LECTURES ON SYMMETRIC SUBGROUPS AND THE FLAG VARIETY

PETER E. TRAPA

OVERVIEW

The first lecture introduces the finite Hecke algebra $\mathcal{H}$ of a Coxeter system as a convolution algebra of functions on the corresponding finite Chevalley group which are invariant by the left and right translation action of a Borel subgroup. This leads naturally to a construction of $\mathcal{H}$ modules from subgroups of the Chevalley group. We then turn to the Lusztig-Vogan formalism of this construction for connected reductive groups over an algebraically closed field of positive characteristic, and the $\mathcal{H}$ modules obtained by considering certain special subgroups, namely symmetric ones. In the end, we obtain an action of $\mathcal{H}$ on a space $\mathcal{M}$ with basis given by irreducible K-equivariant constructible sheaves on the flag variety for G. Here we return to characteristic zero and take to be G a *complex* connected reductive group (as in Adams' lectures) and K once again a symmetric subgroup.

The next goal of these lectures is to compute the Hecke algebra action explicitly in terms of the geometry of K orbits on the flag variety. This will consume most of Lectures 2–4. Along the way, we need to explain why such K orbits should have anything to do with the representation theory of real groups. In Vogan's lectures, for instance, the orbits of the real group itself are the fundamental geometric objects of study. The bridge is given by Matsuki duality and Beilinson-Bernstein localization, which is discussed in Lecture 3.

In Lecture 5, we introduce a canonical basis of $\mathcal{M}$ and use the $\mathcal{H}$ module structure to characterize it. This is the main theorem of [LV]. Finally we explain the representation theoretic significance of this other basis as in [V3].

Date: July 22, 2009.

LECTURE I: MODULES FOR CONVOLUTION ALGEBRAS

Suppose G is a finite group. In a first course on abstract algebra, the group algebra $\mathbb{C}[G]$ is usually defined to consists of formal (complex) linear combinations of the elements $\{e_g \mid g \in G\}$. Multiplication of basis elements is defined in the natural way,

$$e_g e_h = e_{gh},$$

and extended linearly to all of $\mathbb{C}[G]$. The reason why this is a good definition is that the category of finite-dimensional complex representations of G is equivalent to the category of finite-dimensional unital $\mathbb{C}[G]$-modules.

One disadvantage of this definition is that it is not very useful if G is (say) an infinite topological group. The discrete linear combinations of basis elements e_g cannot possibly incorporate the topology of G, and modules over $\mathbb{C}[G]$ cannot capture information about the natural (e.g., continuous) representations of G. So it is helpful to reinterpret the definition of $\mathbb{C}[G]$ (for G finite) in such a way that is well-suited for generalization.

From this point of view, it is better to think of $\mathbb{C}[G]$ (again for G finite) as follows. As a vector space, we may easily identify $\mathbb{C}[G]$ with $\mathcal{F}(G)$, functions on G,

$$\mathbb{C}[G] \longrightarrow \mathcal{F}(G)$$
$$\sum_g c_g g \longrightarrow f,$$

where f is defined so that $f(g) = c_g$. If we trace the algebra structure on $\mathbb{C}[G]$ through this isomorphism, we see that it translates into the following product (denoted $\star$) on $\mathcal{F}(G)$,

$$[f_1 \star f_2](x) = \sum_{g \in G} f_1(g) f_2(g^{-1}x). \tag{1}$$

This in fact is the usual operation of convolution of functions on a group. For instance, everyone learns early on about convolution of L^1 functions on $\mathbb{R}$,

$$(f_1 \star f_2)(x) = \int_{\mathbb{R}} f_1(y) f_2(x-y) dy.$$

It is easy to see the analogy with (1): the sum in (1) is really an integral with respect to the counting measure.

Thus $\mathbb{C}[G]$ is nothing but the convolution algebra of functions on G. Clearly this is a formulation of the group algebra that admits easy generalization beyond finite groups. In the case of topological or Lie groups, one may consider various algebras of function spaces (capturing information about the topology or manifold structure of G) whose modules are then related to various kinds of representations of G (preserving corresponding structures). We will not pursue this point of view at all, but nonetheless it provides some motivations for the constructions we do pursue.

Return to (1). Since the counting measure is (in particular) left invariant, the $\star$ product behaves well with respect to the left regular representations on functions.

More precisely, let L denote the left regular representations on $\mathcal{F}(G)$,

$$(L_x f)(g) = f(x^{-1}g).$$

Then

$$L_x(f_1 \star f_2) = [(L_x f_1) \star f_2] \tag{2}$$

We also have the corresponding fact for the right regular representation R,

$$(R_x f)(g) = f(gx).$$

namely

$$R_x(f_1 \star f_2) = [f_1 \star (R_x f_2)] \tag{3}$$

Thus if H is any subgroup of G, the space $\mathcal{F}(H\backslash G/H)$ of functions which are invariant under the left and actions of H is a subalgebra of $\mathcal{F}(G)$ closed under $\star$.

Exercise 1.1. Verify the assertion of the previous sentence. Show that $\mathcal{F}(H\backslash G/H)$ has a unit. Thus $\mathcal{F}(H\backslash G/H)$ is a complex associative algebra with unit.

Next suppose we are gives another subgroup K of G. Then $\mathcal{F}(H\backslash G/K)$ naturally becomes a modules for $\mathcal{F}(H\backslash G/H)$ via

$$\begin{aligned} \mathcal{F}(H\backslash G/H) \times \mathcal{F}(H\backslash G/K) &\longrightarrow \mathcal{F}(H\backslash G/K) \\ (f,\gamma) &\longrightarrow f \star \gamma. \end{aligned}$$

So subgroups are a natural source of modules.

Ultimately we want to generalize this construction to the setting when F need not be finite. In order to do so, we reinterpret the definition given above which looks much more categorical (and thus easier to generalize)[1]. Suppose X and Y are finite sets and $\pi : X \to Y$ is a map between then. We can then pullback and pushforward function on X and Y

$$\begin{aligned} \pi_* &: \mathcal{F}(X) \longrightarrow \mathcal{F}(Y) \\ \pi^* &: \mathcal{F}(Y) \longrightarrow \mathcal{F}(X) \end{aligned}$$

via

$$[\pi_*(f)](y) = \sum_{g \in \pi^{-1}(y)} f(g),$$

and

$$[\pi_*(h)](x) = h(\pi(x)).$$

Given any set M, consider the projection define projections

$$\pi_{ij} : M \times M \times M \longrightarrow M \times M \quad i,j \in \{1,2,3\},\ i \neq j,$$

via

$$\pi_{ij}(m_1, m_2, m_3) = (m_i, m_j).$$

Then we obtain a product $\star'$ on $\mathcal{F}(M \times M)$ defined for $f, h \in \mathcal{F}(M \times M)$ by

$$f \star h = (\pi_{13})_*[\pi_{12}^* f \otimes \pi_{23}^* h].$$

[1] A nice exposition of related ideas is given in [CG] starting in Section 2.6.

This in fact defines an associative algebra structure on $\mathcal{F}(M \times M)$.

In the next proposition we take $M = G$ for a finite group and relate the associative algebra $(\mathcal{F}(G \times G), \star')$ (which makes no use of the group structure on G) to the group algebra $(\mathcal{F}(G), \star)$ (which does).

Proposition 1.2. *Let G be a finite group. The subspace $\mathcal{F}\left(\Delta(G)\backslash(G \times G)\right)$ of $\mathcal{F}(G \times G)$ which are invariant under the diagonal action $g \cdot (x, y) = (gx, gy)$ of G is closed under $\star'$. The map*

$$\begin{aligned} (\mathcal{F}(G), \star) &\longrightarrow (\mathcal{F}(G \times G), \star') \\ f &\longrightarrow \tilde{f} \end{aligned}$$

where $\tilde{f}$ is determined by requiring

$$\tilde{f}(x, 1) = f(x)$$

is an isomorphism of associative algebras.

Proof. Exercise. □

The nice thing about the proposition is that the $\star'$ operation looks much more amenable to generalization (when replacing function on G by sheaves on G). The next thing to check is that the module constructions above come along for the ride.

Corollary 1.3. *In the setting of Proposition 1.2, let H and K be subgroups of G. The isomorphism of Proposition 1.2 restricts to give an algebra isomorphism*

$$(\mathcal{F}(H\backslash G/H), \star) \longrightarrow \left(\mathcal{F}\left(\Delta(G)\backslash(G \times G)/(H \times H)\right), \star'\right)$$

and a vector space isomorphism

$$\mathcal{F}(H\backslash G/K) \longrightarrow \mathcal{F}\left(\Delta(G)\backslash(G \times G)/(H \times K)\right)$$

Now write

$$\begin{aligned} \pi_{12} &: G/H \times G/H \times G/K \longrightarrow G/H \times G/H \\ \pi_{23}, \pi_{13} &: G/H \times G/H \times G/K \longrightarrow G/H \times G/K. \end{aligned}$$

for the obvious projections. Consider

$$\begin{aligned} \mathcal{F}\left(\Delta(G)\backslash(G \times G)/(H \times H)\right) \times \mathcal{F}\left(\Delta(G)\backslash(G \times G)/(H \times K)\right)& \\ \longrightarrow \mathcal{F}\left(\Delta(G)\backslash(G \times G)/(H \times K)\right)& \end{aligned}$$

defined by

$$(f, h) \mapsto (\pi_{13})_*[\pi_{12}^* f \otimes \pi_{23}^* h]. \tag{4}$$

This makes $\mathcal{F}\left(\Delta(G)\backslash(G \times G)/(H \times K)\right)$ a module for $\left(\mathcal{F}\left(\Delta(G)\backslash(G \times G)/(H \times H)\right), \star'\right)$

Exercise 1.4. Using the identifications of Proposition 1.2, verify that the module constructed in the previous paragraph recovers the module defined after Exercise 1.1.

In general, the group algebra $\mathcal{F}(G)$ is complicated (since multiplication in G is), and for general H, the sets G/H or $H\backslash G/H$ are also complicated. But for suitably nice choices of H one might hope for substantial simplifications. The choice we have in mind (whose details were worked out by Iwahori) takes H to be a Borel subgroup in a finite Chevalley group.

For concreteness, suppose $G = GL(n, \mathbb{F}_q)$, the group of n-by-n invertible matrices over a finite field with $q = p^f$ elements. Let B denote the set of upper triangular matrices. According to the Bruhat decomposition, $B\backslash G/B$ is parameterized by $W = S_n$. A set of representatives of the B-double cosets is simply the permutation matrices, say $A(w)$, for $w \in W$. Given w, let T_w denote the function whose value is $1/|B|$ on the B-double coset containing $A(w)$ and 0 elsewhere. Then $\mathcal{F}(B\backslash G/B)$ is clearly spanned by the various T_w as w ranges over W. The question is how two such elements multiply.

As a first example, suppose $n = 2$, and write $W = \{e, s\}$. The double coset containing $A(e)$ consists of the elements of B, the double coset containing $A(s)$ consists of everything else. It is clear that T_e is the identity in $\mathcal{F}(B\backslash G/B)$. (For this to be true, we needed the $1/|B|$ factor in the definition of T_e.) Meanwhile $(T_s + T_e)$ is the constant function $1/|B|$ on G. So $(T_s + T_e)^2 = \frac{|G|}{|B|}(T_s + T_e)$. For notational simplicity, write $\mathbb{H} = \mathbb{H}(\mathrm{GL}(2, \mathbb{F}_q))$ in place of $\mathcal{F}(B\backslash G/B)$. Then $\mathbb{H}$ is the complex associative algebra with basis T_s and T_e, T_e is the identity in $\mathcal{H}$ (and so we denote it by 1 as usual), and

$$(T_s + 1)^2 = (q + 1)(T_s + 1)$$

since $|G|/|B| = (q + 1)$. A quick exercise shows that an equivalent presentation of $\mathcal{H}$ is given by the relation

$$(T_s + 1)(T_s - q) = 1.$$

If we formally set $q = 1$, we get the complex associative algebra with unit generated by T_s subject to $T_s^2 = 1$. That is, specializing q to 1 gives the group algebra $\mathbb{C}[W]$ of W.

Here is the corresponding abstract definition. (It is more natural to work over $\mathbb{Z}$ instead of $\mathbb{C}$.)

Definition 1.5. Suppose (W, S) is a Coxeter system and q is an indeterminant. Define the Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$ attached to (W, S) to be the associative algebra with unit over the ring of formal Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ generated by $\{T_s \mid s \in S\}$ subject to the relations

(1) $(T_s + 1)(T_s - q) = 0$; and
(2) $\overbrace{T_{s_i} T_{s_j} \cdots}^{m_{ij}} = \overbrace{T_{s_j} T_{s_i} \cdots}^{m_{ij}}$.

The algebra has basis $\{T_w \mid w \in W\}$ where

$$T_w := T_{s_1} \cdots T_{s_k}$$

is well-defined for any reduced expression $w = s_1 \cdots s_k$ of w.

Exercise 1.6. Let $G = \mathrm{GL}(n, \mathbb{F}_q)$. Show that the convolution algebra of B bi-invariant functions on G satisfies the relations of Definition 1.5. (We showed this above for $n = 2$. The general case amounts to checking the braid relations, which reduces to the case of $n = 3$.) More generally, the result holds if G is a finite Chevalley group (and the proof again reduces to rank two).

Remark 1.7. Let $G = \mathrm{GL}(n, \mathbb{F}_q)$ (or any finite Chevalley group). Although we shall not use it, the algebra $\mathcal{F}(B\backslash G/B)$ also arise naturally when decomposing the spherical principal series $X := \mathcal{F}(G/B)$ of right B-invariant functions on G (with the left translation action). Because of (3), convolution (on the right) with an element h of $\mathcal{F}(B\backslash G/B)$ defines an intertwining operator of X with itself,

$$\begin{aligned} \mathcal{F}(B\backslash G/B) &\longrightarrow \mathrm{Hom}_G(X, X) \\ h &\longrightarrow [f \mapsto f \star h]. \end{aligned}$$

In fact, this map is an isomorphism.

The remark suggests that if F is finite, then $\mathcal{F}(B\backslash G/B)$ is important for the study of the finite group G. The same should be true if F is algebraically closed. The formalism after Corollary 1.3 suggests that generalizations to this case might be possible. In fact, Lusztig and Vogan [LV] introduced versions of these constructions for reductive groups G over the algebraic closure $\bar{\mathbb{F}}_q$ when K is a symmetric subgroup. (Later Mars and Springer generalized the construction to a wider class of groups K [MS].) The correct "function-like" objects are $\Delta(G)$-equivariant constructible sheaves of l-adic vector spaces on $G/B \times G/B$ or $G/B \times G/K$ whose stalks satisfy natural purity conditions. The Grothendieck group of the former form an algebra with the product essentially given by (4) where the pullback and pushforward are now operations on sheaves. (Since pushforward is not exact in this setting, one must take Euler characteristics[2].) This algebra turns out to be the Hecke algebra of Definition 1.5, and the formalism of (4) gives modules over it. One of the aims of the next several lectures is to describe this module structure explicitly following [LV] and [V3].

The Grothendieck group of the relevant l-adic sheaves canonically identifies with that of the more elementary category of equivariant constructible sheaves of complex vector spaces. So we need to recall the basics of such sheaves. Let H be a complex reductive algebraic group acting algebraically on a complex algebraic variety X. A sheaf $\mathcal{S}$ (of complex vector spaces, say) is H-equivariant if it is invariant under the pullback of the action map m_h (mapping x to $h \cdot x$) for each $h \in H$. The sheaf is constructible (with respect to the stratification of X by H-orbits) if it is locally constant when restricted to each H orbit on X. (This is analogous to the requirement that a function on X is H invariant.) Let $\mathcal{C}(H, X)$ denote the category of H-equivariant constructible sheaves on X.

The irreducible objects in $\mathcal{C}(H, X)$ admit a simple parametrization. To each such irreducible sheaf $\mathcal{S}$, we first attach its support. Since $\mathcal{S}$ is assumed to be H-equivariant,

[2]Or work in an appropriate derived category.

its support is H invariant. Since $\mathcal{S}$ is also irreducible, its support necessarily is a single H orbit Q on X. So, in fact, $\mathcal{S}$ is locally constant on Q. (A remark on terminology: locally constant sheaves are sometimes called local systems. So, in particular, an irreducible constructible sheaf is a local system on its support.) By following loops around in Q we obtain a representations of $\pi_1(q, Q)$ on the stalk of a base point x in Q. The equivariance condition makes this amount to a representation of the centralizer in H of x which much be trivial on its identity component. Write

$$A(Q) := Z_H(x)/Z_H^\circ(x). \tag{5}$$

Up to isomorphism this of course does not depend on the choice of x. We have thus attached a representations of $A(Q)$ to $\mathcal{S}$. Since $\mathcal{S}$ is irreducible, this representation is irreducible. In fact, the process is reversible and we get

$$\{\text{Irreducible objects in } \mathcal{C}(H, X)\} \leftrightarrow \{(Q, \phi) \mid Q \in H\backslash X, \phi \in A(Q)\hat{}\}.$$

The objects on the right are thus (isomorphism classes of) irreducible H-equivariant local systems on H orbits in X. We denote them by $\mathcal{L}oc_H(X)$.

Return to the setting of Adams' first lecture and suppose G is a connected reductive complex algebraic group with Borel subgroup B. Let K be a symmetric subgroup of G. Then, following [LV], we obtain a geometrically defined action of the Hecke algebra $\mathcal{H}$ of Definition 1.5 (attached to the root data of G) on a $\mathbb{Z}[q, q^{-1}]$-module $\mathcal{M}$ with basis indexed by $\mathcal{L}oc_{\Delta(G)}(G/B \times G/K)$ [3] . This module structure may also be defined from purely representation theoretic considerations as in [V3]. The goal of the next few lectures is to give the explicit computation of this module structure and explain what it has to do with the representation theory (discussed in Vogan's lectures) of the real group $G_\mathbb{R}$ corresponding to K (as in Adams' second lecture). The final lecture will be devoted to the main result of [LV] which gives a characterization of a canonical basis of the $\mathcal{H}$ module.

[3] Restriction from $G/B \times G/K$ to $G/B \times eK$ defines an equivalence of $\mathcal{C}(\Delta(G), G/B \times G/K)$ with $\mathcal{C}(K, G/B)$, and identifies $\mathcal{L}oc_{\Delta(G)}(G/B \times G/K)$ with $\mathcal{L}oc_K(G/B)$. Often we work with these latter local systems, since it is easier to draw pictures of them in low rank.

LECTURE II: THE HECKE MODULE STRUCTURE IN RANK ONE

The computation of the $\mathcal{H}$ action on $\mathcal{M}$ introduced in the last section reduces in large part to four symmetric pairs (G, K): $(\mathrm{SL}(2), \mathrm{SL}(2))$, $(\mathrm{SL}(2), \mathrm{SO}(2))$, $(\mathrm{GL}(2), \mathrm{O}(2))$, $(SL(2) \times \mathrm{SL}(2), \Delta(\mathrm{SL}(2)))$. In the setting of Adams' second lecture, these correspond to the real groups $\mathrm{SU}(2)$, $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{GL}(2, \mathbb{R})$, and $\mathrm{SL}(2, \mathbb{C})$. We begin this lecture by working through these four cases.

Example 2.1. Suppose $(G, K) = (\mathrm{SL}(2) \times \mathrm{SL}(2), \Delta(\mathrm{SL}(2)))$. In this case $G/B = \mathbb{P}^1 \times \mathbb{P}^1$, and the orbits of K are parametrized by the Weyl group $\{e, s\}$ according to the relative position of a pair of lines: there is the closed diagonal orbit Q_e of pairs of the same line, and there is an open orbit Q_s consisting of everything else. The centralizer of any point is connected. So there are only the trivial irreducible local systems on these orbits. We write

$$\mathcal{L}oc_K(G/B) = \{\gamma_e, \gamma_s\}.$$

As mentioned in the previous lecture, the Lusztig-Vogan $\mathcal{H}$ action on the module $\mathcal{M}$ with basis $\mathcal{L}oc_K(G/B)$ is simply the left-regular representation of $\mathcal{H}$,

$$T_s\gamma_e = \gamma_s$$
$$T_s\gamma_s = (q-1)\gamma_s + q\gamma_e.$$

We derived these formulas in Lecture 1 by working over $\mathbb{F}_q$, but virtually the identical derivation holds in the Lustzig-Vogan setting.

Example 2.2. Suppose now $(G, K) = (\mathrm{SL}(2), \mathrm{SO}(2))$. In this case $G/B = \mathbb{P}^1$ and the action of $z \in \mathrm{SO}(2, C) \simeq \mathbb{C}^\times$ takes the line through (a, b) to the line through $(za, z^{-1}b)$. There are thus three orbits: the line Q_+ through $(1, 0)$, the line Q_- through $(0, 1)$), and the open orbit $Q_\circ$ consisting of everything else. The centralizer of K both closed orbits is K itself, which is connected, and so there is only the trivial irreducible local system on closed orbit. Meanwhile, the stabilizer of a point in the open orbit is easily seen to be $\pm 1 \subset \mathbb{C}^\times$. So there are two irreducible local systems on $Q_\circ$, one of which is trivial of course, the other which is nontrivial. We write

$$\mathcal{L}oc_K(G/B) = \{\delta_+, \delta_-, \gamma_\circ, \gamma_\circ'\},$$

where $\gamma_\circ'$ is the nontrivial local system and the notation for the others is obvious. We are going to waive our hands once again at the Lusztig-Vogan $\mathcal{H}$ calculation by working by analogy over $\mathbb{F}_q$ and using the module structure given by (4). Consider first

$$(T_s + 1) \cdot \delta_+.$$

Since δ_+ is trivial, we can think of it as the constant function 1 on Q_+. We need to translate back to $\mathcal{L}oc_{\Delta(G)}(G/B \times G/K)$ to invoke (4). So we should think of δ_+ as the constant function 1 on the corresponding closed orbit in $G/B \times G/K$. Since $(T_s + 1)$ is the constant function 1 on $G/B \times G/B$, (4) gives

$$(T_s + 1)\delta_+ = (\#\text{ of points in } Q_+ \text{ over } \mathbb{F}_q)(\text{constant function 1 on } G/B \times G/K).$$

Reverting back to $\mathcal{L}oc_K(G/B)$, the constant function 1 on $G/B \times G/K$ corresponds to the sum $\delta_+ + \delta_- + \gamma_\circ$. So we get

$$(T_s + 1)\delta_+ = \delta_+ + \delta_- + \gamma_\circ.$$

or

$$T_s\delta_+ = \delta_- + \gamma_\circ. \tag{6}$$

By symmetry,

$$T_s\delta_- = \delta_+ + \gamma_\circ. \tag{7}$$

In fact this is indeed the Lusztig-Vogan action.

Next we try to use the same intuition to approach $(T_s + 1)\gamma_\circ$. We once again conclude

$$(T_s + 1)\gamma_\circ = (\#\text{ of points in } Q_\circ \text{ over } \mathbb{F}_q)(\text{constant function 1 on } G/B \times G/K).$$

Since $\mathbb{P}^1$ has $q+1$ points, $Q_\circ$ has $q-1$, and we have

$$(T_s + 1)\gamma_\circ = (q-1)(\delta_+ + \delta_- + \gamma_\circ)$$

or

$$T_s\gamma_\circ = (q-2)\gamma_\circ + (q-1)(\delta_+ + \delta_-). \tag{8}$$

This is, once again, indeed the Lusztig-Vogan action.

Finally, we turn to $(T_s + 1)\gamma_\circ'$. Since $\gamma_\circ'$ is the Mobius bundle, it has no global sections, and so it behaves like the zero function in the formalism of (4). That is,

$$(T_s + 1)\gamma_\circ' = 0$$

or

$$T_s\gamma_\circ' = -\gamma_\circ', \tag{9}$$

and we once again recover the Lusztig-Vogan action.

Exercise 2.3. Duplicate the analysis for the compact symmetric pair $(\mathrm{SL}(2), \mathrm{SL}(2))$. (Hint: $\mathcal{L}oc_K(G/B)$ consists of a single element.)

Example 2.4. Consider $(G, K) = (\mathrm{GL}(2), \mathrm{O}(2))$. (It would be better conceptually to stick to rank one and instead consider the noncompact symmetric pair for PSL(2). But it notationally easy use the matrices in GL(2).) We once again have $G/B = \mathbb{P}^1$, but this time K has two connected components: its identity component $\mathrm{SO}(2, \mathbb{C})$, and the translate of $\mathrm{SO}(2, \mathbb{C})$ by $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The action of t interchanges the two closed $\mathrm{SO}(2, \mathbb{C})$ orbits Q_+ and Q_- of the previous example. Write

$$Q_\pm = Q_+ \cup Q_-.$$

The centralizer of a point in $Q_\pm$ in K is $\mathrm{SO}(2, \mathbb{C})$, so there is only the trivial irreducible local system on $Q_\pm$.

Meanwhile the action of t preserves the open $\mathrm{SO}(2, \mathbb{C})$ orbit $Q_\circ$, so this is also a K orbit in this case. But the centralizer in K of a point in $Q_\circ$ is now isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ so there are *four* irreducible local systems on the open orbit. Write

$$\mathcal{L}oc_K(G/B) = \{\delta_\pm, \gamma_\circ^{++}, \gamma_\circ^{--}, \gamma_\circ^{-+}, \gamma_\circ^{+-}\}.$$

Here $\gamma_\circ^{++}$ is the trivial local system, $\gamma_\circ^{--}$ has global sections, but neither $\gamma_\circ^{+-}$ nor $\gamma_\circ^{-+}$ do. Because of the nontrivial local system with sections, the heuristic computations we have been making over $\mathbb{F}_q$ are less useful. We simply record the formulas for the Lusztig-Vogan action without saying too much about their actual derivation:

$$T_s\delta_\pm = \delta_\pm + \gamma_\circ^{++} + \gamma_\circ^{--} \tag{10}$$

$$T_s\gamma_\circ^{++} = (q-1)\gamma_\circ^{++} - \gamma_\circ^{--} + (q-1)\delta_\pm \tag{11}$$

$$T_s\gamma_\circ^{--} = (q-1)\gamma_\circ^{--} - \gamma_\circ^{++} + (q-1)\delta_\pm \tag{12}$$

$$T_s\gamma_\circ^{+-} = -\gamma_\circ^{+-} \tag{13}$$

$$T_s\gamma_\circ^{-+} = -\gamma_\circ^{-+}. \tag{14}$$

Remark 2.5. The above rank one examples are already detailed enough to glimpse one of the deepest features of the theory of real groups, the duality of [V4]. Given any module $\mathcal{M}$ for $\mathcal{H}$, define it's dual as

$$\mathcal{M}^* := \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}\left(\mathcal{M}, \mathbb{Z}[q,q^{-1}]\right).$$

Since $\mathcal{H}$ is nonabelian, making $\mathcal{M}^*$ into a $\mathcal{H}$ modules requires choosing an antiautomorphism of $\mathcal{H}$. Concretely, for for $T_s \in \mathcal{H}$ and $\mu \in \mathcal{M}^*$, define

$$T_s \cdot \mu = [-q(T_s)^{-1}]^{\mathrm{tr}}\mu.$$

Next return to the setting of Example 2.2. The span of δ_+, δ_- and $\gamma_\circ$ is invariant under $\mathcal{H}$. Denote this module my $\mathcal{M}_0$. In the setting of Example 2.4, let $\mathcal{N}_0$ denote the span of $\delta_\pm$ and $\gamma_\circ^{++}$ and $\gamma_\circ^{--}$, also an $\mathcal{H}$ module. Let $\mu_\circ$, μ_+, and μ_- denote the corresponding basis of the $\mathcal{H}$ module $\mathcal{N}_0^*$. Consider the map

$$\mathcal{M}_0 \longrightarrow \mathcal{N}_0^*$$

taking δ_+ to μ_+, δ_- to μ_-, and $\gamma_\circ$ to $\mu_\circ$. Then is it is not hard to verify that this is an isomorphism of $\mathcal{H}$ modules. Meanwhile the dual of the $\mathcal{H}$ module $\mathcal{M}_1$ spanned by $\gamma_\circ'$ in Example 2.2 is dual to the $\mathcal{H}$ module in Example 2.3.

Return to the general setting of the $\mathcal{H}$ action on $\mathcal{M}$ defined in [LV] for a symmetric pair (G,K). Then it turns out that $\mathcal{M}$ breaks into a direct sum of $\mathcal{H}$ modules ("blocks") $\mathcal{M}_0, \cdots, \mathcal{M}_k$. For a fixed i, there is a block $\mathcal{N}_i$ in the $\mathcal{H}$ module structure of some *dual* symmetric pair $(G^\vee, K')$ such that $\mathcal{M}_i \simeq \mathcal{N}_i^*$ as $\mathcal{H}$ modules. This is Vogan duality.

In each of the above examples, we saw that $\mathcal{L}oc_K(G/B)$ is finite. This holds in general. Since the action is algebraic, the assertion amounts to saying that K acts on G/B with finitely-many orbits.

Proposition 2.6. *Let (G,K) be a symmetric pair.*

1. *K acts with finitely many orbits on G/B.*
2. *If (G,K) is complex (i.e. of the form $(G \times G, \Delta(G))$), then every element of $\mathcal{L}oc_K(G/B)$ is trivial.*

LECTURE III: REPRESENTATIONS, LOCALIZATION, AND THE STRUCTURE OF $K\backslash G/B$

Let $G_{\mathbb{R}}$ denote the real group corresponding to a symmetric pair (G, K) as in Adams' third lecture. So far we have been dealing with function-like objects on K orbits on G/B. Meanwhile, Vogan has shown in his lectures that it is function-like objects on $G_{\mathbb{R}}$ orbits on G/B which naturally give rise to nice representations of $G_{\mathbb{R}}$. Here is a bridge between the two viewpoints.

Theorem 3.1 (Matsuki Duality). *In the setting of the previous paragraph, there is a natural bijection of $G_{\mathbb{R}}$ orbits on G/B with K orbits on G/B which inverts the closure order on each set.*

Example 3.2. The threorem in a tautology for a complex symmetric pair $(G \times G, \Delta(G))$.

Exercise 3.3. Verify the theorem for $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{R})$ and $\mathrm{GL}(2, \mathbb{R})$.

Next we want to go further and explain briefly how to attach to each irreducible Harish-Chandra module an element of $\mathcal{L}oc_K(G/B)$. (Excellent references for this material are [Mi1] and [Mi2].) Let $\mathcal{D}$ denote the sheaf of algebraic differential operators on G/B, and let $\mathfrak{g}$ denote the Lie algebra of G. Assume first that X is an irreducible $\mathrm{U}(\mathfrak{g})$ module whose annihilator in $\mathrm{U}(\mathfrak{g})$ contains the ideal I generated by the annihilator in the center $\mathrm{Z}(\mathfrak{g})$ of the trivial representation (i.e. suppose the annihilator of X contains the augmentation ideal of $\mathrm{Z}(\mathfrak{g})$). In the terminology of Vogan's lectures, assume X has the same infinitesimal character as the trivial representation. Since $\mathrm{U}(\mathfrak{g})$ acts on G/B by global differential operators the "localization"

$$\mathcal{X} := \mathcal{D} \otimes_{\mathrm{U}(\mathfrak{g})} X$$

makes sense as a sheaf of (quasi-coherent) $\mathcal{D}$ modules. This defines a functor

$$[U(\mathfrak{g})/I]\text{-mod} \longrightarrow \mathcal{D}\text{-mod}.$$

which is, in fact, an equivalence (with inverse given by taking global sections). For general infinitesimal character, a very similar statement holds, but one must use sheaves of twisted differential operators (with the twist corresponding to the infinitesimal character).

Now suppose X is an irreducible $(\mathfrak{g}, K)$ module, as in Vogan's lectures, with trivial infinitesimal character. Then the localization $\mathcal{X}$ will be suitably K-equivariant (and irreducible). So the support $\mathrm{supp}(X) := \mathrm{supp}(\mathcal{X}) \subset G/B$ will be a K-invariant subset of G/B; since $\mathcal{X}$ is irreducible, it will be the closure of a single K orbit which we denote $\mathrm{supp}_\circ(X)$. In this way, we have attached an element of $K\backslash G/B$ to an irreducible Harish-Chandra module for the symmetric pair (G, K). To get an element of the more refined set $\mathcal{L}oc_K(G/B)$ from $\mathcal{X}$ (and hence X) we must consider the $\mathcal{D}$-module inverse image of $\mathcal{X}$ with respect to the inclusion of $\mathrm{supp}_\circ(X)$ into G/B. Since

X is irreducible, this inverse image is a locally free sheaf of $\mathcal{O}_{G/B}$ modules, and hence defines an element of $\mathcal{L}oc_K(G/B)$.

In fact, to motivate the study of the general Hecke algebra action, we need to understand something about the classification of irreducible Harish-Chandra sheaves of $\mathcal{D}$-modules or, by the equivalence of localization, the classification of irreducible $(\mathfrak{g}, K)$ modules with trivial infinitesimal character. For this we shall unfortunately be somewhat vague. Fix $\tau \in \mathcal{L}oc_K(G/B)$ supported on Q in $K\backslash G/B$. Then it is possible to form a "direct image" $\mathcal{I}(Q,\tau)$ of τ with respect to the inclusion of Q in G/B. This is a (coherent) sheaf of $\mathcal{D}$-modules supported on the closure of Q. (One of the main technical results of [V3] precisely identifies its global sections as the Harish-Chandra module of a principal series representation of $G_{\mathbb{R}}$.) The standard module $\mathcal{I}(Q,\tau)$ has a unique irreducible quotient $\mathcal{L}(Q,\tau)$. Each irreducible Harish-Chandra sheaf of $\mathcal{D}$ modules arises in this way, and there are no coincidences among the $\mathcal{L}(Q,\tau)$. In this way $\mathcal{L}oc_K(G/B)$ parametrizes irreducible Harish-Chandra sheaves, and hence irreducible $(\mathfrak{g}, K)$ modules with trivial infinitesimal character.

In the Grothendieck group of Harish-Chandra sheaves of $\mathcal{D}$-modules, we may write

$$[\mathcal{I}(\phi)] = \sum_{\psi \in \mathcal{L}oc_K(G/B)} M_{\phi,\psi}[\mathcal{L}(\psi)],$$

for nonnegative integers $M_{\phi,\psi}$. It is possible to invert this system and write

$$[\mathcal{L}(\phi)] = \sum_{\psi \in \mathcal{L}oc_K(G/B)} m_{\phi,\psi}[\mathcal{I}(\psi)]$$

this time for integers $m_{\phi,\psi}$.

The Kazhdan-Lusztig conjectures for $G_{\mathbb{R}}$ (proved in [V3]) characterize — and compute — the integers $m_{\phi,\psi}$ in terms of the Lusztig-Vogan Hecke algebra action. This will be discussed in the final lecture. To finish the present lecture, we make precise the sense in which the examples of Lecture 2 are indicative of the general case. This is, without a doubt, one of the most basic ideas in these lectures.

Fix a symmetric pair (G, K), chose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, and a Borel subalgebra $\mathfrak{b}$ containing it. Let $\mathfrak{B} \simeq G/B$ denote the variety of Borel subalgebras in $\mathfrak{g}$. For a simple root α of $\mathfrak{h}$ in $\mathfrak{b}$, write P_α for the corresponding parabolic subgroup containing B, and let $\mathfrak{P}_\alpha \simeq G/P_\alpha$ denote the variety of parabolic subalgebras of type α. Write π_α for the projection from $\mathfrak{B}$ to $\mathfrak{P}_\alpha$. This is a $\mathbb{P}^1 \simeq P_\alpha/B$ bundle.

Let Q be an orbit of K on $\mathfrak{B}$ and consider

$$S = S_\alpha(Q) := \pi_\alpha^{-1}\left(\pi_\alpha(Q)\right).$$

Since π_α is in particular K equivariant, S is a union of K orbits on $\mathfrak{B}$. We claim S contains a unique dense K orbit. This is obvious if K is connected: since π_α is a $\mathbb{P}^1$ bundle, S is a K invariant irreducible subset of $\mathfrak{B}$, and so Proposition 2.6 implies it has a dense K orbit. In general, the component group of K acts transitively on the irreducible component of Q, hence S, and the claim once again follows. We write

$$s_\alpha \cdot Q = \text{ unique dense orbit in } S_\alpha(Q).$$

(This actually leads to a well-defined monoid action on $K\backslash G/B$ [?], but we shall not emphasize this.) Since π_α is a fibration (with one dimensional fiber), either

$$\dim(s_\alpha \cdot Q) = \dim(Q) + 1$$

or

$$\dim(s_\alpha \cdot Q) = \dim(Q).$$

In the latter case we say α is in the descent set of Q; in the latter, we say it is not. The terminology is motivated by the following exercise.

Exercise 3.4. Consider a complex symmetric pair $(G \times G, \Delta(G))$. Fix a Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_L \oplus \mathfrak{h}_R$ in $\mathfrak{g} \oplus \mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{b}_L \oplus \mathfrak{b}_R$. Fix $Q \in K\backslash(G \times G)/B$ parametrized by $w \in W$ according to the Bruhat decomposition.

(i) Suppose α_L is a simple root of $\mathfrak{h}$ in $\mathfrak{b}$ whose corresponding root space lies in $\mathfrak{b}_L$. Then α is in the descent set of Q if and only if

$$l(s_\alpha w) < l(w).$$

(ii) Suppose α_L is a simple root of $\mathfrak{h}$ in $\mathfrak{b}$ whose corresponding root space lies in $\mathfrak{b}_L$. Then α is in the descent set of Q if and only if

$$l(w s_\alpha) < l(w).$$

□

In the setting above, given x in $\mathfrak{B}$, let $L_x = \pi_\alpha^{-1}(\pi_\alpha(x))$. Fix an orbit Q of K on G/B and suppose α is not in the descent set of Q. Set

$$Q' = s_\alpha \cdot Q.$$

Then there are exactly three mutually exclusive possibilities[4]:

(b) $Q \cap L_x$ consists of a single point and

$$L_x = (Q \cap L_x) \bigcup (Q' \cap L_x);$$

or, equivalently,

$$S_\alpha(Q) = Q \bigcup Q'.$$

In this case, we say α is a complex root for Q (not in the descent set of Q). Correspondingly we also say α is a complex root for Q' (in the descent set of Q). We write, as in Adams' lectures, $Q = s_\alpha \times Q'$ and $Q' = s_\alpha \times Q$. The basic example of this is Example 2.1.

(d) $Q \cap L_x$ consists of a single point, there is another orbit Q'' (denoted $s_\alpha \times Q$ in Adams' lectures) such that $Q'' \cap L_x$ is a point, and

$$L_x = (Q \cap L_x) \bigcup (Q' \cap L_x) \bigcup (Q'' \cap L_x);$$

[4]The apparently strange labeling of the cases is arranged to match [LV, Lemma 3.5] and [V3, Definition 6.4].

or, equivalently,

$$S_\alpha(Q) = Q \bigcup Q' \bigcup Q''.$$

In this case, we say α is Type I noncompact imaginary for Q and that Q' is the Cayley transform through α of Q or Q''. This time we say α is a Type I real root for Q', and write $s_\alpha \times Q' = Q'$. The basic example is Example 2.2.

(c) $Q \cap L_x$ consists of two points, and

$$L_x = (Q \cap L_x) \bigcup (Q' \cap L_x).$$

or, equivalently,

$$S_\alpha(Q) = Q \bigcup Q'.$$

In this case, we say α is Type II noncompact imaginary for Q. We say α is a Type II real root for Q'' and that Q'' is the Cayley transform of Q through α. We write $s_\alpha \times Q = Q$ and $s_\alpha \times Q' = Q'$. The basic example is Example 2.4.

Suppose next that α *is* in the descent set of Q. Cases (b)–(d) above cover three possibilities, and there is a fourth:

(a) $Q \cap L_x = L_x$ or, equivalently, $Q = Q'$. In this case we say α is compact imaginary for Q and write $Q = s_\alpha \times Q$. The basic example is Example 2.3.

Remark 3.5. The terminology "noncompact imaginary", "real", "complex", "Cayley transform", etc., are motivated by terminology from the representation theory of real groups. This is explained in Adams' lectures, as is the cross action $s_\alpha \times Q$.

All of the information in cases (a)–(d) is available from the output of `atlas`. (This might be explained in Thursday's evening session.) We simply finish with the rank one examples of Lecture 2.

Example 3.6. Let us consider Example 2.1 to begin[5].

```
real: type
Lie type: A1.A1
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): C
main: kgb
there is a unique real form: sl(2,C)
kgbsize: 2
Name an output file (return for stdout, ? to abandon):
0:  0  0  [C,C]   1  1    *  *  e
1:  1  0  [C,C]   0  0    *  *  1,2
```

[5]Atlas output is from version 0.3, the same version used on the web interface.

The output of this last table means there are two elements of $K\backslash G/B$. The second column gives their lengths (which are simply their dimensions minus the dimension of a closed orbit, one in this case). The terms in brackets correspond to the simple roots α_L and α_R (in the obvious notation as in Exercise 3.4). The `[C,C]` in the first row means that both α_L and α_R are complex roots for the orbit Q_0; the `[C,C]` in the second row means the same thing for the orbit Q_1. The next entries 11 in the first row mean, respectively, that $s_{\alpha_L} \times Q_0 = Q_1$ and that $s_{\alpha_R} \times Q_0 = Q_1$; the next entries 11 in the next row mean, respectively, that $s_{\alpha_L} \times Q_1 = Q_0$ and that $s_{\alpha_R} \times Q_1 = Q_0$.

Example 3.7. We next look at the output corresponding to Example 2.2.

```
empty: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): s
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
real: kgb
kgbsize: 3
Name an output file (return for stdout, ? to abandon):
0:  0  0  [n]   1    2  e
1:  0  0  [n]   0    2  e
2:  1  1  [r]   2    *  1
```

This time there are three orbits of dimensions 0, 0, and 1 respectively. The `[n]` and `r` mean that unique simple root α is noncompact imaginary for the first two, and real for the third. The next column says

$$s_\alpha \times Q_0 = Q_1$$
$$s_\alpha \times Q_1 = Q_0$$
$$s_\alpha \times Q_2 = Q_2.$$

The next column says that the Cayley transform of Q_0 and Q_1 through α is Q_2, and the `*` means that the Cayley transform of Q_2 though α is not defined (since α is real for Q_2). The root α is Type I noncompact imaginary for Q_0 and Q_1.

Example 3.8. Here is the output corresponding to Example 2.4.

```
empty: type
Lie type: T1.A1
```

```
elements of finite order in the center of the simply connected group:
Q/Z.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): ss
main: realform
(weak) real forms are:
0: gl(1,R).su(2)
1: gl(1,R).sl(2,R)
enter your choice: 1
real: kgb
kgbsize: 2
Name an output file (return for stdout, ? to abandon):
0:  0  0  [n]   0    1  e
1:  1  1  [r]   1    *  1
```

There are thus two orbits, one of length 0 and one of length 1. The unique simple root α is noncompact imaginary for the (and Type II) for the former, real for the latter, the cross action in α fixes both orbits, the Cayley transform of Q_0 through α is Q_1, and the Cayley transform of Q_1 through α is not defined.

Example 3.9. Finally we continue Exercise 2.3.

```
empty: type
Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): c
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 0
real: kgb
kgbsize: 1
Name an output file (return for stdout, ? to abandon):
0:  0  0  [c]   0    *  e
```

The output is more or less self-explanatory.

Example 3.10. Finally we conclude with a higher rank example of $\mathrm{Sp}(4, \mathbb{R})$.

```
empty: type
Lie type: C2
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es): s
main: realform
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
real: kgb
kgbsize: 11
Name an output file (return for stdout, ? to abandon):
 0:  0  0  [n,n]    1   2     4   5  e
 1:  0  0  [n,n]    0   3     4   6  e
 2:  0  0  [c,n]    2   0     *   5  e
 3:  0  0  [c,n]    3   1     *   6  e
 4:  1  1  [r,C]    4   9     *   *  1
 5:  1  2  [C,r]    7   5     *   *  2
 6:  1  2  [C,r]    8   6     *   *  2
 7:  2  2  [C,n]    5   8     *  10  1,2,1
 8:  2  2  [C,n]    6   7     *  10  1,2,1
 9:  2  1  [n,C]    9   4    10   *  2,1,2
10:  3  3  [r,r]   10  10     *   *  2,1,2,1
```

Consider the row labeled 4. It describes an orbit of length 1 (dimension 2 in this case). The first simple root α (which is short in the Bourbaki labeling) is real, the long simple root β is complex. We see that $s_\beta \times Q_4 = Q_9$, so indeed β is not in the descent set of Q_4. Turning to the row labeled 9, we see that α is now noncompact imaginary for Q_9. The Cayley transform of Q_9 through α is Q_{10}, the open orbit. The root α is Type II (why?).

LECTURE IV: THE HECKE ALGEBRA ACTION IN GENERAL

We need to extend the analysis at the end of the last section from $K\backslash G/B$ to $\mathcal{L}oc_K(G/B)$. Fortunately this is once again no harder than the examples of Lecture 2.

Definition 4.1 ([V3, Definition 6.4])**.** Let (G, K) be a symmetric pair and fix $H \subset B \subset G$. Let $\mathcal{M}$ be the free $\mathbb{Z}[q, q^{-1}]$ module with basis indexed by $\mathcal{L}oc_K(G/B)$. We identify $\tau \in \mathcal{L}oc_K(G/B)$ with the corresponding basis element of $\mathcal{M}$. Let α be a simple root of $\mathfrak{h}$ in $\mathfrak{b}$, and write $s = s_\alpha$ for the corresponding simple root.

(a) Suppose $\gamma \in \mathcal{L}oc_K(G/B)$ is supported on Q of the form considered in (a) of the previous section. Then define
$$T_s\gamma = q\gamma,$$
and $s \times \gamma = \gamma$.

(b1) Suppose $\gamma \in \mathcal{L}oc_K(G/B)$ is supported on Q of the form considered in (b) of the previous section. Then there is a unique K equivariant locally constant sheaf η on $Q \cup Q'$ such that $\eta|_Q$ is γ. (In other words, there is a unique K equivariant locally constant extension η of γ to $S = Q \cup Q'$.) In this case $\gamma' := \eta|_{Q'}$ is irreducible, i.e. is an element of $\mathcal{L}oc_K(G/B)$. Define
$$T_s\gamma = \gamma',$$
$s \times \gamma' = \gamma$, and $s \times \gamma = \gamma'$.

(b2) Suppose $\gamma' \in \mathcal{L}oc_K(G/B)$ is supported on Q' of the form considered in (b) of the previous section. Then there is a unique K equivariant locally constant sheaf η extending γ to $Q \cup Q'$. As before, $\gamma := \eta|_Q \in \mathcal{L}oc_K(G/B)$. Define
$$T_s\gamma' = (q-1)\gamma + q\gamma'.$$

(c1) Suppose $\gamma \in \mathcal{L}oc_K(G/B)$ is supported on Q of the form considered in (c) of the previous section. Then there are two K equivariant locally constant sheaves η_+ and η_- extending γ to $S = Q \cup Q'$. Let $\gamma_+ = (\eta_+)_{Q'}$ and $\gamma_- = (\eta_-)_{Q'}$. Define
$$T_s\gamma = \gamma + \gamma_+ + \gamma_-,$$
$s \times \gamma_\pm = \gamma_\mp$, and $s \times \gamma = \gamma$.

(c2) Suppose $\gamma' \in \mathcal{L}oc_K(G/B)$ is supported on Q' of the form considered in (c) of the previous section. *Assume* there is a K equivariant locally constant sheaf η extending γ' to $S = Q \cup Q'$; in this case η is unique. Write $\gamma = \eta|_Q$. Then there is one other extension K equivariant locally constant extension $\eta' \neq \eta$ of γ to S. Write $\gamma'' = \eta'|_{Q'}$. Define
$$T_s\gamma' = (q-1)\gamma' - \gamma'' + (q-1)\gamma.$$

(d1) Suppose $\gamma \in \mathcal{L}oc_K(G/B)$ is supported on Q of the form considered in (d) of the previous section. Then there is a unique locally constant extension η of γ to $S(Q) = Q \cup Q' \cup Q''$. Write $\gamma' = \eta|_{Q'}$ and $\gamma'' = \eta|_{Q''}$. Define
$$T_s\gamma = \gamma' + \gamma'',$$

$s \times \gamma = \gamma''$, $s \times \gamma' = \gamma'$, and $\gamma' = s \times \gamma'$.

(d2) Suppose $\gamma' \in \mathcal{L}oc_K(G/B)$ is supported on Q' of the form considered in (d) of the previous section. Let η denote the unique K-equivariant locally constant extention to $S(Q') = Q \cup Q' \cup Q''$. Let $\gamma = \eta|_Q$ and $\gamma'' = \eta|_{Q''}$. Define

$$T_s\gamma' = (q-2)\gamma' + (q-1)(\gamma + \gamma'').,$$

(e) Suppose $\gamma' \in \mathcal{L}oc_K(G/B)$ is supported on Q' of the form considered in (c) of the previous section (as in (c2) above), but suppose there is no K equivariant locally constant sheaf η extending γ' to S. Define

$$T_s\gamma' = -\gamma'$$

and $s \times \gamma' = \gamma'$.

Proposition 4.2 ([LV, Lemma 3.5]). *The formulas of Definition 4.1 coincide with the geometrically defined Lusztig-Vogan Hecke algebra action on $\mathcal{M}$.*

Remark 4.3. In rank 1, Definition 4.1 of course reduces to the formulas of Lecture 2.

Remark 4.4. The formulas $s \times \gamma$ generate an action of W on $\mathcal{L}oc_K(G/B)$. After composing with the equivalence of localization (Lecture 3), this is Vogan's cross action ([Vgr, Chapter 8]) defined in the context of coherent continuation of characters.

Rather than deal with generalities, we shall instead consider examples of Definition 4.1 using the output of `atlas`.

LECTURE V: THE KAZHDAN-LUSZTIG CONJECTURES FOR REAL GROUPS

References

[CG] N. Chriss, V. Ginzburg, *Representation Theory and Complex Geometry*, Birhäuster (Boston), 1997.

[KL] Kazhdan, D. and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.*, **53**(1979), no. 2, 165–184.

[LV] Lusztig, G. and Vogan, D., Singularities of closures of K-orbits on flag manifolds, *Invent. Math.* **71**(1983), 365–379.

[MS] Mars, J.G.M. and Springer, T. A., Hecke Algebra Representations related to spherical varieties, *Representation Theory*, **2**(1998), 33–69.

[Mi1] Milicic, D., *Localization and Representation Theory of Reductive Lie Groups* available at http://www.math.utah.edu/~milicic/book.pdf

[Mi2] Milicic, D., Algebraic $\mathcal{D}$-modules and representations of semisimple Lie groups, in *Analytic Cohomology and Penrose Transform*, M. Eastwood, J.A. Wolf, R. Zierau, editors, Contemporary Mathematics, **154** (1993), 133–168.

[Sp] Springer, T., Schubert varieties and generalizations, in *Representation theories and algebraic geometry*, edited by A. Broer, A, Daigneault, G. Sabidussi, Springer Verlag, 1998.

[V3] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups III: Proof of Kazhdan-Lusztig conjecture in the integral case, *Invent. Math.*, **71**(1983), no. 2, 381–417.

[V4] Vogan, D. A., Jr., Irreducible characters of semisimple Lie groups IV: character multiplicity duality, *Duke Math. J.*, **49**(1982), no. 4, 943–1073.

[Vgr] Vogan, D. A., Jr., *Representations of Real Reductive Groups*, Progress in Math., Birkhäuser(Boston), **15**(1981).

Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090
E-mail address: ptrapa@math.utah.edu