Lectures on differential equations in complex domains

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Notes for a graduate course in real and complex analysis

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I. Differential equations

1. Existence and uniqueness of solutions. Let Ω be a domain in $\mathbb{C}$ and a_k, $k = 1, 2, \dots, n$, holomorphic functions on Ω. We consider the following homogeneous differential equation of order n

$$\frac{d^n y}{dz^n} + a_1 \frac{d^{n-1} y}{dz^{n-1}} + \dots + a_{n-1} \frac{dy}{dz} + a_n y = 0$$

on Ω. Let y be a solution of this differential equation in Ω, and define $Y : \Omega \longrightarrow \mathbb{C}^n$ by

$$Y_1 = y,\ Y_2 = \frac{dy}{dz}, \dots,\ Y_n = \frac{d^{n-1} y}{dz^{n-1}}.$$

Then

$$\frac{dY}{dz} = \begin{pmatrix} y' \\ y'' \\ \vdots \\ y^{(n-1)} \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ \vdots \\ y^{(n-1)} \\ -a_1 y^{(n-1)} - a_2 y^{(n-2)} - \dots - a_{n-1} y' - a_n y \end{pmatrix} = AY$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}.$$

Therefore, Y is a solution of the first order system of differential equations

$$\frac{dY}{dz} = AY$$

in Ω. Clearly, if Z is a solution of this system, its first component is a solution of our differential equation.

Therefore, we established the following simple result.

1.1. LEMMA. *The mapping $y \mapsto Y$ is a linear bijection from the vector space of all solutions of the differential equation*

$$\frac{d^n y}{dz^n} + a_1 \frac{d^{n-1} y}{dz^{n-1}} + \dots + a_{n-1} \frac{dy}{dz} + a_n y = 0$$

in Ω, onto the space of all solutions of the first order system

$$\frac{dY}{dz} = AY$$

in Ω.

Therefore instead of studying the space of all solutions of the differential equation, we can study a more general problem of studying the solutions of the first order system

$$\frac{dY}{dz} = AY$$

where $A : \Omega \longrightarrow M_n(\mathbb{C})$ is an arbitrary holomorphic map.

The main result we want to prove is the following theorem.

1.2. Theorem. *Let Ω be a simply connected region in $\mathbb{C}$, $z_0 \in \Omega$ and $A : \Omega \longrightarrow M_n(\mathbb{C})$ a holomorphic map. For any $Y_0 \in \mathbb{C}^n$ there exists a unique holomorphic function $Y : \Omega \longrightarrow \mathbb{C}^n$ such that*

$$\frac{dY}{dz} = AY$$

in Ω, and

$$Y(z_0) = Y_0.$$

Therefore, the linear mapping $Y \mapsto Y(z_0)$ is an isomorphism of the linear space of all solutions of this system in Ω onto $\mathbb{C}^n$. In particular we have the following consequence.

1.3. Corollary. *The linear space of all solutions of the system*

$$\frac{dY}{dz} = AY$$

in a simply connected domain Ω is n-dimensional.

By 1, these results have their analogues for n^{th}-order differential equations.

1.4. Theorem. *Let Ω be a simply connected region in $\mathbb{C}$, $z_0 \in \Omega$. For any complex numbers $y_0, y_1, \dots, y_n$ there exists a unique holomorphic function $y \in H(\Omega)$ such that*

$$\frac{d^n y}{dz^n} + a_1 \frac{d^{n-1} y}{dz^{n-1}} + \ldots + a_{n-1} \frac{dy}{dz} + a_n y = 0$$

in Ω, and

$$y(z_0) = y_0,\ y'(z_0) = y_1, \dots, y^{(n-1)} = y_{n-1}.$$

1.5. Corollary. *The linear space of all solutions of the differential equation*

$$\frac{d^n y}{dz^n} + a_1 \frac{d^{n-1} y}{dz^{n-1}} + \ldots + a_{n-1} \frac{dy}{dz} + a_n y = 0$$

in a simply connected domain Ω is n-dimensional.

Now we shall prove 2. Let $D = D(z_0, R)$ be a disk centered at z_0 and contained in Ω. We shall first consider the solutions on D. Since A is holomorphic on D we can represent it by its Taylor series:

$$A(z) = \sum_{p=0}^{\infty} B_p (z - z_0)^p$$

where $B_p \in M_n(\mathbb{C})$, $p \in \mathbb{Z}$. The solution Y of our system on D should also be represented by its Taylor series

$$Y(z) = \sum_{p=0}^{\infty} T_p (z - z_0)^p$$

with $T_p \in \mathbb{C}^n$, $p \in \mathbb{Z}$. The differential equation

$$\frac{dY}{dz} = AY$$

now leads to

$$\sum_{p=1}^{\infty} p T_p (z - z_0)^{p-1} = \left(\sum_{r=0}^{\infty} B_r (z - z_0)^r \right) \left(\sum_{s=0}^{\infty} T_s (z - z_0)^s \right)$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} B_r T_s (z - z_0)^{r+s} = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} B_{m-k} T_k \right) (z - z_0)^m$$

on D. By changing the index in the first sum we get

$$\sum_{m=0}^{\infty} (m+1) T_{m+1} (z - z_0)^m = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} B_{m-k} T_k \right) (z - z_0)^m$$

on D, which implies that

$$(m+1) T_{m+1} = \sum_{k=0}^{m} B_{m-k} T_k$$

for any $m \in \mathbb{Z}_+$. Therefore,

$$T_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} B_{m-k} T_k$$

are the *recursion relations* for the coefficients. Since $T_0 = Y(z_0) = Y_0$, and each T_{m+1} is expressed by these formulas in terms of $T_0, T_1, \dots, T_m$, we see that Y_0 uniquely determines the coefficients in the expansion. Therefore, the solution Y on D is uniquely determined by its value at z_0. This in turn implies the same assertion for solutions in Ω. This completes the uniqueness part of the proof.

To show the existence on D, it is enough to show that the formal series

$$\sum_{p=0}^{\infty} T_p (z - z_0)^p$$

converges on D, for any initial condition $T_0 = Y_0$. We shall prove this by Cauchy's *majorization method*. For any matrix C we denote by $\|C\|$ the maximum of absolute values of its matrix coefficients. Assume that

$$\|B_p\| \le b_p,$$

for some $b_p \ge 0$, for all $p \in \mathbb{Z}_+$. Consider the power series

$$a(z) = \sum_{p=0}^{\infty} b_p (z - z_0)^p$$

and assume that it converges on some $D' = D(z_0, r)$ with $r \le R$. Then we can consider the first order differential equation

$$\frac{dy(z)}{dz} = n a(z) y(z)$$

on D'. For any $z \in D'$ denote by $[z_0, z]$ the oriented segment connecting z_0 with z. Then

$$F : z \longmapsto \int_{[z_0, z]} a(w)\, dw$$

is a holomorphic function in D' and

$$\frac{dF}{dz} = a(z)$$

for $z \in D'$. This implies that the function

$$y = \|Y_0\| e^{n \int_{[z_0, z]} a(w)\, dw}$$

is holomorphic in D',

$$y(z_0) = \|Y_0\|$$

and

$$\frac{dy}{dz} = \|Y_0\| e^{n \int_{[z_0,z]} a(w)\, dw} na(z) = na(z)y.$$

Therefore, y is the solution of the initial value problem

$$\frac{dy(z)}{dz} = na(z)y(z), \ y(z_0) = \|Y_0\|.$$

Assume that

$$y(z) = \sum_{p=0}^{\infty} t_p (z - z_0)^p$$

is the Taylor series of y. Then we get the recursion relations

$$t_{m+1} = \frac{n}{m+1} \sum_{k=0}^{m} b_{m-k} t_k$$

for all $m \in \mathbb{Z}_+$. Since all b_p are non-negative, $t_0 \geq 0$ implies by induction in m that $t_m \geq 0$ for all $m \in \mathbb{Z}_+$. On the other hand, we see by induction that

$$\|T_p\| \leq t_p$$

for all $p \in \mathbb{Z}$. First, by definition this is true for $m = 0$. If $p \geq 0$, we have

$$\|T_{p+1}\| = \frac{1}{p+1} \left\| \sum_{k=0}^{p} B_{p-k} T_k \right\| \leq \frac{1}{p+1} \sum_{k=0}^{p} \|B_{p-k} T_k\| \\ \leq \frac{n}{p+1} \sum_{k=0}^{p} \|B_{p-k}\| \, \|T_k\| \leq \frac{n}{p+1} \sum_{k=0}^{p} b_{p-k} t_k = t_{p+1}$$

what completes the argument.

This estimate implies that the radius of convergence of the power series

$$\sum_{p=0}^{\infty} T_p (z - z_0)^p$$

is at least equal to r. Therefore, it converges in D'.

Hence, to show the existence of solutions on a disk around z_0 it is enough to find a “good” majorization. For example, for any $r < R$, the function $z \mapsto \|A(z)\|$ is bounded on D'. Fix $r < R$ and $M > 0$ such that $\|A(z)\| \leq M$. By the Cauchy estimates, we have

$$\|B_p\| \leq \frac{M}{r^p}$$

for all $p \in \mathbb{Z}_+$. Hence, we can take $b_p = \frac{M}{r^p}$, $p \in \mathbb{Z}_+$. Then

$$a(z) = \sum_{p=0}^{\infty} b_p (z - z_0)^p = M \sum_{p=0}^{\infty} \left(\frac{z - z_0}{r} \right)^p = \frac{M}{1 - \frac{z - z_0}{r}} = \frac{Mr}{r - (z - z_0)},$$

for $z \in D'$. Therefore, the power series

$$\sum_{p=0}^{\infty} T_p (z - z_0)^p$$

converges in D'. Since $r < R$ was arbitrary, we finally conclude that this power series converges in D. This completes the proof of the theorem for D.

It remains to prove the existence for Ω. This follows from the monodromy theorem. Let $z \in \Omega$ be arbitrary and let $\gamma : [a, b] \longrightarrow \Omega$ be a path connecting z_0 with z. Since γ^* is compact, there exists $R > 0$ such that all open disks of radius R with center in γ^* lie in Ω. Also, we can find a finite family $D_0, D_1, \dots, D_n$ of disks of radius $\frac{R}{2}$, such that the center z_j of D_j is in D_{j-1} for $j = 1, 2, \dots, n$, and $z_n = z$. Since the disk of radius R centered at z_j contains D_{j-1}, by the previous result, we can find find solutions $Z_0, Z_2, \dots, Z_n$ of our system on disks $D_0, D_1, \dots D_n$ such that

(i) $Z_0(z_0) = Y_0$;
(ii) the function element (Z_j, D_j) is a direct continuation of the element (Z_{j-1}, D_{j-1}) for $j = 1, 2, \dots, n$.

Therefore (Z_0, D_0) allows analytic continuation along γ. Since Ω is simply connected, by the monodromy theorem Z_0 extends to a holomorphic map from Ω into $\mathbb{C}^n$. Also, it is evident that this map is a solution of our system. This completes the proof of 2.

2. Fundamental matrix. Let Ω be a simply connected domain in $\mathbb{C}$, $A : \Omega \longrightarrow M_n(\mathbb{C})$ a holomorphic map and

$$\frac{dY}{dz} = AY$$

a first order system in Ω. Fix a base point z_0. Let $e_1, e_2, \dots, e_n$ be the canonical basis of $\mathbb{C}^n$, i. e.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \dots, e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \ e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then, by 2, we can find solutions $S_1, S_2, \dots, S_{n-1}, S_n$ of our system in Ω satisfying the following initial conditions

$$S_1(z_0) = e_1, \ S_2(z_0) = e_2, \ \dots, S_{n-1}(z_0) = e_{n-1}, S_n(z_0) = e_n.$$

Let $S : \Omega \longrightarrow M_n(\mathbb{C})$ be the holomorphic function such that its columns are $S_1, S_2, \dots, S_n$. Then S satisfies the differential equation

$$\frac{dS}{dz} = AS$$

in Ω, and

$$S(z_0) = I,$$

where $I \in M_n(\mathbb{C})$ is the identity matrix. Clearly, by 1.2, S is uniquely determined by these properties. We call S the *fundamental matrix* of the system

$$\frac{dY}{dz} = AY$$

in Ω for the base point z_0.

Evidently, the solution Y of our system for the initial condition $Y(z_0) = Y_0$ is given by

$$Y(z) = S(z)Y_0$$

for $z \in \Omega$. The columns $S_1, S_2, \dots, S_n$ of S are linearly independent solutions of our system. Hence, by 1.3, they form a basis of the vector space of all solutions in Ω. By 1.2, their evaluations $S_1(z), S_2(z), \dots, S_n(z)$ are linearly independent vectors in $\mathbb{C}^n$ for any $z \in \Omega$. In other words, we have the following result.

2.1. Proposition. *Let S be the fundamental matrix of the system*

$$\frac{dY}{dz} = AY$$

in Ω. Then $S(z) \in \mathrm{GL}(n, \mathbb{C})$ for any $z \in \Omega$.

Actually, we can calculate the determinant of the fundamental matrix S. Let

$$\Delta(z) = \det S(z)$$

for $z \in \Omega$. Then Δ is a holomorphic function in Ω and $\Delta(z_0) = 1$. Let $\mathfrak{S}_n$ be the permutation group of $\{1, 2, \dots, n\}$, and $\epsilon : \mathfrak{S}_n \longrightarrow \{-1, 1\}$ the parity homomorphism. Then

$$\Delta(z) = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) S_{1\sigma(1)}(z) S_{2\sigma(2)}(z) \dots S_{n\sigma(n)}(z)$$

for any $z \in \Omega$. Hence, we have

$$\begin{aligned}
\frac{d\Delta(z)}{dz} &= \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) \frac{d}{dz} \left(S_{1\sigma(1)}(z) S_{2\sigma(2)}(z) \dots S_{n\sigma(n)}(z) \right) \\
&= \sum_{i=1}^{n} \left(\sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{i-1\sigma(i-1)}(z) \frac{dS_{i\sigma(i)}(z)}{dz} S_{i+1\sigma(i+1)}(z) \dots S_{n\sigma(n)}(z) \right) \\
&= \sum_{i=1}^{n} \left(\sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{i-1\sigma(i-1)}(z) \left(\sum_{k=1}^{n} A_{ik}(z) S_{k\sigma(i)}(z) \right) \dots S_{n\sigma(n)}(z) \right) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik}(z) \left(\sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) S_{1\sigma(1)}(z) \dots S_{i-1\sigma(i-1)}(z) S_{k\sigma(i)}(z) \dots S_{n\sigma(n)}(z) \right).
\end{aligned}$$

If $k \neq i$ the inner sum represents the expression for the determinant with equal i^{th} and k^{th} rows. Therefore, these terms vanish and we get

$$\begin{aligned}\frac{d\Delta(z)}{dz} &= \sum_{i=1}^{n} A_{ii}(z)\left(\sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)S_{1\sigma(1)}(z)\dots S_{i-1\sigma(i-1)}(z)S_{i\sigma(i)}(z)S_{i+1\sigma(i+1)}(z)\dots S_{n\sigma(n)}(z)\right)\\ &= \sum_{i=1}^{n} A_{ii}(z)\,\det S(z) = \operatorname{Tr} A(z)\,\Delta(z).\end{aligned}$$

2.2. Lemma. *The determinant Δ of the fundamental matrix S of the system*

$$\frac{dY}{dz} = AY$$

satisfies the differential equation

$$\frac{d\Delta}{dz} = \operatorname{Tr} A\,\Delta$$

in Ω.

Since Ω is simply connected, the integral along a path γ in Ω connecting z_0 to z

$$\int_\gamma \operatorname{Tr} A(w)\,dw$$

doesn't depend on the choice of γ. Hence we can put

$$\int_{z_0}^{z} \operatorname{Tr} A(w)\,dw = \int_\gamma \operatorname{Tr} A(w)\,dw.$$

This integral is a holomorphic function of z, and

$$\frac{d}{dz}\int_{z_0}^{z} \operatorname{Tr} A(w)\,dw = \operatorname{Tr} A(z)$$

for $z \in \Omega$. Therefore,

$$\Delta(z) = e^{\int_{z_0}^{z} \operatorname{Tr} A(w)\,dw}$$

for $z \in \Omega$.

II. Systems with regular singularities

1. Functions of moderate growth. Let $D = D(0,R) = \{z \in \mathbb{C} \mid |z| < R\}$ be the disk in $\mathbb{C}$ of radius R centered at 0. Denote by $D^* = D - \{0\}$ the corresponding punctured disk. Let $\tilde{D}^*$ be the universal cover of D^* and $p : \tilde{D}^* \longrightarrow D^*$ the corresponding projection. We can realize $\tilde{D}^*$ as the half-plane $\{t \in \mathbb{C} \mid \operatorname{Re} t < \log R\}$ and $p(t) = e^t$. Fix a base point z_0 in D^* and $t_0 \in \tilde{D}^*$ such that $p(t_0) = z_0$. For any $m \in \mathbb{Z}$ we define the map $T_m : \tilde{D}^* \longrightarrow \tilde{D}^*$ by $T_m(t) = t + 2\pi i m$ for $t \in \tilde{D}^*$. Then $p(T_m(t)) = p(t)$ for any $t \in \tilde{D}^*$, and $m \mapsto T_m$ is the map of the fundamental group $\pi_1(D^*) = \mathbb{Z}$ into the group of deck transformations of $\tilde{D}^*$.

By abuse of language, we call holomorphic functions on $\tilde{D}^*$ "multivalued" holomorphic functions on D^*. Holomorphic functions f on D^* correspond in this identification to functions of the form $\tilde{f} = f \circ p$.

Let $C = \{re^{i\theta} \mid 0 < r < R,\ \theta_0 \le \theta \le \theta_1\}$ be a sector of D^* for some $\theta_0, \theta_1 \in \mathbb{R}$ such that $\theta_1 - \theta_0 < 2\pi$. We say that a function f on C has *moderate growth at* 0 if there exist $\epsilon > 0$, $c > 0$ and $k \in \mathbb{Z}_+$ such that

$$|f(z)| \le c \frac{1}{|z|^k}$$

for $z \in C$ and $|z| \le \epsilon$. A holomorphic function on D^* has moderate growth at 0 if and only if it has at most a pole at 0.

The strip $\tilde{C} = \{t \in \mathbb{C} \mid \operatorname{Re} t < \log R, \theta_0 \le \operatorname{Im} t \le \theta_1\} \subset \tilde{D}^*$ evenly covers C. We say that a "multivalued" holomorphic function f on D^* has *moderate growth at* 0 if all its restrictions to such strips $\tilde{C}$ are pullbacks of functions of moderate growth on sectors C. Examples of such functions are: z^α for any $\alpha \in \mathbb{C}$ — it is actually the function $e^{\alpha t}$ on $\tilde{\mathbb{C}}^* = \mathbb{C}$, $\log z$ — it is actually the function t on $\tilde{\mathbb{C}}^* = \mathbb{C}$.

The following result is evident.

1.1. Lemma. *All "multivalued" holomorphic functions of moderate growth on D^* form a ring.*

Since $\tilde{D}^*$ is simply connected, any holomorphic function on $\tilde{D}^*$ is derivative of some other holomorphic function on $\tilde{D}^*$. This implies that for any "multivalued" holomorphic function f on D^* there exists a "multivalued" holomorphic function g on D^* such that $z\frac{dg}{dz} = f$.

1.2. Lemma. *Let f be a "multivalued" holomorphic function on D^*. Then the following conditions are equivalent:*

(i) *f has moderate growth at 0;*
(ii) *$z\frac{df}{dz}$ has moderate growth at 0.*

Proof. (i)$\Rightarrow$(ii) If f has moderate growth at 0, this means that the corresponding function $\tilde{f}$ on $\tilde{D}^*$ satisfies

$$|\tilde{f}(t)| \le c e^{-k \operatorname{Re} t}$$

on each strip $\tilde{C}$. Let $\epsilon > 0$ be small and $\tilde{C}'$ the strip corresponding to the sector $C' = \{re^{i\theta} \mid 0 < r < e^{-\epsilon}R,\ \theta_0 + \epsilon \le \theta \le \theta_1 - \epsilon\}$. By Cauchy estimates applied to the circle of radius ϵ around $t \in \tilde{C}'$ we see that

$$\left|\frac{d\tilde{f}}{dt}\right| \leq \frac{c}{\epsilon}e^{-k(\operatorname{Re} t+\epsilon)} \leq c' e^{-k \operatorname{Re} t},$$

hence

$$\left|z\frac{df}{dz}\right| \leq c' \frac{1}{|z|^k}$$

on C'. Since C and ϵ were arbitrary, $z\frac{df}{dz}$ has moderate growth at 0.

(ii)⇒(i) In this case, we have $z\frac{df}{dz}$ has moderate growth at 0, i. e.

$$\left|\frac{d\tilde{f}}{dt}\right| \leq ce^{-k \operatorname{Re} t}$$

on $\tilde{C}$. Let $t_0, t_1 \in \tilde{C}$ with $\operatorname{Re} t_0 \leq \operatorname{Re} t_1$ and $\operatorname{Im} t_0 = \operatorname{Im} t_1$. Integrating along the line γ connecting t_0 with t_1 we get

$$|\tilde{f}(t_1) - \tilde{f}(t_0)| \leq \left|\int_\gamma \frac{d\tilde{f}}{dt}dt\right| \leq c\int_{\operatorname{Re} t_0}^{\operatorname{Re} t_1} e^{-ks}ds = \frac{c}{k}(e^{-k \operatorname{Re} t_0} - e^{-k \operatorname{Re} t_1}).$$

By leaving $\operatorname{Re} t_1$ fixed we get

$$|\tilde{f}(t_0)| \leq ce^{-k \operatorname{Re} t_0}$$

for sufficiently large $c > 0$ and $t_0 \in \tilde{C}$ with $\operatorname{Re} t_0$ sufficiently negative. This implies that f has moderate growth at 0. □

Let $A \in M_n(\mathbb{C})$. We define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

Then, $t \mapsto e^{tA}$ is a holomorphic map from $\mathbb{C}$ into $\mathrm{GL}(n, \mathbb{C})$. Clearly,

$$\frac{de^{tA}}{dt} = Ae^{tA} = e^{tA}A.$$

Moreover, if $B \in M_n(\mathbb{C})$ is another matrix commuting with A, we have

$$e^A e^B = e^{A+B}.$$

Let $N \in M_n(\mathbb{C})$ be a nilpotent matrix such that $N^n = 0$ and $N^{n-1} \neq 0$. Then N is equivalent to the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

i. e. the matrix $\lambda I + N$ is equivalent to the Jordan cell matrix with eigenvalue λ. Now

$$e^{t(\lambda I+N)} = e^{\lambda t} e^{tN} = e^{\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!} t^k N^k = e^{\lambda t} \sum_{k=0}^{n-1} \frac{1}{k!} t^k N^k,$$

i. e. the matrix coefficients of this matrix are linear combinations of functions of the form $t^k e^{\lambda t}$, $k \in \mathbb{Z}_+$. Since every matrix is equivalent to a direct sum of Jordan cell matrices, we conclude that the matrix coefficients of e^{tA} are linear combinations of functions of the form $t^k e^{\lambda t}$, where $k \in \mathbb{Z}_+$ and λ is an eigenvalue of A.

We can view e^{tA} as a "multivalued" holomorphic map z^A from $\mathbb{C}^*$ into $\mathrm{GL}(n, \mathbb{C})$. Its matrix coefficients are linear combinations of "multivalued" holomorphic functions $z^\lambda \log^k z$, where $k \in \mathbb{Z}_+$ and λ is an eigenvalue of A. This immediately implies that z^A has moderate growth at 0.

2. First order systems on a punctured disk. Let $A : D^* \longrightarrow M_n(\mathbb{C})$ be a holomorphic map. We consider the system of first order differential equations

$$\frac{dU}{dz} = AU \tag{1}$$

on D^*. Let Ω be a simply connected open neighborhood of z_0 in D^*, and let $\tilde{\Omega}$ be the simply connected neighbourhood of t_0 which evenly covers Ω. Then any local solution Y of (1) in Ω lifts to the holomorphis function $\tilde{Y} = Y \circ p$ in $\tilde{\Omega}$. Since Y can be analytically continued along any path in D^*, the monodromy theorem implies that $\tilde{Y}$ extends to a holomorphic function in $\tilde{D}^*$. In particular, this implies that the lifting $\tilde{S}$ of the fundamental matrix S of (1) in Ω extends to a holomorphic function on $\tilde{D}^*$. We denote it by the same letter. Therefore, the restricitions of function $t \mapsto (\tilde{S} \circ T_1)(t) = \tilde{S}(t + 2\pi i)$ to $\tilde{\Omega}$ is a lifting of a holomorphic function in Ω which satisfies the same differential equation as S. Hence, $t \mapsto \tilde{S}(t + 2\pi i)\tilde{S}(t_0 + 2\pi i)^{-1}$ is the lifting of a function satisfying the same differential equation as S and also has the value $\tilde{S}(t_0 + 2\pi i)\tilde{S}(t_0 + 2\pi i)^{-1} = I$ at t_0. Therefore, it is the lifting to $\tilde{\Omega}$ of S on Ω. This implies that

$$\tilde{S}(t) = \tilde{S}(t + 2\pi i)\tilde{S}(t_0 + 2\pi i)^{-1}$$

for any $t \in \tilde{D}^*$. Therefore,

$$\tilde{S}(t + 2\pi i) = \tilde{S}(t)\tilde{S}(t_0 + 2\pi i)$$

for any $t \in \tilde{D}^*$. Let $R \in M_n(\mathbb{C})$ be such that

$$M = \tilde{S}(t_0 + 2\pi i) = e^{2\pi i R}.$$

The matrix M is called the *monodromy* of (1). Then, consider the function $t \mapsto \tilde{S}(t)e^{-tR}$. Then

$$\tilde{S}(t + 2\pi i)e^{-(t+2\pi i)R} = \tilde{S}(t)\tilde{S}(t_0 + 2\pi i)e^{-2\pi i R}e^{-tR} = \tilde{S}(t)e^{-tR}$$

for all $t \in \tilde{D}^*$. Therefore, this function is invariant under deck transformations. It follows that there exists a holomorphic map $P : D^* \longrightarrow M_n(\mathbb{C})$ such that

$$\tilde{S}(t)e^{-tR} = P(e^t)$$

for all $t \in \tilde{D}^*$. Since the fundamental matrix is always a regular matrix, P is actually taking values in $\mathrm{GL}(n, \mathbb{C})$. Hence,

$$\tilde{S}(t) = P(e^t)e^{tR}$$

for all $t \in \tilde{D}^*$. Formally we write that the "multivalued" function S on D^* is given as

$$S(z) = P(z)z^R.$$

Therefore we proved the following result.

2.1. Proposition. *Let M be the monodromy of the system (1). Then for any $R \in M_n(\mathbb{C})$ such that $M = e^{2\pi i R}$, there exists a holomorphic map $P : D^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ such that*

$$S(z) = P(z)z^R.$$

This result has the following consequence.

2.2. Corollary. *There exists a "multivalued" solution of the system (1) of the form $z^\alpha F(z)$ where $F : D^* \longrightarrow \mathbb{C}^n$ is a holomorphic map and $e^{2\pi i\alpha}$ is an eigenvalue of the monodromy matrix M.*

Proof. Let $M = e^{2\pi i R}$ for some $R \in M_n(\mathbb{C})$. Let v be an eigenvector of R and denote by α its eigenvalue. Then $z^R v = z^\alpha v$, hence

$$S(z)v = P(z)z^R v = z^\alpha P(z)v = z^\alpha F(z). \quad \square$$

Now we study an example which will play a critical role later. Let $R \in M_n(\mathbb{C})$. Consider

$$\frac{dV}{dz} = \frac{R}{z}V. \tag{2}$$

on $\mathbb{C}^*$.

2.3. Lemma.

(i) *The fundamental matrix of (2) is given by*

$$S(z) = C_0 z^R$$

where C_0 is a constant regular matrix.

(ii) *The monodromy of (2) is given by*

$$M = e^{2\pi i R}.$$

Proof. (i) Clearly,

$$\frac{dC_0 z^R}{dz} = C_0 \frac{dz^R}{dz} = C_0 \frac{R}{z} z^R.$$

If we put $C_0 = z_0^{-R} = e^{-t_0 R}$, C_0 commutes with R. Hence, we have

$$\frac{dS}{dz} = \frac{R}{z} C_0 z^R = \frac{R}{z} S$$

and

$$S(z_0) = C_0 z_0^R = I.$$

(ii) We have

$$S(z_0 e^{2\pi i}) = C_0 z_0^R e^{2\pi i R} = e^{2\pi i R},$$

which implies that $M = e^{2\pi i R}$ is the monodromy of (2). □

Let R' be another matrix such that $M = e^{2\pi i R'}$. Then 1. implies that the fundamental matrix of (2) can be written as $P(z) z^{R'}$. This implies that there exists a holomorphic function $Q : \mathbb{C}^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ such that

$$z^R = Q(z) z^{R'}$$

on $\mathbb{C}^*$. Since the matrix coefficients of z^R and $z^{R'}$ are functions of moderate growth at 0 we conclude that Q is of moderate growth at 0, i. e. it has at most a pole at 0. This implies that

$$z^{-R} = Q\left(\frac{1}{z}\right) z^{-R'}$$

and again $z \mapsto Q(\frac{1}{z})$ is of moderate growth at 0. Therefore, it has at most a pole at 0. It follows that the matrix coefficients of Q are rational functions with possibe poles at 0, i. e. they are linear combinations of powers of z.

If we differentiate the equality

$$Q(z) = z^R z^{-R'},$$

we get

$$\frac{dQ}{dz} = \frac{R}{z} Q - Q \frac{R'}{z}.$$

Hence, we have the following result.

2.4. Lemma. *Let $R, R' \in M_n(\mathbb{C})$ be such that $e^{2\pi i R} = e^{2\pi i R'}$. Then there exists a map $Q : \mathbb{C}^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ with the following properties:*

(i) *the matrix coefficients of Q are linear combinations of powers of z;*

(ii)

$$\frac{dQ}{dz} = \frac{R}{z} Q - Q \frac{R'}{z}$$

on $\mathbb{C}^*$.

3. Systems with regular singularities. We consider the system of differential equations (1) on D^*. We say that this system is *equivalent* to the system

$$\frac{dV}{dz} = BV, \tag{3}$$

where $B : D^* \longrightarrow M_n(\mathbb{C})$ is holomorphic, if there is a holomorphic map $\Phi : D^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ with at most a pole at 0 satisfying the differential equation

$$\frac{d\Phi}{dz} = B\Phi - \Phi A$$

on D^*.

We claim that this relation is an equivalence relation. First we remark that the formula for inverse of a matrix implies that $\Phi^{-1} : z \mapsto \Phi(z)^{-1}$ is a holomorphic map from D^* into $\mathrm{GL}(n, \mathbb{C})$ and that it has at most a pole at 0. Also, by differentiating the relation $\Phi(z)\Phi(z)^{-1} = I$ we get that

$$\frac{d\Phi}{dz}\Phi^{-1} = -\Phi\frac{d\Phi^{-1}}{dz}$$

which implies that

$$\Phi\frac{d\Phi^{-1}}{dz} = -\frac{d\Phi}{dz}\Phi^{-1} = -B + \Phi A \Phi^{-1} = \Phi(A\Phi^{-1} - \Phi^{-1}B),$$

and

$$\frac{d\Phi^{-1}}{dz} = A\Phi^{-1} - \Phi^{-1}B$$

on D^*. This implies that our relation is symmetric.

Assume that $C : D^* \longrightarrow M_n(\mathbb{C})$ is a holomorphic map and consider the system

$$\frac{dW}{dz} = CW. \tag{4}$$

Assume that it is equivalent to the second system, i. e. that there exists a a holomorphic map $\Psi : D^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ with at most a pole at 0 satisfying the differential equation

$$\frac{d\Psi}{dz} = C\Psi - \Psi B$$

on D^*. Then the map $\Psi\Phi : D^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ has at most a pole at 0 and

$$\frac{d\Psi\Phi}{dz} = \frac{d\Psi}{dz}\Phi + \Psi\frac{d\Phi}{dz} = (C\Psi - \Psi B)\Phi + \Psi(B\Phi - \Phi A) = C\Psi\Phi - \Psi\Phi A.$$

Therefore, our relation is also transitive.

To se the actual meaning of this equivalence relation, assume that Y is a solution of the first system on an open subset Ω of D^*, i. e.

$$\frac{dY}{dz} = AY$$

on U. Then

$$\frac{d\Phi Y}{dz} = \frac{d\Phi}{dz} Y + \Phi \frac{dY}{dz} = (B\Phi - \Phi A)Y + \Phi AY = B\Phi Y,$$

i. e. ΦY is a solution of the second system on Ω. Therefore, the systems are equivalent if there exists a holomorphic map $\Phi : D^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ with at most pole at 0 which maps solutions of one system into the solutions of the other system.

Now we can reformulate the result of 2.3. and 2.4.

3.1. Lemma. *Let $R, R' \in M_n(\mathbb{C})$ be such that $e^{2\pi i R} = e^{2\pi i R'}$. Then the systems*

$$\frac{dU}{dz} = \frac{R}{z} U$$

and

$$\frac{dV}{dz} = \frac{R'}{z} V$$

on D^* are equivalent, and their monodromy is

$$M = e^{2\pi i R} = e^{2\pi i R'}.$$

Consider now two equivalent systems

$$\frac{dU}{dz} = AU$$

and

$$\frac{dV}{dz} = BV$$

on D^*. Assume that $\Phi : D^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ gives the equivalence. If S_A is the fundamental matrix of the first system,

$$S_B(z) = \Phi(z) S_A(z) \Phi(z_0)^{-1}$$

is the fundamental matrix of the second system. Really,

$$S_B(z_0) = \Phi(z_0) S_A(z_0) \Phi(z_0)^{-1} = \Phi(z_0)\Phi(z_0)^{-1} = I$$

and

$$\begin{aligned} \frac{dS_B(z)}{dz} &= \frac{d\Phi(z)}{dz} S_A(z)\Phi(z_0)^{-1} + \Phi(z) \frac{dS_A(z)}{dz} \Phi(z_0)^{-1} \\ &= (B(z)\Phi(z) - \Phi(z)A(z)) S_A(z)\Phi(z_0)^{-1} + \Phi(z)A(z)S_A(z)\Phi(z_0)^{-1} \\ &= B(z)\Phi(z)S_A(z)\Phi(z_0)^{-1} = B(z)S_B(z), \end{aligned}$$

what proves our assertion. This implies that the monodromy M_B of the second system is equal to

$$M_B = S_B(z_0 e^{2\pi i}) = \Phi(z_0) S_A(z_0 e^{2\pi i}) \Phi(z_0)^{-1} = \Phi(z_0) M_A \Phi(z_0)^{-1}$$

where M_A is the monodromy of the first system. Therefore, we proved the following result.

3.2. Proposition. *Equivalent systems on D^* have equivalent monodromies.*

Therefore, there is a well-defined map, given by the monodromy map, from the equivalence classes of first order systems of rank n on D^* into conjugacy classes in $\mathrm{GL}(n, \mathbb{C})$.

We say that a system

$$\frac{dU}{dz} = AU,$$

where $A : D^* \longrightarrow M_n(\mathbb{C})$ is a holomorphic map, has a *regular singularity* at 0 if all its "multivalued" solutions have moderate growth at 0. For example, by 2.3. the system

$$\frac{dV}{dz} = \frac{R}{z}V$$

has a regular singularity at 0.

3.3. Lemma. *Let*

$$\frac{dU}{dz} = AU$$

be a system on D^* with regular singularity at 0. Then any system equivalent to it also has a regular singularity at 0.

Proof. Let

$$\frac{dV}{dz} = BV$$

be a system equivalent to the first one. Then there exists a function $\Phi : D^* \longrightarrow \mathrm{GL}(n, \mathbb{C})$ with at most a pole at 0 such that all solutions of the second system have the form ΦU, for a solution U of the first system. Since Φ has moderate growth at 0, this implies that all solutions of the second systems have moderate growth at 0. □

Therefore, having regular singularity at 0 is a property which depends on the equivalence class only.

3.4. Theorem. *Let*

$$\frac{dU}{dz} = AU$$

be a system on D^* with a regular singularity at 0. Let M be its monodromy and $R \in M_n(\mathbb{C})$ such that $M = e^{2\pi i R}$. Then this system is equivalent to the system

$$\frac{dV}{dz} = \frac{R}{z}V$$

Proof. Let S be the fundamental matrix of this system. By 2.1. it has the form $S(z) = P(z)z^R$. Since our system has regular singularity at 0, its fundamental matrix has moderate growth at 0. Hence, $P(z) = S(z)z^{-R}$ has at most a pole at 0. Also

$$\frac{dP(z)}{dz} = \frac{dS(z)}{dz}z^{-R} + S(z)\frac{dz^{-R}}{dz} = A(z)S(z)z^{-R} - S(z)\frac{R}{z}z^{-R} = A(z)P(z) - P(z)\frac{R}{z}$$

and our systems are equivalent. □

An immediate consequence is the following fundamental result.

3.5. THEOREM. *The monodromy map defines a bijection between equivalence classes of systems of rank n on D^* with regular singularity at* 0 *and the conjugacy classes in* $\mathrm{GL}(n,\mathbb{C})$.

PROOF. Let $M \in \mathrm{GL}(n,\mathbb{C})$ and $R \in M_n(\mathbb{C})$ such that $e^{2\pi iR} = M$. By a previous remark the system

$$\frac{dV}{dz} = \frac{R}{z}V$$

has a regular singularity at 0. By 2.3. its monodromy is equal to M. Therefore, the map is surjective.

By the preceding theorem and 2.3, every system of rank n on D^* with a regular singularity at 0 is equivalent to a system of this form with the same monodromy. Therefore it is enough to show that the systems

$$\frac{dV}{dz} = \frac{R}{z}V$$

and

$$\frac{dW}{dz} = \frac{R'}{z}W,$$

such that their monodromies $M = e^{2\pi iR}$ and $M' = e^{2\pi iR'}$ belong to the same conjugacy class in $\mathrm{GL}(n,\mathbb{C})$, are equivalent. Assume that $M' = TMT^{-1}$ with $T \in \mathrm{GL}(n,\mathbb{C})$. Then the second system is equivalent to the system

$$\frac{dU}{dz} = \frac{T^{-1}R'T}{z}U$$

with monodromy $e^{2\pi T^{-1}R'T} = T^{-1}e^{2\pi iR'}T = T^{-1}M'T = M$. By 3.1. it follows that this system is equivalent to the first one. $\square$

Finally, we want to prove the following useful criterion for a system to have a regular singularity at 0.

3.6. THEOREM. *Let*

$$z\frac{dU}{dz} = AU$$

be a system on D^* with a holomorphic map $A : D \longrightarrow M_n(\mathbb{C})$. Then this system has a regular singularity at 0.

PROOF. By shrinking D a bit we can assume that $\|A(z)\|$ is bounded on D.

Let U be a solution of this system in a sector defined by $C = \{re^{i\theta} \mid 0 < r < R,\ \theta_0 \leq \theta \leq \theta_1\}$ for some $\theta_0, \theta_1 \in \mathbb{R}$ such that $\theta_1 - \theta_0 < 2\pi$. Then $\tilde{C} = \{t \in \mathbb{C} \mid \mathrm{Re}\, t < \log R, \theta_0 \leq \mathrm{Im}\, t \leq \theta_1\} \subset \tilde{D}^*$ evenly covers C. Therefore we can pull U to a holomorphic function $U \circ p$ with walues in $\mathbb{C}^n$. Let U_j be the j^{th} component of U. Then, if we put $s = \mathrm{Re}\, t$, we have

$$\left|\frac{\partial(U_j \circ p)}{\partial s}\right| = \left|\frac{d(U_j \circ p)}{dt}\right| = \left|\frac{dU_j}{dz}e^t\right| = \left|z\frac{dU_j}{dz}\right| = \left|\sum_{k=1}^{n} A_{jk}(z)U_k(z)\right| \leq M\|U(z)\|.$$

Therefore,

$$\left|\frac{\partial |U_j\circ p|^2}{\partial s}\right| = \left|\frac{\partial (U_j\circ p)}{\partial s}\overline{(U_j\circ p)} + (U_j\circ p)\frac{\partial \overline{(U_j\circ p)}}{\partial s}\right|$$
$$= 2\left|\frac{\partial (U_j\circ p)}{\partial s}\right|\cdot |U_j\circ p| \le M\|U(z)\|^2 \le M\left(\sum_{k=1}^{n}|U_j\circ p|^2\right).$$

If we put

$$F=\sum_{k=1}^{n}|U_j\circ p|^2$$

we get

$$\left|\frac{\partial F}{\partial s}\right| \le nMF$$

and

$$\left|\frac{\partial \log F}{\partial s}\right| \le nM.$$

This implies that

$$-nM \le \frac{\partial \log F}{\partial s} \le nM,$$

and by integration from s_0 to s_1, $s_0\le s_1$, we get

$$-nM(s_1-s_0)\le \log F(s_1+i\theta)-\log F(s_0+i\theta)\le nM(s_1-s_0),$$

i. e.

$$|\log F(s_1+i\theta)-\log F(s_0+i\theta)|\le nM|s_1-s_0|$$

for all $s_0+i\theta, s_1+i\theta\in\tilde{C}$. Hence, if we fix s_1 we get

$$|\log F(s_0+i\theta)|\le nM|s_0|+M',$$

uniformly in $\theta_0\le\theta\le\theta_1$, for sufficiently large $M'>0$. This implies that

$$\log F(t)\le -nM\operatorname{Re}t+M'$$

for $t\in\tilde{C}$ with $\operatorname{Re}t\le 0$. For some sufficiently large $c>0$, we finally have

$$0\le F(t)\le c\left|e^{-nMt}\right|$$

for all $t\in\tilde{C}$ with $\operatorname{Re}t\le 0$. Hence, near 0 in C we have

$$\|U(z)\|\le d\frac{1}{|z|^k}$$

for some sufficiently large $d>0$ and $k\in\mathbb{Z}_+$. This implies that U is of moderate growth at 0. $\square$

4. Fuchs' theorem. Now we want the following remarkable theorem due to Fuchs.

4.1. THEOREM. *Let*

$$P = a_0 \frac{d^n}{dz^n} + a_1 \frac{d^{n-1}}{dz^{n-1}} + \ldots + a_{n-1} \frac{d}{dz} + a_n$$

be a differential operator with holomorphic coefficients on D. Assume that a_0 has no zeros in D except maybe at 0. *Then the following statements are equivalent:*

(i) *all "multivalued" solutions of the differential equation $Py = 0$ on D^* have moderate growth at* 0*;*

(ii) *the functions $\frac{a_k}{a_0}$ have at most a pole of order k at* 0 *for $k = 1, 2, \ldots, n$.*

We start the proof with the following remark.

4.2. LEMMA. *Let $D = z\frac{d}{dz}$. Then*

(i)

$$D^n = z^n \frac{d^n}{dz^n} + \sum_{i=1}^{n} c_i z^{n-i} \frac{d^{n-i}}{dz^{n-i}}$$

with $c_i \in \mathbb{Z}$;

(ii)

$$z^n \frac{d^n}{dz^n} = D^n + \sum_{j=1}^{n} d_j D^{n-j}$$

with $d_j \in \mathbb{Z}$.

PROOF. (i) Clearly, the assertion is true for $n = 1$. Also, $D(z^k) = kz^k$ for any $k \in \mathbb{Z}_+$. Therefore,

$$D\left(z^k \frac{d^k}{dz^k}\right) = z^{k+1} \frac{d^{k+1}}{dz^{k+1}} + kz^k \frac{d^k}{dz^k}$$

for any $k \in \mathbb{Z}_+$. Hence, if we assume that the assertion holds for $n - 1$, we get

$$D^n = DD^{n-1} = D\left(z^{n-1} \frac{d^{n-1}}{dz^{n-1}} + \sum_{i=1}^{n-1} c_i \frac{d^{n-1-i}}{dz^{n-1-i}}\right)$$
$$= D\left(z^{n-1} \frac{d^{n-1}}{dz^{n-1}}\right) + \sum_{i=1}^{n-1} c_i D\left(\frac{d^{n-1-i}}{dz^{n-1-i}}\right),$$

and the relation follows from the previous formula.

(ii) follows immediately from (i). □

Therefore, by dividing the differential equation $Py = 0$ with a_0 and multiplying by z^n, we get the differential equation

$$z^n \frac{d^n y}{dz^n} + \left(z \frac{a_1}{a_0}\right) z^{n-1} \frac{d^{n-1} y}{dz^{n-1}} + \ldots + \left(z^{n-1} \frac{a_{n-1}}{a_0}\right) z \frac{dy}{dz} + \left(z^n \frac{a_n}{a_0}\right) y = 0.$$

The condition (ii) in 1. is equivalent with the condition that all coefficients $z^k \frac{a_k}{a_0}$, $k = 1, 2, \dots, n$, have removable singularities at 0.

Therefore, 2. implies that if the condition (ii) holds the equation $Py = 0$ can be written as

$$D^n y + b_1 D^{n-1} y + \dots + b_{n-1} D y + b_n y = 0$$

where b_k, $k = 1, 2, \dots, n$, are holomorphic on D. Applying 2. in the opposite direction, we see that if the equaton can be written in this for with holomorphic b_k, $k = 1, 2, \dots, n$, P satisfies the condition (ii).

Define

$$Y_1 = y,\ Y_2 = Dy,\ \dots\ ,\ Y_n = D^{n-1} y,$$

and Y as the column vector with components $Y_1, Y_2, \dots, Y_n$. Then

$$DY_1 = Y_2,\ DY_2 = Y_3,\ \dots\ ,\ DY_{n-1} = Y_n,\ DY_n = -b_1 Y_n - b_2 Y_{n-1} - \dots - b_{n-1} Y_2 - b_n Y_1,$$

i. e.

$$z \frac{dY}{dz} = BY$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -b_n & -b_{n-1} & -b_{n-2} & \dots & -b_2 & -b_1 \end{pmatrix}.$$

By 3.6. this system on D^* has a regular singularity at 0. Hence, its solutions have moderate growth at 0. This proves that (ii)⇒(i) in 1.

Now we want to prove the converse. The proof is by induction in degree of P. Assume that all solutions of $Py = 0$ have moderate growth at 0. By 2.2. there exists a "multivalued" solution $u(z) = z^\alpha f(z)$ where $\alpha \in \mathbb{C}$ and f is holomorphic on D^*. Since y has moderate growth at 0, f has at most a pole at 0 and by changing α we can actually assume that f is holomorphic on D and $f(0) \neq 0$. Also, by shrinking D if necessary we can assume in addition that f has no zeros in D.

Assume first that the degree of P is 1. In this case, $P = D + b_1$. Therefore,

$$0 = P(u) = D(z^\alpha f) + b_1 z^\alpha f = \alpha z^\alpha D(f) + b_1 z^\alpha f = z^\alpha (\alpha D(f) + b_1 f).$$

Therefore,

$$b_1 = \alpha \frac{D(f)}{f}$$

and it is holomorphic in D. This proves the assertion in this case.

Consider the differential equation $P(uv) = 0$ with $\deg P > 1$. Clearly,

$$D(uv) = D(u)v + uD(v),$$

hence by induction

$$D^k(uv) = \sum_{j=0}^{k} \binom{k}{j} D^{k-j}u\, D^j v$$

for $k \in \mathbb{Z}_+$. This implies that, if we put $b_0 = 1$, we have

$$P(uv) = D^n(uv) + b_1 D^{n-1}(uv) + \ldots + b_{n-1}D(uv) + b_n(uv) = \sum_{k=0}^{n} b_{n-k}D^k(uv)$$

$$= \sum_{k=1}^{n} b_{n-k} \sum_{j=0}^{k} \binom{k}{j} D^{k-j}u\, D^j v + b_n uv = P(u)v + \sum_{k=1}^{n}\sum_{j=1}^{k} \binom{k}{j} b_{n-k} D^{k-j}u\, D^{j-1}(Dv)$$

$$= \sum_{k=0}^{n-1}\sum_{j=0}^{k} \binom{k+1}{j+1} b_{n-k-1} D^{k-j}u\, D^j(Dv)$$

$$= \sum_{j=0}^{n-1} \left(\sum_{k=j}^{n-1} \binom{k+1}{j+1} b_{n-k-1} D^{k-j}u \right) D^j(Dv)$$

$$= \sum_{j=0}^{n-1} \left(\sum_{p=0}^{n-j-1} \binom{p+j+1}{j+1} b_{n-j-1-p} D^p u \right) D^j(Dv)$$

after relabeling the indices. Since

$$D(u) = D(z^\alpha f) = D(z^\alpha)f + z^\alpha D(f) = \alpha z^\alpha f + z^\alpha D(f) = z^\alpha(\alpha f + D(f))$$

by induction we see that for any $j \in \mathbb{Z}_+$ we have

$$D^j(u) = z^\alpha h_j$$

where h_j is holomorphic in D and $h_0 = f$. Therefore,

$$P(uv) = z^\alpha \left(\sum_{j=0}^{n-1} \left(\sum_{p=0}^{n-j-1} \binom{p+j+1}{p} b_{n-j-1-p} h_p \right) D^j(Dv) \right)$$

and $P(uv) = 0$ is equivalent to

$$\sum_{j=0}^{n-1} d_{n-1-j} D^j(Dv) = 0$$

with $d_0 = 1$ and

$$d_k = \sum_{p=0}^{k} \binom{p+n-k}{p} b_{k-p} h_p$$

$$= b_k h_0 + \sum_{p=1}^{k} \binom{p+n-k}{p} b_{k-p} h_p = b_k f + \sum_{p=1}^{k} \binom{p+n-k}{p} b_{k-p} h_p$$

for $k = 1, 2, \dots, n-1$. Therefore, all solutions v of $P(uv) = 0$ have the form $z^{-\alpha}\frac{1}{f}y$ where y is a solution of $P(y) = 0$. By our assumption, all solutions of $P(y) = 0$ have moderate growth at 0. Therefore, all solutions v of $P(uv) = 0$ have moderate growth at 0. By 1.2. all functions Dv have also moderate growth at 0. Let w be a "multivalued" solution of the equation

$$\sum_{j=0}^{n-1} d_j D^j w = 0,$$

then there exists a "multivalued" holomorphic function v such that $Dv = w$. Hence, w must have moderate growth at 0. By the induction assumption it follows that the coefficients d_k are holomorphic in D. By induction in k, this implies that all b_k, $k = 1, 2, \dots, n$, are holomorphic in D^*. This completes the proof of the implication (i)⇒(ii).

5. Formal solutions. Let $\mathbb{C}[[z]]$ be the ring of formal series, i. e. the ring consisting of series

$$\sum_{p=0}^{\infty} a_p z^p$$

where $a_p \in \mathbb{C}$ and $a_p = 0$ for p sufficiently negative. Clearly, the addition

$$\sum_{p=0}^{\infty} a_p z^p + \sum_{p=0}^{\infty} b_p z^p = \sum_{p=0}^{\infty} (a_p + b_p) z^p$$

and multiplication by a complex number

$$\lambda \left(\sum_{p=0}^{\infty} a_p z^p \right) = \sum_{p=0}^{\infty} \lambda a_p z^p$$

and the multiplication

$$\left(\sum_{p=0}^{\infty} a_p z^p \right) \left(\sum_{q=0}^{\infty} b_q z^q \right) = \sum_{s=0}^{\infty} \left(\sum_{k=0}^{s} a_k b_{s-k} \right) z^s$$

are well-defined operations in $\mathbb{C}[[z]]$.

Let A be the complex vector space with the basis $\{z^\alpha \mid \alpha \in \mathbb{C}\}$. Then we can define a multiplication $A \times A \longrightarrow A$ via

$$z^\alpha \, z^\beta = z^{\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{C}$. One can check that this defines a commutative ring structure on A.

Let B be the complex vector space with the basis $\{\log^k z \mid k \in \mathbb{Z}_+\}$. Then we can define a multiplication $B \times B \longrightarrow B$ via

$$\log^k z \, \log^l z = \log^{k+l} z$$

for any $k, l \in \mathbb{Z}$. One can check that this defines a commutative ring structure on B.

Now, $A \otimes_{\mathbb{C}} B \otimes_{\mathbb{C}} \mathbb{C}[[z]]$ is a commutative ring. Let I be its ideal generated by elements of the form $z \otimes 1 \otimes 1 - 1 \otimes 1 \otimes z$. The ring

$$L = (A \otimes_{\mathbb{C}} B \otimes_{\mathbb{C}} \mathbb{C}[[z]])/I$$

is called the ring of *formal logarithmic series*. Elements of L are finite sums of the type

$$\Phi = \sum_{\alpha,k} z^\alpha \ \log^k z \ \Phi_{\alpha,k}$$

where $\Phi_{\alpha,k}$ are formal power series. We say that this expression is *reduced* if $\Phi_{\alpha,k} \neq 0$ and $\Phi_{\beta,l} \neq 0$ implies that $\alpha - \beta \notin \mathbb{Z}$. Clearly, every Φ can be represented by a reduced expression.

5.1. Lemma. *Let $\Phi \in L$. If*

$$\Phi = \sum_{\alpha,k} z^\alpha \ \log^k z \, \Phi_{\alpha,k}$$

is a reduced expression, the following assertions are equivalent:

(i) $\Phi = 0$*;*
(ii) $\Phi_{\alpha,k} = 0$ *for all* $\alpha \in \mathbb{C}$ *and* $k \in \mathbb{Z}_+$*.*

Proof. Clearly, (ii) implies (i).

To prove the converse, first define an automorphism ψ of A by

$$\psi(z^\alpha) = e^{2\pi i \alpha} z^\alpha$$

for $\alpha \in \mathbb{C}$. This automorphism defines an automorphism of the ring $A \otimes_{\mathbb{C}} B \otimes_{\mathbb{C}} \mathbb{C}[[z]]$ which acts as identity on the second and third factor. This automorphisms leaves $z \otimes 1 \otimes 1 - 1 \otimes 1 \otimes z$ fixed, hence it leaves I invariant. It follows that it defines an automorphism Ψ of L which satisfies

$$\Psi(z^\alpha \log^k z \Phi) = e^{2\pi i \alpha} z^\alpha \ \log^k z \, \Phi$$

for any $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}_+$ and formal series Φ.

Therefore

$$0 = \Phi = \sum_{\alpha} z^\alpha \left(\sum_k \log^k z \, \Phi_{\alpha,k} \right),$$

and each term in the first sum is an eigenvector of Ψ for the eigenvalue $e^{2\pi i \alpha}$. Since all of these eigenvalues are mutually different by our assumption, we conclude that

$$\sum_k \log^k z \, \Phi_{\alpha,k} = 0$$

for all $\alpha \in \mathbb{C}$.

Now we can define an automorphism ω of B by

$$\omega(\log z) = c \log z$$

where $c \in \mathbb{R}_+^*$, and extend to an automorphism of the ring $A \otimes_{\mathbb{C}} B \otimes_{\mathbb{C}} \mathbb{C}[[z]]$ which acts as identity on the first and third factor. Again, this automorphism acts as identity on $z \otimes 1 \otimes 1 - 1 \otimes 1 \otimes z$, hence it leaves the ideal I invariant. Therefore, it induces an automorphism Ω of L given by

$$\Omega(z^\alpha \log^k z \; \Phi) = c^k z^\alpha \log^k z \; \Phi$$

for any $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}_+$ and formal series Φ.

Therefore, each term in the sum

$$\sum_k \log^k z \, \Phi_{\alpha,k} = 0$$

is an eigenvector of Ω with eigenvalue c^k. Since c is a positive number, all its powers are mutually different and $\Phi_{\alpha,k} = 0$. $\square$

We can define the action of $\frac{d}{dz}$ on L by

$$\frac{d}{dz}\left(z^\alpha \log^k z \sum_{p=0}^{\infty} a_p z^p\right) = (\alpha z^{\alpha-1} \log^k z + k z^{\alpha-1} \log^{k-1} z) \sum_{p=0}^{\infty} a_p z^p + z^\alpha \log^k z \sum_{p=0}^{\infty} p a_p z^{p-1}$$

for any $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}_+$.

5.2. Lemma. *Let Φ be a formal logarithmic series such that $\frac{d\Phi}{dz} = 0$. Then Φ is a constant.*

Proof. Let

$$\Phi = \sum_{\alpha,k} z^\alpha \;\log^k z \; \Phi_{\alpha,k}$$

be a reduced expression of Φ. In this case,

$$0 = \frac{d\Phi}{dz} = \sum_\alpha z^{\alpha-1} \sum_k \left((\alpha \log^k z + k \log^{k-1} z)\Phi_{\alpha,k} + z \;\log^k z \frac{d\Phi_{\alpha,k}}{dz}\right).$$

By 1, this immediately implies that for each α we have

$$0 = \sum_k \left((\alpha \log^k z + k \log^{k-1} z)\Phi_{\alpha,k} + z \;\log^k z \frac{d\Phi_{\alpha,k}}{dz}\right)$$
$$= \sum_k \log^k z \left(\alpha \Phi_{\alpha,k} + (k+1)\Phi_{\alpha,k+1} + z\frac{d\Phi_{\alpha,k}}{dz}\right).$$

For a fixed α, take the largest k with $\Phi_{\alpha,k} \neq 0$. Then $\Phi_{\alpha,k+1} = 0$, hence

$$\alpha \Phi_{\alpha,k} + z \frac{d\Phi_{\alpha,k}}{dz} = 0.$$

On the other hand, if $\Phi_{\alpha,k} = \sum_{p=0}^{\infty} a_p z^p$ we have

$$\alpha \Phi_{\alpha,k} + z \frac{d\Phi_{\alpha,k}}{dz} = 0,$$

and

$$0 = \alpha \sum_{p=0}^{\infty} a_p z^p + \sum_{p=1}^{\infty} p a_p z^p = \sum_{p=0}^{\infty} (\alpha + p) a_p z^p.$$

Hence $a_p \neq 0$ implies $\alpha + p = 0$. Hence, if $\alpha \notin -\mathbb{Z}$, we have $\Phi_{\alpha,k} = 0$. Therefore, $\Phi_{\alpha,k} \neq 0$ implies that $\alpha = -s \in -\mathbb{Z}$ and $\Phi_{-s,k} = a z^s$ for some $a \in \mathbb{C}$. Now,

$$-s \Phi_{-s,k-1} + k a z^s + z \frac{d\Phi_{-s,k-1}}{dz} = 0.$$

Therefore, if $\Phi_{-s,k-1} = \sum_{p=0}^{\infty} b_p z^p$, we get

$$0 = -s \sum_{p=0}^{\infty} b_p z^p + k a z^s + \sum_{p=0}^{\infty} p b_p z^p = \sum_{p=0}^{\infty} (p - s) b_p z^p + k a z^s.$$

This implies that $(p - s) b_p = 0$ for $p \neq s$, i. e. $b_p = 0$ in this case. Also, $ka = 0$. Therefore, $k = 0$. It follows finally that $\Phi_{\alpha,k} \neq 0$ implies that $\alpha = -s \in -\mathbb{Z}$, $k = 0$ and $\Phi_{-s,0} = a z^s$ for some $a \in \mathbb{C}$. Hence, $\Phi = z^{-s} \Phi_{-s,0} = a$. □

We say that a formal logarithmic series Φ is *convergent* if there exists a reduced expression

$$\Phi = \sum_{\alpha,k} z^\alpha \log^k z \; \Phi_{\alpha,k}$$

such that the formal power series $\Phi_{\alpha,k}$ converge in some disk D around 0. Clearly, if one reduced expression of Φ has this property all other reduced expresions have it too.

The next result claims that in the case of a regular singularity formal solutions of a first order system are automatically convergent.

5.3. Theorem. *Let*

$$\frac{dU}{dz} = AU$$

be a first order system on D^* with a regular singularity at 0. *Let*

$$F(z) = \sum_{\alpha,k} z^\alpha \log^k z \; F_{\alpha,k}$$

be a reduced expression of a formal logarithmic series which is a formal solution of this system. Then formal power series $F_{\alpha,k}$ converge in D.

PROOF. Let $S(z) = P(z)z^R$ be the fundamental matrix of our system. Then its inverse is given by

$$S(z)^{-1} = z^{-R}P(z)^{-1},$$

hence its matrix coefficients are formal logarithmic series. This implies that the matrix coefficients of $S(z)^{-1}F(z)$ are formal logarithmic series. Also,

$$\frac{d(S(z)^{-1}F(z))}{dz} = \frac{dS(z)^{-1}}{dz}F(z) + S(z)^{-1}\frac{dF(z)}{dz} = \frac{dS(z)^{-1}}{dz}F(z) + S(z)^{-1}A(z)F(z).$$

Moreover, by differentiation of $S(z)^{-1}S(z) = I$, we get

$$\frac{dS(z)^{-1}}{dz}S(z) = -S(z)^{-1}\frac{dS(z)}{dz} = -S(z)^{-1}A(z)S(z),$$

what leads to

$$\frac{dS(z)^{-1}}{dz} = -S(z)^{-1}A(z),$$

and finally to

$$\frac{d(S(z)^{-1}F(z))}{dz} = -S(z)^{-1}A(z)F(z) + S(z)^{-1}A(z)F(z) = 0.$$

By 1, we conclude that $S(z)^{-1}F(z) = C_0 \in \mathbb{C}^n$ and $F(z) = S(z)C_0$. Therefore, F is convergent. □

6. Frobenius method. In this section we shall discuss a method for solving differential equations near regular singular points due to Frobenius. We shall restrict ourselves to the treatment of a second order differential equation

$$P(y) = \frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y = 0$$

on D^*. By Fuchs' theorem, p has at most a pole of order 1 at 0, and q at most a pole of order 2 at 0. Let

$$zp(z) = \sum_{r=0}^{\infty} a_r z^r$$

and

$$z^2q(z) = \sum_{s=0}^{\infty} b_s z^s$$

be the corresponding Taylor series in D. We want to find a formal solution y of the equation of the form

$$y(z) = y(\lambda, z) = z^\lambda \sum_{t=0}^{\infty} c_t z^t = \sum_{t=0}^{\infty} c_t z^{t+\lambda}.$$

We have

$$\begin{aligned}
&z^2 y'' + zp(z)zy' + z^2 q(z) y \\
&= \sum_{t=0}^{\infty} (t+\lambda)(t+\lambda-1) c_t z^{t+\lambda} + \left(\sum_{r=0}^{\infty} a_r z^r\right)\left(\sum_{t=0}^{\infty} (t+\lambda) c_t z^{t+\lambda}\right) + \left(\sum_{s=0}^{\infty} b_s z^s\right)\left(\sum_{t=0}^{\infty} c_t z^{t+\lambda}\right) \\
&= \sum_{t=0}^{\infty} (t+\lambda)(t+\lambda-1) c_t z^{t+\lambda} + \sum_{t=0}^{\infty}\left(\sum_{k=0}^{t} (t-k+\lambda) a_k c_{t-k}\right) z^{t+\lambda} + \sum_{t=0}^{\infty}\left(\sum_{l=0}^{t} b_l c_{t-l}\right) z^{t+\lambda} \\
&\qquad = \sum_{t=0}^{\infty}\left((t+\lambda)(t+\lambda-1)c_t + \sum_{k=0}^{t}((t-k+\lambda)a_k + b_k)c_{t-k}\right) z^{t+\lambda}.
\end{aligned}$$

Denote by

$$f(\lambda) = \lambda(\lambda-1) + \lambda a_0 + b_0$$

the *indicial polynomial* of our equation at 0. Assume that c_t are rational functions in λ satisfying

$$\begin{aligned}
0 &= (t+\lambda)(t+\lambda-1)c_t + \sum_{k=0}^{t}((t-k+\lambda)a_k + b_k)c_{t-k} \\
&= \big((t+\lambda)(t+\lambda-1) + (t+\lambda)a_0 + b_0\big)c_t + \sum_{k=1}^{t}((t-k+\lambda)a_k + b_k)c_{t-k}
\end{aligned}$$

for $t \in \mathbb{N}$. Then

$$c_t = \frac{\sum_{k=1}^{t}((t-k+\lambda)a_k + b_k)c_{t-k}}{f(\lambda+t)}$$

for $t \in \mathbb{N}$, and all coefficients are uniquely determined by c_0 using these recursion relations. Also, in this case we get

$$P(y) = z^2 y'' + zp(z)zy' + z^2 q(z) y = f(\lambda) c_0 z^\lambda.$$

The equation

$$f(\lambda) = \lambda(\lambda-1) + \lambda a_0 + b_0 = 0$$

If r is a root of the indicial equation such that $r+\mathbb{N}$ doesn't contain any other root, $\lambda = r$ is a regular point of all c_t if it is a regular point of c_0. Therefore, if we put $c_0 = 1$, we see that $y(r, z)$ is a formal solution of our equation.

There are two possibilities for the roots of the indicial equation.

(A) The difference of the roots r and s of the indicial equation is not an integer. In this case we immediately see that by putting $c_0 = 1$ and

$$y_1(z) = y(r, z) = z^r f_1(z),\ \ y_2(z) = y(s, z) = z^s f_2(z)$$

we get two formal solutions of our differential equation with formal power series f_1 and f_2. By 5.3, we see that f_1 and f_2 converge in D and these solutions are actual solutions of our equation in D^*. Also, they are clearly linearly independent since $r - s \notin \mathbb{Z}$.

(B) The difference of the roots $r - s \in \mathbb{Z}_+$. In this case we can get one solution corresponding to the root r by putting $c_0 = 1$:

$$y_1(z) = y(r, z) = z^r f_1(z)$$

and as before we conclude that f_1 is a convergent power series in D. It remains to determine another, linearly independent solution. There are two slightly different cases:

(B1) Assume in addition that $r = s$. Then $f(\lambda) = (\lambda - r)^2$. Hence if we put $c_0 = 1$ and

$$P\left(\frac{\partial y}{\partial \lambda}\right) = \frac{\partial P(y)}{\partial \lambda} = f'(\lambda)z^\lambda + f(\lambda)z^\lambda \log z = (\lambda - r)(2z^\lambda + (\lambda - r)z^\lambda \log z).$$

Hence

$$y_2(z) = \left.\frac{dy(\lambda, z)}{d\lambda}\right|_{\lambda=r}$$

is also a formal solution of this equation. To find its form we remark that

$$\frac{\partial y}{\partial \lambda} = z^\lambda \log z \sum_{t=0}^{\infty} c_t z^t + z^\lambda \sum_{t=0}^{\infty} \frac{\partial c_t}{\partial \lambda} z^t,$$

hence

$$y_2(z) = \log z\, y_1(z) + z^r f_2(z)$$

where f_2 is a formal power series. As before, we conclude that it converges in D. Clearly, this solution is linearly independent from y_1.

(B2) Assume that $r \neq s$. Then $f(\lambda) = (\lambda - r)(\lambda - s)$ and $t_0 = r - s \in \mathbb{N}$. Therefore, $f(s+t_0) = 0$ and if we solve the recursion relations with $c_0 = 1$ we see that c_{t_0} can have a pole at s, and we cannot get a formal solution by evaluating $y(\lambda, z)$ at $\lambda = s$. To eliminate this problem we put $c_0 = \lambda - s$. In this case $c_0, c_1, c_2, \dots, c_{t_0-1}$ contain $\lambda - s$ as a factor. Since they all have a zero at s, c_{t_0} is regular at s and all c_t, $t > t_0$, are regular at s. By evaluating $y(\lambda, z)$ at s we would get a formal solution

$$Y(z) = z^s \sum_{t=0}^{\infty} c_t(s) z^t = z^s \sum_{t=t_0}^{\infty} c_t(s) z^t = z^r \sum_{t=0}^{\infty} c_{t+t_0}(s) z^t$$

since all coefficients $c_0, c_1, \dots, c_{t_0-1}$ would vanish. By 5.3, it is a converges. On the other hand,

$$P\left(\frac{\partial y}{\partial \lambda}\right) = \frac{\partial P(y)}{\partial \lambda} = f'(\lambda)c_0 z^\lambda + f(\lambda)c_0' z^\lambda + f(\lambda)c_0 z^\lambda \log z$$
$$= f'(\lambda)(\lambda - s)z^\lambda + f(\lambda)z^\lambda + f(\lambda)(\lambda - s)z^\lambda \log z,$$

hence

$$\frac{\partial y}{\partial \lambda} = z^\lambda \log z \sum_{t=0}^{\infty} c_t z^t + z^\lambda \sum_{t=0}^{\infty} \frac{\partial c_t}{\partial \lambda} z^t,$$

evaluated at $\lambda = s$ is also a formal solution. Since $\frac{\partial c_0}{\partial \lambda} = 1$ we see that this solution has the form

$$y_2(z) = \log z \, Y(z) + z^s f_2(z)$$

where f_2 is a convergent series in D with $f_2(0) = 1$, hence it is not proportional to y_1. Therefore, every solution is a linear combination of y_1 and y_2. In particular,

$$z^r \sum_{t=0}^{\infty} c_{t+t_0}(s) z^t = Y = c_1 y_1 + c_2 y_2 = c_1 z^r f_1(z) + c_2 z^s f_2(z) + c_2 \log z Y(z).$$

Since there are no terms involving $\log z$ on the left side this implies that $c_2 = 0$, and Y is proportional to y_1. Therefore,

$$y_2(z) = a \log z y_1(z) + z^s f_2(z)$$

for some $a \in \mathbb{C}$.

Remark. The eignvalues of the monodromy in the case (A) are $e^{2\pi i r}$ and $e^{2\pi i s}$ and correspond to eigenvectors y_1 and y_2. Therefore, in this case the monodromy is a semisimple matrix. In case (B) the monodromy has one eigenvalue $e^{2\pi i r} = e^{2\pi i s}$. In the case (B1) it is not semisimple, while in the case (B2) it is semisimple if and only if the constant a is zero.

6. Bessel equation. As an example, we consider now the *Bessel equation*

$$z^2 y'' + z y' + (z^2 - \rho^2) y = 0$$

where $\rho \in \mathbb{C}$. Clearly, this differential equation has only 0 as a singular point in $\mathbb{C}$, and this is a regular singular point. Therefore, we can apply the Frobenius method to find solutions in $\mathbb{C}^*$. Lrt

$$y = y(\lambda, z) = z^\lambda \sum_{p=0}^{\infty} c_p z^p$$

$\lambda \in \mathbb{C}$. Then

$$
\begin{aligned}
&z^2y'' + zy' + (z^2-\rho^2)y \\
&= \sum_{p=0}^{\infty}(p+\lambda)(p+\lambda-1)c_p z^{p+\lambda} + \sum_{p=0}^{\infty}(p+\lambda)c_p z^{p+\lambda} + \sum_{p=0}^{\infty} c_p z^{p+\lambda+2} - \rho^2 \sum_{p=0}^{\infty} c_p z^{p+\lambda} \\
&= \sum_{p=0}^{\infty}\big((p+\lambda)^2-\rho^2\big)c_p z^{p+\lambda} + \sum_{p=2}^{\infty} c_{q-2} z^{q+\lambda} \\
&= (\lambda^2-\rho^2)c_0 z^{\lambda} + \big((\lambda+1)^2-\rho^2\big)c_1 z^{\lambda+1} + \sum_{p=2}^{\infty}\Big(\big((p+\lambda)^2-\rho^2\big)c_p + c_{p-2}\Big) z^{p+\lambda}.
\end{aligned}
$$

Assume that $c_1 = 0$ and that

$$\big((p+\lambda)^2-\rho^2\big)c_p + c_{p-2} = 0$$

for all $p \geq 2$. Then we have

$$c_{2p+1} = 0$$

for $p \in \mathbb{Z}_+$ and

$$c_p = -\frac{c_{p-2}}{(p+\lambda)^2-\rho^2}$$

for $p \geq 2$, and

$$z^2y'' + zy' + (z^2-\rho^2)y = (\lambda^2-\rho^2)c_0 z^{\lambda}.$$

It remains to find even coefficients c_{2p}, $p \in \mathbb{Z}_+$. We have

$$c_{2p} = -\frac{c_{2(p-1)}}{(2p+\lambda)^2-\rho^2} = -\frac{c_{2(p-1)}}{(2p+\lambda-\rho)\,(2p+\lambda+\rho)} = -\frac{c_{2(p-1)}}{2^2\,(p+\frac{\lambda-\rho}{2})\,(p+\frac{\lambda+\rho}{2})}.$$

By induction we see that

$$c_{2p} = \frac{(-1)^p}{2^{2p}}\;\frac{\Gamma(\frac{\lambda-\rho}{2}+1)\,\Gamma(\frac{\lambda+\rho}{2}+1)}{\Gamma(\frac{\lambda-\rho}{2}+p+1)\,\Gamma(\frac{\lambda+\rho}{2}+p+1)}\;c_0$$

for $p \in \mathbb{Z}_+$.

Assume now that $\operatorname{Re}\rho \geq 0$. The indicial equation is $\lambda^2 = \rho^2$, so its roots are ρ and $-\rho$. This implies that one solution of the equation is

$$
\begin{aligned}
z^{\rho}\sum_{p=0}^{\infty} c_{2p}(\rho) z^{2p} &= z^{\rho}\sum_{p=0}^{\infty}(-1)^p \frac{\Gamma(\rho+1)}{\Gamma(p+1)\,\Gamma(\rho+p+1)}\;c_0\left(\frac{z}{2}\right)^{2p} \\
&= z^{\rho}\sum_{p=0}^{\infty}(-1)^p \frac{\Gamma(\rho+1)}{p!\,\Gamma(\rho+p+1)}\;c_0\left(\frac{z}{2}\right)^{2p}.
\end{aligned}
$$

If we put

$$c_0 = \frac{1}{2^\rho \Gamma(\rho+1)}$$

we get that one solution is given by

$$J_\rho(z) = \sum_{p=0}^{\infty} (-1)^p \frac{1}{p!\,\Gamma(\rho+p+1)} \left(\frac{z}{2}\right)^{\rho+2p}.$$

Since $\frac{1}{\Gamma}$ is an entire function, this defines a formal series for arbitrary $\rho \in \mathbb{C}$. This formal series is always a formal solution of the Bessel equation, hence by 5.3. it is convergent. The function J_ρ is called the ρ^{th} *Bessel function*. If $\rho \notin -\mathbb{N}$, $\frac{1}{\Gamma(\rho+1)} \neq 0$, hence the leading coefficients of J_ρ and $J_{-\rho}$ are nonzero. This implies that the solutions J_ρ and $J_{-\rho}$ of the Bessel differential equation are not proportional for for $\rho \notin -\mathbb{Z}_+$, i. e. the arbitrary solution of this equation has the form

$$y = C_1\, J_\rho + C_2\, J_{-\rho}.$$

The functions $\Gamma(\rho+p+1)$ have a first order pole for $p = 0, 1, \dots, n-1$, at $\rho = -n$, $n \in \mathbb{Z}$. Therefore, the corresponding coefficients are all zero. It follows that

$$\begin{aligned} J_{-n}(z) &= \sum_{p=n}^{\infty} (-1)^p \frac{1}{p!\,\Gamma(-n+p+1)} \left(\frac{z}{2}\right)^{-n+2p} \\ &= (-1)^n \sum_{q=0}^{\infty} (-1)^q \frac{1}{(q+n)!\,q!} \left(\frac{z}{2}\right)^{n+2q} = (-1)^n \sum_{q=0}^{\infty} (-1)^q \frac{1}{q!\,\Gamma(n+q+1)} \left(\frac{z}{2}\right)^{n+2p}, \end{aligned}$$

i. e.

$$J_{-n} = (-1)^n J_n$$

for $n \in \mathbb{Z}_+$. Therefore, we have to determine another linearly independent solution of Bessel equation for integral $\rho = n$.

Assume first that $\rho = 0$. Then, if we put $c_0 = 1$, we get

$$y = z^\lambda \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma(\frac{\lambda}{2}+1)^2}{\Gamma(\frac{\lambda}{2}+p+1)^2} \left(\frac{z}{2}\right)^{2p}.$$

Let

$$d_p = \frac{\Gamma(\frac{\lambda}{2}+1)^2}{\Gamma(\frac{\lambda}{2}+p+1)^2}.$$

Then $d_0 = 1$ and

$$\begin{aligned} \left.\frac{\partial d_p}{\partial \lambda}\right|_{\lambda=0} &= 2\frac{1}{p!}\,\frac{\partial}{\partial \lambda}\left(\frac{\Gamma(\frac{\lambda}{2}+1)}{\Gamma(\frac{\lambda}{2}+p+1)}\right)\Bigg|_{\lambda=0} \\ &= 2\frac{1}{p!}\,\frac{\partial}{\partial \lambda}\left(\frac{1}{(\frac{\lambda}{2}+1)(\frac{\lambda}{2}+2)\dots(\frac{\lambda}{2}+p)}\right)\Bigg|_{\lambda=0} = -\frac{1}{p!^2}\sum_{q=1}^{p}\frac{1}{q}, \end{aligned}$$

for $p \in \mathbb{N}$. Hence,

$$\left.\frac{\partial y}{\partial \lambda}\right|_{\lambda=0} = \log z \sum_{p=0}^{\infty} (-1)^p \frac{1}{\Gamma(p+1)^2} \left(\frac{z}{2}\right)^{2p} - \sum_{p=1}^{\infty} (-1)^p \frac{1}{p!^2} \left(\sum_{q=1}^{p} \frac{1}{q}\right) \left(\frac{z}{2}\right)^{2p}$$

$$= \log z J_0(z) + \sum_{p=1}^{\infty} (-1)^{p+1} \frac{1}{p!^2} \left(\sum_{q=1}^{p} \frac{1}{q}\right) \left(\frac{z}{2}\right)^{2p}.$$

This implies that a solution of the Bessel equation linearly independent from J_0 for $\rho = 0$ is given by

$$\log z \; J_0(z) + \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p!^2} \left(\sum_{q=1}^{p} \frac{1}{q}\right) \left(\frac{z}{2}\right)^{2p}.$$

It remains to treat the case $\rho = n \in \mathbb{N}$. As we remarked

$$y(\lambda, z) = z^{\lambda} \sum_{p=0}^{\infty} (-1)^p \frac{\Gamma(\frac{\lambda-n}{2}+1)\,\Gamma(\frac{\lambda+n}{2}+1)}{\Gamma(\frac{\lambda-n}{2}+p+1)\,\Gamma(\frac{\lambda+n}{2}+p+1)} \, c_0 \left(\frac{z}{2}\right)^{2p}.$$

Denote

$$d_p = \frac{\Gamma(\frac{\lambda-n}{2}+1)\,\Gamma(\frac{\lambda+n}{2}+1)}{\Gamma(\frac{\lambda-n}{2}+p+1)\,\Gamma(\frac{\lambda+n}{2}+p+1)} \, c_0$$

for $p \in \mathbb{Z}_+$. Then

$$d_p = \frac{c_0}{(\frac{\lambda-n}{2}+1)(\frac{\lambda-n}{2}+2)\dots(\frac{\lambda-n}{2}+p)(\frac{\lambda+n}{2}+1)(\frac{\lambda+n}{2}+2)\dots(\frac{\lambda+n}{2}+p)},$$

hence, if $p \geq n$ the first factor has a first order pole at $\lambda = -n$. If we put

$$c_0 = -2^{n-1} \, (n-1)! \, (\lambda + n),$$

we eliminate this pole. Also, we get $d_p(-n) = 0$ for $p < n$. On the other hand, for $p \geq n$ we get

$$d_p(-n) = -\frac{2^n (n-1)!}{p!} \frac{1}{(-n+1)(-n+2)\dots(-2)\cdot(-1)\cdot 1\cdot 2\dots(p-n)}$$

$$= 2^n (-1)^n \frac{1}{p!(p-n)!}.$$

Therefore,

$$y_2(z) = 2^n \log z \; z^{-n} \sum_{p=n}^{\infty} (-1)^{p+n} \frac{1}{p!(p-n)!} \left(\frac{z}{2}\right)^{2p} + z^{-n} \sum_{p=0}^{\infty} (-1)^p \frac{\partial d_p}{\partial \lambda}(-n) \left(\frac{z}{2}\right)^{2p}$$

$$= \log z \sum_{q=0}^{\infty} (-1)^q \frac{1}{(q+n)!q!} \left(\frac{z}{2}\right)^{2q+n} + 2^{-n} \sum_{p=0}^{\infty} (-1)^p \frac{\partial d_p}{\partial \lambda}(-n) \left(\frac{z}{2}\right)^{2p-n}$$

$$= \log z J_n(z) + 2^{-n} \sum_{p=0}^{\infty} (-1)^p \frac{\partial d_p}{\partial \lambda}(-n) \left(\frac{z}{2}\right)^{2p-n}.$$

Now, for $0 \leq p \leq n-1$,

$$\frac{\partial d_p}{\partial \lambda}(-n) = -\frac{2^{n-1}(n-1)!}{p!} \frac{1}{(-n+1)(-n+2)\dots(-n+p)} = (-1)^{p-1}\frac{2^{n-1}(n-p-1)!}{p!}.$$

For $p = n$, we have

$$\frac{\partial d_p}{\partial \lambda}(-n) = (-1)^{n-1}\frac{2^{n-1}}{n!}\sum_{q=1}^{n}\frac{1}{q};$$

and for $p > n$, we have

$$\frac{\partial d_p}{\partial \lambda}(-n) = (-1)^{n-1}\frac{2^{n-1}}{p!(p-n)!}\left(\sum_{q=1}^{p}\frac{1}{q} + \sum_{q=1}^{p-n}\frac{1}{q}\right).$$

This finally leads to

$$\begin{aligned} y_2(z) = \log z J_n(z) - \frac{1}{2}\sum_{p=0}^{n-1}\frac{(n-p-1)!}{p!}\left(\frac{z}{2}\right)^{2p-n} \\ -\frac{1}{2}\frac{1}{n!}\left(\sum_{q=1}^{n}\frac{1}{q}\right)\left(\frac{z}{2}\right)^{n} - \frac{1}{2}\sum_{p=n+1}^{\infty}(-1)^{p+n}\frac{1}{p!(p-n)!}\left(\sum_{q=1}^{p}\frac{1}{q}+\sum_{q=1}^{p-n}\frac{1}{q}\right)\left(\frac{z}{2}\right)^{2p-n} \\ = \log z J_n(z) - \frac{1}{2}\sum_{p=0}^{n-1}\frac{(n-p-1)!}{p!}\left(\frac{z}{2}\right)^{2p-n} \\ -\frac{1}{2}\frac{1}{n!}\left(\sum_{q=1}^{n}\frac{1}{q}\right)\left(\frac{z}{2}\right)^{n} - \frac{1}{2}\sum_{p=1}^{\infty}\frac{(-1)^p}{p!(p+n)!}\left(\sum_{q=1}^{p+n}\frac{1}{q}+\sum_{q=1}^{p}\frac{1}{q}\right)\left(\frac{z}{2}\right)^{2p+n}. \end{aligned}$$

Therefore, for $\rho \notin \mathbb{Z}$ the monodromy of the Bessel equation is semisimple with eigenvalues $e^{\pm 2\pi i\rho}$, and for $\rho \in \mathbb{Z}$ the monodromy is not semisimple and its eigenvalue is $e^{2\pi i\rho}$.