Algorithms for Representation Theory of Real Reductive Groups

Lectures at Snowbird, June 2006
Lectures 1-4

Jeffrey Adams and Fokko du Cloux

More detailed notes are available at
`www.liegroups.org`

Outline of Lectures

Lecture 1:

- Root data and reductive groups
- Automorphisms and inner classes
- Real forms and involutions
- Basic Data
- Extended groups
- Strong involutions

Lecture 2:

- Representations of strong involutions
- Translation families
- L-homomorphisms
- L-data
- Parametrization of representations in terms of L-data

Lecture 3:

- The flag variety $K\backslash G/B$
- The one-sided parameter space $\mathcal{X}$
- Structure of $\mathcal{X}$
- Twisted involutions in the Weyl group and $\mathcal{I}_W$
- Fibers of the map $\mathcal{X} \to \mathcal{I}_W$

Lecture 4:

- W action on $\mathcal{X}$
- Cartan subgroups and strong real forms
- Cayley transforms
- The parameter space $\mathcal{Z}$ and the parametrization of representations

We recommend making some simplying assumptions the first time through. At various times we will assume:

- G is semisimple
- G is simply connected
- G is adjoint
- $\mathrm{Out}(G) = 1$
- $\gamma = 1$
- $x^2 = 1$
- All of the above

For example if G is semisimple $\mathrm{Out}(G)$ is a subgroup of the automorphism group of the Dynkin diagram, and equal to it if G is adjoint or simply connected. Also the spaces $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are all finite.

The main topic of these talks is:

Problem: Compute the irreducible representations of a real reductive algebraic group *explicitly*.

This is in principle done, by work of Langlands, complete by Knapp, Zuckerman and Vogan. Even for the experts this is a difficult calculation in even small groups, not to mention E_8.

These talks report on work of Fokko du Cloux.

His work, and these talks, are described boxed:

$$\boxed{\text{math}} \longrightarrow \boxed{\text{algorithms}} \longrightarrow \boxed{\text{software}}$$

The math involves work of many people, most recently Adams, Barbasch and Vogan. The algorithm is largely due to Fokko du Cloux, based on conversations with Adams and Vogan. The `atlas` software is 100% due to Fokko.

Example 1 The complex group E_8 has three real forms, which together have

$$1 + 1 + 3,150 + 73,410 + 73,410 + 453,010$$
$$= 603 - 032$$

irreducible representations with infinitesimal character ρ (the same as that of the trivial representation).

Of these 3,733, or .62%, are unitary (this was computed by Scott Crofts).

The number of irreducible representations is closer to the $\sqrt{|W|}$ than $|W|$, which is very good news: $|W| = 696,729,600$, which would be too many representations to handle easily, even by computer.

The **Atlas of Lie Groups and Representations** is a project to compute the unitary dual of real groups. One of the main goals of the project is to make information available to the general mathematical audience, much like the Atlas of Finite Groups.

An early version of the `atlas` software is availabe on the atlas web site, www.liegroups.org. We encourage you to download it and try it yourself. These talks will outline the algorithm behind the software. In some accompanying evening talks I will demonstrate the software itself.

Note: a windows version of the software is now available.

Here is an example of the kind of combinatorial object we will be discussing.

Theorem 2 Let $G = Sp(2n, \mathbb{C})$ and $G^\vee = SO(2n+1, \mathbb{C})$.

The irreducible representations of $Sp(2n, \mathbb{R})$ with infinitesimal character ρ are parametrized by:

$$\{(x, y\}/G \times G^\vee$$

where

$$\begin{aligned} x &\in \mathrm{Norm}_G(H), \quad x^2 = -I \\ y &\in \mathrm{Norm}_{G^\vee}(H^\vee) \quad y^2 = I \\ \theta^t_{x,\mathfrak{h}} &= -\theta_{y,\mathfrak{h}^\vee} \end{aligned} \tag{3}$$

Here $H \subset G$ is a Cartan subgroup, $\theta_{x,\mathfrak{h}}(X) = \mathrm{Ad}(x)(X)$ for $X \in \mathfrak{h}$. Similarly $H^\vee \subset G^\vee$ and $\theta_{y,h^\vee}$. There is a natural duality between $\mathfrak{h}$ and $h^\vee$, and $\theta^t_{x,\mathfrak{h}} \in \mathrm{End}(\mathfrak{h}^\vee)$.

For example for $Sp(4)$ the set under discussion has 18 elements.

The basic setting is a connected reductive algebraic group G. This is defined by its root data $(X, \Delta, X^\vee, \Delta^\vee)$ where

(1) H is a Cartan subgroup

(2) $X = X^*(H)$ is the character lattice

(3) Δ is the set of roots

(4) $X^\vee = X_*(H)$ is the co-character lattice

(5) $\Delta^\vee$ is the set of co-roots

There is a standard exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

A splitting datum is a set $(H, B, \{X_\alpha\})$ where H is a Cartan subgroup, B is a Borel subgroup, and $\mathcal{X}_\alpha$ is a set of simple root vectors. Given such a splitting datum we obtain a splitting of the exact sequence, taking $\mathrm{Out}(G)$ to the stabilizer of the splitting.

A real form of G is the fixed points $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ of an anti-holomorphic involution σ, or the involution σ itself. Instead of anti-holomorphic involutions we prefer to work with holomorphic ones.

Lemma 4 *Given an anti-holomorphic involution σ there is a holomorphic involution θ satisfing $\theta\sigma = \sigma\theta$ and $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$.*

The correspondence $\sigma \leftrightarrow \theta$ is a bijection, between $G(\mathbb{C})$ conjugacy classes of σ's and θ's.

Definition 5 *We say involutions θ, θ' are inner if they have the same image* *in* $Out(G)$*.*

Fix $\gamma \in Out(G)$, $\gamma^2 = 1$*. The inner class of* γ *is the set of involutions* θ *mapping to* γ*.*

Let $\Gamma = \{1, \sigma\}$ be the Galois group of $\mathbb{C}/\mathbb{R}$.

Definition 6 *Given* (G, γ) *we let*

$$G^\Gamma = G \rtimes \Gamma$$

where the action of $\sigma \in \Gamma$ *is by* $\phi_S(\gamma)$ *for some splitting* ϕ_S*.*

Recall

(G, γ), $\gamma^2 = 1$

$$G^\Gamma = G \rtimes \Gamma \ni \delta = 1 \times \sigma$$

$$\theta_\delta(g) = \delta g \delta^{-1}$$

Suppose

$$\theta^2 =, \quad p(\theta) = \gamma \in \mathrm{Out}(G)$$

Then

$$\begin{aligned}\theta(g) &= (h\delta)g(h\delta)^{-1} \quad h \in G \\ &= h\theta_\delta(g)h^{-1}\end{aligned}$$

Definition 7

$$x = h\delta \in G^\Gamma \backslash G, x^2 \in Z(G)$$

define

$$\theta_x(h) = xhx^{-1}$$

Example: $G = GL(n, \mathbb{C})$

$x = \mathrm{diag}(\overbrace{1, \ldots, 1}^{p}, \overbrace{-1, \ldots, -1}^{q})$

$G^{\theta_x} = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$

Corresponding real form is $U(p, q)$, $K = U(p) \times U(q)$, $K(\mathbb{C}) = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$

Definition 8 *A "strong involution" or "strong real form" of* G *is* $x \in G^\Gamma \backslash G$, $x^2 \in Z(G)$

$$\mathcal{I}(G, \gamma) = \{\textit{strong real forms}\}/G$$

Proposition 9

$$\mathcal{I}(G, \gamma) \leftrightarrow \{\textit{involutions in inner class } \gamma\}/G$$

This is a bijection if G *is adjoint.*

Example: $G = SL(2)$ $(\gamma = 1)$

$$\mathcal{I}/G = \{I, -I, \mathrm{diag}(i, -i)\}$$

These map to the (ordinary) real forms:

$$\pi I \rightarrow SU(2)$$

$$\mathrm{diag}(i, -i) \rightarrow SL(2, \mathbb{R})$$

It is helpful to think of these strong real forms as:

$$I \rightarrow SU(2, 0)$$

$$-I \rightarrow SU(0, 2)$$

$$\mathrm{diag}(i, -i) \rightarrow SU(1, 1)$$

Representations

A representation of $G = SL(2,\mathbb{R})$ is an action on a Hilbert space

Algebraic version: a $(\mathfrak{g}, K)$-module where

$$\mathfrak{g} = \mathrm{Lie}_{\mathbb{C}}(G) = \mathfrak{sl}(2,\mathbb{C})$$
$$K = SO(2,\mathbb{C}) \simeq \mathbb{C}^{\times}$$

Here K is the complexification of the maximal compact subgroup $SO(2)$ of $SL(2,\mathbb{R})$. The representations of $SO(2)$ and the algebraic representations of $SO(2,\mathbb{C})$ are the same.

A $(\mathfrak{g}, K)$ module is a representation of $\mathfrak{g}$, and an algebraic representation of K, with a compatibility condition.

Definition 10 *A representation of a strong real from x of G is a $(\mathfrak{g}, K)$-module. We say $(x, \pi) \simeq (x', \pi')$ if there exists $g \in G(\mathbb{C})$ such that $gxg^{-1} = x', \pi^g \simeq \pi'$.*

Recall $K_x = G^{\theta_x}$

Key Example: $G = SL(2)$

$$x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$\pi =$ Holomorphic discrete series with $K-$ types $2, 4, 6, \ldots$

$\pi^* =$ Anti-holomorphic discrete series with $K-$ types $-2, -4, -6, \ldots$

$$\{(x, \pi), (x, \pi^*)\}$$

is a discrete series L-packet for x, i.e. $SL(2, \mathbb{R})$

Now let

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then $wxw^{-1} = -x, \pi^w = \pi^*$, i.e.

$$(x, \pi^*) \equiv (-x, \pi)$$

So we can write our L-packet

$$\{(x, \pi), (-x, \pi)\}$$

Translation

An infinitesimal character for G is given by a semisimple conjugacy class in $\mathfrak{g}^*$. Write λ for an infinitesimal character. We say λ is regular if $\langle\lambda, \alpha^\vee\rangle \neq 0$ for all roots α.

Let

$$\mathcal{M}(x, \lambda) = \{(x, \pi) \,|\, \pi \text{ has infinitesimal character } \lambda$$

(up to equivalence).

This is a finite set.

Theorem 11 *If* λ, λ' *are regular and* $\lambda - \lambda' \in X^*(H)$ *then*

$$\mathcal{M}(x, \lambda) \simeq \mathcal{M}(x', \lambda)$$

Fix (G, γ)

If $G \leftrightarrow (X, \Delta, X^\vee, \Delta^\vee)$, let $G^\vee$ be the dual group: given by root data $(X^\vee, \Delta^\vee, x, \Delta)$

There is a natural definition of $\gamma^\vee = -w_0\gamma^t \in \mathrm{Out}(G^\vee)$

Define $G^\Gamma = G \rtimes \Gamma$ and $G^{\vee\Gamma} = G^\vee \rtimes \Gamma$

Remark 12 $G^{\vee\Gamma}$ is the L-group of G (not obvious from this definition)

For example if γ is the inner class of the split real form of G, then $\gamma^\vee = 1$

The Langlands classification says:

Associated to an admissible homomorphism

$$\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$$

and a real form $G(\mathbb{R})$ of G in the given inner class is an L-packet, which is a finite set Π_ϕ of representations of $G(\mathbb{R})$

Definition 13 *A one sided L-datum for (G,γ) is a pair $(y, B_1^\vee)$ where y is a strong involution for $(G^\vee,\gamma^\vee)$ and $B_1^\vee$ is a Borel subgroup of $G^\vee$.*

A complete one sided L-datum is a triple $(y, B_1^\vee, \lambda)$ with $e^{2\pi i\lambda} = y^2 \in Z(G^\vee)$.

Given $S = (y, B_1^\vee, \lambda)$ let

$$\phi_S : W_{\mathbb{R}} \to G^{\vee\Gamma}$$

be defined by

$$\phi_S(z) = z^{\lambda_1}\overline{z}^{\mathrm{Ad}(y)\lambda_1}$$

$$\phi_S(j) = e^{-\pi i\lambda} y$$

Here

$$W_{\mathbb{R}} = \langle \mathbb{C}^\times, j \rangle$$

where $jzj^{-1} = \overline{z}$ and $j^2 = -1$.

Note: we have chosen a y-stable Cartan subgroup $H_1^\vee \subset B_1^\vee$, and chosen $g \in G^\vee$ so that $gH^\vee g^{-1} = H_1^\vee$ and $\lambda_1 = \mathrm{Ad}(g)\lambda$ is $B_1^\vee$-dominant.

Definition 14

$$\mathcal{P} = \{\textit{one sided L-data } (y, B_1^\vee)\}/G^\vee$$

$$\mathcal{P}_c = \{\textit{complete one sided L-data } (y, B_1^\vee, \lambda)\}/G$$

The next result follows easily.

Theorem 15
$\mathcal{P}_c \leftrightarrow \{$*L-packets for strong real forms of* G

with regular integral infinitesimal charact

$\leftrightarrow \{\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}\}/G^{\vee}$

and

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Note: Let $\{x_i \,|\, i \in I\}$ be a set of representatives of $\mathcal{I}/G$. By definition an "L-packet for strong real forms of G" is a union of L-packets as $i \in I$.

Example 16 $G = SL(2)$, for a certain ϕ,

$$\Phi_\phi = \{(I, \mathbb{C}), (-I, \mathbb{C}), (x, \pi), (x, \pi^*) = (-x, \pi)\}$$

where $x = \text{diag}(i, -i)$.

Here

$$\begin{aligned} (I, \mathbb{C}) &= \text{trivial representation of } SU(2 \\ (-I, \mathbb{C}) &= \text{trivial representation of } SU(0 \\ (x, \pi) &= \text{holomorphic discrete series of} \\ (-x, \pi) = (x, \pi^*) &= \text{holomorphic discrete series of} \\ \{(x, \pi), (x, \pi^*)\} &= \text{L-packet for } SU(1, 1) \end{aligned}$$

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$$\{(x, \pi), (x, \pi^*)\} = \text{L-packet for } SU(1, 1)$$

Fix (G, γ), gives $(G^\vee, \gamma^\vee)$

$$G^\Gamma = G \rtimes \Gamma$$

$$G^{\vee\Gamma} = G^\vee \rtimes \Gamma$$

Recall one-sided L-data:

$$\mathcal{P}(G^\vee, \gamma^\vee) = \{(y, B_1^\vee)\}/G^\vee$$

y is a strong involution

$B_1^\vee \subset G^\vee$ is a Borel subgroup

and complete one-sided L-data:

$$\mathcal{P}_c(G^\vee, \gamma^\vee) = \{(y, B_1^\vee, \lambda)\}/G^\vee$$

$\lambda \in \mathfrak{h}^\vee, \exp(2\pi i\lambda) = y^2 \in Z(G^\vee)$

Theorem 27

$$\mathcal{P}_c(G^\vee, \gamma^\vee) \leftrightarrow \{\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}\}/G^\vee$$

$\leftrightarrow \{$*L-packets for strong real forms of* G*, regular integral infinitesimal character*$\}$

$\mathcal{P}(G^\vee, \gamma^\vee) \leftrightarrow$

$\{$*translation families of L-packets for strong real forms of* G*, with regular integral infinitesimal character*$\}$

First step in our algorithm: combinatorial description of $\mathcal{P}(G^\vee, \gamma^\vee)$

By symmetry (and to avoid $^\vee$ clutter) define

$$\mathcal{P}(G,\gamma) = \{(x, B_1)\}/G$$

We may as well conjugate B_1 to B:

$$g : (x, B_1) \to (x', B)$$

We can furthermore conjugate by B to take x to the normalizer of H:

$$b : (x', B) \to (x'', B)$$

We can still conjugate by H.

This gives the primary combinatorial construction.

Definition 28 *Fix* (G, γ) *and therefore* $(G^\vee, \gamma^\vee)$ *Define* $\mathcal{X} = \mathcal{X}(G^\vee, \gamma^\vee)$*:*

$$\begin{aligned}\mathcal{X} &= (\mathcal{I} \cap N^\Gamma)/H \\ &= \{x \in \mathrm{Norm}_{G^\Gamma \backslash G}(H) \,|\, x^2 \in Z(G)\}/H\end{aligned}$$

For example if $\gamma^\vee = 1$ this can be identified with

$$\begin{aligned}\mathcal{X} &= (\mathcal{I} \cap N)/H \\ &= \{g \in \mathrm{Norm}_G(H) \,|\, g^2 \in Z\}/H\end{aligned}$$

Note: W acts on $\mathcal{X}$ by conjugation

Proposition 29 *There is a natural bijection*

$$\mathcal{X}(G^\vee, \gamma^\vee) \leftrightarrow \mathcal{P}(G^\vee, \gamma^\vee)$$

so $\mathcal{X}$ parametrizes translation families of maps of the Weil group.

Example: $G = SL(2)$

$$N = \{\mathrm{diag}(z, 1/z)\} \cup \{\begin{pmatrix} 0 & z \\ -\frac{1}{z} & 0 \end{pmatrix}$$

$$\mathcal{I} \cap N = \{\pm I, \pm\mathrm{diag}(i, -i)\} \cup \{\begin{pmatrix} 0 & z \\ -\frac{1}{z} & 0 \end{pmatrix}$$

H acts on the second term by $z \to z^{-1}$, so

$$\mathcal{X} = \{I, -I, t, -t, w\}$$

$(t = \mathrm{diag}(i, -i)$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$

Suppose $G = PGL(2)$ and $G^\vee = SL(2)$

Then $\mathcal{P}(G^\vee, 1)$ gives translation families of maps $\phi : W_\mathbb{R} \longrightarrow SL(2)$.

Take $\lambda \in \mathbb{Z} + \frac{1}{2}$ with $e^{2\pi i\lambda} = y^2$

(1) $y = \epsilon I, y^2 = I, \lambda = n$:

$$\begin{aligned} \phi(z) &= \mathrm{diag}(|z|^{2n}, |z|^{-2n}) \\ \phi(j) &= \epsilon(-I)^n \end{aligned}$$

(2)$y = \epsilon t, y^2 = -1, \lambda = n + \frac{1}{2}$:

$$\begin{aligned} \phi(z) &= \mathrm{diag}(|z|^{2n+1}, |z|^{-2n-1}) \\ \phi(j) &= \epsilon(-I)^n \end{aligned}$$

(3) $y = w, y^2 = -1, \lambda = n + \frac{1}{2}$:

$$\begin{aligned} \phi(z) &= \mathrm{diag}((z/\overline{z})^{2n+1}, (z/\overline{z})^{-2n-1}) \\ \phi(j) &= w \end{aligned}$$

(1), (2) $\leftrightarrow$ principal series

(3) $\leftrightarrow$ discrete series

(1) $y = \epsilon I, y^2 = I, \lambda = n$:

$$\phi(z) = \mathrm{diag}(|z|^{2n}, |z|^{-2n})$$
$$\phi(j) = \epsilon(-I)^n$$

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(3) $y = w, y^2 = -1, \lambda = n + \frac{1}{2}$:

$$\phi(z) = \mathrm{diag}((z/\overline{z})^{2n+1}, (z/\overline{z})^{-2n-1})$$
$$\phi(j) = w$$

Relation with K-orbits on the flag variety

Recall we conjugated (x, B_1) to (x', B) to get $\mathcal{X}$. Instead we now conjugate x to a fixed set of representatives of $\mathcal{I}/G$.

Choose

$$\{x_i \,|\, i \in I\} \leftrightarrow \mathcal{I}/G$$

If G is semisimple this is a finite set.

For $i \in I$ let $\theta_i = \mathrm{int}(x_i), K_i = G^{\theta_i}$

$$\mathcal{P}(G,\gamma) = \{(x, B_1)\}/G$$

We may conjugate x to some x_i:

$$g : (x, B_1) \to (x_i, B_2)$$

We are still allowed to conjugate by K_i on the $B's$, i.e. the flag variety G/B:

This gives:

$$\boxed{\mathcal{X}(G,\gamma) = \mathcal{P}(G,\gamma) = \cup_i K_i\backslash G/B}$$

the union of the flag varieties of the strong real forms of G.

Symmetrize the picture:

Definition 30 *An L-datum for* (G,γ) *is:*

$$(x, B_1, y, B_1^\vee)$$

where

(1) x *is a strong real form of* G,

(2) B_1 *is a Borel subgroup of* G,

(3) y *is a strong real form of* $G^\vee$,

(4) $B_1^\vee$ *is a Borel subgroup of* $G^\vee$

(5) $(\theta_{x,\mathfrak{h}})^t = -\theta_{y,\mathfrak{h}^\vee}$

$$\mathcal{L} = \{L - data\}/G \times G^\vee$$

(Note: choose $g : H \to H_1, B \to B_1$. Then $\theta_{x,\mathfrak{h}} = \theta_x \in \mathrm{Aut}(\mathfrak{h}_1)$ carried over to H by g, and $\theta_{y,\mathfrak{h}^\vee}$ similarly.)

Theorem 31

$\mathcal{L} \leftrightarrow$ {*translation families of irreducible representations of strong real forms of* G *with regular integral infinitesimal character*}

Note: define

$$\mathcal{L}_c = \{x, B_1, y, B_1^\vee, \lambda)\}/G \times G^\vee$$

similarly, resulting theorem without the "translation families".

Sketch:

$$(y, B_1^\vee, \lambda) \to \phi \to \Pi_\phi$$

an L-packet of representations of strong real forms of G

The element x gives a strong real form, and the choice of B_1 is what is needed to pick out a single representation of the strong real form x in this L-packet.

A bit more precisely:

$$\phi : W_{\mathbb{R}} \to H_1^{\vee\Gamma} \hookrightarrow G^{\vee\Gamma}$$

Assume G is semisimple and simply connected. Then $H_1^{\vee}$ is isomorphic to the L-group of H_1. This isomorphism is not canonical: it depends on the choice of B_1. This choice then gives a character of $H_1(\mathbb{R})$, and this gives a representation of the strong real form x of G. [General case: algebraic covering group of H_1 plays a role.]

Our parameter set is

$$\begin{aligned}\mathcal{L} \subset & \mathcal{P}(G,\gamma) \times \mathcal{P}(G^\vee,\gamma^\vee) \\ &= \mathcal{X}(G,\gamma) \times \mathcal{X}(G^\vee,\gamma^\vee)\end{aligned}$$

We want to study it in more detail. This really only involves $\mathcal{X}(G,\gamma)$ and $\mathcal{X}(G^\vee,\gamma^\vee)$ separately.

Consider $\mathcal{X}(G,\gamma)$.

For simplicity assume $\gamma = 1$

Recall

$$\begin{aligned}\mathcal{X} &= (\mathcal{I} \cap N)/H \\ &= \{x \in N, x^2 \in Z\}/H\end{aligned}$$

Let

$\mathcal{I}_W = \{w \in W \mid w^2 = 1\}$

The map $N \to W$ induces

$$\boxed{p : \mathcal{X} \to \mathcal{I}_W}$$

Example: SL(2)

$$\mathcal{X} = \{I, -I, t, -t, w\}$$

$$\begin{aligned} I, -I, t, -t &\to 1 \\ w &\to s_\alpha \end{aligned}$$

$$\mathcal{X}_w = \text{fiber over } w \in \mathcal{I}_W$$

$$\mathcal{X}(z) = \{x \in \mathcal{X} \mid x^2 = z \in Z\}$$

Recall W acts on $\mathcal{X}$ by conjugation

$$w \cdot \mathcal{X}_v = \mathcal{X}_{wvw^{-1}}$$

Proposition 32

(1) $\mathcal{X} \leftrightarrow \cup_i K_i \backslash G/B$

(2) $\mathcal{I}/W \leftrightarrow \{$*Cartan subgroups in* $G_{qs}\}$

(3) $\mathcal{X}/W \leftrightarrow \cup_i \{$*Cartan subgroups in* $G_{x_i}\}$

(4) $\mathcal{X}_w(z) \simeq [{}^\vee H(\mathbb{R})/H^\vee(\mathbb{R})^0]^\vee$

(5) $\mathcal{X}_1/W \leftrightarrow \{$*strong real forms*$\}$

Some loose ends

1) Some of the combinatorics related to $K\backslash G/B$ may be found in *The Bruhard order on symmetric varieties*, R. W. Richardson and T. A. Springer, Geometriae Dedicata, 1990.

2) The most complete version of this picture is the book by Adams, Barbasch and Vogan. However:

a) In ABV G^Γ is defined to be $G \rtimes \Gamma$ where Γ acts by an anti-holomorphic involution

b) You won't find the parameters $(x, B_1, y, B_1^\vee)$, or $\mathcal{X}(G,\gamma)$ there. See *Lifting of Characters*, Birkhauser 101 (Proceedings of the Bowdoin confererence), 1991. This is a more friendly introduction to the program, only considering regular integral infinitesimal character.

Members of the Atlas project

Fokko du Cloux
4 Place Gabriel Péri
Lyon, 69007, France
ducloux@desargues.univ-lyon1.fr

Jeffrey Adams
Dan Barbasch
Birne Bineger
Bill Casselman
Dan Ciubotaru
Fokko du Cloux
Tatiana Howard
Wai-Ling Lee
Alfred Noel
Alessandra Pantano
Annegret Paul
Siddhartha Sahi
Susana Salamanca
John Stembridge
Peter Trapa
David Vogan
Marc van Leeuwen
Jiu-Kang Yu

These slides, and revised versions of the notes will appear at `www.liegroups.org` (and the conference web site)

Recall some constructions:

$$(G,\gamma), G^\Gamma = G \rtimes \Gamma$$

$$\begin{aligned}\mathcal{X} &= \{x \in \mathrm{Norm}_{G^\Gamma\backslash G}(H) \mid x^2 \in Z(G)\}/H \\ &= \{x \in \mathrm{Norm}_G(H) \mid x^2 \in Z(G)\}/H \quad (\gamma = 1)\end{aligned}$$

$$\begin{aligned}\mathcal{I}_W &= \{w \in W^\Gamma\backslash W \mid w^2 = 1\} \\ &= \{w \in W \mid w^2 = 1\} \quad (\gamma = 1)\end{aligned}$$

$$p : \mathcal{X} \to \mathcal{I}_W$$

$$\phi : \mathcal{X} \ni x \to x^2 \in Z^\Gamma$$

$$\mathcal{X}_\tau = p^{-1}(\tau) \quad \tau \in \mathcal{I}_W$$

$$\mathcal{X}(z) = \phi^{-1}(z) \quad z \in Z^\Gamma$$

$$\mathcal{X}_\tau(z) = \mathcal{X}_\tau \cap \mathcal{X}(z)$$

W acts on $\mathcal{X}$ and $\mathcal{I}_W$ by conjugation

Proposition 33

(1) $\mathcal{X} = \cup_i \mathcal{X}_i \leftrightarrow \cup_i K_i \backslash G/B$

(2) $\mathcal{I}_W/W \leftrightarrow \{\textit{Cartan subgroups in } G_{qs}\}$

(3) $\mathcal{X}/W \leftrightarrow \cup_i \{\textit{Cartan subgroups in } G_{x_i}\}$

(4) $X_\tau(z) \simeq [{}^\vee H(\mathbb{R})/H^\vee(\mathbb{R})^0]^\vee$

(5) $\mathcal{X}_\tau/W \leftrightarrow \{\textit{strong real forms containing Cart}$

(6) $\mathcal{X}_\delta/W \leftrightarrow \{\textit{all strong real forms}\}$

(7) $W(K_x, H) \simeq \mathit{Stab}_W(x)$

(8) W^τ *acts on* $\mathcal{X}_\tau$

In particular $|\mathcal{X}_\tau(z)| = 2^k$

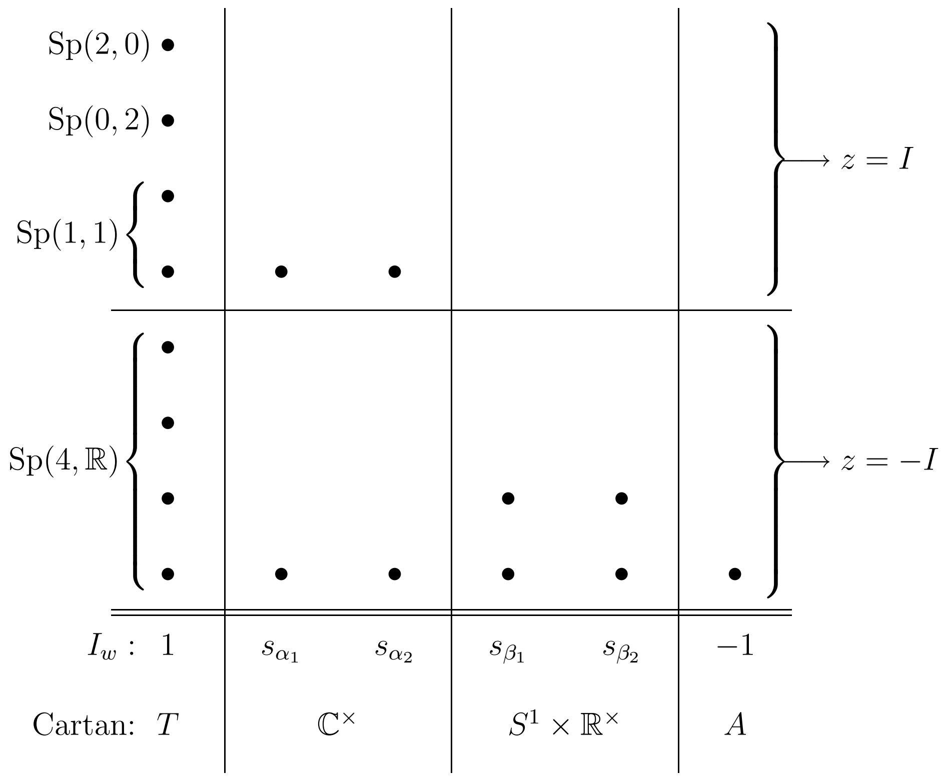

Figure 1: X for $G = \mathrm{Sp}(4)$

(Thanks to Les Saper for this slide)

Return to the symmetric setting:
$(G,\gamma),(G^\vee,\gamma^\vee),\mathcal{X}=\mathcal{X}(G,\gamma),\mathcal{X}^\vee=\mathcal{X}(G^\vee,\gamma^\vee)$

Definition 34

$$\begin{aligned}\mathcal{Z} &= \mathcal{X}\times_0\mathcal{X}^\vee\\ &= \{(x,y)\in\mathcal{X}\times\mathcal{X}^\vee \mid \theta^t_{x,\mathfrak{h}} = -\theta_y\}/G\times G^\vee\end{aligned}$$

Note:

$$\mathcal{Z} = \cup_i K_i\backslash G/B \times_0 \cup_j K_j^\vee\backslash G^\vee B^\vee$$

Theorem 35 There is a natural bijection between $\mathcal{Z}$ and the set of translation families of irreducible representations of strong real forms of G with regular integral infinitesimal character.

Corollary 36

(1) Fix a set $\Lambda \subset P_{reg}$ of representatives of $P/X^*(H)$. Then there is a natural bijection between $\mathcal{Z}$ and the union, over $\lambda \in \Lambda$, of irreducible representations of strong real forms of G, with infinitesimal character λ.

(2) Suppose G is semisimple and simply connected. Then there is a natural bijection between $\mathcal{Z}$ and the irreducible representations of strong real forms of G with infinitesimal character ρ.

(3) Suppose G is adjoint, and fix a set $\Lambda \subset P_{reg}$ of representatives of P/R. Then there is a natural bijection between $\mathcal{Z}$ and the irreducible representations of real forms of G, with infinitesimal character in Λ.

Example: $SL(2)/PGL(2)$

Orbit	x	x^2	θ_x	G_x	λ	rep	Orbit	y	y^2	θ_y	$G_y^\vee$	λ	rep
$\mathcal{O}_{2,0}$	I	I	1	$SU(2,0)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	2ρ	$PS_{\mathbb{C}}$
$\mathcal{O}_{0,2}$	-I	I	1	$SU(0,2)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	2ρ	PS_{sgn}
$\mathcal{O}_+$	t	-I	1	$SU(1,1)$	ρ	DS_+	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	ρ	$\mathbb{C}$
$\mathcal{O}_-$	-t	-I	1	$SU(1,1)$	ρ	DS_-	$\mathcal{O}'_*$	w	I	-1	$SO(2,1)$	ρ	sgn
$\mathcal{O}_*$	w	-I	1	$SU(1,1)$	ρ	$\mathbb{C}$	$\mathcal{O}'_+$	t	I	-1	$SO(2,1)$	ρ	DS
$\mathcal{O}_*$	w	I	1	$SU(1,1)$	ρ	PS_{odd}	$\mathcal{O}'_{3,0}$	I	I	1	$SO(3)$	ρ	$\mathbb{C}$

Definition 37 A *block* of representations is the $\mathbb{Z}$-span of irreducible representations given by the equivalence relation $\operatorname{Ext}(X, Y) \neq 0$.

A block $\mathcal{B}$ is where the Kazhdan-Lusztig polynomials live: $\mathcal{B}$ has two bases: of irreducible representations, or of standard representations. The Kazhdan-Lusztig polynomials compute each standard module as a sum of irreducible modules, and vice-versa.

The program computes Kazhdan-Lusztig polynomials. We've computed them for every group up to rank 8, with the exception of the block of size $453,060$

Vogan duality

Theorem 38 Given a block $\mathcal{B}$ for a real form $G(\mathbb{R})$ of G, there is a real form of $G^\vee$, and a block $\mathcal{B}^\vee$ for $G^\vee(\mathbb{R})$ which is "dual" to $\mathcal{B}$.

Dual means: the Kazhdan-Lusztig matrices for $\mathcal{B}$ and $\mathcal{B}^\vee$ are transposes.

Theorem 39 Vogan duality is:

$$(x, y) \to (y, x)$$

Example: $Sp(4)/SO(3,2)$

Block of $Sp(4,\mathbb{R})$:

```
 0( 0,6): 1 2  ( 4, *)  ( 6, *)  [i1,i1]  0
 1( 1,6): 0 1  ( 4, *)  ( *, *)  [i1,ic]  0
 2( 2,6): 3 0  ( 5, *)  ( 6, *)  [i1,i1]  0
 3( 3,6): 2 3  ( 5, *)  ( *, *)  [i1,ic]  0
 4( 4,4): 4 8  ( *, *)  ( *, *)  [r1,C+]  1  1
 5( 5,4): 5 9  ( *, *)  ( *, *)  [r1,C+]  1  1
 6( 6,5): 7 6  ( *, *)  ( *, *)  [C+,r1]  1  2
 7( 7,2): 6 7  ( *, *)  (10,11)  [C-,i2]  2  121
 8( 8,3): 9 4  (10, *)  ( *, *)  [i1,C-]  2  212
 9( 9,3): 8 5  (10, *)  ( *, *)  [i1,C-]  2  212
10(10,0): 10 11( *, *)  ( *, *)  [r1,r2]  3  2121
11(10,1)  11 10( *, *)  ( *, *)  [rn,r2]  3  2121
```

Dual block of $SO(3,2) = PGSp(4,\mathbb{R})$:

```
 0(0,10): 1 0  ( 2, *)  ( 3, 4)  [i1,i2]  0
 1(1,10): 0 1  ( 2, *)  ( *, *)  [i1,ic]  0
 2(2, 7): 2 7  ( *, *)  ( *, *)  [r1,C+]  1  1
 3(3, 8): 5 4  ( *, *)  ( *, *)  [C+,r2]  1  2
 4(3, 9): 6 3  ( *, *)  ( *, *)  [C+,r2]  1  2
 5(4, 4): 3 5  ( *, *)  ( 8,10)  [C-,i2]  2  121
 6(4, 5): 4 6  ( *, *)  ( 9,11)  [C-,i2]  2  121
 7(5, 6): 7 2  ( 8, 9)  ( *, *)  [i2,C-]  2  212
 8(6, 0): 9 10 ( *, *)  ( *, *)  [r2,r2]  3  2121
 9(6, 1): 8 11 ( *, *)  ( *, *)  [r2,r2]  3  2121
10(6, 2): 10 8 ( *, *)  ( *, *)  [rn,r2]  3  2121
11(6, 3): 11 9 ( *, *)  ( *, *)  [rn,r2]  3  2121
```

Relation with Vogan's talks

$$\begin{tikzcd}
\mathcal{M}(\mathfrak{g}, K, \lambda) \arrow[rr, "\text{Local Langlands}"] \arrow[drr, "\text{Vogan Duality}"'] & & \mathcal{P}(G^\vee/B^\vee, K^\vee)_{\lambda^\vee} \arrow[d, "\text{Localization}"] \\
& & \mathcal{M}(\mathfrak{g}^\vee, K^\vee, \lambda^\vee)
\end{tikzcd}$$

As discussed by Vogan, the top line conjecturally exists over any local field. It takes the

Kazhdan-Lusztig matrix for $\mathcal{M}(\mathfrak{g}, K, \lambda)$

to the transpose of the

Kazhdan-Lusztig matrix for $\mathcal{P}(G^\vee/B^\vee, K^\vee)_{\lambda^\vee}$

The right arrow only exists over $\mathbb{R}$, and is what makes our picture entirely symmetric in $G, G^\vee$

What is left to do?

(1) Non-integral infinitesimal character λ:
Replace $G^\vee$ with

$$G^\vee_\lambda = \mathrm{Cent}^\vee_G(\lambda)$$

This has root system

$$\langle \lambda, \alpha^\vee \rangle$$

(2) Singular infinitesimal character λ
Use translation principle from regular infinitesimal character. This has a big kernel, and there should be a better way directly at λ

(3) K-types and representations of G

Implement K, irreducible representations of K, actual representations of G (rather than translation families) and their K-types, including lowest K-types

Note: K is *not* necessarily the real points of a *connected* algebraic group. What does it mean to describe K?

David Vogan and Alfred Noel are working on this

(3) **Compute the unitary dual**

References