# Weighted orbital integrals

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# Definition

$F$ local field of characteristic 0.

$G$ connected reductive group over $F$
(rather its $F$-rational points),

$A_G$ maximal $F$-split torus in center of $G$,
its Lie algebra $\mathfrak{a}_G$ for $F = \mathbb{R}$.

$G^1$ common kernel of all continuous homomorphisms $G \to \mathbb{R}$.
The latter are of the form $g \mapsto \lambda(H)$ with $\lambda \in \mathfrak{a}_G^*$, where $g = g^1 \exp H$, $g^1 \in G^1$, $H \in \mathfrak{a}_G$.

For general $F$, $\mathfrak{a}_G^* := \mathrm{Hom}(G, \mathbb{R})$ with dual space $\mathfrak{a}_G$, characterize $H = H_G(g)$ by above condition for all $\lambda$.

Action of $i\mathfrak{a}_G^*$ on unitary dual $\Pi(G)$:
$\pi_\lambda(g) := e^{\lambda(H_G(g))}\pi(g)$.

**Example:** $G = \mathrm{GL}(n, F)$, $A_G = F^\times$, $\mathfrak{a}_G = \mathbb{R}$,
$H_G(g) = \frac{1}{n} \log |\det g|$.

Parabolic subgroup $P$ has Levi decomposition $P = MN$. Fix special maximal subgroup $K$, then $G = PK = MNK$,
write $H_P(mnk) := H_M(m)$.

$\mathcal{P}(M)$ set of parabolics $P$ with given Levi component $M$;
is in bijection with set of chambers $\mathfrak{a}_P^+$ in $\mathfrak{a}_M$.

$v_M(x) := \mathrm{vol}_{\mathfrak{a}_M/\mathfrak{a}_G} \mathrm{conv}\{-H_P(x) : P \in \mathcal{P}(M)\}$.

**Definition 1** *If $f \in \mathcal{C}(G)$ and $m \in M$ s. t. $G_m \subset M$,*

$J_M(m, f) := |D(m)|^{1/2} \int_{G_m \backslash G} f(x^{-1}mx) v_M(x) \, d\dot{x}$,

*where $D(m) := \det_{\mathfrak{g}_s \backslash \mathfrak{g}}(\mathrm{Id} - \mathrm{Ad}(s))$, $s$ semisimple component of $m$.*

$$v_M(x) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} \frac{e^{-\lambda(H_P(x))}}{\theta_P(\lambda)},$$

$\theta_P^{-1}$ Fourier transform of characteristic function of ${}^+\mathfrak{a}_P$.

For $\mathrm{rk}_F\, G = 1$, $\mathcal{P}(M) = \{P, \bar{P}\}$,

$$J_M(m, f) = -|D^M(m)|^{1/2} \delta_P(m) \times \int_K \int_N f(k^{-1}mn'k) \rho_P(H_{\bar{P}}(n)) \, dn' \, dk,$$

where $mn' = n^{-1}mn$,
$\delta_P(m) = e^{\rho_P(H_M(m))} = \det(\mathrm{Ad}_{\mathfrak{n}}(m))^{1/2}$.
If $G = \mathrm{GL}(2, F)$, $M$ diagonal, then $H_{\bar{P}} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = -\log \|(1, x)\|$ for $K$-invariant norm on $F^2$.

# Invariant Fourier transform

$\widehat{f}(\pi) := \operatorname{tr} \breve{\pi}(f)$ for $f \in \mathcal{C}(G)$ and $\pi \in \Pi_{\text{temp}}(G)$, where

$$\pi(f) = \int_G f(g)\pi(g)\, dg.$$

$$\text{Then} \qquad \operatorname{tr} \pi(f) = \int_G \Theta_\pi(g) f(g)\, dg,$$

$\Theta_\pi$ analytic on $G_{\text{reg}} := \{g \in G : G_g \text{ is a torus}\}$, $\Theta_{\breve{\pi}} = \bar{\Theta}_\pi$.

**Trace Paley-Wiener theorem:** $f \mapsto \widehat{f}$ is open, continuous surjection of $\mathcal{C}(G)$ onto explicit "Schwarz space" $\mathcal{I}(G)$ of functions

$$\phi : \Pi_{\text{temp}}(G) \to \mathbb{C}.$$

From $\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$ we see

$$\widehat{f_1 * f_2} = \widehat{f_2 * f_1}.$$

**Definition 2** *The tempered distribution* $I : \mathcal{C}(G) \to \mathbb{C}$ *has the Fourier transform* $\hat{I} : \mathcal{I}(G) \to \mathbb{C}$ *if* $\quad \hat{I}(\widehat{f}) = I(f)$ *for all* $f$.

$I$ necessarily invariant: $I(f_1 * f_2) = I(f_2 * f_1)$.

**Example 1:** $I(f) = J_G(e, f) = f(e)$, $\hat{I}$ Plancherel measure.

# Tempered dual

All $\pi \in \Pi_{\mathrm{temp}}(G)$ are constituents of $\mathrm{Ind}_P^G \sigma$ for some Levi $L$, $\sigma \in \Pi_2(L)$ (square integrable mod $A_L$) and $P \in \mathcal{P}(L)$.

For $\phi \in \mathcal{I}(G)$, $\phi(\mathrm{Ind}_P^G \sigma_\lambda)$ is Schwartz function of $\lambda \in i\mathfrak{a}_L^* / \operatorname{Stab} \sigma$.
Compatibility for reducible $\mathrm{Ind}_P^G \sigma_\lambda$.

For $P' = LN' \in \mathcal{P}(L)$, intertwining operator $J_{P'|P}(\sigma_\lambda) : \mathrm{Ind}_P^G \sigma_\lambda \rightsquigarrow \mathrm{Ind}_{P'}^G \sigma_\lambda$, $\psi \mapsto \psi'$,

$$\psi'(g) = \int_{N \cap N' \backslash N'} \psi(n'g)\, dn'$$

for $\Re\lambda$ in some cone in $\mathfrak{a}_L^*$.
Compact picture: $\psi|_K$.
Meromorphic continuation for $K$-finite $\psi$.

Meromorphic normalizing factors $r_{P'|P}$, so that $J_{P'|P}(\sigma_\lambda) = r_{P'|P}(\sigma_\lambda) R_{P'|P}(\sigma_\lambda)$ with $R_{P'|P}(\sigma_\lambda)$ regular for $\lambda = 0$,

$$R_{P''|P'}(\sigma) R_{P'|P}(\sigma) = R_{P''|P}(\sigma).$$

For $w \in W_L$ and generic $\sigma$,
$\mathrm{Ind}_P^G \sigma \sim \mathrm{Ind}_{Pw}^G \sigma \sim \mathrm{Ind}_P^G (w\sigma) =: \sigma^G$.

$\Theta_{\sigma^G}$ vanishes on $G_{\text{ell}}$ (the set of $g$ not contained in any parabolic over $F$).

**Example 2:** $I(f) = J_G(l, f)$, $l \in L \cap G_{\text{reg}}$, $\operatorname{supp} \phi \subset \{\sigma^G : \sigma \in \Pi_2(L)\}$.
$\hat{I}(\phi) = \hat{J}_G(l, \phi)$ vanishes unless $l \in L_{\text{ell}}$, in which case it equals

$$|D^L(l)|^{1/2} \sum_{\sigma \in \Pi_2(L)/i\mathfrak{a}_L^*} \int_{i\mathfrak{a}_L^*/\operatorname{Stab}\sigma} \Theta_{\sigma_\lambda}(l)\phi(\sigma_\lambda^G)\, d\lambda.$$

$\hat{J}_G(g, \phi)$ found for $F = \mathbb{R}$ and all $g$ by Herb.

Arthur introduced set $T(G)$ of virtual tempered representations, so that $\phi \in \mathcal{I}(G)$ is determined by $\phi(\tau)$, $\tau \in T(G)$.

$T_{\text{ell}}(G) = \{\tau \in T(G) : \Theta_\tau|_{G_{\text{ell}}} \neq 0\}$.
Any $\phi$ is determined by $\phi(\tau^G)$ for $\tau \in T_{\text{ell}}(L)$.

$$\mathcal{I}(G) \cong \widehat{\bigoplus_{[L]}} \mathcal{C}(T_{\text{ell}}(L))^{W_L}.$$

**Example:** For $\mathrm{SL}(2, \mathbb{R})$, replace limits of discrete series $\pi_1$, $\pi_2$ by $\tau_\pm = \pi_1 \pm \pi_2$.
$\tau_- \in T_{\text{ell}}(G)$.

# Weighted characters

$L$ Levi in $G$, $\sigma \in \Pi_{\text{temp}}(L)$,
In compact picture of $\text{Ind}_P^G \sigma$:

$$\mathcal{R}_P(\sigma) := \lim_{\lambda \to 0} \sum_{P' \in \mathcal{P}(L)} \frac{R_{P'|P}(\sigma)^{-1} R_{P'|P}(\sigma_\lambda)}{\theta_P(\lambda)}.$$

If $L$ maximal, then
$\mathcal{R}_P(\sigma) = -\frac{1}{\theta_P(\lambda)} R_{\bar{P}|P}(\sigma)^{-1} \frac{d}{dz} R_{\bar{P}|P}(\sigma_{z\lambda})|_{z=0}$.

$\phi_L(f, \sigma) := \text{tr}(\text{Ind}_P^G(\check{\sigma}, f)\mathcal{R}_P(\check{\sigma})) \quad$ indep. of $P$.

Then $\phi_L : \mathcal{C}(G) \to \mathcal{I}(L)$ (at least for $F = \mathbb{R}$), $\phi_G(f) = \hat{f}$.

Replace $R_{P'|P}$ by $J_{P'|P}$ to get meromorphic $\mathcal{J}_P(\sigma_\lambda)$ and $\phi_P(f, \sigma_\lambda)$ for $f \in \mathcal{H}(G)$ (Hecke-algebra).

# Invariant distributions

For maximal Levi $M$ in $G$, write

$$J_M(m,f) = I_M(m,f) + \hat{J}_M^M(m, \phi_M(f))$$

Then $I_M(m)$ is invariant. Explicit if $M$ also minimal, i. e., $M/A_M$ compact. Version for $f \in \mathcal{H}(G)$:

$$J_M(m,f) = I_P(m,f) + |D^M(m)|^{1/2} \int_{\Pi_{\text{temp}}(M)} \Theta_{\sigma_\varepsilon}(m) \phi_P(f, \sigma_\varepsilon)\, d\sigma,$$

where $\varepsilon \in \mathfrak{a}_P^+$ s. t. $\phi_M(\sigma_\lambda, f)$ has no poles for $0 < \Re\lambda \leq \varepsilon$.

**Theorem 1 (Arthur)** *There are invariant distributions $I_M^L(m)$ on $L$ for all Levi subgroups $L \supset M$ s. t.*

$$\begin{aligned} J_M(m,f) &= \sum_{L \supset M} \hat{I}_M^L(m, \phi_L(f)) \\ &= I_M(m,f) + \sum_{\substack{L \supset M \\ L \neq G}} \hat{I}_M^L(m, \phi_L(f)). \end{aligned}$$

**Theorem 2 (Arthur)** *There exist smooth functions*

$$\Phi_{M,L} : (M \cap G_{\text{reg}}) \times T^G_{\text{disc}}(L) \to \mathbb{C}$$

*such that*

$$\hat{I}_M(m,\phi) = \sum_{[L]} \int_{W_L \backslash T^G_{\text{disc}}(L)} \Phi_{M,L}(m,\tau)\phi(\tau^G)\,d\tau.$$

*If $\pi \in \Pi_2(G) \subset T_{\text{ell}}(G)$, then $\Phi_{M,G}(m,\pi)$ vanishes unless $m \in G_{\text{ell}}$, in which case it equals*

$$(-1)^{\dim \mathfrak{a}^G_M} |D(m)|^{1/2} \Theta_\pi(m),$$

*where $a^G_M = \mathfrak{a}_M/\mathfrak{a}_G$.*

Case $M = G$ see above.
By descent, $\Phi_{M,L}$ also known for
$\dim \mathfrak{a}^G_M + \dim \mathfrak{a}^G_L \leq \dim \mathfrak{a}^G_T$, where $T = G_m$.

# Differential equations

Let $F = \mathbb{R}$. $\pi \in \Pi(G)$ has infinitesimal character $\chi_\pi : Z(\mathfrak{g}) \to \mathbb{C}$.

Fix maximal torus $T$, Levi $L$, $\tau \in T_{\mathrm{ell}}(L)$.

Set $\chi = \chi_{\tau^G}$, $\Phi_M(t) = \Phi_{M,L}(t,\tau)$ for $M \supset T$.
$\chi_{\tau^G}(z) = \chi_\tau(z_L)$.

**Theorem 3 (Arthur)** *There exist smooth functions*

$$\partial_M^{M'} : T \cap G_{\mathrm{reg}} \to \mathrm{Hom}(Z(\mathfrak{m}'), U(\mathfrak{t}))$$

*such that*

$$\chi(z)\Phi_M(t) = \sum_{M' \supset M} \partial_M^{M'}(t, z_{M'})\Phi_{M'}(t)$$

*for $z \in Z(\mathfrak{g})$. In particular, $\partial_G^G(t,z) = z_T$.*

For parabolic $P = M_P N \supset T$,

$T_P := \{t \in T : |t^\alpha| > 1 \ \forall \text{ roots } \alpha \text{ of } T \text{ in } N\}$.

**Theorem 4 (W.H.)** *The system applied to $(\Phi_M)_{M \supset M_P}$ is holonomic on $T_{\mathbb{C}} \cap G_{\mathbb{C},\mathrm{reg}}$ and has a regular singularity at $\infty$ on $T_{\mathbb{C},P}$.*
*For every $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ with $\chi_\lambda = \chi$ (where $\chi_\lambda(z) = z_T(\lambda)$) there is a unique solution $\Psi = \Psi^{P,\lambda}$ on $\tilde{T}_{\mathbb{C},P}$ s. t.*

- $\Psi_G(\exp H) = e^{\lambda(H)}$,
- $\Psi_M(t) \to 0$ *as* $t \underset{P}{\to} \infty$ *if* $M \neq G$.

*For sufficiently regular $\chi$, every solution is of the form*

$$\Phi_M(t) = \sum_{\chi_\lambda = \chi} \sum_{M' \supset M} c_{M'}(\lambda) \Psi_M^{P \cap M', \lambda}(t).$$

**Problems:** Describe $\Psi_M(t)$.
Find $c_{M'}(\lambda) = c^P_{M',L}(\lambda, \tau)$ for $\Phi_M(t) = \Phi_{M,L}(t, \tau)$ on $T_P$.

From a theorem of Arthur (2006) we get

**Theorem 5** *If $L \subset P \neq G$, then*

$$\Phi_{P,L}(t,\tau) \to 0 \quad \text{as } t \underset{P}{\to} \infty.$$

The Fourier transforms $\Phi_{M,L}(t,\tau)$ satisfy jump relations at singular $t$.

To determine $c_{M',L}(\lambda,\tau)$, we need jump relations for $\Psi_M(t)$.

Solved for $\mathrm{rk}_{\mathbb{R}} G = 1$, $G = \mathrm{GL}(3,\mathbb{R})$.

**Theorem 6 (W.H.)** *If $G$ is arbitrary, $T$ split, $P$ minimal and $\sigma(t) = t^\lambda$ sufficiently regular, then*

$$\Phi_{P,T}(t,\sigma) = \sum_{\lambda' \in W_T \lambda} \Psi_T^{P,\lambda'}(t).$$

# Explicit Fourier transforms

**Theorem 7 (W.H.)** *If $M$ is maximal, then*

$$\Psi_M^{P,\lambda}(t) = t^\lambda \sum_\alpha \eta(\check\alpha) b(\lambda(\check\alpha), t^{-\alpha}),$$

*sum over roots $\alpha$ of $\mathfrak{t}_\mathbb{C}$ s. t. $\mathfrak{g}_\alpha \subset \mathfrak{n}_\mathbb{C}$ but $\mathfrak{g}_\alpha \not\subset \mathfrak{m}_\mathbb{C}$,*

$$b(s,z) = \sum_{n=1}^\infty \frac{z^n}{n+s} = z\int_0^1 \frac{x^s}{1-zx}\,dx$$

*($|z|<1$, $\Re s > -1$).*

*If $G = \mathrm{GL}(3,\mathbb{R})$, $P$ upper triangular, $M = T$ diagonal, then*

$$\Psi_T^{P,\lambda}(t) = t^\lambda\Big(\tilde b(\lambda_{23},\lambda_{13},t_{32},t_{21}) + \tilde b(\lambda_{12},\lambda_{13},t_{21},t_{32})$$
$$+ b(\lambda_{12},t_{31})b(\lambda_{23},t_{32}) + b(\lambda_{12},t_{21})b(\lambda_{23},t_{31})\Big),$$

*where $t_{ij} = t_i/t_j$, $\lambda_{ij} = \lambda_i - \lambda_j$,*

$$\tilde b(s_1,s_2,z_1,z_2) = \sum_{n_2=1}^\infty \sum_{n_1=n_2}^\infty \frac{z_1^{n_1} z_2^{n_2}}{(s_1+n_1)(s_2+n_2)}$$
$$= z_1 z_2 \int_0^1\int_0^1 \frac{x_1^{s_1} x_2^{s_2}}{(1-z_1x_1)(1-z_1z_2x_1x_2)}\,dx_1\,dx_2,$$

*($|z_i|<1$, $\Re s_i > -1$).*