The Combinatorics of W-Graphs

Computational Theory of Real Reductive Groups Workshop

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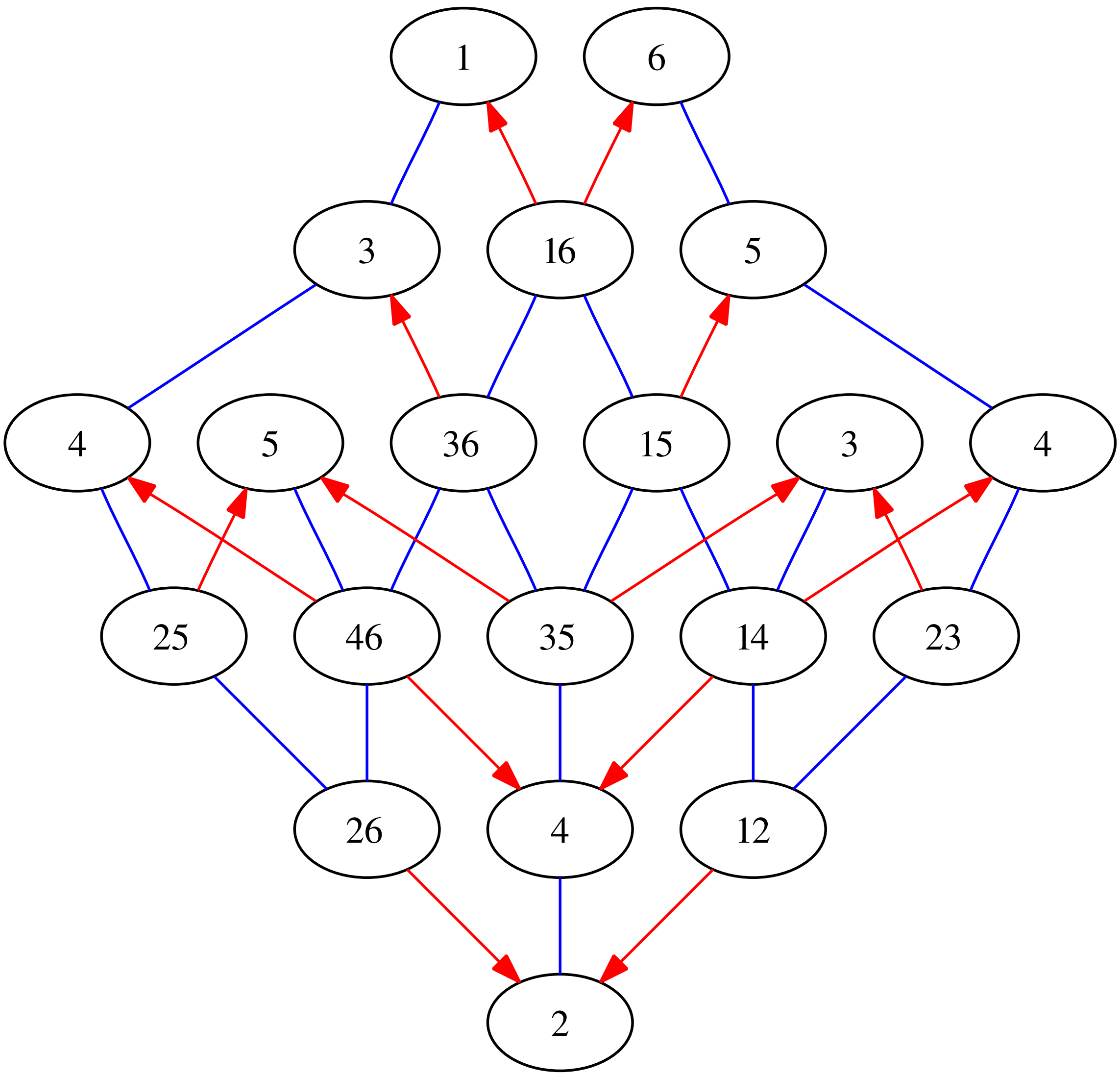


1. What is a W-Graph?

Let (W, S) be a Coxeter system, $S = \{s_1, \ldots, s_n\}$.

For us, W will always be a finite Weyl group.

Let $\mathcal{H} = \mathcal{H}(W, S) =$ the associated Iwahori-Hecke algebra over $\mathbb{Z}[q^{\pm 1/2}]$.

$$= \langle T_1, \ldots, T_n \mid (T_i - q)(T_i + 1) = 0, \text{ braid relations} \rangle.$$

DEFINITION. An S-**labeled graph** is a triple $\Gamma = (V, m, \tau)$, where

- V is a (finite) vertex set,
- $m : V \times V \to \mathbb{Z}[q^{\pm 1/2}]$ (i.e., a matrix of edge-weights),
- $\tau : V \to 2^S = 2^{[n]}$.

NOTATION. Write $m(u \to v)$ for the (u, v)-entry of m.

Let $M(\Gamma) =$ free $\mathbb{Z}[q^{\pm 1/2}]$-module with basis V.

Introduce operators T_i on $M(\Gamma)$:

$$T_i(v) = \begin{cases} qv & \text{if } i \notin \tau(v), \\ -v + q^{1/2} \sum_{u : i \notin \tau(u)} m(v \to u) u & \text{if } i \in \tau(v). \end{cases}$$

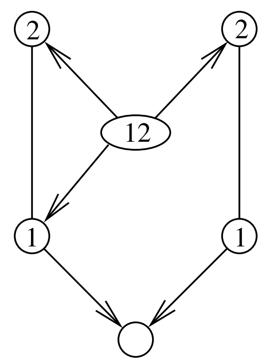
DEFINITION (K-L). Γ is a W-**graph** if this yields an $\mathcal{H}$-module.

NOTE: $(T_i - q)(T_i + 1) = 0$ (always), so W-graph $\Leftrightarrow$ braid relations.

$$T_i(v) = \begin{cases} qv & \text{if } i \notin \tau(v), \\ -v + q^{1/2} \sum_{u: i \notin \tau(u)} m(v \to u) u & \text{if } i \in \tau(v). \end{cases} \tag{1}$$

REMARKS.

- Kazhdan-Lusztig use T_i^t, not T_i.
- Restriction: for $J \subset S$, $\Gamma\big|_J := (V, m, \tau\big|_J)$ is a W_J-graph.
- At $q = 1$, we get a W-representation.
- However, braid relations at $q = 1 \not\Rightarrow W$-graph:



- If $\tau(v) \subseteq \tau(u)$, then (1) does not depend on $m(v \to u)$.

CONVENTION. WLOG, all W-graphs we consider will be **reduced**:

$$m(v \to u) = 0 \text{ whenever } \tau(v) \subseteq \tau(u).$$

DEFINITION. A W**-cell** is a strongly connected W-graph.

For every W-graph Γ, $M(\Gamma)$ has a filtration whose subquotients are cells.

Typically, cells are not irreducible as $\mathcal{H}$-reps or W-reps.

However (Gyoja, 1984): every irrep of W may be realized as a W-cell.

2. The Kazhdan-Lusztig W-Graph

$\mathcal{H}$ has a distinguished basis $\{C_w : w \in W\}$ (the Kazhdan-Lusztig basis). The left and right action of T_i on C_w is encoded by a $W \times W$-graph

$$\Gamma_{LR} = (W,\ m,\ \tau_{LR}):$$

- $\tau_{LR}(v) = \tau_L(v) \cup \tau_R(v)$, where

$$\tau_L(v) = \{i_L : \ell(s_i v) < \ell(v)\}, \quad \tau_R(v) = \{i_R : \ell(v s_i) < \ell(v)\}$$

- m is determined by the Kazhdan-Lusztig polynomials:

$$m(u \to v) = \begin{cases} \mu(u,v)+\mu(v,u) & \text{if } \tau_{LR}(u) \not\subseteq \tau_{LR}(v), \\ 0 & \text{if } \tau_{LR}(u) \subseteq \tau_{LR}(v), \end{cases}$$

where $\mu(u,v) =$ coeff. of $q^{(\ell(v)-\ell(u)-1)/2}$ in $P_{u,v}(q)$ ($= 0$ unless $u \leqslant v$).

Remarks.

- Hard to compute $\mu(x,y)$ without first computing $P_{x,y}(q)$.
- Restricting Γ_{LR} to the left action (say) yields a W-graph Γ_L.
- The cells of Γ_L decompose the regular representation of $\mathcal{H}$.
- Every two-sided K-L cell C has a "special" W-irrep associated to it that occurs with positive multiplicity in each left K-L cell $\subset C$.
- In type A, every left cell is irreducible, and the partition of W into left and right cells is given by the Robinson-Schensted correspondence.

The representation theory connection (complex groups):

- K-L "Conjecture": $P_{w_0 x, w_0 y}(1) =$ multiplicity of L_y in M_x,
- Vogan: $\mu(x,y) = \dim \operatorname{Ext}^1(M_x, L_y)$,

where M_w=Verma module with h.w. $-w\rho - \rho$, $L_w =$ simple quotient.

3. W-Graphs for Real Groups

There is a similar story for real groups:

Let $K =$ complexification of the maximal compact subgroup of $G_{\mathbb{R}}$.

Irreps can be assigned to K-orbits on G/B (complex case: $W \approx B\backslash G/B$).

There are K-L-V polynomials $P_{x,y}(q)$ generalizing K-L polynomials.

The top coefficients $\mu(x, y)$ encode a W-graph structure Γ_K on $K\backslash G/B$.

Usually Γ_K will break into more than one component (block).

Example. In the split real form of E_8, the W-graph has 6 blocks, the largest of which has 453,060 vertices and 104 cells.

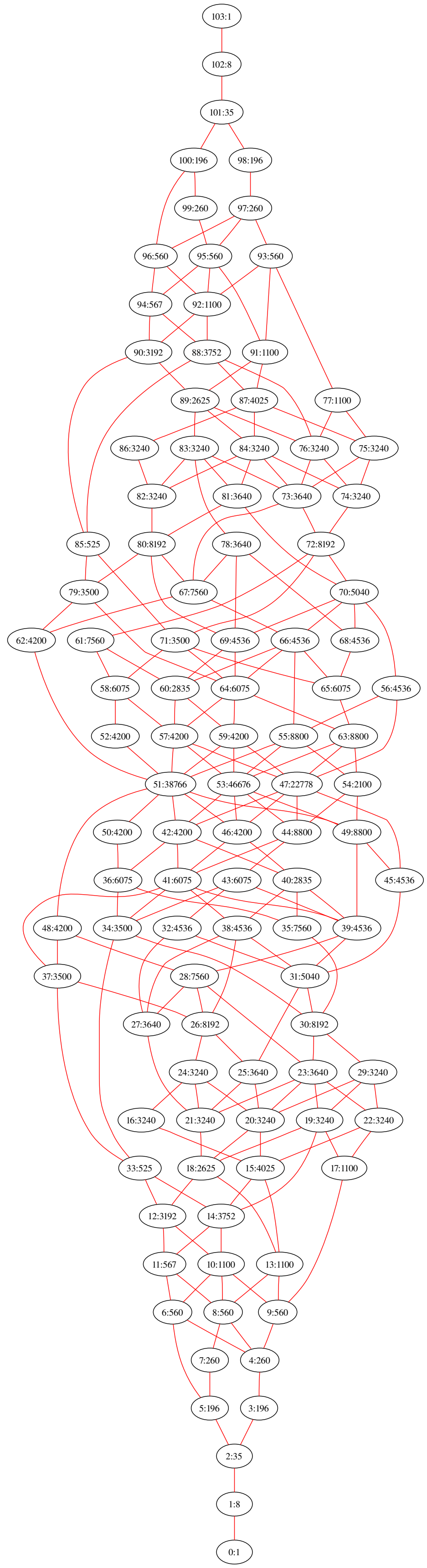
Cells for real groups often appear as cells of Γ_L. Not always.

Example. $G_{\mathbb{C}}$ as a real group.

It has Weyl group $W \times W$; its $W \times W$-graph is Γ_{LR}.

Main Points.

- The most basic constraints on these W-graphs are sufficiently strong that combinatorics alone can lend considerable insight into the structure of W-graphs and cells for real and complex groups.
- Sufficiently deep understanding of the combinatorics can yield constructions of W-cells without needing to compute K-L(-V) polynomials.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4. Admissible W-Graphs

Three observations about the W-graphs for real and complex groups:

(1) They have nonnegative integer edge weights.

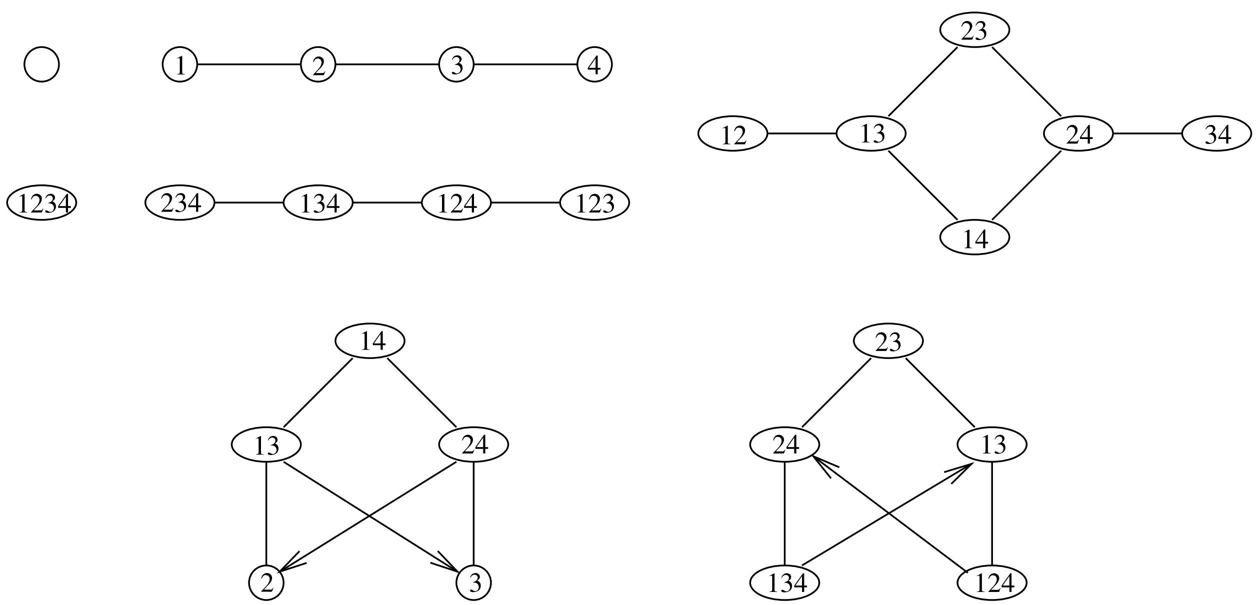
(2) They are **edge-symmetric**; i.e.,

$$m(u \to v) = m(v \to u) \ \text{ if } \tau(u) \not\subseteq \tau(v) \text{ and } \tau(v) \not\subseteq \tau(u).$$

(3) They are bipartite. (If $\mu(u, v) \neq 0$, then $\ell(u) \neq \ell(v)$ mod 2.)

Definition. A W-graph is **admissible** if it satisfies (1)–(3).

Example. The admissible A_4-cells:

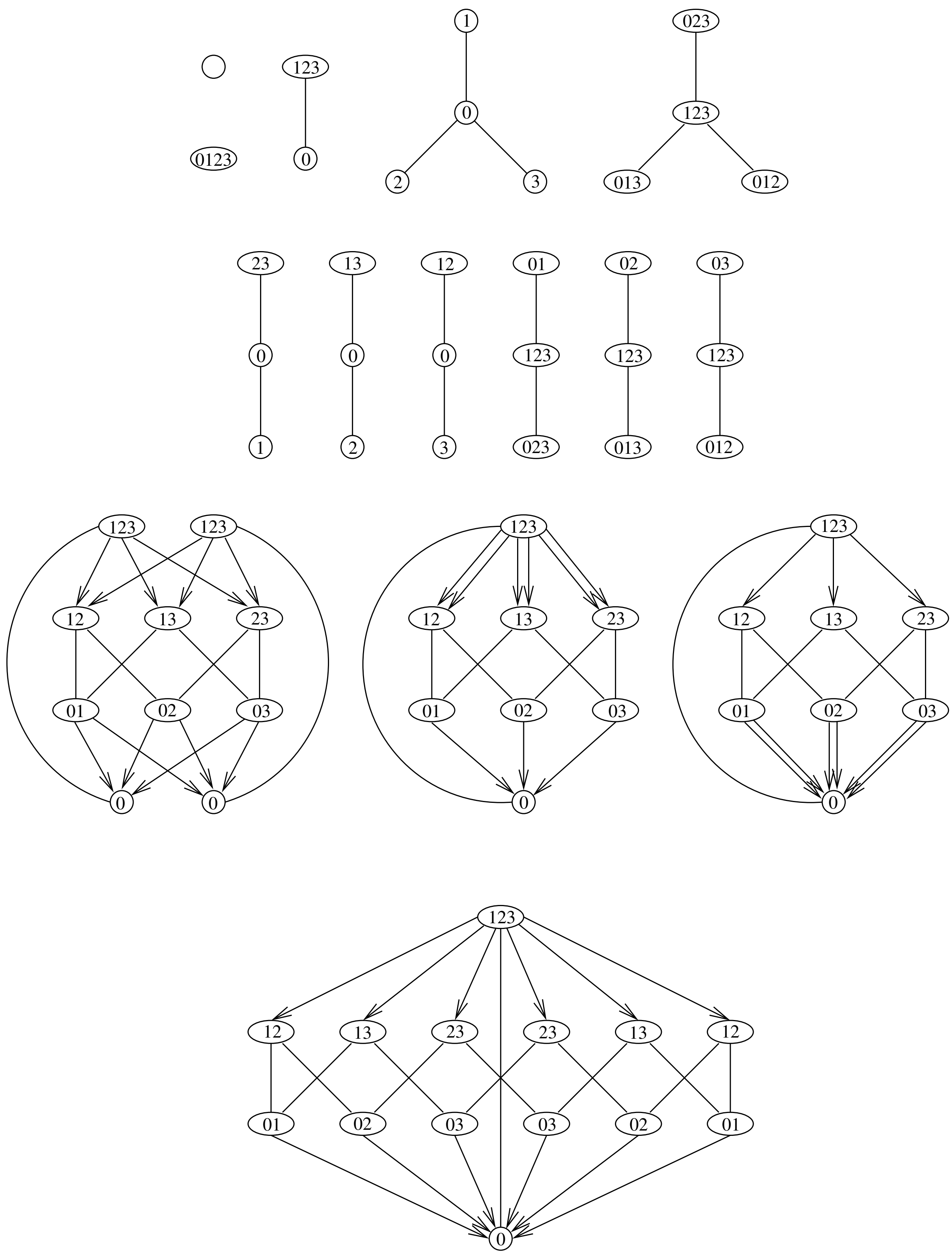


All of these are K-L cells; none are synthetic.

Question. *Is every admissible A_n-cell a K-L cell?* (Confirmed for $n \leqslant 9$.)

Caution. McLarnan-Warrington: Interesting things happen in A_{15}.

The admissible D_4-cells (three are synthetic):



5. Some Interesting Questions

PROBLEM 1. *Are there finitely many admissible W-cells?*

- Confirmed for $A_1, \ldots, A_9$, B_2, B_3, D_4, D_5, D_6, E_6, G_2.
- What about $W_1 \times W_2$-cells? More about this in Part II.

PROBLEM 2. *Classify/generate all admissible W-cells.*

PROBLEM 3. *How can we identify which admissible cells are synthetic?*

- Example: If Γ contains no "special" W-rep, then Γ is synthetic.
- Regard non-synthetics as closed under Levi restriction.

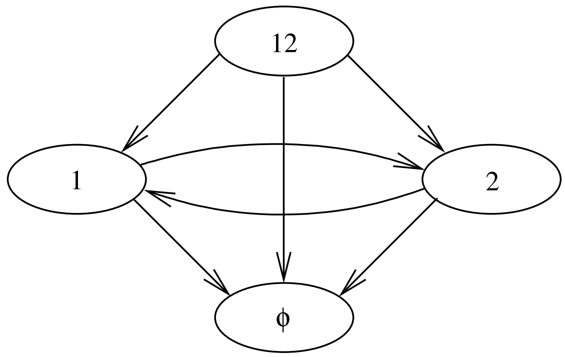
PROBLEM 4. *Understand "compressibility" of W-cells and W-graphs.*

- A given W-cell or W-graph should be reconstructible from a small amount of data. (Possible approaches: binding and branching rules.)

6. The Admissible Cells in Rank 2

Consider $W = I_2(p)$ (dihedral group), $2 \leqslant p < \infty$.

Given an $I_2(p)$-graph, partition the vertices according to τ:



Focus on non-trivial cells: $\tau(v) = \{1\}$ or $\{2\}$ for all $v \in V$.

The edge weight matrix will then have a block structure: $m = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$.

The conditions on m are as follows:

- $p = 2$: $m = 0$.
- $p = 3$: $m^2 = 1$ (i.e., $AB = BA = 1$).
- $p = 4$: $m^3 = 2m$.
- $p = 5$: $m^4 - 3m^2 + 1 = 0$.

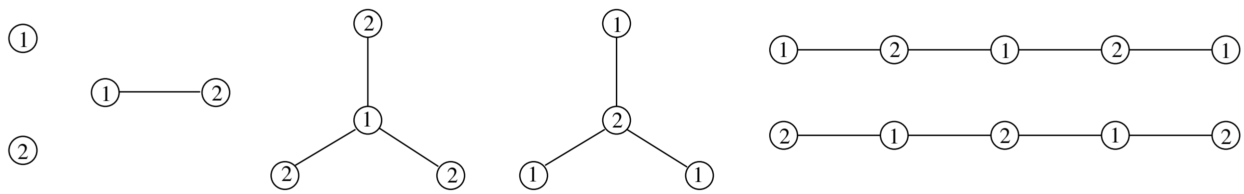
$\vdots$

Remarks.

- If we assume only $\mathbb{Z}$-weights, no classification is possible (cf. $p = 3$).
- Edge symmetry $\Leftrightarrow m = m^t$.
- When $p = 3$, edge weights $\in \mathbb{Z}^{\geqslant 0} \Rightarrow$ edge symmetry, but not in general.

Theorem 1. *A 2-colored graph is an admissible $I_2(p)$-cell iff it is a properly 2-colored A-D-E Dynkin diagram whose Coxeter number divides p.*

Example. The Dynkin diagrams with Coxeter number dividing 6 are A_1, A_2, D_4, and A_5. Therefore, the (nontrivial) admissible G_2-cells are



Remark. The nontrivial K-L cells for $I_2(p)$ are paths of length $p-2$.

Fact (Vogan; cf. Problem 3). In a Levi restriction of type $B_2 = I_2(4)$, all nontrivial B_2-cells in Γ_K are paths of length 2.

Proof Sketch. Let Γ be any properly 2-colored graph.

Let $\phi_p(t)$ be the Chebyshev polynomial such that $\phi_p(2\cos\theta) = \dfrac{\sin p\theta}{\sin\theta}$.

Then Γ is an $I_2(p)$-cell $\Leftrightarrow \phi_p(m) = 0$

$\Leftrightarrow m$ is diagonalizable with eigenvalues $\subset \{2\cos(\pi j/p) : 1 \leqslant j < p\}$.

Now assume Γ is admissible ($m = m^t$, $\mathbb{Z}^{\geqslant 0}$-entries).

If Γ is an $I_2(p)$-cell, then $2 - m$ is positive definite.

Hence, $2 - m$ is a (symmetric) Cartan matrix of finite type.

Conversely, let A be any Cartan matrix of finite type (symmetric or not).

Then the eigenvalues of A are $2 - 2\cos(\pi e_j/h)$, where $e_1, e_2, \dots$ are the exponents and h is the Coxeter number. $\square$

7. Combinatorial Characterization

What are the graph-theoretic implications of the braid relations?

THEOREM 2. *An admissible S-labeled graph is a W-graph if and only if the following properties are satisfied:*

- *the Compatibility Rule,*
- *the Simplicity Rule,*
- *the Bonding Rule, and*
- *the Polygon Rule.*

THE COMPATIBILITY RULE (applies to all W-graphs for all W):
If $m(u \to v) \neq 0$, *then*

every $i \in \tau(u) - \tau(v)$ *is bonded to every* $j \in \tau(v) - \tau(u)$.

Necessity follows from analyzing commuting braid relations.

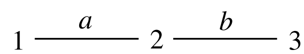
REFORMULATION: Define the **compatibility graph** $\mathrm{Comp}(W, S)$:

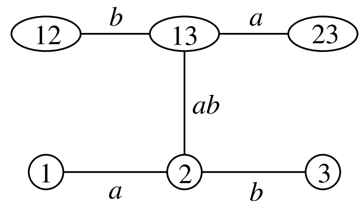
- vertex set $2^S = 2^{[n]}$,
- edges $I \to J$ when

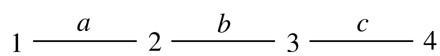
$$I \not\subseteq J \text{ and every } i \in I - J \text{ is bonded to every } j \in J - I.$$

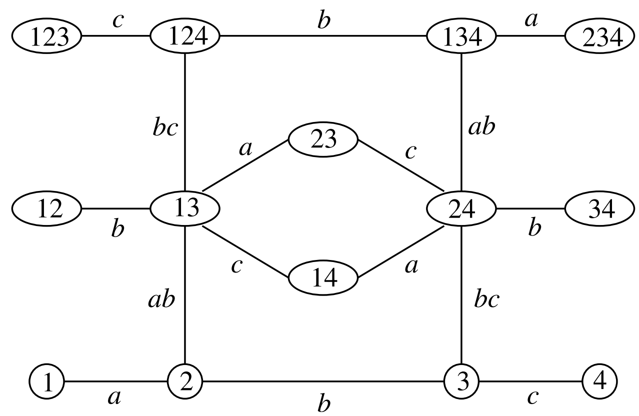
Compatibility means that $\tau : \Gamma \to \mathrm{Comp}(W, S)$ is a graph morphism.

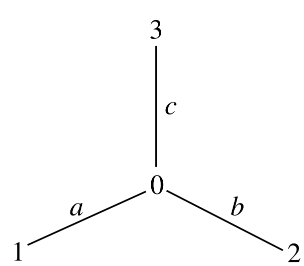
Compatibility graphs for A_3, A_4, and D_4

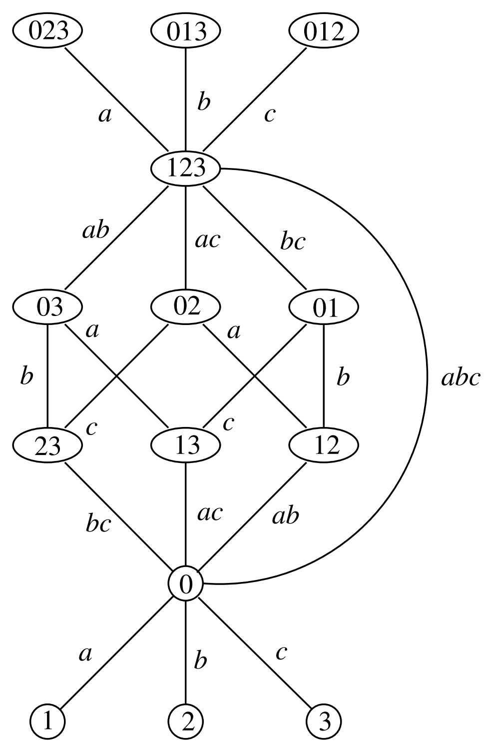












THE SIMPLICITY RULE:

Every edge $u \to v$ is either

- an **arc**: $\tau(u) \supsetneq \tau(v)$ (and there is no edge $v \to u$), or
- a **simple edge**: $m(u \to v) = m(v \to u) = 1$

Necessity follows from Theorem 1.

THE BONDING RULE:

If $s_i s_j$ has order $p_{ij} \geqslant 3$, then the cells of $\Gamma|_{\{i,j\}}$ must be

- singletons with $\tau = \varnothing$ or $\tau = \{i, j\}$, and
- *A*-*D*-*E* Dynkin diagrams with Coxeter number dividing p_{ij}.

Necessity again follows from Theorem 1.

EXAMPLE. If $p_{ij} = 3$, then the nontrivial cells in $\Gamma|_{\{i,j\}}$ are $\{i\} — \{j\}$.

Equivalently (for bonds with $p_{ij} = 3$): if $i \in \tau(u)$, $j \notin \tau(u)$ then there is a unique vertex v adjacent to u such that $i \notin \tau(v)$, $j \in \tau(v)$.

REMARK. The Compatibility, Simplicity, and Bonding Rules suffice to determine all admissible A_3-cells.

The Polygon Rule:

[Compare with G. Lusztig, *Represent. Theory* **1** (1997), Prop. A.4.]

Define

$$V^{ij} := \{v \in V : i \in \tau(v),\ j \in \tau(v)\},$$

$$V^i_j := \{v \in V : i \in \tau(v),\ j \notin \tau(v)\},$$

$$V_{ij} := \{v \in V : i \notin \tau(v),\ j \notin \tau(v)\}.$$

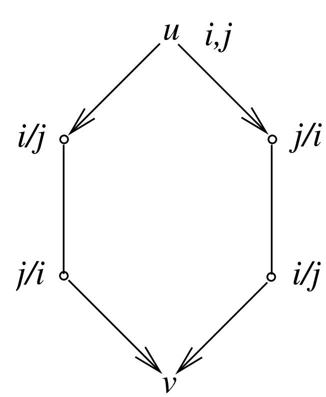
A path $u \to v_1 \to \cdots \to v_{r-1} \to v$ is *alternating of type* (i, j) if

$$u \in V^{ij},\ v_1 \in V^i_j,\ v_2 \in V^j_i,\ v_3 \in V^i_j,\ v_4 \in V^j_i,\ \ldots,\ v \in V_{ij}.$$

Set $N^r_{ij}(u, v) := \sum m(u \to v_1)m(v_1 \to v_2) \cdots m(v_{r-1} \to v)$
(sum over all r-step alternating paths of type (i, j)).
Then:

$$N^r_{ij}(u, v) = N^r_{ji}(u, v) \quad \text{for } 2 \leqslant r \leqslant p_{ij}.$$

Example. 3-step alternating paths



Remark. The Polygon Rule is **quadratic** in the arc weights.

8. Direct Products

Does the classification of admissible $W_1 \times W_2$-cells reduce to W_1 and W_2?

Not obviously. Not all cells are direct products.

Let $\Gamma = (V,\ m,\ \tau_1 \cup \tau_2)$ be an admissible $W_1 \times W_2$-graph.

Fact. Every edge $u \to v$ has one of three flavors:

- Type 1: $\tau_1(u) \not\subseteq \tau_1(v)$, $\tau_2(u) = \tau_2(v)$
- Type 2: $\tau_1(u) = \tau_1(v)$, $\tau_2(u) \not\subseteq \tau_2(v)$
- Type 12: $\tau_1(u) \supsetneq \tau_1(v)$, $\tau_2(u) \supsetneq \tau_2(v)$

Type 2 edges (and no others) are deleted when restricting Γ to W_1.

Hence, τ_2 is constant on W_1-cells.

Key Question. *Are there no arcs* **between** *cells in the W_1-restriction of a $W_1 \times W_2$-***cell** *Γ?*

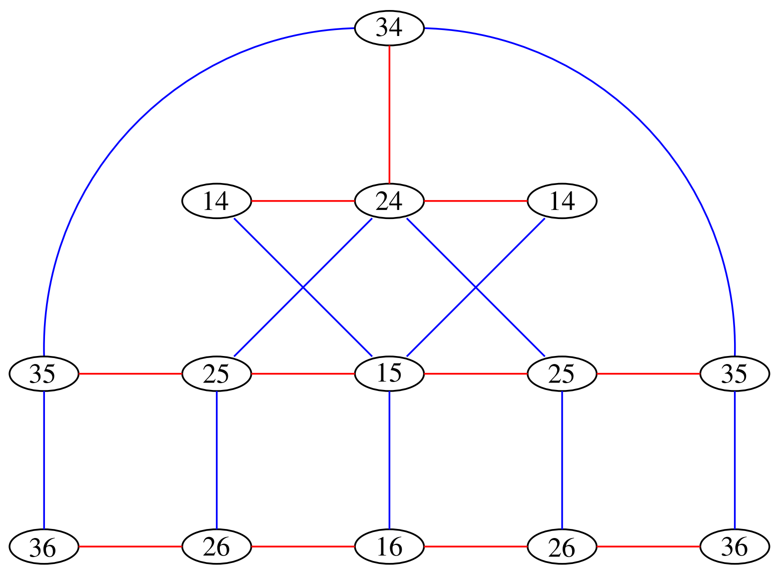
True for two-sided K-L cells. If true for a general $W_1 \times W_2$-cell Γ, then

- Type 12 edges cannot exist within Γ.
- Every W_1-cell in Γ meets every W_2-cell.
- Bounds the number admissible cells for $W_1 \times W_2$ in terms of W_1, W_2.
- Every W_1-cell in Γ has the same τ_1-support.

Even if the answer is negative, something weaker is true.

Fact. The τ_1-support of Γ equals the τ_1-support of an admissible W_1-cell.

An admissible (K-L) $B_3 \times B_3$-cell



9. A Strategy for Resolving the Key Question

Consider two properties of an arbitrary admissible W-graph $\Gamma = (V, m, \tau)$:

PROPERTY A. *If* Γ_1 *and* Γ_2 *are cells of* Γ *such that* $\Gamma_1 < \Gamma_2$ *in the induced partial order, then* $\tau(\Gamma_1) \neq \tau(\Gamma_2)$.

PROPERTY B. *If* Γ_1 *and* Γ_2 *are cells of* Γ *such that* $\Gamma_1 < \Gamma_2$ *in the induced partial order and* $\tau(\Gamma_1) = \tau(\Gamma_2)$, *then there is a third cell* Γ_3 *such that* $\Gamma_1 < \Gamma_3 < \Gamma_2$ *and* $\tau(\Gamma_3) \not\subseteq \tau(\Gamma_1) = \tau(\Gamma_2)$.

- (Easy) Property A implies Property B.
- Property B affirmatively resolves the Key Question.
- Property A holds for the left K-L graph Γ_L. False in general.
- Property B has been confirmed for all low-rank admissible cells.

N.B. If Property B holds for W_1, then the Key Question has an affirmative answer for all $W_1 \times W_2$-cells, for all choices of W_2.

10. Support Families

It is natural to partition W-cells into families according to their τ-support. Any two left K-L cells either

- belong to the same two-sided cell, and
- have the same τ-support, and
- contain the same "special" W-irrep,

or

- belong to distinct two-sided cells, and
- have unequal τ-support, and
- have no W-irreducibles in common.

NOTE. The τ-support of an admissible W-cell

- need not match the τ-support of a left K-L cell, and
- need not contain a special W-irrep (a synthetic marker).

QUESTION. *For each τ-support $\mathcal{T} \subset 2^S$, is there a W-irrep $\sigma = \sigma(\mathcal{T})$ such that every admissible W-cell with τ-support $\mathcal{T}$ contains a copy of σ?*

Assuming the Key Question has an affirmative answer, if $\Gamma_1, \dots, \Gamma_l$ are W-cells that appear in some admissible $W \times W'$-cell for some W', then they must have a W-irrep in common.

11. Molecular Components of W-Graphs

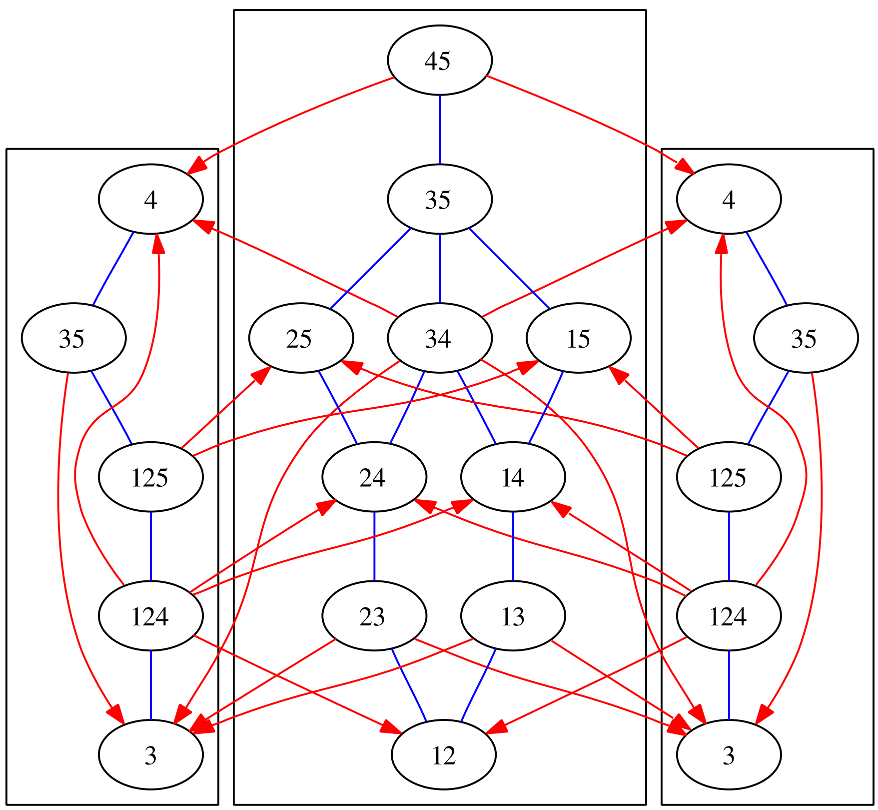
Recall the Simplicity Rule: every edge $u \to v$ is either

- an **arc**: $\tau(u) \supsetneq \tau(v)$ (and there is no edge $v \to u$), or
- a **simple edge**: $m(u \to v) = m(v \to u) = 1$

Definition. A **molecular component** of an admissible W-graph Γ is a subgraph whose simple edges form a single connected component.

Remark. All K-L cells in type A have only one molecular component.

A D_5-cell with three molecular components:



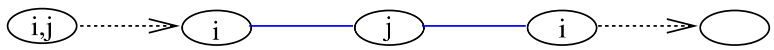
Classification strategy: first classify molecules, then classify all of the ways they may be glued together into (admissible) cells.

12. Synthesizing Molecules

IDEA #1: We can "easily" generate S-labeled graphs that satisfy the Compatibility, Simplicity, and Bonding Rules. No arc worries.

ISSUE: There are too many.

Need the Polygon Rule. Recall that it involves alternating (i, j)-paths:



FACT. Let (u, v, r, i, j) be an instance of the Polygon Rule (initial point u, terminal point v, path length r). Then

- if $r = 2$ and there is $k \in \tau(v) - \tau(u)$, or
- if $r = 3$ and there is $k, l \in \tau(v) - \tau(u)$ such that k is not bonded to i and l is not bonded to j, or
- if $r \geqslant 3$ and there is $k \in \tau(v) - \tau(u)$ such that k is not bonded to i or j,

then the resulting constraint is **linear** in weights of arcs.

An alternating path with only one arc can only involve the molecular components containing the two endpoints.

CONCLUSION: These instances of the Polygon Rule can be imposed **locally**.

SO: add the **Local** Polygon Rule as a constraint on molecular components.

13. Stable Molecules

DEFINITION. An S-labeled graph that satisfies the Compatibility, Simplicity, Bonding, and Local Polygon Rules is **molecular**.

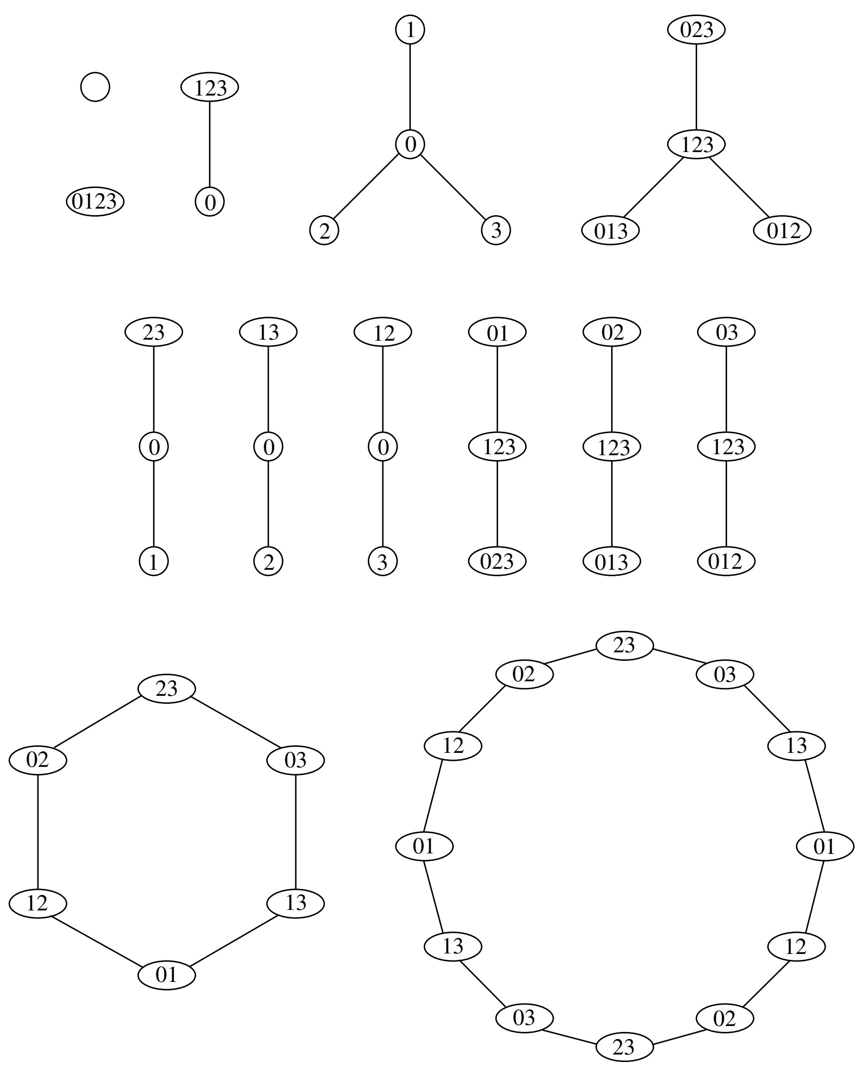
- If it has only one molecular component, it is a **molecule**.
- If it occurs in some admissible W-graph, it is **stable**.

For $n \leqslant 9$, the A_n-molecules are precisely the K-L cells!

There do exist unstable molecules. Sometimes infinitely many.

But in all cases so far, they have manageable structure.

The stable D_4-molecules:



14. Binding Spaces

Given a list of (stable) W-molecules, what are all of the (stable) molecular graphs that can be obtained by binding them together?

Focus on pairs of molecules, say Γ_1 and Γ_2.

Regard every inclusion $\tau(v_1) \supsetneq \tau(v_2)$ as a potential arc $v_1 \to v_2$.

Danger: Admissible graphs must be bipartite!

Work in a category of **molecules-with-parity**:

every vertex has a parity, edges connect vertices of opposite parity.

Molecules are connected, so each affords two parity choices.

Notation: $\Gamma \mapsto -\Gamma$ (parity-reversing operator).

Definition. A **binding space** is the vector space $B(\Gamma_1 \to \Gamma_2)$ of weight assignments for arcs $\Gamma_1 \to \Gamma_2$ that satisfy the Local Polygon Rule.

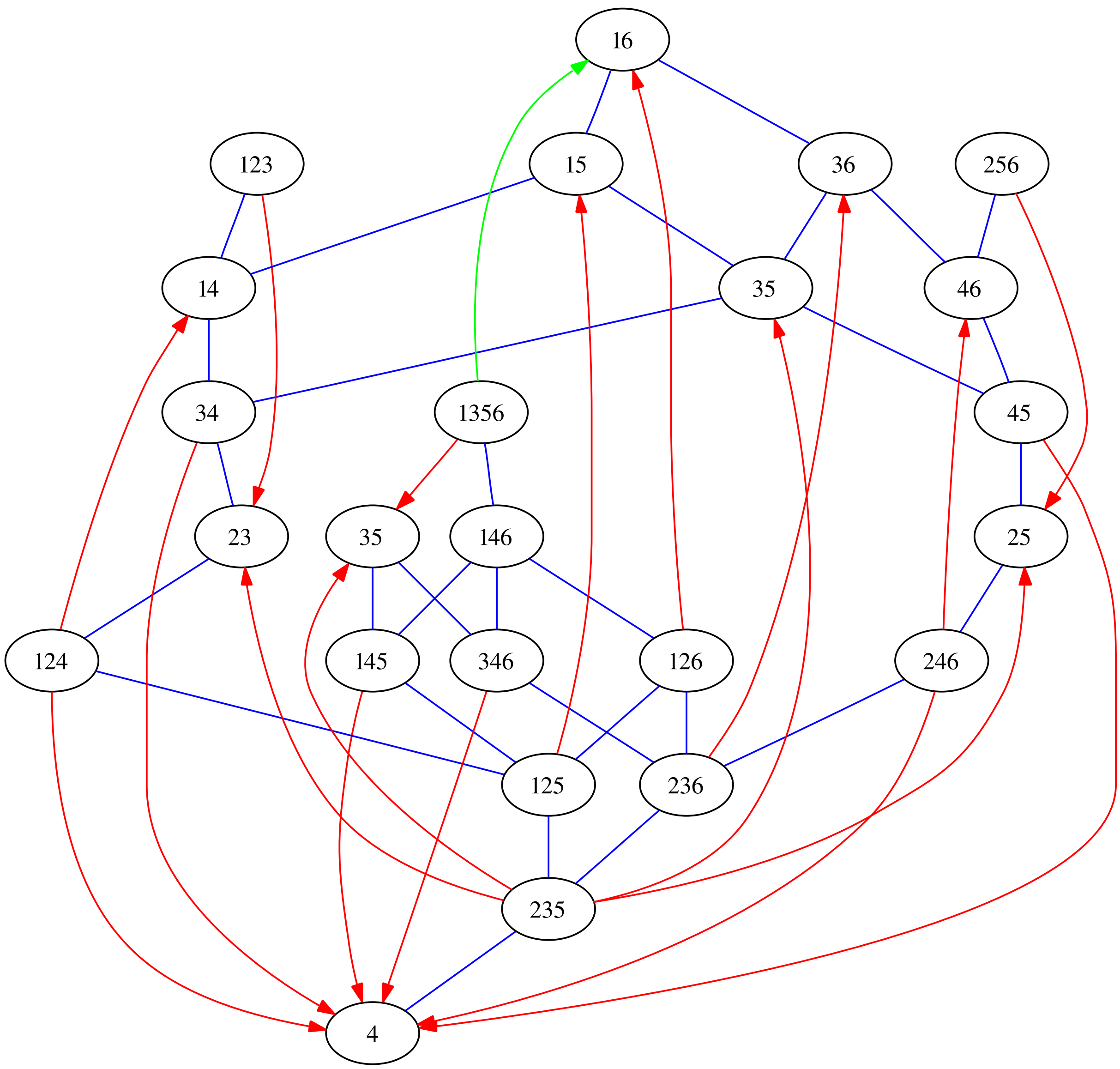
- Depends only on the simple edges of Γ_1 and Γ_2.
- In simply-laced cases (at least), there is no torsion.
- Often, $\dim B(\Gamma_1 \to \Gamma_2) = 0$ or 1.
- Self-binding: $B(\Gamma \to \Gamma)$ (even), $B(\Gamma \to -\Gamma)$ (odd).

Definition. A binding is **stable** if it occurs in some admissible W-graph.

Note. Each W-molecule Γ also has an **internal** binding space $B(\Gamma)$.

- $B(\Gamma)$ may be identified with an affine translate of $B(\Gamma \to \Gamma)$.

Example. An E_6-molecule with $\dim B(\cdot) = 1$:



15. Binding Families

DEFINITION. The **bindability graph** $\mathrm{BG}(W)$ is the directed graph with

- vertices corresponding to W-molecules
- edges $\Gamma \to \Gamma'$ whenever $\dim B(\pm\Gamma \to \pm\Gamma') > 0$.

Similarly, there is a **stable** bindability graph $\mathrm{BG}_{\mathrm{st}}(W)$.

Break $\mathrm{BG}(W)$ or $\mathrm{BG}_{\mathrm{st}}(W)$ into strongly connected components.

NOTE. Every admissible W-cell is obtained by binding together one or more W-molecules from some strongly connected component of $\mathrm{BG}(W)$.

- The same holds for $\mathrm{BG}_{\mathrm{st}}(W)$.
- This provides another natural way to partition W-cells into families.
- The resulting binding families of W-cells are partially ordered.
- For every admissible W-graph Γ, there is an order-preserving map

$$\phi(\Gamma) : \{\text{cells of } \Gamma\} \to \{\text{binding families of } W\text{-cells}\}.$$

QUESTIONS.

- Is $\phi(\Gamma_L)$ surjective (i.e., does every binding family contain a K-L cell)?
- Are the fibers of $\phi(\Gamma_L)$ unions of 2-sided cells?
- Is every binding family a union of support families?
- Are the binding families mutually orthogonal (as W-modules)?
- Is there a “special” molecule that occurs in every W-cell in a family?

Binding families of W-cells for $W = D_5$, D_6, and E_6.