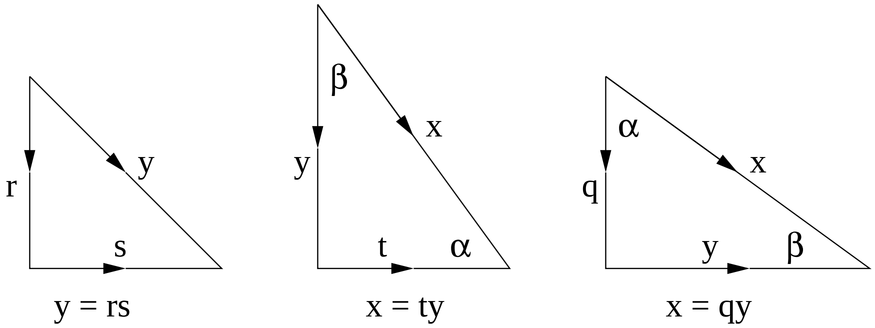

FIGURE 7. The metric 2-cells of K.

- $[(q,-1),(y,1)]$ of length $\pi/2$; ($qy = x$ is reduced)
- $[(y,-1),(x,-1)]$ of length β; ($y < x$)

Thus, if we consider all unordered pairs of vertices $\{\,(w,\epsilon),(w',\epsilon')\,\}$ up to their length and signs $\{\,(\ell(w),\epsilon),(\ell(w'),\epsilon')\,\}$, we get exactly nine different 1-cells in L. This list is complete because every 1-cell in L necessarily arises from a link of one of the three different oriented, metric 2-cells in Figure 7 .

We repeat this analysis for the 2-cells of L. Each 2-cell of L is a simplex. Each such simplex arises from the link of v_0 in a 3-cell of K.

The 3-cells of K are in one to one correspondence with the allowable expressions of length three. For each allowable expression $\lambda := (r,s,t)$, we get four 2-cells in L by considering $Lk(v_0, q_\lambda(C^3_\lambda)) = \bigsqcup_{i=1,\dots,4} Lk(v_i, C^3_\lambda)$. (Refer to Figure 8.)

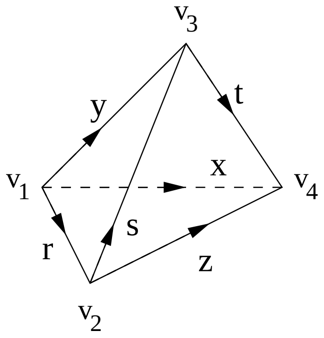

FIGURE 8. Each vertex of a 3-cell contributes a different 2-cell to the link.

We enumerate the 2-cells of L (refer to Figure 9). Counter-clockwise from the upper left corner, they are, respectively, the links of v_1, v_3, v_4, and v_2. These are illustrated in the order shown so as to suggest how the 2-cells will fit together in L. Two 2-cells are glued along a face

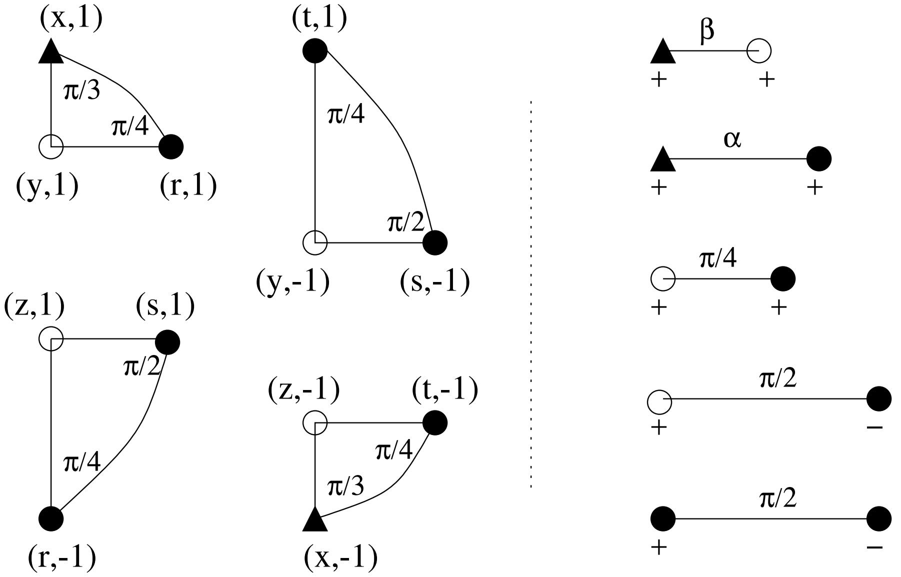

FIGURE 9. The metric 2-cells of L.

if and only if they have the same vertices (labelled by the same allowable element and sign). In particular, such vertices must have the same length. We have illustrated the length of a vertex as follows: a reflection is symbolized by a solid circle, a rotation by an open circle, and an element of length three by a solid triangle. When convenient, we indicate the sign of the vertex by adding a + or − symbol to the diagram as in the list to the right.

In the right column, there is a list of the lengths of the edges. We can recover our complete listing of 1-cells in L (up to length and sign of the vertices) if we change all the + signs to − signs. Note that the bottom 1-cell does not give rise to a new 1-cell if we change the signs—it is characterized as a pair of vertices of length one with opposite signs.

The 2-cells in L are spherical triangles. From the spherical law of cosines or by considering the dihedral angles between the faces of the model polyhedral 3-cell of K, one can compute their angles. The measures of the angles in each spherical triangle is indicated beside each vertex. The unlabelled angles are understood to be $\pi/2$.

Again, we can list the vertices of each 2-cells in ascending order with respect to the ordering on $Allow(W) \times \{\pm 1\}$:

- $Lk(v_1, C^3) = [(r,1),(y,1),(x,1)]$; (algebraically: $r < y < x$)
- $Lk(v_2, C^3) = [(r,-1),(s,1),(z,1)]$; ($rz = x$ and $s < z$)
- $Lk(v_3, C^3) = [(s,-1),(y,-1),(t,1)]$; ($s < y$ and $yt = x$)
- $Lk(v_4, C^3) = [(t,-1),(z,-1),(x,-1)]$; ($t < z < x$)

Proposition 8.2. *Given vertices* $\{(w_1, \epsilon_1), \dots, (w_3, \epsilon_3)\}$. *These vertices span a 2-cell in* $L \iff$

1. *all the vertices have the same sign and the vertices are totally ordered:* $w_i \leq w_j \leq w_k$ *for some permutation* (i, j, k) *of* $(1, 2, 3)$, *or*
2. *exactly two vertices,* $w_i \leq w_j$, *are positive and* $w_k w_j = x_T$ *for some*
   $x_T \in Allow(W; 3)$, *or*
3. *exactly two vertices,* $w_i \leq w_j$, *are negative and* $w_j w_k = x_T$ *for some*
   $x_T \in Allow(W; 3)$.

In each of the last two, the negative vertex right multiplied by the positive vertex gives an allowable element.

Proposition 8.3. L *is an* M_1*-simplicial complex.*

Proof. We have seen that every cell of L is a simplex. The simplices are glued according to the labelling of their vertices. Each simplex injects into L as none of the faces of a given simplex have the same vertex set. From the classification of all such cells, we see that the intersection of two simplices is another simplex. Moreover, the metric on each simplex is compatible along the intersection. □

Corollary 8.4. *Let* $T \in \mathcal{S}$ *and let* K_T *be the Brady complex of* (W_T, T) *together with the total ordering of* S *restricted to* T. *Denote the link of* v_0 *by* L_T. *Then* L_T *is subcomplex of* L*; moreover, the inclusion* $L_T \to L$ *restricts to an isometry on each face.*

9. CAT(0) SPACES AND THE LINK CONDITION

Let κ be a real number. Let $D_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$ and let $D_\kappa = \infty$ if $\kappa \leq 0$. A metric space, (X, d), is D_κ*-geodesic* if every two points $x, y \in X$ of distance less than D_κ may be joined by a geodesic segment. (Though these geodesics, in general, are not unique, we will conveniently denote such a segment by $[x, y]$.) Note that each model space, M^n_κ, is D_κ-geodesic.

Definition. Let (X, d) be a D_κ-geodesic metric space. A triangle $\Delta = [x, y] \cup [y, z] \cup [x, z]$ satisfies the *CAT(*κ*) inequality* if for each point p in the arc $[y, z]$, $d(x, p) \leqq |\bar{x} - \bar{p}|$, where $\bar{x}$ and $\bar{p}$ are the comparison points on a comparison triangle $\bar{\Delta} \subset M^2_\kappa$. If every triangle in X of perimeter $< 2D_\kappa$ satisfies the CAT(κ) inequality, we say that X is a *CAT(*κ*) space.*

A geodesic metric space (X, d) is said to bee *locally CAT(κ)* if each point has a open neighborhood in which all triangles satisfy the CAT(κ) inequality. Locally CAT(κ) spaces are said to have *curvature* $\leq \kappa$.

By the comparison theorems of Riemannian geometry, if X is a Riemannian manifold of non-positive sectional curvature and d is the intrinsic length metric, then X is a locally CAT(0) space. The study of $CAT(0)$ spaces, thus, extends the study of non-positive curvature to the more general setting of metric spaces.

We will show that the universal covering space of K_Γ is a CAT(0) space whenever Γ defines a three dimensional FC Artin group. Thus, $A_\Gamma \cong \pi_1(K_\Gamma)$ will act geometrically on $\widetilde{K_\Gamma}$ by deck transformations, proving the Main Theorem.

The followin two key theorems reduce the question of whether $\widetilde{K_\Gamma}$ is CAT(0) to the question of whether the link L_Γ is CAT(1).

Theorem. *(Local to Global) An M_κ-polyhedral complex K, with Shapes(K) finite, is (globally) CAT(κ) if and only if K is locally CAT(κ) and contains no isometrically embedded circles of length less than $2D_\kappa$. In particular, an M_0-polyhedral complex is CAT(0) if and only if it is locally CAT(0) and simply connected.*

Theorem. *(Link Condition) A M_κ-polyhedral complex K, with Shapes(K) finite, is a locally CAT(κ) space if and only if for every vertex v of K, the geometric link, $Lk(v, K)$, is CAT(1) space.*

The Local to Global theorem is analagous to the Cartan-Hadamard Theorem of differential geometry. The Link Condition, on the other hand, is essentially the statement that the κ-cone (in the sense of Berestovski) on the link of a vertex is CAT(κ) $\iff$ the link is CAT(1); so, in particular, a (sufficiently small) neighborhood of the vertex is CAT(κ) $\iff$ the link is CAT(1). We refer the reader to [BH] for proofs of the above theorems.

Together, these theorems reduce the question of whether K_Γ is locally CAT(0) to the question of whether the link L_Γ contains any isometrically embedded circles of length $< 2\pi$. Such circles are parametrized by closed (local) geodesics:

Definition. Let X be a geodesic metric space. A path $\gamma : [a, b] \to X$ is a *local geodesic* if it is locally an isometric embedding. This path defines a *closed local geodesic* if $\gamma(a) = \gamma(b)$ and the induced map from $[a, b]/(a \sim b) \to X$ defines a local isometric embedding with respect to the quotient metric. The image of a closed local geodesic of length $< 2\pi$ is called a *short loop*.

In the case of a finite M_κ-complex X, a path defines a local geodesic if and only if for each $a \leq t \leq b$, the distance in $Lk(\gamma(t), X)$ between the incoming and outgoing unit vectors is $\geq \pi$. This is a practical way to decide if a given path is locally geodesic because such a path must necessarily "look" like a geodesic in M_κ^n when restricted to an open cell.

Remark. When the context is clear, we will blur the distinction between the the path $\gamma : [a, b] \to X$ and its trace, $\gamma([a.b]) \subset X$. So, given a subspace $Y \subseteq X$, we might say that γ "intersects" or "meets" Y. Likewise, we may say that $\gamma \cap Y = \emptyset$ if γ does not meet Y.

10. OUTLINE OF THE PROOF OF THE MAIN THEOREM

Suppose Γ defines a three dimensional FC Artin group. Let $K = K_\Gamma$ be its Brady complex. We argue that K is a locally CAT(0) space:

First, verify that L_Γ is locally CAT(1). According to the Link Condition, this will follow if the link of every vertex in L_Γ is CAT(1). Using the Local to Global theorem, this reduces to a one dimensional computation; namely, we should check that the link of every vertex of L_Γ does not contain any short loops.

Second, verify that L_Γ is (globally) CAT(1). Again, using the Local to Global theorem, this reduces to arguing that L_Γ does not contain any short loops.

Finally, by the Link Condition, K_Γ is locally CAT(0); and, so, by the Local to Global theorem, its universal covering space is (globally) CAT(0). Thus, $A_\Gamma \cong \pi_1(K)$ acts geometrically (via deck transformations) on a CAT(0) space. In other words, A_Γ is CAT(0).

As noted before, the difficulty is to understand the geodesics in L_Γ. We begin with a study of its local geometry.

11. LINKS OF EDGES

Lemma 11.1. *Suppose (w, ϵ) is a vertex of L_Γ and suppose E_w is the edge in K labelled by the allowable element w. Then $Lk((w, \epsilon), L)$ is isometric to $Lk(E_w, K)$, the geometric link of the edge E_w.*

The geometric link of an edge in an M_0-complex K is, by definition, the M_1-complex determined by all initial unit tangent vectors of geodesics in K which are orthogonal to the edge. More formally, it is the M_1-complex whose cells are defined as follows: Suppose C_λ is convex polyhedral cell of K which contains an edge E_λ that is attatched to E. Then there is a (possibly empty) cell, C_λ^-, of $Lk(E, K)$ which is, by defintion, the convex M_1-cell determined by the inital unit tangent vectors of geodesic rays from $x \in int(E_\lambda)$ into C_λ which are orthogonal (in M_0^n) to E_λ. (Here, we choose x to be some interior point of E_λ.) Every cell of $Lk(E, K)$ arises in this way. Two such cells $C_{\lambda(1)}^-$ and $C_{\lambda(2)}^-$ are glued along the common face $(C_{\lambda(1)} \cap C_{\lambda(2)})^-$ whenever $C_{\lambda(1)}$ and $C_{\lambda(2)}$ are both attached to the edge $E \subset K$. The metric on $Lk(E, K)$ is defined by the intrinsic pseudometric.

Proof. (of 11.1) Let $(w, 1) \in L$. Let $E_w \subset K$ be the edge of K labelled by the allowable element w. The initial unit tangent vector of a geodesic ray based at v_0 that traverses E_w with the same orientation ($\epsilon = 1$) defines $(w, 1)$. Choose a point x in the interior E_w which lies in the initial segment of the geodesic ray. The points of $Lk(E_w, K)$ correspond to initial unit tangent vectors of rays based at x which are orthogonal to E_w. If x is sufficiently close to v_0, then we may regard these vectors as tangent to the spheres of radius $d(v_0, x)$ centered at v_0 appearing in each cell that attaches to the edge E_w. Dilating these spheres until they have radius one, we can identify these vectors (points in $Lk(E_w, K)$) with the initial unit tangent vectors of geodesics in $Lk((w, 1), L)$ based at $(w, 1)$. This defines a continuous map $Lk(E_w, K) \to Lk((w, 1), L)$. This map is easily seen to have a continuous inverse by reversing the process—given an initial tangent vector of a geodesic ray in $Lk((w, 1), L)$, for each cell that attaches to E_w, regard the vector as being tangent to the unit tangent sphere at v_0. Then shrink the unit tangent sphere until it meets an interior point of the edge attaching to E_w. The scaled sphere, regardless of how much it is scaled, determines the same point in $Lk(E_w, K)$.

To see that this map is an isometry, it suffices to check that corresponding edges are assigned the same (spherical) length. A pair of adjacent vertices in $Lk(E_w, K)$ occurs whenever there are two faces F_1 and F_2 of cell C which are attached at E_w: $F_1 \cap F_2$ is glued to E_w. The distance in C between these two vertices is defined to be the dihedral angle between the faces. On the other hand, if we consider the hyperplanes supported by each F_i ($C \subset M_0^n$) and consider their intersection with the appropriate unit tangent sphere, we can see that the corresponding vertices of $Lk((w, 1), L)$ are separated by a distance equal to this dihedral angle. The argument for $\epsilon = -1$ is the same. In particular, we have shown that $Lk((w, 1), L) \cong Lk(E_w, K) \cong Lk((w, -1), L)$. □

Notation. The link L is a simplicial complex; so, the vertices in $Lk((w, \epsilon), L)$ are naturally labelled by another vertex in L, i.e. they can be labelled by an element $(w, \epsilon) \in Allow(W) \times \{\pm 1\}$. This, in turn, defines a labelling of the vertices in the link of an edge of K: Given $E_w \subset K$, the label of a vertex of $Lk(E_w, K)$ is chosen to be the same as the label on the corresponding vertex of $Lk((w,1),K)$. This is illustrated in Figure 10.

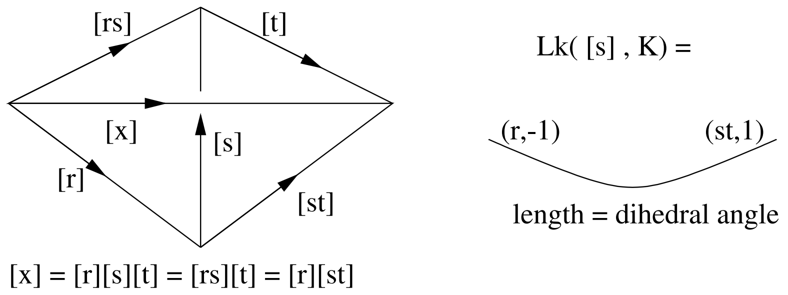

FIGURE 10. Labeling the vertices in the link of an edge of K.

Now we begin an analysis of the link of an edge E_w. There are three cases according to the length of w.

Case 1: Let $w \in \mathit{Allow}(W)$ have length one. So, w is a reflection; write $w = r$. We can enumerate the vertices of $Lk(E_r, K)$ by finding all the 2-cells which are attached to E_r. These 2-cells correspond to precisely those allowable expressions of length two (w_1, w_2) for which $r = w_1$ or w_2. Thus, (r, w_2) gives rise to the vertex $(rw_2, 1) \in Lk(E_r, K)$ and (w_1, r) gives rise to the vertex $(w_1, -1) \in Lk(E_r, K)$. We have the following characterization:

(1) $(w,1) \in Lk(E_r, K) \iff r < w$, and
(2) $(w,-1) \in Lk(E_r, K) \iff wr$ is allowable.

Similarly, we can enumerate all the edges of $Lk(E_r, K)$ by considering all three cells which are attached to E_r. These correspond to allowable expressions (w_1, w_2, w_3) in which $r = w_i$ for some i. As we will be most interested in which edges share a common vertex, we will use a more suitable notation. The allowable expression (w_1, w_2, w_3) is a product of three reflections, one of which is r. Their product is an allowable element $x = x_T$, for some $T \in \mathcal{S}$. To understand the general picture, it suffices to consider three overlapping R-reduced words: $x = pqr = qrs = rst$. These words define allowable expressions (p, q, r), etc. The edges in $Lk(E_r, L)$ corresponding to these expressions are listed below (refer to Figure 11):

(1) (p, q, r) gives rise to an edge from $(pq, -1)$ to $(q, -1)$ in the link of E_r, i.e. an edge from $(xr, -1)$ to $(q, -1)$, where $q < xr$. This edge has length $\pi/4$.

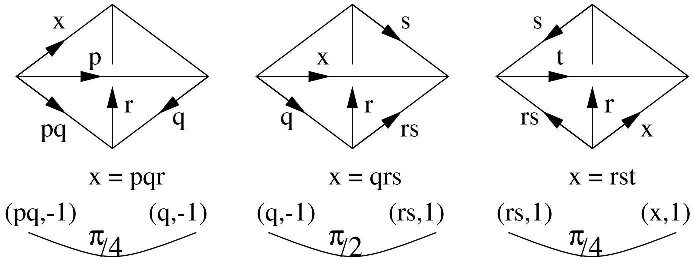

Figure 11. The edges of $Lk(E_r, L)$.

(2) (q, r, s) gives rise to an edge from $(q, -1)$ to $(rs, 1)$ in the link of E_r, i.e. an edge from $(q, -1)$ to $(y, 1)$, where $qy = x$ and $r < y$. This edge has length $\pi/2$.

(3) (r, s, t) gives rise to an edge from $(rs, 1)$ to $(rst, 1)$ in the link of E_r, i.e. an edge from $(y, 1)$ to $(x, 1)$, where $r < y < x$. This edge has length $\pi/4$.

Lemma 11.2. *Let (W, S) be three dimensional Coxeter system with a total ordering S. Let $T \in \mathcal{S}$ have cardinality three. Suppose $r < x_T$. Then $Lk(E_r, K_T)$ is isometric to the spherical suspension $\{(x_T, 1), (xr, -1)\} * \{p_1, \ldots, p_k\}$, where each p_i corresponds to a unique allowable rotation $y \in Allow(x_T; 2)$ such that $r < y$.*

$X * Y$ denotes the spherical join of two M_1-complexes. (Refer to [BH] for precise definitions.) When X has just two elements this is called the spherical suspension. The spherical join of two CAT(1) spaces is a CAT(1) space. So, as a corollary, we have the following:

Corollary 11.3. *Let (W, S) be a three dimensional Coxeter system with a total ordering of S and let $T \in \mathcal{S}$. Then $Lk(E_r, K_T)$ is CAT(1).*

Proof. (of Lemma 11.2) There is a unique allowable element of length three, namely $x = x_T$; and there is a unique allowable element of length two y such that $yr = x$, namely $y = xr$. Among the vertices in $Lk(E_r, K_T)$, $(x, 1)$ is the only vertex of length three and $(xr, -1)$ is the only negative vertex of length two. There are no positive vertices of length one. So, the remaining vertices are of the form $(y, 1)$ where $r < y < x$ or $(q, -1)$ where $qr \in Allow(x_T)$. Given $r < y < x$, we can write $x = rst$, reduced, where $y = rs$. By shifting, we get a reduced word $x = (rstsr)(r)(s)$; let $q = rstsr = xy^{-1}$. So, the vertices $(y, 1)$ and $(q, -1)$ arise in pairs; and $qy = x$. From the reduced expressions $x = rst = qrs = pqr$, where $pq = xr$, we obtain edges in $Lk(E_r, K_T)$ from $(x, 1)$ to $(y, 1)$ to $(q, -1)$ to $(xr, -1)$. (Refer to the characterization of the edges in Figure 11. In Figure 12, we have illustrated an alternate point of view: the link of the link). These are all the edges. If (w, ϵ) is adjacent to $(x, 1)$, then, by the characterization above, $r < w < x$ and $\epsilon = 1$. Suppose $r < y$; then (w, ϵ) is adjacent to $(y, 1)$ if and only if $y < w$ and $\epsilon = 1$ or $wy = x$ and $\epsilon = -1$. So, w is, respectively, either x or xy^{-1}. The vertices adjacent to $(yr, -1)$ or $(xr, -1)$ can be characterized by similar arguments.

Now, let p_y be the midpoint of the edge from $(y, 1)$ to $(xy^{-1}, -1)$. The distance from p_y to $(x, 1)$ or to $(xr, -1)$ is exactly $\pi/2$. Thus, $Lk(E_r, K_T)$ is the spherical suspension over these points with poles $(x, 1)$ and $(xr, -1)$. □

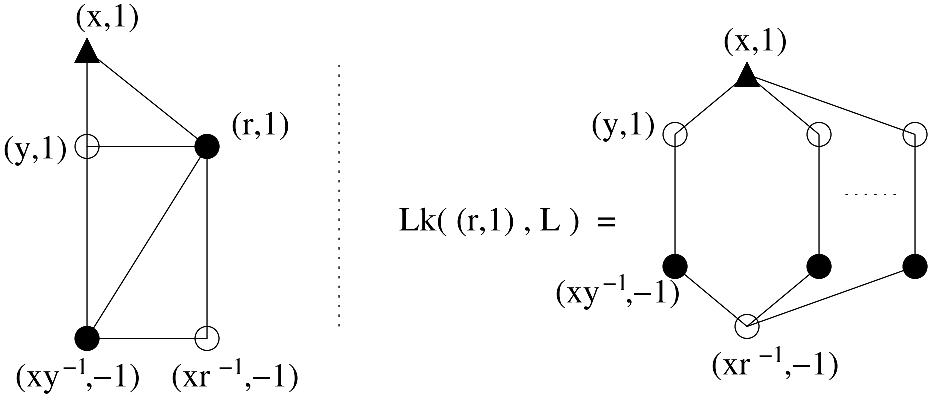

FIGURE 12. The link of the link is isometric to the link of the edge.

To describe the link of E_r in the entire complex K, we consider the intersections of subcomplexes.

Lemma 11.4. *Let $T, T' \in \mathcal{S}$ be distinct and of cardinality three. Suppose $r < x_T, x_{T'}$. Then $Lk(E_r, K_T) \cap Lk(E_r, K_T)$ is empty if $|T \cap T'| < 2$ and is equal to $\{(x_{T\cap T'}, 1), (x_{T\cap T'}r, -1)\}$ if $|T \cap T'| = 2$. In the latter case, the two vertices in the intersection lie distance π apart in each of the links.*

Proof. Suppose $(w, 1)$ belongs to each link. As $x_T \neq x_{T'}$, w must have length < 3. By the characterization of vertices, we have that $r < w$ and $w < x_T, x_{T'}$. So, by Proposition 3.9, $w = x_{T\cap T'}$ and $\ell(w) = 2$. In particular, $|T \cap T'| = 2$. On the other hand, if $(w, -1)$ is a vertex common to each link, then wr has length $= 2$ and $wr < x_T, x_{T'}$. Again, by Proposition 3.9, $wr = x_{T\cap T'}$.

To see that these two common points lie distance π apart in each link, simply observe that they are not joined by an edge. (The product

$x_{T\cap T'} r x_{T\cap T'}$ would have to equal both x_T and $x_{T'}$). Thus, the two points lie on different great arcs in the suspension. From the description of the suspension, it is easy to see that they are joined by a geodesic of length π. □

Theorem 11.5. *Let* (W, S) *be a three dimensional FC Coxeter system together with a total ordering of* S*. Suppose* $r \in Allow(W; 1)$ *is an allowable reflection. Then* $Lk(E_r, K)$ *is CAT(1).*

Proof. As $Lk(E_r, K)$ is a 1-complex, it suffices to show that it contains no short loops (length $< 2\pi$). Suppose that γ is a geodesic loop. If γ is contained in a subcomplex $Lk(E_r, K_T)$ for some $T \in \mathcal{S}$, then γ has length equal to 2π. This follows from Lemma 11.2. So, we may assume that γ meets the interior of two or more subcomplexes of the form $Lk(E_r, K_T)$, where $|T| = 3$ and $T \in \mathcal{S}$. The loop γ consists of two or more *segments* which join vertices common to these subcomplexes. (We define a *segment* to be any edge path which joins two vertices which could possibly belong to two distinct complexes of the form $Lk(E_r, K_T)$ with $|T| = 3$ and $T \in \mathcal{S}$. The vertices common to these subcomplexes will be called *singular* to refelect the fact that geodesics do not admit unique continuations through them.) The segments in $Lk(E_r, K)$ take one of the following three forms:

1. an edge path from $(y, 1)$ to $(x_T, 1)$ to $(y', 1)$, where $y, y' < x_T$
2. an edge (path) from $(y, 1)$ to $(x_T y^{-1}, -1)$, where $y < x_T$
3. an edge path from $(x_T y^{-1}, -1)$ to $(x_T r, -1)$ to $(x_T y'^{-1}, -1)$, where $y, y' < x_T$.

All of these edge paths have length $\pi/2$. So, we only need to rule out loops consisting of two or three segments. (One segment cannot form a loop— the path would not be geodesic.)

An edge path of two segments must consist of segments of the same form, and these segments must lie in a distinct subcomplexes. The first and third are not possible (use Proposition 3.9— the common vertices would be the same). The second would not be geodesic— it traverses the same edge twice.

An edge path of three segments must lie in either two or three distinct subcomplexes. In the first case, let the complexes be indexed by T and T'. There are two common singular vertices along the path, and each is labelled by an element of $W_{T\cap T'}$. The vertices cannot have the same sign (as before, use Proposition 3.9). Then the negative vertex right multiplied by the positive vertex defines an allowable element of length three. This is impossible— the product belongs to $W_{T\cap T'}$.

In the second case, there are three spherical subsets, T, T', and T'', and there are three common vertices. These subsets intersect pairwise in subsets of cardinality two (Lemma 11.4). In light of the FC condition and the hypothesis of dimension three, the link complex spanned by T, T', and $T'' \in \Gamma$ must form a complex consisting of three triangles with a common edge. (Refer to Figure 13 where we have illustrated the ways in which T, T', and T'' might intersect.) It follows that $|T \cap T' \cap T''| = 2$. An edge loop which consists entirely of segments of the first type or entirely of segments of the second type gives rise to three common rotations: $x_{T\cap T'}, x_{T'\cap T'}, x_{T'\cap T''}$. But the indices are all the same. So, the path cannot be geodesic. And edge loop which uses different segments must join either two positive and one negative common vertex or two negative and one positive vertex. In the case of two positive common vertices, they are each labelled by some $x_{T(i)\cap T(j)}$; but $|T \cap T' \cap T''| = 2$, so they are the same. So, the path is not geodesic. In the other case, the vertices are labelled by elements of the form $x_{T(i)\cap T(j)}r$. Again, they are the same, and the path cannot be geodesic.

Thus, every edge loop must consist of at least four segments, i.e. every geodesic loop has length $\geq 2\pi$. So, by the Local to Global theorem, $Lk(E_r, K)$ is CAT(1). □

Remark. Observe that if we omit the FC condition, the K, with its given metric, need not be locally CAT(0). For instance, let W_Γ be defined by a link complex Γ which is the boundary of a three simplex. There is a short loop of length $3\pi/2$ in $Lk(E_r, K)$ which consists of segments of the first form.

Case 2: Now we turn to the second type of edge in K, where E_w is labelled by an allowable element of length 2 (a rotation). Write $w = y$. As before, we characterize the vertices and edges in the complex $Lk(E_y, K)$. Each vertex corresponds to a 2-cell labelled by an allowable expression (w_1, w_2) which is attached to E_y. Thus, $y = w_1, w_2$, or w_1w_2. The 2-cell labelled by (y, w_2) contributes the vertex $(yw_2, 1) \in Lk(E_y, K)$. Similarly, (w_1, y) contributes $(w_1, -1)$, and (w_1, w_2), where $y = w_1w_2$ contributes $(w_1, 1)$. So, the vertices fall into exactly one of the following categories:

(1) $(x, 1) \in Lk(E_y, K) \iff y < x$,
(2) $(q, -1) \in Lk(E_y, K) \iff qy \in Allow(W; 3)$.
(3) $(r, 1) \in Lk(E_y, K) \iff r < y$.

As usual, the letters q, r, s, and t represent reflections and x represents an allowable element of length three (a Coxeter element for some $T \in \mathcal{S}$ such that $|T| = 3$).

Edges correspond to allowable expressions (w_1, w_2, w_3) of length three for which $y = w_1w_2$ or $y = w_2w_3$. It suffices to consider two R-reduced expressions of length three: $x = qrs = rst$, where $y = rs$. The lengths of the edges are determined by the dihedral angles between faces. The corresponding 3-cells in K give rise to the following edges in $Lk(E_r, K)$:

(1) A reduced expression $x = qrs = qy$ corresponds to an edge from $(q, -1)$ to $(r, 1)$, i.e. and edge from $(xy^{-1}, -1)$ to $(r, 1)$ where $r < y < x$. This edge has length $\pi/2$.
(2) A reduced expression $x = rst = yt$ corresponds to an edge from $(r, 1)$ to $(x, 1)$, where $r < y < x$. This edge has length $\pi/2$.

Lemma 11.6. *Let (W, S) be a three dimensional Coxeter system with a total ordering of S. Let $T \in \mathcal{S}$ have cardinality three. Suppose $y < x_T$ is an allowable rotation. Then $Lk(E_y, K_T)$ is isometric to the spherical suspension $\{(x_T, 1), (x_T y^{-1}, -1)\} * \{(r, 1) : r < y\}$. Thus, $Lk(E_y, K_T)$ is CAT(1).*

Proof. $(w, 1) \in Lk(E_y, K_T)$ if and only if $w < y$ or $y < w$. Only $w = x_T$ satisfies the second condition, and only reflections $w = r < y$ satisfies the first. $(w, -1) \in Lk(E_y, K_T)$ if and only if (w, y) is an allowable expression of length three if and only if $w = x_T y^{-1}$; so this vertex is unique. We have already enumerated all the edges between such vertices: there is an edge of length $\pi/2$ from $(x_T, 1)$ to each $(r, 1)$ such that $r < y$ and there is an edge of length $\pi/2$ from each $(r, 1)$ to $(x_T y^{-1}$ because $r < y < x$. Thus, we have described $Lk(E_y, K_T)$ as a spherical suspension with poles $(x_T, 1)$ and $(x_T y^{-1}, -1)$. It is the suspension of two CAT(1) spaces (discrete sets); so it is CAT(1). □

The singular points in $Lk(E_y, K)$ are easy to describe:

Lemma 11.7. *Let (W, S) be a three dimensional Coxeter system together with a total ordering of S. Let $T, T' \in \mathcal{S}$ be distinct and of cardinality three. Suppose $y < x_T, x_{T'}$. Then $Lk(E_y, K_T) \cap Lk(E_y, K_{T'}) = \{(r, 1) : r < y\} = R_{T \cap T'}$. These common points lie distance π apart in each of the links.*

Proof. As $x_T \neq x_{T'}$, the common vertices must have the form $(r, 1)$ where $r < y$. These points lie at the equator of the suspensions; thus, two such points are always distance π apart. In fact, by Proposition 3.9, we have that $y = x_{T \cap T''}$. Every reflection in $R_{T \cap T''}$ is y-allowable. □

Theorem 11.8. *Let (W, S) be a three dimensional Coxeter system together with a total ordering of S. Suppose $y \in Allow(W; 2)$ is an allowable rotation. If y is a maximal element ($\nexists\, w \in Allow(W)$ such that $w > y$), then $Lk(E_y, K)$ is the discrete set $\{(r, 1) : r < y\}$ and $y = x_T$ for some $T \in \mathcal{S}$ of cardinality two. If y is not a maximal element, then*

$Lk(E_y, K) \cong \{(x_T, 1), (x_T y^{-1}, -1) : y < x_T\} * \{(r, 1) : r < y\}$. In either case, $Lk(E_y, K)$ is CAT(1).

Proof. The case of y maximal is obvious from the characterization of vertices. A discrete space is CAT(1). If y is not maximal, then there are subcomplexes $Lk(E_y, K_T)$ such that $y < x_T$. The union of these subcomplexes is $Lk(E_y, K)$. Their common intersection is precisely the set $\{r < y : r \in R_T\}$. The description of $Lk(E_y, K)$ as a spherical suspension follows. In particular, the complex is CAT(1). □

Remark. We have shown that if y is not maximal, then $Lk(E_y, K)$ has diameter π. That is, every pair of points in the complex lie distance $\leq \pi$ apart. This follows from the fact that the complex is a spherical suspension. If y is maximal, then $Lk(E_y, K)$ is discrete. These observations will be useful later on for describing the paths in L which are locally geodesic at the vertex $(y, \epsilon) \in L$.

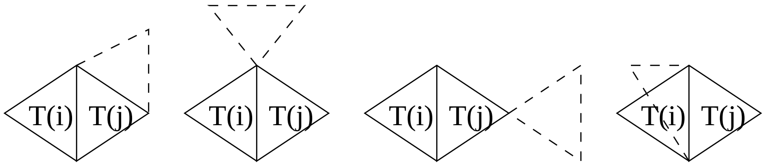

FIGURE 13. How three (maximal) spherical subsets can define a three dimensional FC Artin group.

Case 3: Finally, consider the third type of edge in K, where E_w is labelled by an allowable element of length three, i.e. $w = x_T$ for some spherical subset of cardinality three. Write $x_T = x$. The vertices of $Lk(E_x, K)$ correspond to allowable expression of length two (w_1, w_2) such that $w_1 w_2 = x$. So, the vertices have the form $(w_1, 1)$, where $w_1 < x$. These vertices are labelled by either a reflection $r < x$ or a rotation $y < x$. There is an edge from $(r, 1)$ to $(y, 1)$ if and only if $r < y < x$; the length of this edge is $\pi/3$. Observe that $Lk(E_x, K) = Lk(E_x, K_T)$.

Theorem 11.9. *Let (W, S) define a three dimensional FC Coxeter system together with a total ordering of S. Let $x = x_T \in Allow(W; 3)$. Then $Lk(E_x, K)$ is CAT(1).*

Proof. As the link is equal to the subcomplex $Lk(E_x, K_T)$, the argument is identical to T. Brady's argument in [Br1]. Brady's observation is that the complex is bipartite graph all of whose edges have length $\pi/3$. So, it suffices to eliminate short loops which are made up of two or four edges. A loop of two edges would not be geodesic. A loop of four edges corresponds to the algebraic problem $r, q < y, z < x$. The reflections, r and q, and the rotations, y and z, belong to the finite three generator Coxeter group W_T. Considering geometric representation $\sigma : W_T \to O(\mathbb{R}^3)$, we see that the intersection of the hyperplanes corresponding to r and q is the axis of rotation for both y and z. There is only one such allowable rotation; so $y = z$ and the path is not geodesic. Thus, every loop is made up of at least six edges, and, hence, has length $\geq 2\pi$. □

Thus, we have verified that the link of each edge in K is a CAT(1) space. By Lemma 11.1 and the Link Condition, we conclude that

Theorem 11.10. *Suppose Γ defines a three dimensional FC Artin group together with a total ordering of the generating set. Then L_Γ is locally CAT(1).*

The remainder of the paper is dedicated to proving that L_Γ is globally CAT(1) whenever Γ defines a three dimensional FC Artin group. First, we show that there do not exist short loops inside the 1-skeleton of L. Next, we will show that if γ is an isometrically embedded loop of minimal length in L, then it can be “rotated”, preserving it's length, into the 1-skeleton of L. Thus, the analysis of the 1-skeleton of L is sufficient to rule out all short loops that might occur in L.

12. Basic gluing of CAT(1) spaces

Recall that a subspace Y of a geodesic metric space (X, d) is *r-convex* if every pair of points $x, y \in Y \subset X$ such that $d(x, y) < r$ may be joined by a geodesic segment, and, moreover, every such segment lies in Y.

Theorem 12.1. *(Basic Gluing of CAT(1) Spaces) Let X_1 and X_2 be CAT(1) spaces and let Y be a complete metric space. Suppose we are given π-convex subspaces $A_i \subset X_i$ and isometries $\phi_i : Y \to Y_i \subset X_i$ for $i = 1, 2$. Then the space obtained by gluing X_1 and X_2 along Y, denoted by $X := X_1 \sqcup_Y X_2$, is CAT(1).*

The proof may be found in [BH]. The idea is to use Aleksandrov's Lemma. Basically, the lemma says that if two triangles satisfy the CAT(1) inequality, then so does a triangle obtained gluing the two given triangles together along an isometric edge. The hypotheses of Theorem 12.1 guarantee that every triangle in X of perimeter $< 2\pi$ may be decomposed into two triangles which each lie in either X_1 or X_2.

By applying Basic Gluing, we will prove that L_Γ is CAT(1) whenever the Coxeter graph Γ is sufficiently simple. For such a Coxeter graph, it will be relatively easy to decide that the subcomplexes Y_i are π-convex. More precisely, successive application of either of the following two lemmas will apply:

Lemma 12.2. *Let X_i be CAT(1) and let $Y_i \subset X_i$, $i = 1, 2$. Assume that Y_1 and Y_2 are finite discrete sets of the same cardinality. If every pair of points $x, y \in Y_i$ are distance $\geq \pi$ apart in X_i, then Basic Gluing applies to $X_1 \sqcup_{Y_1 = Y_2} X_2$.*

Proof. Each Y_i is trivially a π-convex subspace of X_i. Apply Basic Gluing. ☐

Lemma 12.3. *Let Y and X_1 be connected CAT(1) spaces. Suppose we are given a continuous bijection $\phi_1 : Y \to Y_1 \subset X_1$ which takes local geodesics to local geodesics. If Y has diameter $\leq \pi$ then ϕ_1 is an isometry and Y_1 is a π-convex subspace of X_1.*

Proof. Let $x, y \in Y$. Let λ parameterize a geodesic segment from x to y. Then $\phi_1 \circ \lambda$ parameterizes a locally geodesic segment in X of length $\leq \pi$. As X is CAT(1), this segment is, in fact, a geodesic. (In a CAT(1) space, local geodesics of length $\leq \pi$ are (global) geodesics.) Hence, ϕ is an isometry. So, every pair of points in Y_1 may be joined by a geodesic which lies in Y_1. As geodesics in a CAT(1) space of length $< \pi$ are unique, every geodesic, which joins a pair of points in Y_1 which are distance $< \pi$ apart in X_1, is contained in Y_1. Hence, Y_1 is a π-convex subspace of X_1. ☐

13. Simple three dimensional FC Artin systems

We will prove that if Γ is sufficiently simple, then L_Γ is CAT(1). We begin by recalling what is known about the (global) curvature of L_T, the link of the Brady complex K_T associated to a spherical Coxeter group W_T.

Theorem 13.1. *(T. Brady, J. McCammond) If (W_T, T) defines a spherical Coxeter group with one, two, or three generators, then L_T is CAT(1).*

The one generator case is trivial, the two generator case was studied by T. Brady and J. McCammond in [BM], and the three generator case is the subject of T. Brady's article [Br1]. Each link decomposes as a spherical suspension:

If $|T| = 2$, then $L_T \cong \{\, (y, 1), (y, -1) \,\} * \{\, p_r : r < y \,\}$, where $y = x_T$. Thus, L_T is CAT(1). The point p_r is the midpoint of the edge

$[(yr,-1),(r,1)]$. The longitudinal arcs in the suspension are the union of three edges: $[(y,-1),(yr,-1)]\cup[(yr,-1),(r,1)]\cup[(r,1),(y,1)]$. In particular, L_T has diameter π.

If $|T|=3$, then $L_T\cong\{(x,1),(x-1)\}*Lk(E_x,K_T)$, where $x=x_T$. Thus, L_T is CAT(1).

Now, consider the intersection of such subcomplexes in L:

Lemma 13.2. *Let (W,S) be a Coxeter system of dimension ≤ 3. Suppose $T,T'\in\mathcal{S}$ are distinct and $T\cap T'\neq\emptyset$. Then the vertices common to L_T and $L_{T'}$ are the elements of $Allow(x_{T\cap T'})\times\{\pm1\}$.*

Proof. Suppose (w,ϵ) is a vertex of L_T and $L_{T'}$. Then $w\le x_T,x_{T'}$. If $\ell(w)=|T\cap T'|$, then, by Proposition 3.9, $w=x_{T\cap T'}$. Otherwise, $\ell(w)<|T\cap T'|\le 2$; so w is a reflection in $R_T\cap R_{T'}=R_{T\cap T'}$. Every such reflection is $x_{T\cap T'}$-allowable. ☐

Lemma 13.3. *Let (W,S) be a Coxeter system of dimension ≤ 3. Suppose $T,T'\in\mathcal{S}$ are distinct and $|T\cap T'|=2$. Then the edges common to L_T and $L_{T'}$ are precisely the edges of $L_{T\cap T'}$. If $|T\cap T'|<2$, then there are no common edges.*

Proof. The second statement is just a dimension count. For the first statement, we only need to show that there are no more edges than those in $L_{T\cap T'}$. If there is an edge $[(w,\epsilon),(w',\epsilon)]$ in common, then $w,w'\le x_{T\cap T'}$. So, each is either a reflection in $R_{T\cap T'}$ or equal to $x_{T\cap T'}$. Let $y:=x_{T\cap T'}$. According to the characterization of edges, there are, naively, five possible types:

$$[(r,-1),(y,-1)],\quad [(r,-1),(q,1)],\quad [(r,1),(y,1)]$$
$$[(y,-1),(q,1)],\quad [(r,-1),(y,1)],$$

where $r,q\le y$. The first three (left to right) necessarily lie in $L_{T\cap T'}$ and the final two are not possible. (The product of the negative multiplied on the right by the positive defines an allowable element of length three; but the product belongs to $W_{T\cap T'}$.) ☐

Combining the above two lemmas, we conclude that the subcomplex $L_T\cap L_{T'}$, with its intrinsic metric, is isometric to $L_{T\cap T'}$.

Lemma 13.4. *Let (W,S) be an FC Coxeter system of dimesion ≤ 3. Totally order S, and let K be the Brady complex and L its link. Let $T\in\mathcal{S}$ and let L_T be the link of K_T. Assume that L is CAT(1).*

1. *If $|T|=1$, then L_T is a finite discrete set and every pair of points $x,y\in L_T\subset L$ are distance $\ge\pi$ apart.*
2. *If $|T|=2$, then L_T has diameter $\le\pi$ and the inclusion $L_T\to L$ takes local geodesics to local geodesics.*

The two cases in the Lemma 13.4 correspond to the two cases where we can apply basic gluing (Lemmas 12.2 and 12.3). At first glance, Lemma 13.4 is only applicable in the base cases where (W, S) is a spherical Coxeter system of three or fewer generators; for, only in these, cases do we know that the links are CAT(1). But, given two such links, L_{T_1} and L_{T_2} with $|T_1 \cap T_2| = 1$ or 2, we can apply Basic Gluing along the subcomplex $L_{T_1 \cap T_2}$. Thus, we deduce that $L_{T_1 \cup T_2}$ is CAT(1). This process can continue, but only if we glue along a common spherical subset with one or two generators. In terms of link complexes, if we know that L_{Γ_1} and L_{Γ_2} are CAT(1) and $\Gamma_1 \cap \Gamma_2$ is a single vertex or a single edge, then we can deduce that $L_{\Gamma_1 \cup \Gamma_2}$ is CAT(1). The link complex of an arbitrary three dimensional FC Artin group cannot be so easily described; for instance, the link complex might be the cone on a loop with four or more edges. Nonetheless, this method will be sufficient to detect all short loops in L.

Definition. Let (W, S) be a Coxeter system. We say that $T \in \mathcal{S}$ is a *maximal* spherical subset if it is not a proper subset of any $T' \in \mathcal{S}$. A subcomplex indexed by a maximal spherical subset is called a *maximal subcomplex.* This descriptor may be applied to the complexes L_T or $Lk(E_w, K_T)$.

Proof. (of Lemma 13.4) Let L be the link of the Brady complex of an FC Coxeter system (W, S) of dimension ≤ 3. Let $T \in \mathcal{S}$ have a single element: $T = \{r\}$. Then, $L_T = \{(r, 1), (r, -1)\}$. We need to show that these points lie distance $\geq \pi$ apart from one another inside L. As we are assuming that L is CAT(1) and as every local geodesic of length $\leq \pi$ is a (global) geodesic, it suffices to find a locally geodesic segment from $(r, 1)$ to $(r, -1)$. If T is maximal, then $(r, 1)$ and $(r, -1)$ are connected components of L, and, so, they lie distance $\geq \pi$ apart. Suppose $T \subset T' = \{r, s\}$. Let $y := x_{T'}$.

If T' is maximal, then the connected component of $L_{T'}$ is a 1-complex. Every path along the edges which does not double back on itself is a local geodesic. We find a path γ consisting of the following adjacent edges:

$$[(r, 1), (y, 1)] \cup [(ry, 1), (y, 1)] \cup [(r, -1), (ry, 1)].$$

This path has length $\pi/4 + \pi/4 + \pi/2 = \pi$. Note that as $r < y$, $\ell(ry)$. (Either $y = rs$ or $y = sr$ according to the total order; thus, $ry = s$ or $ry = rsr$.)

If T' is not maximal, then we consider $T'' = \{r, s, t\}$. The connected component of $L_{T''}$ is a piecewise spherical 2-complex. In fact, the same path above remains locally geodesic. We only need to verify that the path γ is locally geodesic at vertices $(y, 1)$ and $(ry, 1)$. We appeal to our analysis of the links $Lk((y,1),L) \cong Lk(E_y,K)$ and $Lk((ry,1),L) \cong Lk(E_{ry},K)$:

There is a (locally geodesic) path in $Lk(E_y,K)$ of length π which joins $(r,1)$ to $(ry,1)$. As $Lk(E_y,K)$ is CAT(1), this path is a geodesic. Thus, the distance between $(r,1)$ and $(ry,1)$ is equal to π. Thus, γ is locally geodesic at $(y,1)$.

Likewise, there is a locally geodesic path in $Lk(E_{ry},K)$ of length π which joins $(y,1)$ to $(r,-1)$. As $Lk(E_{ry},1)$ is CAT(1), this path is a geodesic. Thus, the distance between $(y,1)$ and $(r,-1)$ is equal to π. Thus, γ is locally geodesic at $(ry,1)$. We conclude that γ is a geodesic in L; and, hence, $(r,1)$ and $(r,-1)$ are distance π apart in L.

Now suppose that $T \in \mathcal{S}$ has two elements: $T = \{r,s\}$. As we have already remarked above, L_T has diameter π. To show that the inclusion $L_T \to L$ takes local geodesics to local geodesics, it suffices to compute distances in the links of the link or, equivalently, in the links of the edges. Let (w,ϵ) be a vertex of L_T. We want to show that if (w',ϵ') and (w'',ϵ'') are vertices adjacent to (w,ϵ) in L_T, then the distance between (w',ϵ') and (w'',ϵ'') in $Lk(E_w,K)$ is equal to π. As $Lk(E_w,K)$ is CAT(1), it suffices to find a locally geodesic path from (w',ϵ') to (w'',ϵ'') of length π. There are several cases, but the computations are straightforward and similar to the above calculations. $\square$

Definition. Let Γ define a three dimensional FC Artin group A_Γ together with a total ordering of the generating set S. We say that Γ is *simple* if there are at most three maximal spherical subsets in $\mathcal{S}$.

Theorem 13.5. *Let Γ define an FC Artin group of dimension ≤ 3. If Γ is simple, then L_Γ is CAT(1).*

Proof. Let $T_1,\dots,T_k$ be the maximal spherical subsets, where $k \leq 3$. Each maximal spherical subset defines a top dimensional simplex in Γ. As Γ has dimension ≤ 2, these subsets define either a vertex, an edge, or a 2-simplex. If every simplex is a vertex, then L is discrete and, hence, trivially CAT(1). Suppose that Γ has least one simplex of dimension > 0 and assume that Γ is connected. We observe that $L_\Gamma = \bigcup L_{T_i}$. Moreover, L_Γ may be constructed by gluing (in some order) the T_i's so that each gluing occurs along either a single vertex or a single edge. This follows from the FC hypothesis. Refer to Figure 13 to visualize all such possibilities. There are at most two such stages of gluing, and at each stage, the pieces which are glued are of the form L_{T_i} or $L_{T_i \cup T_j}$. These are glued along intersections of the form $L_{T_i \cap T_j}$ or $L_{T_1 \cap T_2 \cap T_3}$. The links arise from Coxeter systems of dimension ≤ 3 with at most two maximal spherical subsets. The intersections arise from spherical Coxter groups with one or two generators. Thus, we may apply Lemma 13.4 at each stage of the gluing. The resulting complex, namely L_Γ, is CAT(1). If Γ has more than one connected component, then the links of these components are disjoint in L_Γ. A disjoint union of CAT(1) spaces is obviously CAT(1). $\square$

Lemma 13.6. *If L and L' are M_1-simplicial complexes and $L' \subset L$ is a subcomplex, then every closed geodesic $\gamma : S^1 \to L$ such that $\gamma(S^1) \subset L'$ defines a closed geodesic in L' with respect to its intrinsic metric. Therefore, if L' is CAT(1) with respect to its intrinsic metric, then γ has length $\geq 2\pi$.*

Proof. A path in $L' \subset L$ which minimizes the distance in L between points in its image clearly minimizes the distance between these points in L'. Thus, a geodesic in L with image in L' defines a geodesic in L'. In particular, if L' is CAT(1) with respect to its intrinsic metric, then the length of this geodesic must be $\geq 2\pi$. $\square$

We will use this lemma to rule out the existence of short loops in L_Γ. Applying Lemma 13.6 in conjuction with Theorem 13.5, we see that no short loop can be contained in any subcomplex $L_{\Gamma'} \subset L_\Gamma$ such that Γ' is simple.

14. LOCALLY GEODESIC EDGE LOOPS IN L_Γ

By a *locally geodesic edge loop*, we mean the image of a locally isometrically embedded loop which lies entirely in the 1-skeleton of L. We begin by proving that certain certain edge paths in $L^{(1)}$ are not locally geodesic.

Lemma 14.1. *A locally geodesic edge path in L cannot contain a subpath of the form*

$$x_{T(1)} \to y_1 \to x_{T(2)} \to y_2 \to x_{T(3)},$$

where each $T(i) \in \mathcal{S}$ has cardinality 3 and each y_i is allowable of length 2.

The signs of the vertices of this path are all the same; strictly speaking, we mean $(x_{T(1)}, \epsilon)$, etc. In the proof, we assume that all the vertices have positive sign. Thus, we have only labelled the vertices by their allowable element. The argument for the case where the vertices have a negative sign is the same.

Proof. As L is a simplicial complex, there are unique edges joining the vertices. Clearly, a local geodesic cannot "double back" along an edge just traversed. So, we must have that $x_{T(1)} \neq x_{T(2)}$ and $x_{T(2)} \neq x_{T(3)}$.

Using the same reasoning, we see that y_1 and y_2 must be distinct. According to Proposition 3.9, $y_1 = x_{T(1)\cap T(2)}$ and $y_2 = x_{T(2)\cap T(3)}$.

Suppose that $T(2) = \{a \prec b \prec c\}$ and $x_{T(2)} = abc$. Then each y_i must be one of ab, bc or ac. These fit together to make the following 2-cell in $L_{T(2)}$:

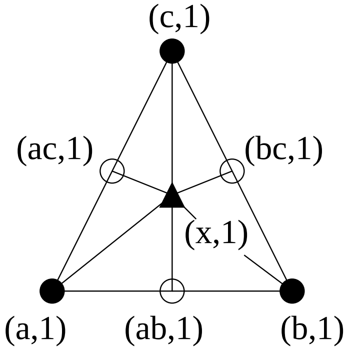

FIGURE 14. The 2-cells of L form an all-right spherical triangle.

y_1 and y_2 must be different allowable rotations of length two from among ab, bc, and ac. Regardless of which particular rotations they are in W_{T_2}, the path $y_1 \to x_{T(2)} \to y_2$ makes an angle of $2\pi/3$ at $x_{T(2)}$; and so, this path is not locally geodesic. So, this configuration cannot appear as a subpath of a locally geodesic edge path in L. $\square$

Lemma 14.2. *Let $L = L_\Gamma$, where Γ defines a three dimensional Coxeter system together with a total ordering of its generating set. Suppose γ is a local geodesic in L which passes through a vertex (y, ϵ), where y is an allowable rotation. Then, either y is not contained in any W_T such that $T \in \mathcal{S}$ and $|T| = 3$, or the distance between incoming and outgoing unit tangent vectors is equal to π.*

Proof. This follows from the fact that $Lk(E_y, K)$ is a suspension— the link has diameter π or is discrete (Theorem 11.8). $\square$

Lemma 14.2 implies that every local geodesic in L which extends an edge from (w, ϵ) to (y, ϵ') (and through this vertex) must first traverse another edge in L. Moreover, if w is a reflection and $\epsilon = \epsilon'$, then the other edge lies in the same subcomplex, L_T, which contained the initial edge. In particular, an edge path of the form $[(r, \epsilon), (y, \epsilon)] \cup [(r', \epsilon), (y, \epsilon)]$ stays within any subcomplex L_T, where $y \leq x_T$. Thus, in some sense, the vertices in L of length two are not "singular".

Remark. There are several different ways to extend a path geodesically through a vertex (r, ϵ) of length one or a vertex (x, ϵ) of length three;

for, the links $Lk(E_r, K)$ and $Lk(E_x, K)$ do not have diameter π. These vertices will be called *singular vertices.*

By a *segment* in L, we mean one of the geodesic edge paths appearing in Figure 15.

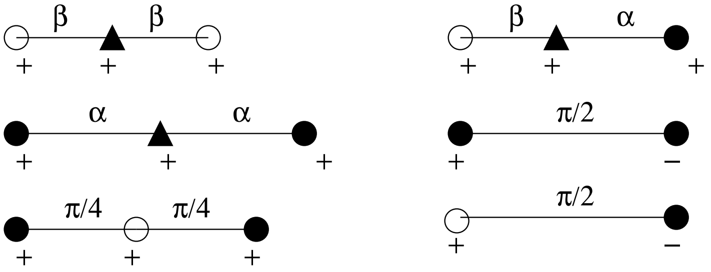

FIGURE 15. The six segments in $L^{(1)}$.

The black squares denote vertices of length one (reflections), the white squares denote vertices of length two (rotations), and the white triangles denote vertices of length three. We have displayed the sign of the vertices. Four of the segments have vertices with the same sign; two have vertices of opposite sign. These polarities may be reversed, changing all positive signs to negative signs. Note that $\alpha + \beta = \pi/2$. It is also helpful to keep in mind the following estimates: $\beta < \pi/4 < \alpha < \pi/2$, $4\beta > \pi/2$, and $\alpha + \beta = \pi/2$.

Proposition 14.3. *Every locally geodesic edge loop in L can be decomposed into segments which meet only at vertices.*

Proof. Every possible edge path consisting of two adjacent edges can be formed by using the list of segments with the exception of the following two:

(1) $\bullet - \circ - \triangle$; all the vertices in this configuration have the same sign. The two edges make a right angle at the center vertex. (The path belongs to the boundary of the 2-cell $[(r,1),(y,1),(x,1)]$.)

(2) $\bullet - \circ - - \bullet$; the two ends have opposite signs and the double dash denotes an edge of length $\pi/2$. The two edges make a right angle at the center vertex. (The edges belong to the boundary of the 2-cell $[(q,-1),(r,1),(y,1)]$.)

So, neither of these edge paths can appear in a locally geodesic edge loop. $\square$

We now try to list all locally geodesic edge loops which have length $< 2\pi$. Observe that each segment in Figure 15 which contains a vertex

x of length three is entirely contained in the subcomplex L_T, where $x = x_T$. Similarly, the segment $\bullet - \circ - \bullet$ where the middle vertex is the allowable rotation y, is entirely contained in the subcomplex $L_{T(y)}$, where $T(y)$ is the smallest spherical subset $T \in \mathcal{S}$ such that $y \in W_T$. Finally, the remaining two segments are labelled by vertices of opposite sign. Suppose the negative vertex is labelled by the allowable element w_1, and the positive vertex is labelled by the allowable element w_2. Then $w := w_1 w_2$ is allowable and the segment is contained in the subcomplex $L_{T(w)}$. Thus, given a locally geodesic edge loop γ, we can study the subcomplexes which it meets by decomposing this loop into segments.

Applying Lemma 13.6 in conjuction with Theorem 13.5, we deduce that γ cannot be a short loop in L unless it consists of at least four segments. If there are three or fewer segments, then the Artin system defined by the union of the subsets T corresponding to the segments would define a simple Artin system. A short loop of at least four segments must contain at least on one segments of the form $\circ - \blacktriangle - \circ$.

But now consider what happens along the path γ at the end vertices of $\circ - \blacktriangle - \circ$. Suppose that the middle vertex of length three is labelled by x_T and that the end vertices are labelled by y_1 and y_2. At each end vertex, (y_i, ϵ), either the next segment of the path lies in the same maximal subcomplex L_T or it lies in a distinct maximal subcomplex $L_{T'}$, where $T' \in \mathcal{S}$ and $|T'| = 3$.

If the segment of γ issuing from y_2 lies in a different maximal subcomplex, then γ contains the edgepath $\circ - \blacktriangle - \circ - \blacktriangle - \circ$. Applying Lemmas 14.1 and 14.2, we see that the segments of γ issuing from the ends of this edgepath are forced to be edges of length $\pi/2$ with vertices of sign opposite to y_2. So, γ contains the edgepath $\bullet - - \circ - \blacktriangle - \circ - \blacktriangle - \circ - - \bullet$. The length of this edgepath is $\pi + 4\beta$. But, this path cannot form a loop– it is contained in the subcomplex determined by two maximal complexes (those indexing the vertices of length three). So, there must be at least one more segment. But, the next segment must have length $\pi/2$. Thus, *gamma* has length $> 3\pi/2 + 4\beta > 2\pi$.

In the other case, the segments of γ which issue from y_1 and y_2 lie in the same maximal subcomplex determined by x_T; and γ contains the edge path $\bullet - - \circ - \blacktriangle - \circ - - \bullet$. The end vertices have sign opposite to that of x_T. γ must contain at least three more segments, else it lies in a subcomplex defined by a simple Artin group. But, the length of such a loop must be $> 2\pi$. This completes the proof of the following theorem:

Theorem 14.4. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set. Let L_Γ be the link of Brady's complex K_Γ. Then L_Γ does not contain any short loops in its* 1*-skeleton.*

15. DEVELOPING GALLERIES ONTO THE SPHERE

The following definitions are due to M. Elder and J. McCammond [EM] and [EM2].

Definition. Let $\gamma : [a, b] \to X$ define a local geodesic in an M_1-simplicial complex X. Let $(\sigma_1, \dots, \sigma_k)$ be the sequence of closed simplices $\sigma \subset X$ such that $\mathring{\sigma} \cap \gamma \neq \emptyset$ (if σ is a vertex, then we define $\mathring{\sigma} = \sigma$). These simplices are ordered according to the order in which γ meets each one. Let $\mathcal{G}(\gamma)$ denote the M_1-simplicial complex defined by gluing σ_i to σ_j if σ_i is a proper face of σ_j and $j = i - 1$ or $i + 1$, where $1 \leq i, j \leq k$. This complex is called the *linear gallery* determined by γ.

If γ defines a closed local geodesic, then we give the sequence of closed simplices $(\sigma_1, \dots, \sigma_k)$ a cyclic ordering and define an M_1-complex as before but allow the first and last cells to be glued along their common face. Such a gallery is called a *circular gallery.*

To each circular gallery $\mathcal{G}$ there is an associated linear gallery $\mathcal{G}'$ obtained by "cutting open $\mathcal{G}$ along σ_i". Consider the cyclically ordered sequence of closed cells, $(\sigma_1, \dots, \sigma_k)$. Choose a fixed σ_i, and consider the sequence $(\sigma_i, \dots, \sigma_k, \sigma_1, \dots, \sigma_{i-1}, \sigma_i)$. We define $\mathcal{G}'$ to be the linear gallery defined by this sequence.

For each gallery $\mathcal{G}$, there is a unique locally geodesic path (or loop) defined by gluing the maps $\gamma|_{sigma_i}$. The resulting path is called the *lift* of γ.

Theorem 15.1. *(M. Elder, J. McCammond) Let γ be a local geodesic path (or loop) in a* 2*-dimensional M_1-complex X. Then the interior of the linear (or circular) gallery γ immerses into X and retracts onto the lift of γ.*

The proof of Theorem 15.1 can be found in [EM2].

Definition. (Developing a circular gallery onto the sphere) Let $\gamma : [0, h] \to X$ define a closed local geodesic. Choose a parametrization so that $\gamma(0)$ belongs to an edge or vertex of X. Let $\mathcal{G}$ be the gallery determined by γ, and let $\mathcal{G}'$ be the linear gallery obtained by cutting open $\mathcal{G}$ along the closed cell containing $\gamma(0)$ in its interior. Denote the lift of γ by $\hat{\gamma}$. Fix a point p (pole) in the unit sphere and fix an oriented great arc from p to the antipodal point $-p$. Let $\phi : \mathcal{G}' \to M_1^2$

be defined by mapping $\hat{\gamma}(0)$ onto the midpoint of the oriented great arc. If we insist that $\hat{\gamma}$ define a local geodesic and that it make an (oriented) angle of 90 degrees with the oriented great arc, then the map ϕ is uniquely determined. ϕ is called a *developing map.* We say that ϕ *develops* $\mathcal{G}$ onto the sphere.

The key fact about a the developing map is that, by construction, the lift of γ is mapped to a great arc (or circle) on the unit sphere. In particular, if $\phi(\hat{\gamma})$ meets any other great arc in two points, then γ must have length $\geq \pi$.

Local geodesics in $L = L_\Gamma$ develop in a very special way onto the 2-sphere. Let $\mathbb{S}^2$ denote the unit 2-sphere, M_1^2, together with the following simplicial structrure: First divide the sphere into eight spherical triangles by intersecting with the usual coordinate planes. Each of these triangles is a spherical triangle with all lengths and angles measuring 2π (an *all-right* triangle). Then, pass to the barycentric subdivision. The resulting M_1-simplicial complex has 48 spherical triangle, each of which is isometric to the 2-cell of L with edge lengths $\beta, \pi/4$, and α (the 2-cells labelled by allowable elements satisfying $r < y < x$). The other 2-cells of L are isometric to subcomplexes of $\mathbb{S}^2$ (refer to Figure 16). The 2-cells with edge lengths $\pi/4, \pi/2, \pi/2$ are isometric to one half of an all-right triangle; it is isometric to a subcomplex of $\mathbb{S}^2$ formed by three adjacent 2-simplices.

Recall that the vertices of L of length one or three are called singular vertices. The link of these vertices in L have diameter $> \pi$.

Proposition 15.2. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose $\gamma : [0, h] \to L$ is a local geodesic. Assume that either γ does not meet any singular vertices or that such vertices only occur at its endpoints. Then γ determines a gallery which develops onto a subcomplex of $\mathbb{S}^2$.*

Proof. Let $\mathcal{G}$ be the gallery determined by γ. If γ is closed, then let $\mathcal{G}'$ be the linear gallery obtained by cutting open $\mathcal{G}$ along the closed cell containing $\gamma(0)$ in its interior. Choose a point p (pole) in $\mathbb{S}^2$ and an oriented great arc from p to the antipodal point $-p$ so that the initial cell of the gallery maps to a simplex. We may adjust the choice of pole and oriented arc so that the first top dimensional cell crossed by γ develops onto an isometric simplex in $\mathbb{S}^2$. (The dimensions of the cells in $\mathcal{G}$ alternate going up and down. Insist that the larger of the first two cells be mapped to a simplex).

Once this choice is made, the developing map is determined by the condition that the lift of γ be geodesic in $\mathbb{S}^2$. But as γ does not meet

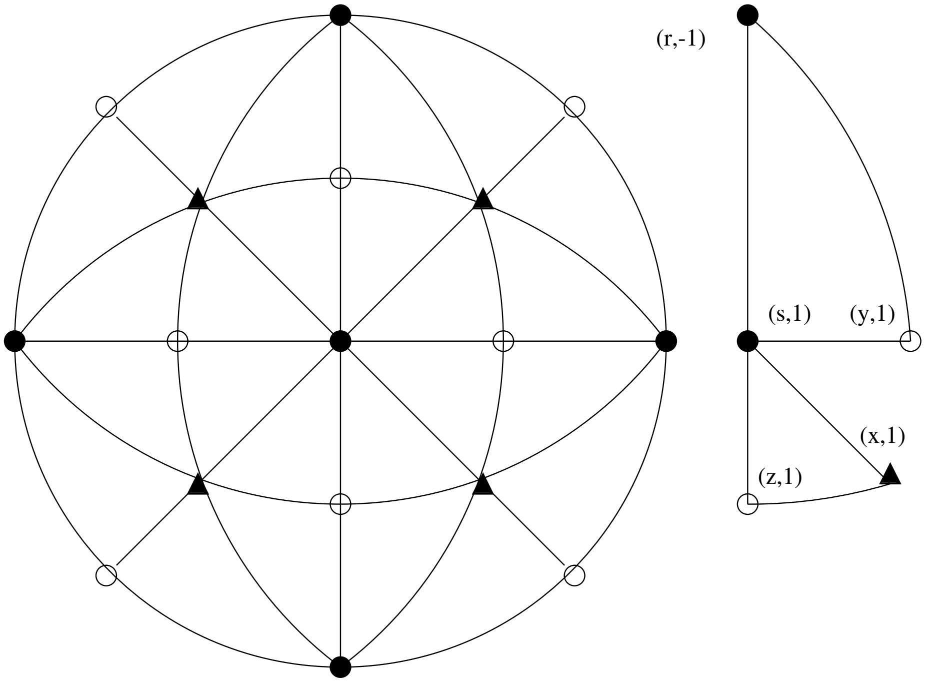

FIGURE 16. A top view of the simplicial complex $\mathbb{S}^2$. To the right are shown the two types of spherical 2-simplices which occur in L. Each simplex is isometric to a subcomplex of $\mathbb{S}^2$

any singular points between time 0 and h, the gallery $\mathcal{G}'$ develops onto a subcomplex of $\mathbb{S}^2$. This follows from the fact that the link of any nonsingular point in L has components which are discrete or of diameter π. We have already observed that the link of a vertex labelled by an allowable rotation has this property (Theorem 11.8). And it is easy to see that the link of an interior point of an edge or 2-cell of L has this property. □

Remark. If γ meets a singular vertex in its interior, it may well make an angle larger than π at this vertex. The resulting cells in gallery near this vertex need not develop onto a subcomplex of $\mathbb{S}^2$.

Every geodesic which does not lie entirely in the 1-skeleton of L admits a parametrization so that it begins in one of the following *general positions*:

(1) There exists a $\delta > 0$ such that $\gamma(0)$ is a vertex of length three and $\gamma(t) \notin L^{(1)}$ for all $0 < t < \delta$.

(2) There exists a $\delta > 0$ such that $\gamma(0)$ belongs to a segment of type $\bullet - \circ - \bullet$ or $\bullet - - \bullet$ and $\gamma(t) \notin L^{(1)}$ for all $0 < t < \delta$.

In Figure 17, we have sketched the intial few cells of galleries (cut-open and developed onto $\mathbb{S}^2$) determined by local geodesics of L beginning in general position. Either we develop the local geodesic beginning at a vertex of length three or we develop beginning at one of the points in the large boundary arc (a great arc of the sphere).

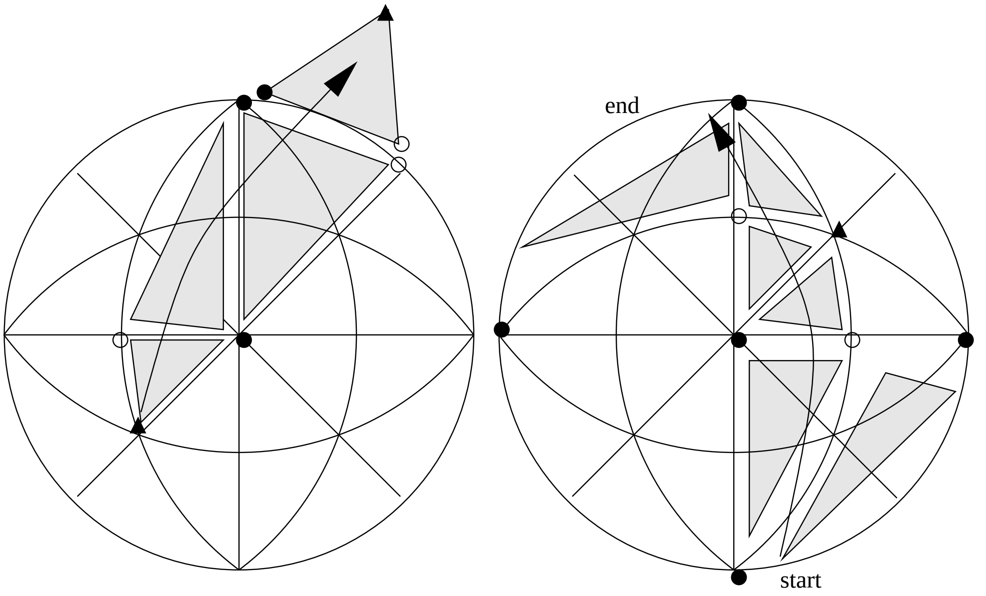

FIGURE 17. Typical galleries of local geodesics in general position.

16. EXTRA-SHORT LOOPS

Definition. A closed local geodesic is called an *extra-short loop* if it has length $\leqq \pi$.

Suppose γ is an extra short closed geodesic in L, parametrized so that it begins in general position. Then, the gallery of γ develops onto at most three all-right triangles as depicted in Figure 18. Fix one such all-right triangle Δ. Then, the simplices in L which develop onto Δ all belong to the same maximal subcomplex L_T for some $T \in \mathcal{S}$. This follows from the fact that the only edges which are common to two distinct maximal subcomplexes belong to a segment of type $\bullet - \circ - \bullet$ or $\bullet - - \bullet$ (use Lemma 13.3). The spherical subset T is determined by either a vertex of length three or by the product of a length one and a length two vertex with opposite sign.

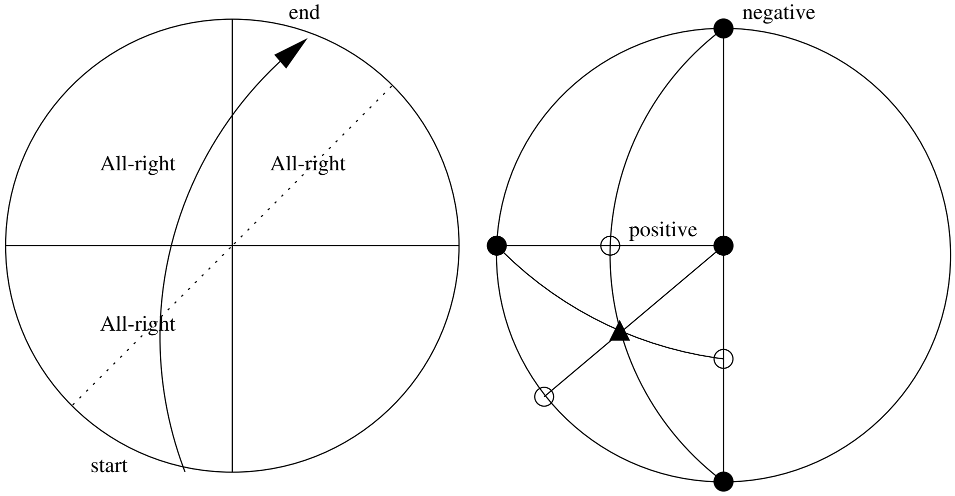

Figure 18. The all-right triangles encode the maximal subcomplexes L_T which contain the local geodesic γ. A local geodesic beginning on the boundary meets at most 3 all-right triangles before traveling distance π (right sphere). A vertex of length 3 determines a unique maximal spherical subcomplex; the product of a length 1 vertex and a length 2 vertex of opposite sign also determines such a subcomplex.

Proposition 16.1. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose γ is a closed local geodesic which does not lie entirely in the 1-skeleton of L. Then, γ has length $> \pi$.*

Proof. Choose a parametrization of γ so that it is in one of the general positions. As in the above discussion, either $\gamma(0)$ is a vertex of length three or $\gamma(0)$ belongs to an edge of the form $\bullet - \circ - \bullet$ or $\bullet - - \bullet$. Then cut-open and develop the gallery onto $\mathbb{S}^2$. If γ had length $\leq \pi$, then, the lift of γ lies in at most three all right triangles, corresponding to, at most, three maximal spherical subcomplexes in L. So, by Theorem 13.5, γ must have length $\geq 2\pi$. ☐

Theorem 16.2. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose $\gamma : [0, h] \to L$ is a closed local geodesic. Then, γ has length $> \pi$. In other words, L_Γ does not contain any extra-short loops.*

Proof. By Theorem 14.4, we may assume that γ does not lie entirely in the 1-skeleton of L. Thus, Proposition 16.1 applies. $\square$

Proposition 16.3. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set, and let $L = L_\Gamma$ be the link of Brady's complex K_Γ. Suppose γ is a local geodesic which joins two singular vertices in L. If γ is not an edge path, then it has length $\geq \pi$.*

Proof. We are assuming that $\gamma(0)$ is a singular vertex and that γ is not an edge path. Thus, γ begins in general position. Develop its gallery onto $\mathbb{S}^2$. Then observe, using Figure 16 as aid, that the only geodesics which join (potentially) singular vertices are either edge paths or have length $= \pi$. (Use the fact that the lift of γ must be a great arc in $\mathbb{S}^2$.) $\square$

17. Shrinking and rotating local geodesics

It remains to show that L does not contain any isometrically embedded circles of length $< 2\pi$ which do not lie entirely in the 1-skeleton. The arguments are inspired by an alternate characterization of CAT(1) spaces due to B. Bowditch [Bow]. We would like to thank J. McCammond for first bringing Bowditch's work to our attention. The actual implementation of Bowditch's ideas are in the spirit of the curvature testing techniques in [EM] and, especially, the more recent paper by M. Elder, J. McCammond, and J.Meier [EMM].

The following theorems of B. Bowditch [Bow] apply to $X = L_\Gamma$:

Theorem 17.1. *(Bowditch) Let X be a compact locally CAT(1) space. If X is not CAT(1), then there exists a minimal length closed geodesic of length $< 2\pi$. Moreover, a closed local geodesic of minimal length is, in fact, a closed geodesic.*

Theorem 17.2. *(Bowditch) Let X be a compact locally CAT(1) space. If γ is a closed local geodesic in X of length $< 2\pi$, then γ may be freely homotoped via non-length increasing paths to constant loop. γ is said to be* shrinkable.

Theorem 17.3. *(Bowditch) Let X be a compact locally CAT(1) space. If γ is a loop in X, then either γ is shrinkable or γ freely homotoped via non-length increasing paths to closed geodesic α.*

Remark. The above three theorems use a reformulation of the locally CAT(1) condition in terms of the length of a minimal closed geodesic. He defines a space to be ϵ-CAT(1) if every triangle of perimeter $< 2\epsilon$ satisfies the CAT(1) inequality. (So, π-CAT(1) is equivalent to

CAT(1).) To prove Theorem 17.1, he shows that X is ϵ-CAT(1) and contains an isometrically embedded circle of length 2ϵ. (The analogous 2ϵ in differential geometry is the systole.) For Theorems 17.2 and 17.3 he uses the Birkhoff curve shortening process. This process takes a closed loop and iterates the process by which we subdivide the loop into segments, join the midpoints of adjacent segments by geodesics, and consider this new loop as the next input. The difficulty is to decide when this process converges.

We apply Bowditch's theorems to a minimal length local geodesic γ in L_Γ. If L_Γ is CAT(1), we are done; otherwise, by Theorem 17.1, such minimal closed geodesic exists and has length $< 2\pi$. We will derive a contradiction.

Definition. Let γ be a closed local geodesic of length $\geq \pi$ in L_Γ. Suppose $\alpha \subset \gamma$ is an arc of length equal to π. A *rotation* of α is a constant length homotopy of α which leaves endpoints fixed. The loop γ' obtained by removing the arc α and replacing it by the rotated arc is said to be *obtained by rotating the arc α*. In particular, γ and γ' have the same length.

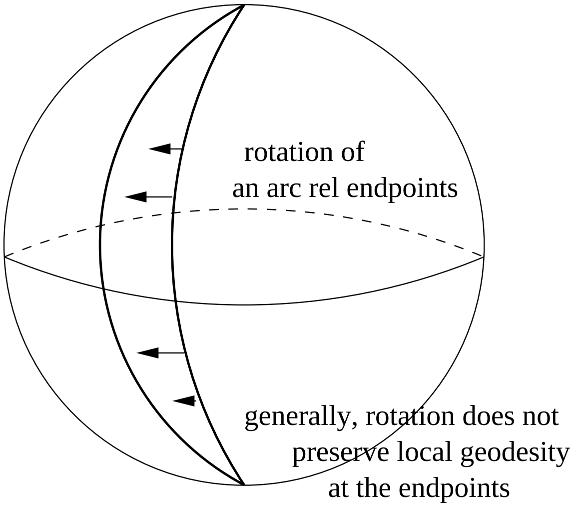

FIGURE 19. An arc $\alpha \subset \gamma$ of length π may be rotated.

Let L_Γ be the link of a Brady complex K_Γ for a three dimensional FC Artin group. Suppose γ is a minimal length closed geodesic of length $< 2\pi$. By Theorem 14.4, we may assume γ is not contained in the 1-skeleton of L. By Theorem 16.2, we may assume that the length of γ is greater than π. Choose an arc α in γ of length π which is not contained in the 1-skeleton. This is possible because the singular vertices are distance at least distance π apart (Proposition 16.3). The singular vertices are the only points in L where a locally geodesic path may switch from an arc contained in the 1-skeleton to an arc which is an edge path. Moreover, we may choose an arc α which admits a parametrization which begins in general position.

If one of the endpoints of α is not a singular point, then we may rotate the arc α by a small amount. First, developing the gallery determined by α onto the two sphere. Then rotate the lift of α by a small amount within the developed gallery. Length is preserved by this homotopy because the lift of α is, by construction, a great arc in $\mathbb{S}^2$. This rotation induces a rotation of α in L because the gallery determined by α immerses into $\mathbb{S}^2$ (Theorem 15.1).

But, the loop γ' obtained by rotation fails to be geodesic at the singular endpoint of α. (This need not be the case if the endpoints were singular!) Choose small balls about each endpoint so that γ' meets each ball in two points. Then join each pair of points by geodesics. The resulting loop has length strictly less than the length of γ'. Moreover, we may realize this reduction by a sequence of homotopies which do not increase length and leave the endpoints fixed. Composing these homotopies, we see that we have homotoped γ through non-length increasing paths to a path of strictly smaller length. This contradicts the minimality of γ.

On the other hand, if we have chosen an arc α in γ which joins singular points, then the path may be rotated into the 1-skeleton. (Use Figure 16 as a visual aid.) If $\alpha(0)$ is a vertex of length one, then we may rotate α into the central line or the boundary arc, whichever is closer. (By general position, the lift of α begins at a vertex in the boundary.) Both the central line and the boundary arc in $\mathbb{S}^2$ correspond to an edge in L. (Figure 17 is particularly instructive.) As the cells of the gallery are developed onto the sphere, the central arc lies in the image. There are other cases besides the gallery shown there. However, by symmetry, the arc begins along the boundary in the region labelled “start”. Because the arc is great arc, it must terminate in the opposite region of the boundary labelled “end”. If $\alpha(0)$ is a vertex of length three, then we may rotate α into either of the arcs which bound the typical gallery as shown in Figure 17.

The only possible obstruction to continuing to rotate an arc occurs when a rotated arc meets a vertex. If the rotated arc is already an edge path, there is no need to rotate further. We never have need to rotate through a singular vertex; for we have already observed that the only paths which join singular vertices which are less than π apart are in fact edge paths. In the present case, $\alpha(0)$ is singular; if the rotation α' ever met a singular point in its interior, then all of α' would be an edge path. So, we would stop rotating.

On the other hand, it is possible that a rotated arc meet a vertex of length two. This is okay, however— the link is either discrete or a suspension. In the first case, the arc must be an edge path; in the second case, it is clear that rotation may be continued (see Figure 20).

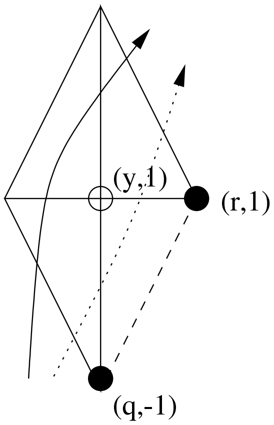

FIGURE 20. Rotating a locally geodesic arc through a vertex of length two. The 2-cell in question always exists because qr is allowable.

Now consider the rotated loop γ'. If γ' is not geodesic at the endpoints, is may be shortened, contradicting minimality. So, instead we have obtained a geodesic loop which now contains an arc of length π which is contained in the 1-skeleton of L. If there were an arc of γ' which was not contained in the 1-skeleton, then the arc must join singular vertices. But, then γ' must have length at least 2π. Whereas, if γ' is contained in the 1-skeleton, it must have length at least 2π. Thus, we have shown that L_Γ does not contain any short loops.

Theorem 17.4. *Let Γ define a three dimensional FC Artin system together with a total ordering of the generating set. Then the link L_Γ is CAT(1).*

18. RESULTS AND CONCLUDING REMARKS

The proof of Theorem 17.4 was the final ingredient in the outline of the proof of the Main Theorem:

Theorem 18.1. *Let Γ define a 3 dimensional FC Artin system, and fix a total ordering of $S = vert(\Gamma)$. Then the link $L_\Gamma = Lk(K_\Gamma, v_0)$ is*

CAT(1); and, moreover, the Artin group, $A_\Gamma \cong \pi_1(K_\Gamma, v_0)$, is CAT(0): it acts geometrically on the universal cover of K_Γ by deck transformations.

The proof given also works for FC Artin systems of dimension less than three. Two dimensional FC Artin groups were shown to be CAT(0) by T. Brady and the same cell complex K, but with a different metric: every edge was assigned length one, so that boundary of every 2-cell was an equilateral triangle. Checking the link condition was equivalent to deciding if L contained any edge loops of fewer than six edges (in the link, the edges have length $\pi/3$).

Our study of the partial ordering on W by reflection length left several open questions. Related to these questions is the idea of a "dual" theory of Coxeter groups. The question of whether Coxeter groups admit a classification purely in terms of reflection length was asked by D. Bessis in [Be]. For finite Coxeter groups, he defines an *abstract finite reflection group* to be a group W together a generating set R and a faithful linear representation $\rho : W \to V \cong \mathbb{R}^n$. The generating set R (reflections) is characterized as precisely those elements $w \in W$ such that $\operatorname{codim}(\ker(\rho(w) - \mathrm{Id})) = 1$. But, it is unclear what is the correct definition of an infinite abstract reflection group. The natural question to ask in the present context, is whether such a clarification is related to finding $K(\pi, 1)$ spaces for Artin groups. In spite of these prospects for future research, the question of whether Artin groups are CAT(0) remains open.

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