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# Spherical unitary representations for split groups

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# 1 Basic examples

## 1.1 Graded Hecke algebra of type $A_1$

Let $\mathbb{H} = \mathbb{H}(A_1)$ be the algebra generated over $\mathbb{C}$ by $s$ and $\alpha$ subject to the relations

$$\begin{aligned} &s^2 = 1 \\ &s \cdot \alpha + \alpha \cdot s = 2. \end{aligned}$$

Denote $\mathbb{A} = Sym(\mathbb{C}\alpha)$. As a $\mathbb{C}$-vector space $\mathbb{H}(A_1) = \mathbb{C}\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{A}$, where $\mathbb{Z}/2\mathbb{Z} = \{1, s\}$.

The algebra $\mathbb{H}$ has a $*$-operation defined on generators by

$$\begin{aligned} &s^* = s \\ &\alpha^* = -\alpha + 2s. \end{aligned}$$

We say that an $\mathbb{H}$-module $U$ is *hermitian* (*unitary*) if it admits a hermitian form (positive definite) $\langle\ ,\ \rangle$ such that

$$\langle x \cdot u_1, u_2 \rangle + \langle u_1, x^* \cdot u_2 \rangle = 0, \quad x \in \mathbb{H},\ u_1, u_2 \in U.$$

(The characters of $\mathbb{A}$ are determined by the action of $\alpha$.) Let $\mathbb{C}_\nu$ denote the character of $\mathbb{A}$ on which $\alpha$ acts by $\nu$.

Define the principal series

$$X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu, \quad \nu \geq 0.$$

Consider the element

$$r_\alpha = s \cdot \alpha - 1.$$

**Lemma 1.1.1.** *The element $r_\alpha$ satisfies the following relations*

$$\alpha \cdot r_\alpha = r_\alpha \cdot (-\alpha) \text{ and } s \cdot r_\alpha = r_\alpha \cdot (-s).$$

Then we immediately have the following result.

**Proposition 1.1.2.** *The map $A(\nu) : X(\nu) \to X(-\nu)$, given by*

$$A(\nu)(x \otimes 1_\nu) = x \cdot r_\alpha \otimes 1_{-\nu},$$

*is an intertwining operator.*

It is a general fact that an invariant hermitian form on a module is equivalent with an intertwining operator between the module and its hermitian dual.

As a $\mathbb{Z}/2\mathbb{Z}$-representation,

$$X(\nu) = triv \oplus sgn = span\{(1+s) \otimes 1_\nu, (1-s) \otimes 1_\nu\}.$$

Note that $(1+s) \cdot r_\alpha = (1+s)(\alpha - 1)$ and $(1-s) \cdot r_\alpha = (1-s)(-\alpha - 1)$. So the hermitian form corresponding to $A(\nu)$ has matrix

$$\begin{pmatrix} a_{triv}(\nu) & 0 \\ 0 & a_{sgn}(\nu) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1-\nu}{1+\nu} \end{pmatrix},$$

where $a_\tau(\nu)$ denote the normalized operators on $\mathbb{Z}/2\mathbb{Z}$-types. (The normalization is such that on the trivial $\mathbb{Z}/2\mathbb{Z}$-type, the operator is identically 1.)

In conclusion, $X(\nu)$, $\nu \geq 0$, has a unique quotient $L(\nu)$, which is unitary for $0 \leq \nu \leq 1$. (At $\nu = 1$, $L(1) = triv$.)

## 1.2 $\mathbf{SL(2,\mathbb{R})}$

Let $G$ be the group $SL(2,\mathbb{R})$, $B = AN$ the Borel subgroup ($A$ is the maximal split torus) and $K = SO(2)$ the maximal compact subgroup. Then $\widehat{K} \cong \mathbb{Z}$.

Consider the spherical principal series

$$X_B(\nu) = Ind_B^G(e^\nu \otimes 1),\ \ \nu \geq 0.$$

(In Prof. Trapa's table, this is denoted by $P_+(\nu)$.)

The Langlands quotient $L(\nu)$ is unitary for $0 \leq \nu \leq 1$. ($L(1)$ is the trivial representation.) Recall that as a $K$-representation,

$$X_B(\nu)|_K = \sum_{m\in\mathbb{Z}} (2m).$$

There is an (integral) intertwining operator

$$A(\nu) : X_B(\nu) \to X_B(-\nu),$$

which is normalized so that it is identically 1 on the trivial $K$-type. One can compute the restriction of $A(\nu)$ on each $K$-type. Since the $K$-types are one-dimensional, these restrictions are scalars. A classical computation shows that these scalars are

$$A_{(2m)}(\nu) = \frac{1-\nu}{1+\nu} \cdot \frac{3-\nu}{3+\nu} \cdot \dots \cdot \frac{2|m|-1-\nu}{2|m|-1+\nu}.$$

**Remark.** Note that

$$A_{(2)}(\nu) = a_{sgn}(\nu) = \frac{1-\nu}{1+\nu},$$

and the (unitary) complementary series is the same in the two cases.

# 2 Generalization

## 2.1 Graded Hecke algebra

Let $(\mathcal{X}, \Pi, \check{\mathcal{X}}, \check{\Pi})$ be a based root datum, with $\Delta$ the roots and $\check{\Delta}$ the coroots, $W$ the Weyl group. Set $\mathfrak{a} = \check{\mathcal{X}} \otimes_{\mathbb{Z}} \mathbb{C}$ and $\check{\mathfrak{a}} = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{C}$. Similarly, define $\mathfrak{a}_{\mathbb{R}}, \check{\mathfrak{a}}_{\mathbb{R}}$.

**Definition 2.1.1.** *(Lusztig) The graded Hecke algebra is the vector space $\mathbb{H} = \mathbb{C}W \otimes \mathbb{A}$, where $\mathbb{A} = Sym(\check{\mathfrak{a}})$, subject to the commutation relation*

$$s_\alpha \cdot \omega = s_\alpha(\omega) \cdot s_\alpha + \omega(\check{\alpha}), \quad \textit{for all } \alpha \in \Pi, \omega \in \check{\mathfrak{a}}.$$

As in the $A_1$ case, $\mathbb{H}$ has a $*$-operation, so it makes sense to define hermitian and unitary modules.

**Remark.** The problem of classifying the unitary representations with Iwahori fixed vectors of split $p$-adic groups can be reduced to the problem of identifying the unitary dual of graded Hecke algebras $\mathbb{H}$.

Some facts about $\mathbb{H}$:

1. (Bernstein,Lusztig) The center of $\mathbb{H}$ is $\mathbb{A}^W$.

2. All simple $\mathbb{H}$-modules are finite dimensional, and the *central characters* are parametrized by $W$-orbits in $\mathfrak{a}$.

3. The $\mathbb{H}$-modules have a Kazhdan-Lusztig classification.

4. (Barbasch-Moy) For every $w \in W$, with reduced expression $w = s_{\alpha_1} \dots s_{\alpha_m}$, one can define the element $r_w = r_{\alpha_1} \dots r_{\alpha_m}$, which does not depend on the reduced decomposition.

Let $X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu$ be the principal series. Assume $\nu \in \mathfrak{a}_\mathbb{R}$ is dominant, i.e., $\langle \alpha, \nu \rangle \geq 0$, for all $\alpha \in \Pi$.

**Definition 2.1.2.** *The $\mathbb{H}$-module $U$ is called spherical if $Hom_W[triv, U] \neq 0$.*

The spherical modules (with real central character) are precisely the (unique) Langlands quotients $L(\nu)$ of $X(\nu)$ with $\nu$ dominant.

Let $w_0$ be the longest Weyl group element. Define the (Barbasch-Moy) intertwining operator

$$A(\nu) : X(\nu) \to X(w_0\nu), \quad x \otimes 1_\nu \mapsto x \cdot r_{w_0} \otimes 1_{w_0\nu}.$$

Then $L(\nu)$ is hermitian if and only if $w_0\nu = -\nu$. Assume this is the case.

If $(\tau, V_\tau)$ is a $W$-type, $A(\nu)$ defines hermitian operators

$$\begin{aligned} &a_\tau(\nu) : Hom_W[V_\tau, X(\nu)] \to Hom_W[V_\tau, X(-\nu)] \\ &a_\tau(\nu) : (V_\tau)^* \to (V_\tau)^*, \end{aligned}$$

by the Frobenius reciprocity. Normalize them so that $a_{triv}(\nu) = Id$. The normalization factor is $(-1)^{|\Delta^+|} \prod_{\alpha \in \Delta^+} (1 + \langle \alpha, \nu \rangle)$.

**Proposition 2.1.3.** *A spherical parameter $\nu$ is unitary if and only if $w_0\nu = -\nu$ and $a_\tau(\nu)$ is positive semidefinite for all $\tau \in \widehat{W}$.*

If $w_0$ has a reduced decomposition $w_0 = s_1 s_2 \cdots s_n$, then the operators $a_\tau(\nu)$ have a decomposition

$$a_\tau(\nu) = a_{\tau,1}(w_1\nu) \cdot a_{\tau,2}(w_2\nu) \cdots a_{\tau,n}(w_n\nu),$$

where $w_i = s_{n-i+1} \ldots s_n$. Each simple operator $a_{\tau,i}(\nu)$ is induced from an $\mathbb{H}(A_1)$-operator and corresponds to a simple root $\alpha_i$. Explicitly,

$$a_{\tau,i}(\nu) = \begin{cases} 1 & \text{on the } (+1)\text{-eigenspace of } s_{\alpha_i} \text{ of } V_\tau^* \\ \frac{1-\langle \alpha_i, \nu \rangle}{1+\langle \alpha_i, \nu \rangle} & \text{on the } (+1)\text{-eigenspace of } s_{\alpha_i} \text{ of } V_\tau^* \end{cases}.$$

## 2.2 Split real groups

Let $B = AN$ be a Borel subgroup, $A$ maximal split torus, $K$ maximal compact. Set $M = A \cap K$. As before, let $X_B(\nu)$ denote the spherical principal series $X_B(\nu) = Ind_B^G(e^\nu \otimes 1)$, where $\nu \in \mathfrak{a}_{\mathbb{R}}^*$, and $\nu$ is dominant.

There is a (Knapp-Zuckerman) normalized intertwining operator

$$A(\nu) : X_B(\nu) \to X_B(-\nu).$$

The Langlands quotient $L(\nu)$, which is spherical, is hermitian if and only if $w_0\nu = -\nu$. If this is the case, for every $K$-type $(\mu, V_\mu)$, $A(\nu)$ induces operators:

$$\begin{aligned} &A_\mu(\nu) : Hom_K[V_\mu, X_B(\nu)] \to Hom_K(V_\mu, X_B(-\nu)) \\ &A_\mu(\nu) : (V_\mu^*)^M \to (V_\mu^*)^M, \end{aligned}$$

by Frobenius reciprocity. The normalization is such that $A_{triv}(\nu) = Id$.

The Weyl group $W = N_G(A)/A \cong N_K(A)/M$, so for every $K$-type $(\mu, V_\mu)$, the space $(V_\mu^*)^M$ is naturally a $W$-type. Denote it by $\tau(\mu)$.

The Barbasch-Vogan idea of *petite* $K$-types is to identify a class of $K$-types $\mu$ such that the operators

$$A_\mu(\nu) = a_{\tau(\mu)}(\nu).$$

(As it will follow from the calculation, the Weyl group operators are for the Hecke algebra of the *dual* root datum.)

The operator $A(\nu)$, and consequently $A_\mu(\nu)$, have a (Gindikin-Karpelević) decomposition into operators $A(s_\alpha, \nu)$ relative to a reduced decomposition of $w_0$.

For each simple root of $A$ in $G$, consider the root homomorphism $\Psi_\alpha : SL(2, \mathbb{R}) \to G$. Via $\Psi_\alpha$, the compact group $SO(2)$ embeds into $K$. Therefore, the $K$-type $(\mu, V_\mu)$ has a decomposition into $\Psi_\alpha(SO(2))$ isotypic components:

$$V_\mu = \otimes_{j \in \mathbb{Z}} V_\mu(j).$$

The action of $M$ preserves $V_\mu(j) + V_\mu(-j)$ and it has fixed vectors if and only if $j$ is even. On the spaced of $M$-fixed vectors of $V_\mu(2m) + V_\mu(-2m)$, as in the $SL(2, \mathbb{R})$ case, the operator $A_\mu(s_\alpha, \nu)$ is

$$A_\mu(s_\alpha, \nu) = \prod_{1 \leq j \leq |m|} \frac{2j - 1 - \langle \check{\alpha}, \nu \rangle}{2j - 1 + \langle \check{\alpha}, \nu \rangle}.$$

**Definition 2.2.1.** *A $K$-type $(\mu, V_\mu)$ is called petite if for every simple root $\alpha$, the decomposition of $V_\mu$ into $\Psi_\alpha(SO(2))$-types contains only the representations $(j)$, $|j| \leq 3$.*

The following result is an immediate consequence.

**Proposition 2.2.2** (Barbasch,Vogan)**.** *If $(\mu, V_\mu)$ is a petite $K$-type, then $A_\mu(\nu) = a_{\tau(\mu)}(\nu)$, where the second operator is the Hecke algebra of $\check{G}$.*

The condition of being petite is very restrictive. For example, for a group $G$, few $W$-types occur in $\tau(\mu)$ for $\mu$ petite $K$-types.

Barbasch identified all the petite $K$-types (and their corresponding $W$-types) for split real groups. There are also extensions of this idea: nonspherical principal series (Barbasch-Pantano), nonlinear covers of split real groups (Adams-Barbasch-Paul-Trapa-Vogan), $U(p, q)$ (Barbasch).

**Example.** If $G = SL(n, \mathbb{R})$, $K = SO(n)$, $W = S_n$, examples of petite $K$-types are $\mu = (\underbrace{2, 2, \ldots, 2}_{k}, 0, \ldots, 0)$, $k \leq \left[\frac{n}{2}\right]$, which has $\tau(\mu) = (n - k, k)$.

# 3 The spherical unitary dual

## 3.1 Relevant $W$-types

Let us return to the setting of the Hecke algebra $\mathbb{H}$. We need to determine the spherical unitary dual of $\mathbb{H}$. In addition, in order to be able to use the calculations for real split groups, one must find a set of *relevant* $W$-types which detect unitarity *and* come from petite $K$-types.

Let $\mathfrak{g}$ denote the complex Lie algebra attached to $\mathbb{H}$. Recall that by the Springer correspondence, to every nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, one attaches a subset of $\widehat{W}$. If $e \in \mathcal{O}$, define the *height* of $\mathcal{O}$ to be

$$ht(\mathcal{O}) = \max\{\ell \geq 0 : ad(e)^{\ell} \neq 0\}.$$

**Definition 3.1.1.** *A $W$-type $\tau$ is called relevant if the nilpotent orbit $\mathcal{O}$ corresponding to $\tau$ in the Springer's correspondence has height $ht(\mathcal{O}) \leq 4$.*

Then we have the first form of the answer for the spherical unitary dual problem.

**Theorem 3.1.2.** *A spherical parameter $\nu$ for the Hecke algebra $\mathbb{H}$ is unitary if and only if $a_{\tau}(\nu)$ is positive semidefinite for all relevant $W$-types $\tau$.*

This result was proved in the classical cases by Barbasch, in the exceptional cases by Barbasch-C.

**Theorem 3.1.3** (Barbasch)**.** *Every relevant $W$-type comes from a petite $K$-type of the split real group.*

**Corollary 3.1.4.** *A spherical parameter $\nu$ for the real split group $G$ is unitary* **only if** *it is unitary for the Hecke algebra associated to $\check{G}$.*

For classical real split groups this condition is also sufficient, as proved by Barbasch.

## 3.2 Explicit description

We are still in the setting of the graded Hecke algebra $\mathbb{H}$.

**Definition 3.2.1.** *A spherical parameter $\nu$ is called generic if the principal series $X(\nu)$ is irreducible.*

The module $X(\nu)$ is reducible if and only if $\langle\alpha, \nu\rangle = 1$, for some positive root $\alpha$.

Let us denote by $SU_0$ the set of unitary spherical generic parameters. This set can be described explicitly (combinatorially).

**Theorem 3.2.2.** *The set of unitary spherical generic parameters $SU_0$ is a union of $k$ simplices (alcoves) in the dominant Weyl chamber, where:*

$$\begin{array}{ll} A_n\text{:} & k = 1 \\ B_n\text{:} & k = 2^{[(n-1)/2]} \\ C_n\text{:} & k = 1 \\ D_n\text{:} & k = 2^{[(n-2)/2]} \\ G_2\text{:} & k = 2 \\ F_4\text{:} & k = 2 \\ E_6\text{:} & k = 2 \\ E_7\text{:} & k = 8 \\ E_8\text{:} & k = 16. \end{array}$$

Note that the root systems above refer to the Hecke algebra, so they are the dual root systems of the split real group.

Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. Any $e \in \mathcal{O}$ can be embedded into a Lie triple $\{e, h, f\}$. The centralizer of the Lie triple in $\mathfrak{g}$ is a reductive Lie subalgebra. Denote it by $\mathfrak{z}(\mathcal{O})$.

To every dominant spherical parameter $\nu \in \mathfrak{a}_{\mathbb{R}}$, one can attach uniquely a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$. The orbit $\mathcal{O}$ is the unique $G$-orbit meeting the 1-eigenspace of $ad(\nu)$ in a dense orbit. (It is the orbit attached in the Kazhdan-Lusztig classification to the Iwahori-Matsumoto dual of the spherical module parametrized by $\nu$.)

One partitions the spherical unitary dual into pieces $CS(\mathcal{O})$ parametrized by nilpotent orbits. Note that by definition $CS(0) = SU_0$.

Let $Exc$ denote the following set of nilpotent orbits:

$$Exc = \{\underbrace{A_1\widetilde{A}_1}_{F_4}, \underbrace{A_2 3A_1}_{E_7}, \underbrace{A_4A_2A_1, A_4A_2, D_4(a_1)A_2, A_3 2A_1, A_2 3A_1, 4A_1}_{E_8}\}.$$

(The notation is as in the Bala-Carter classification.)

Then the spherical unitary dual of $\mathbb{H}$ can be described as follows.

**Theorem 3.2.3** (Barbasch,Barbasch-C.)**.**

1. *If* $\mathcal{O} \notin Exc$, *then*
$$CS(\mathcal{O}) = SU_0(\mathfrak{z}(\mathcal{O})).$$

2. *If* $\mathcal{O} \in Exc$, *and* $\mathcal{O} \neq (4A_1 \subset E_8)$, *then* $CS(\mathcal{O}) \subsetneq SU_0(\mathfrak{z}(\mathcal{O}))$.

3. *If* $\mathcal{O} = (4A_1 \subset E_8)$, *then* $CS(\mathcal{O}) \supsetneq SU_0(\mathfrak{z}(\mathcal{O}))$.