Notes on nilpotent orbits
Computational Theory of Real Reductive Groups Workshop

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Contents

Chapter 1

Background

In Chapter 1 we gather background material on linear algebra and Lie algebras that may get used in the lectures.

1.1 Linear algebra

The theory of Lie algebras is heavily based on a few core results in linear algebra. This section details some of those results. Throughout we assume that vector spaces are over the complex numbers $\mathbb{C}$.

1.1.1 Bilinear forms

This subsection should be skipped for now and used as a reference.

The Cartan-Killing form on a Lie algebra plays an important technical role. It is a symmetric, bilinear form on the Lie algebra (viewed as a vector space).

Let V denote a vector space over $\mathbb{C}$. A **symmetric, bilinear** form is a map $V \times V \to \mathbb{C}$, written as (v, w) for $v, w \in V$. Bilinear means it is linear in each factor and symmetric means that $(v, w) = (w, v)$. Once we choose a basis for V, any bilinear form can be expressed in terms of matrix multiplication as

$$(v, w) = v^T A w,$$

where v and w are written as column vectors and where A is an $n \times n$ matrix with $n = \dim(V)$. The bilinear form is symmetric if and only if A is a symmetric matrix.

Usually we only care about bilinear forms up to a change of basis (much as happens for linear transformations). This amounts to classifying symmetric matrices up to transformation of A into $B^T AB$, where B is an invertible matrix (why?). Now any symmetric matrix can be conjugated to a diagonal matrix by an orthogonal matrix (why?) and recall that orthogonal matrices B satisfy $B^{-1} = B^T$. Hence any symmetric, bilinear forms can be written using a diagonal matrix after choosing the basis for V appropriately. Next, considering $B^T AB$ where A and B are diagonal, we can remove any perfect square divisor of the entries of A. In other words, over $\mathbb{C}$ the matrix A is equivalent to a diagonal matrix where every entry is 0 or 1.

Exercise 1.1.

1. Show that the number of zeros and ones obtained from this process is determines and is uniquely determined by A. In other words, the number of inequivalent symmetric bilinear forms over $\mathbb{C}$ is equal to $n+1$, where n is the dimension of V.

2. What is the situation for classifying symmetric, bilinear forms over the real numbers $\mathbb{R}$? (this is Sylvester's Theorem)

Often we are interested in **non-degenerate** forms. This means that if $(v, w) = 0$ for all $w \in V$, then $v = 0$. Note that this implies the matrix A above is equivalent to the identity matrix, i.e. there are no zeros. Note that over $\mathbb{C}$, there are vectors which are orthogonal to themselves, once the vector space has dimension at least two. For example, taking the usual form on $\mathbb{C}^2$, then $v = (1, i)$ satisfies $(v, v) = 0$. On the other hand, it is still the case that the orthogonal space to a subspace still has the expected dimension. Namely,

Proposition 1.1.1. *Let V be vector space of dimension n with a non-degenerate, symmetric, bilinear form. Let $U \subset V$ be a subspace of dimension m. Define the orthogonal space*

$$U^\perp := \{v \in V \mid (v, u) = 0 \textit{ for all } u \in U\}.$$

Then $U^\perp$ has dimension $n - m$.

Proof. For each $u \in U$, consider the linear map $T_u : V \to \mathbb{C}$ given by $T_u(v) := (v, u)$. These are just elements of the dual vector space V^* of linear maps from V to $\mathbb{C}$. Notice that $U^\perp$ is in the kernel of each T_u, so we can in fact consider T_u as an element in $(V/U^\perp)^*$. Since the form is non-degenerate, if u is nonzero, then T_u cannot be the zero element of $(V/U^\perp)^*$; otherwise, $(v, u) = T_u(v) = T_u(\bar{v}) = 0$ for all $v \in V$, a contradiction. Here, $\bar{v}$ denotes the image of v in $V/U^\perp$.

Notice that the bilinearity of the form means that $T_{au_1+bu_2} = aT_{u_1} + bT_{u_2}$ where $a, b \in \mathbb{C}$. This and the fact that T_u is nonzero whenever u is nonzero implies that the map $U \to (V/U^\perp)^*$ given by $u \to T_u$ is an injective linear map. This gives the inequality $m \leq n - \dim(U^\perp)$, or

$$\dim(U^\perp) \leq n - m.$$

On the other hand, choose a basis $u_1, u_2, \ldots, u_m$ of U. Then $U^\perp$ is the intersection of the kernels of the T_{u_i}. Each kernel has dimension $n - 1$ and so the intersection of m such subspaces must have dimension at least $n - m$.

The two inequalities mean that $\dim U^\perp$ must be $n - m$. □

Exercise 1.2. Fix a non-degenerate symmetric, bilinear form on V. Define the orthogonal group $\mathrm{O}(V)$ to be the set of linear transformations $g : V \to V$ that preserve the form, i.e.

$$\mathrm{O}(V) := \{g \mid (v, w) = (g.v, g.w) \text{ for all } v, w \in V.$$

1. Show that $\mathrm{O}(V)$ is a subgroup of $\mathrm{GL}(V)$, the group of invertible linear endomorphisms of V.

2. Suppose that we have another non-degenerate symmetric, bilinear form $(\cdot, \cdot)'$ on V and define $\mathrm{O}'(V)$ using this form. Show that $\mathrm{O}(V)$ and $\mathrm{O}'(V)$ are conjugate subgroups of $\mathrm{GL}(V)$. In particular, they are isomorphic. (Hint: use the classification of such forms over $\mathbb{C}$).

1.1.2 Jordan decomposition

Recall that a matrix A is **nilpotent** if $A^k = 0$ for some positive integer k. A matrix A is called **semisimple** if A has a basis of eigenvectors, i.e. A is diagonalizable [1].

Theorem 1.1.2 (Jordan decomposition for matrices)**.** *The Jordan decomposition says that every matrix A can be written uniquely as*

$$A = N + S$$

where N is nilpotent, S is semisimple, and N and S commute.

Moreover, there is another important property that is often used in Lie theory: a subspace U satsifies $A(U) \subset U$ if and only if $S(U) \subset U$ and $N(U) \subset U$.

The Jordan decomposition can be proved directly or it can be deduced from the Jordan canonical form of a matrix. The latter says that A is similar to a block diagonal matrix built out of Jordan blocks

$$\begin{bmatrix} \mu & 1 & \dots & 0 & 0 & 0 \\ 0 & \mu & 1 & \dots & 0 & 0 \\ 0 & 0 & \mu & 1 & \dots & 0 \\ \dots & & & & & \\ 0 & \dots & 0 & 0 & \mu & 1 \\ 0 & 0 & \dots & 0 & 0 & \mu \end{bmatrix},$$

where μ is an eigenvalue of A. Moreover the Jordan blocks that appear determine and are uniquely determined by the similarity class of A. In other words, A and B are conjugate under $\mathrm{GL}(V)$ if and only if they have the same Jordan blocks in their Jordan decomposition.

Exercise 1.3. In this exercise, we examine the nilpotent matrices, up to conjugation.

1. Show that a nilpotent matrix has only 0 as an eigenvalue and then use the Jordan canonical form to show that each conjugacy classes of nilpotent $n \times n$ matrices corresponds to a unique partition of n.

2. Suppose that N is a nilpotent $n \times n$ matrix with Jordan blocks of size $\lambda_1 \geq \dots \geq \lambda_k$. Compute the rank of N^k in terms of the partition $[\lambda_i]$.

1.2 Some key facts about Lie algebras

We will use the notation $\mathfrak{g}$ for a Lie algebra. Recall that a Lie algebra is a vector space, which is equipped with a product $[\cdot, \cdot]$, called the bracket. The bracket is bilinear and antisymmetric. It also satisifies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$$

for all $X, Y, Z \in \mathfrak{g}$.

The prototypical Lie algebra is the endomorphisms of a vector space V, equipped with the bracket

$$[X, Y] = XY - YX,$$

[1]Over a non-algebraically closed field, we would say that A is semisimple if it is diagonalizable over the algebraic closure of the field

where the product on the right is composition of endomporphisms (i.e. usual matrix multiplication if we choose a basis of V). This Lie algebra is denoted $\mathfrak{gl}(V)$ or $\mathfrak{gl}_n(\mathbb{C})$ if we pick a basis of V. The trace zero matrices are also a Lie algebra, denoted $\mathfrak{sl}(V)$ or $_n(\mathbb{C})$.

For $X \in \mathfrak{g}$, we write ad_X for the linear map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_X(Y) = [X, Y]$. Then the Jacobi identity says that ad_X is a derivation, i.e.

$$\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)].$$

Moreover, since $\mathrm{ad}_X \in \mathfrak{gl}(V)$, we have a map $\mathfrak{g} \to \mathfrak{gl}(V)$ given by $X \to \mathrm{ad}_X$ and this map is actually a homomorphism of Lie algebras: it is a linear map and also satisfies

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y],$$

where the bracket on the left is computed in $\mathfrak{g}$ and the bracket on the right side is in $\mathfrak{gl}(V)$ where it equals $\mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X$. The fact that $X \to ad_X$ is a homomorphism is equivalent to the Jacobi identity.

Exercise 1.4. Verify that $X \to \mathrm{ad}_X$ is a homomorphism of Lie algebras.

Exercise 1.5. Write $\mathfrak{g}_X$ for the centralizer of X in $\mathfrak{g}$. Write $[\mathfrak{g}, X]$ for the set of elements of the form $[Y, X]$ for $Y \in \mathfrak{g}$. Show that

$$\dim(\mathfrak{g}_x) + \dim([\mathfrak{g}, X]) = \dim \mathfrak{g}$$

(Hint: relate $\mathfrak{g}_X$ and $[\mathfrak{g}, X]$ to the image and kernel of ad_X.)

1.2.1 Cartan-Killing form and semisimplicity

In the classification of Lie algebras, a key technical role is played by the Cartan-Killing form. This is defined to be the symmetric, bilinear form on $\mathfrak{g}$ given by $(X, Y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y)$. This is symmetric by properties of trace and bilinear since ad is linear. We use the notation $\kappa(X, Y)$ for the Killing form.

The Killin form is invariant with respect to $\mathfrak{g}$ in the following sense:

$$\kappa([X, Y], Z) = \kappa(X, [Y, Z]).$$

The following is the heart of the first month of a course in Lie algebras:

Theorem 1.2.1. *The following are equivalent for a Lie algebra:*

1. *The Cartan-Killing form is non-degenerate.*

2. $\mathfrak{g}$ *is semisimple, meaning that there are no nonzero solvable ideals in* $\mathfrak{g}$

3. $\mathfrak{g}$ *is a direct sum of simple subalgebras, where a Lie algebra is simple means that it contains no proper nonzero ideals.*

Using this theorem and the theory of root systems (among other tools), the simple Lie algebras can be classified into one of four infinite families A_n, B_n, C_n, or D_n, or 5 exceptional Lie algebras G_2, F_4, E_6, E_7 or E_8. The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of traceless $n \times n$ matrices is labeled by A_{n-1}.

1.2.2 Jordan decomposition

The Jordan decomposition carries over to semisimple Lie algebras. An element $S \in \mathfrak{g}$ is called semisimple if ad_S is semisimple (as a linear map from the vector space underlying $\mathfrak{g}$ to itself). Similarly, $N \in \mathfrak{g}$ is nilpotent if ad_N is nilpotent.

Theorem 1.2.2 (Jordan decomposition for $\mathfrak{g}$). *For $X \in \mathfrak{g}$ we can write X uniquely as*

$$X = X_s + X_n$$

where $X_s \in \mathfrak{g}$ is semisimple, $X_n \in \mathfrak{g}$ is nilpotent and $[X_s, X_n] = 0$.

The element X_s is called the semisimple part of X and X_n is called the nilpotent part of X.

The decomposition has two important properties:

- It behaves well under Lie algebra homomorphisms: if $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homorphism, then
$$\phi(X)_s = \phi(X_s)$$
and
$$\phi(X)_n = \phi(X_n).$$
- It coincides with the usual Jordan decomposition in $\mathfrak{gl}(V)$. In other words, whether we think of $X \in \mathfrak{gl}(V)$ as an element of the Lie algebra $\mathfrak{gl}(V)$ and use Theorem 1.2.2 or we think of it as a regular old matrix and use Theorem 1.1.2, we get the same decomposition.

Chapter 2

Jacobson-Morozov Theorem

2.1 $\mathfrak{sl}_2(\mathbb{C})$

The semisimple Lie algebra of smallest dimension is $\mathfrak{sl}_2(\mathbb{C})$, the 2×2 matrices of trace zero. It is of dimension three, with a basis given by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

These elements satisfy the relations

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H. \tag{2.1}$$

Notice that E and F are nilpotent matrices and so also nilpotent elements of $\mathfrak{sl}_2(\mathbb{C})$. And H is a semisimple matrix and a semisimple element of $\mathfrak{sl}_2(\mathbb{C})$. Or we could have seen this directly by writing the matrix for ad_H in terms of the (ordered) basis E, H, F of $\mathfrak{g}$:

$$\mathrm{ad}_H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Exercise 2.1. Write the matrices of $\mathrm{ad}_E, \mathrm{ad}_F$ in terms of the basis E, F, H. Write the Cartan-Killing form using this basis and show that it is non-degenerate. This shows that $\mathfrak{sl}_2(\mathbb{C})$ is semisimple.

On the other hand, it is easy to show that $\mathfrak{sl}_2(\mathbb{C})$ is simple directly. Here's the sketch, let I be an ideal. Then $\mathrm{ad}_H(I) = I$ and so ad_H restricted to I is semisimple (a fact from linear algebra). So I must be a sum of eigenspaces for H. Now use the action of E and F to show that if I is proper, then it must be zero.

2.2 Representations of $\mathfrak{sl}_2(\mathbb{C})$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and its representations are the central building blocks in Lie theory. Recall that a representation of $\mathfrak{g}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ for some vector space V. This amounts to finding matrices $e, h, f \in \mathfrak{gl}(V)$ satsifying the relations in 2.1.

The classification of representations of $\mathfrak{sl}_2(\mathbb{C})$ has two components:

- Every representation is the direct sum of irreducible representations (this is true for all semisimple Lie algebras over the complex numbers).
- There is a unique irreducible representation (up to isomorphism) of dimension n for each positive integer n.

2.3 The irreducible representation of dimension n

Let's now explicitly construct the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. Consider the $n \times n$ nilpotent matrix

$$e = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots \\ 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Next we seek a semisimple h such that $[h, e] = 2$. If we try to find h that is diagonal, we see that

$$h = \begin{bmatrix} k & 0 & \dots & 0 & 0 & 0 \\ 0 & k-2 & 0 & \dots & 0 & 0 \\ 0 & 0 & k-4 & 0 & \dots & 0 \\ \dots \\ 0 & \dots & 0 & 0 & k-2n+4 & 0 \\ 0 & 0 & \dots & 0 & 0 & k-2n+2 \end{bmatrix}.$$

Exercise 2.2. Verify the above calculation for h. Show that if $[e, f] = h$, then h must have trace zero. Show that $k = n - 1$ makes the trace of h equal to zero.

Next to find f with $[h, f] = -2f$, we have that f must take the form

$$f = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & 0 & 0 & \dots & 0 \\ \dots \\ 0 & \dots & 0 & a_{n-2} & 0 & 0 \\ 0 & 0 & \dots & 0 & a_{n-1} & 0 \end{bmatrix}.$$

Exercise 2.3. Show that $[e, f] = h$ is satisfied exactly when $a_i = i(n - i)$. Thus we have constructed a representation of $\mathfrak{sl}_2$ of dimension n. Verify that this is indeed an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$.

Notice the following corollary of our construction:

Corollary 2.3.1. *For every nilpotent matrix $e \in \mathfrak{gl}_n(\mathbb{C})$, there exists $h, f \in \mathfrak{gl}_n(\mathbb{C})$ satisfying the relations 2.1. This shows that every nilpotent matrix can be embedded in a copy of $\mathfrak{sl}_2(\mathbb{C})$ sitting inside of $\mathfrak{gl}_n(\mathbb{C})$*

Proof. If we can do this for a nilpotent matrix e, then we can also do this for any conjugate geg^{-1} where $g \in GL_n(\mathbb{C})$. This is because we can conjugate h and f by g and obtain the necessary matrices. Hence we might as well take e to be in Jordan form. But for each Jordan block, we figured out the appropriate h and f above. If we do this for each block and put the blocks together, we get the relevant matrices h and f for e. □

For example, if

$$e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(built from two Jordan blocks), then

$$h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

work to satisfy the requirements of the corollary.

2.4 Jacobson-Morozov Theorem

The corollary in the previous section holds in general for a semisimple Lie algebra $\mathfrak{g}$:

Theorem 2.4.1 (Jacobson-Morozov)**.** *Given a nilpotent element $e \in \mathfrak{g}$, there exists $h, f \in \mathfrak{g}$ satisfying the relations 2.1. In other words, e belongs to a copy of $\mathfrak{sl}_2(\mathbb{C})$ sitting inside of $\mathfrak{g}$ as a subalgebra.*

One proof of this use the result for $\mathfrak{gl}_n(\mathbb{C})$. Another proof to be sketched in the lectures is by induction on the dimension of $\mathfrak{g}$.

Chapter 3

Classifying nilpotent orbits

The standing assumption is that $\mathfrak{g}$ is semisimple. Our goal in these lectures is to classify the orbits of nilpotent elements in $\mathfrak{g}$ under the action of the group G (of connected automorphisms of $\mathfrak{g}$). We can think of G either as an algebraic group (algebraic variety with a compatible group structure) or a Lie group (manifold with compatible group structure).

3.1 Dynkin-Kostant classification

To classify the nilpotent G-orbits (that is, the nilpotent elements up to the action of G), we use the Jacobson-Morozov theorem: Let $\mathcal{A}_{hom}$ denote the G-conjugacy classes of Lie algebra homomorphisms from $\mathfrak{s}l_2(\mathbf{C})$ to $\mathfrak{g}$. That is, two homomorphism ϕ, ϕ' are conjugate if we there exists $g \in G$ such that $\phi = phi' \circ \mathrm{Ad}(g)$, where $\mathrm{Ad}(g)$ denotes the automorphism of $\mathfrak{g}$ determined by g.

Let $H = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $E = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ be elements of $\mathfrak{s}l_2(\mathbf{C})$ as before.

Consider the map $\Omega : \mathcal{A}_{hom} \to \{\text{nilpotent } G\text{-orbits in } \mathfrak{g}\}$ given by $\Omega(\phi) = \mathrm{Ad}(G)\phi(E)$. This is a bijection: surjectivity is just the Jacobson-Morozov theorem and injectivity follows from a theorem of Kostant.

Let $\Upsilon : \mathcal{A}_{hom} \to \{\text{semi-simple } G\text{-orbits in } \mathfrak{g}\}$ be the map $\Upsilon(\phi) = \mathrm{Ad}(G)\phi(H)$. A theorem of Mal'cev shows that Υ is injective.

Conclusion: nilpotent orbits in $\mathfrak{g}$ are parametrized by the image of Υ. This set was completely determined by Dynkin. The classification of nilpotent orbits in terms of these semisimple elements is called the Dynkin-Kostant classification.

Pick a basis of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ for the root system of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$. Assume that $h \in \mathfrak{h}$ belongs to a conjugacy class in the image of Υ. By using the action of the Weyl group, we can assume that $\alpha_i(h) \geq 0$ for all i. Attaching these n numbers to the Dynkin diagram of $\mathfrak{g}$ is what we call the **weighted Dynkin diagram** of the corresponding nilpotent orbit $\mathcal{O}$.

A few facts:

- $\alpha_i(h)$ are integers. This follows from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$.
- Moreover, $\alpha_i(h) \in \{0, 1, 2, \}$. This is one way to prove there are a finite number of nilpotent orbits.

- Let $\mathfrak{g}_i$ denote the i-eigenspace for ad_h. Then another consequence of the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ is that:
$$\dim(\mathcal{O}) = \dim \mathfrak{g} - (\dim \mathfrak{g}_0 + \dim \mathfrak{g}_1).$$
The dimension of $\mathfrak{g}_i$ can be computed from the weighted Dynkin diagram of $\mathcal{O}$.

For example, in G_2 we find (using techniques discussed later) that there are 5 nilpotent orbits. Their weighted Dynkin diagrams are

$$0 => 0, 0 => 1, 1 => 0, 2 => 0, 2 => 2$$

and the dimensions of these orbits are
$$0, 6, 8, 10, 12.$$
It is a general fact that orbits in $\mathfrak{g}$ always have even dimension.

3.2 Bala-Carter Theorem

to be continued....

Chapter 4

Classical groups

4.1 Parametrizing orbits by partitions

4.2 Partial order on orbits