RELATING REAL AND P-ADIC KAZHDAN-LUSZTIG POLYNOMIALS

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Abstract. Fix an integral semisimple element λ in the Lie algebra $\mathfrak{g}$ of a complex reductive algebraic group G. Let L denote the centralizer of λ in G and let $\mathfrak{g}(-1)$ denote the -1-eigenspace of $\mathrm{ad}(\lambda)$ in $\mathfrak{g}$. Under a natural hypothesis (which is always satisfied for classical groups), we embed the closure of each L orbit on $\mathfrak{g}(-1)$ into the closure of an orbit of a symmetric subgroup K containing L on a partial flag variety for G. We use this to relate the local intersection homology of the latter orbit closures to the former orbit closures. This, in turn, relates multiplicity matrices for split real and p-adic groups. We also describe relationships between "microlocal packets" of representations of these groups.

1. introduction

The main result of this paper, Theorem 3.1, relates certain Kazhdan-Lusztig polynomials that arise in the representation theory of real and p-adic groups. Since these polynomials encode the multiplicities of irreducible representations in standard representation, we thus relate these two kinds of multiplicities. Under favorable circumstances (which are always satisfied for $\mathrm{GL}(n)$ and $\mathrm{Sp}(2n)$, for example), our results imply that the decomposition matrix for certain unipotent representations of a split p-adic group is a submatrix (in a variety of different ways) of the decomposition matrix for representations of a split real group.

In more detail, suppose λ is a semisimple element in the Lie algebra $\mathfrak{g}$ of a complex reductive algebraic group G. Let L denote the centralizer in G of λ. Then, for $c \in \mathbb{C}^\times$, L acts with finitely many orbits on the c-eigenspace of $\mathrm{ad}(\lambda)$ [Vi]. Thus one can consider local intersection homology Poincaré polynomials for the closures of L orbits. (An elementary argument reduces matters to the case of λ integral and $c = -1$ and we will consider this case henceforth; see Remark 2.8.) Lusztig [Lu4, Lu5] established a finite effective algorithm to compute these polynomials. Roughly speaking, their values at 1 give multiplicities of irreducible unipotent representations in standard representations of the split p-adic form of the Langlands dual of G [Lu3].

On the other hand, let K denote the identity component of the fixed points in G of the automorphism θ obtained by conjugation by $\exp(i\pi\lambda)$. Then K acts with finitely many orbits on the partial flag variety $\mathcal{P}$ consisting of conjugates of the sum of the nonnegative eigenspaces of $\mathrm{ad}(\lambda)$. Once again, one can consider local intersection homology Poincaré polynomials. Vogan [Vo3] described a finite effective algorithm to compute them which has been implemented in the software package `atlas`. Evaluating these polynomials at 1 gives multiplicities of irreducibles in standard representation of the identity component of the split real form of the Langlands dual of G [Vo4, ABV].

The algorithms of [Lu5] and [Vo3] are completely different. Nonetheless, Theorem 3.1 says that certain polynomials that they compute separately are the same. Under a certain hypothesis (see (2.2)) we define a map ϵ, a kind of truncated exponential map, from the

-1-eigenspace $\mathfrak{g}(-1)$ of $\mathrm{ad}(\lambda)$ to $\mathcal{P}$, and use it to define an injection of L orbits on $\mathfrak{g}(-1)$ to K orbits on $\mathcal{P}$; see Definition 2.2. This gives rise to a restriction φ of irreducible local K-equivariant local system on $\mathcal{P}$ to irreducible L equivariant local systems on $\mathfrak{g}(-1)$. We prove in Theorem 3.1 that

$$P_{\varphi(\psi),\varphi(\gamma)} = P_{\psi,\gamma}; \tag{1.1}$$

here the polynomial on the left-hand side is relevant for p-adic group representations, and the polynomial on the right-hand side is relevant for real group representations. In terms of representation theory, φ can be thought of as matching an irreducible unipotent representation of a split p-adic group with an irreducible Harish Chandra module for a split real group, and (1.1) implies that their respective multiplicities in certain standard modules (also matched by φ) are the same.

Equation (1.1) generalizes the main geometric result for $\mathrm{GL}(n)$ of Ciubotaru-Trapa [CiT]; see Remark 3.3. In Remark 3.4, we also give analogous results equating the p-adic polynomials with Kazhdan-Lusztig polynomials arising from category $\mathcal{O}$, generalizing results of Zelevinsky [Z] for $\mathfrak{gl}(n)$ to all classical groups.

There are several important hypotheses to highlight. As we indicated above, the existence of the map ϵ depends on (2.2). As explained in Example 2.4, (2.2) always holds in the classical groups, but Example 2.6 shows that it cannot hold in certain exceptional cases. Moreoever, when (2.2) holds, the definition of ϵ depends on a certain choice of ordering of the set $\mathscr{P}$ in Definition 2.2. Different choices lead to different maps ϵ, and therefore to different maps φ on local systems. Nonetheless (1.1) hold for all such choices. In other words, depending on the choices made, we possibly identify the p-adic polynomial on the left-hand side of (1.1) with different instances of the real polynomial on the right-hand side. The dependence on the choice of ordering is perhaps disappointing, since one might have hoped for a canonical relationship. On the other hand, in practice different choices lead to different matchings that reveal interesting nontrivial coincidences among the polynomials in question.

Note also that we have assumed K to be the identity component of the fixed points of θ. This is perhaps unnatural from the point of view of representations of algebraic groups, and requires explanation. If we had instead worked with the potentially disconnected $K' = G^\theta$, the orbits of K' on $\mathcal{P}$ can be reducible, and different irreducible components can contribute to the local intersection cohomology in ways that make the left-hand side of (1.1) a *summand* of the right-hand side. In any particular case, this is tractable to understand, but general statements in the presence of this kind of disconnectedness are somewhat cumbersome. See Remark 3.5.

Finally, in addition to matching local intersection homology polynomials, Theorem 3.1 shows that φ also matches microlocal geometric information. A number of interesting consequences for ABV micro-packets of representations are sketched in Section 6.

2. Matching of orbits

Let G be a complex connected reductive algebraic group with Lie algebra $\mathfrak{g}$. Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ and fix a semisimple element $\lambda \in \mathfrak{t}$ which is integral and weakly dominant in the sense that the pairing of λ with any coroot for $\mathfrak{t}$ in $\mathfrak{n}$ is a non-negative integer. (See Remark 2.8 for a discussion of the nonintegral case.) For each $i \in \mathbb{Z}$, write

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [\lambda, x] = ix\},$$

the i-eigenspace for $\mathrm{ad}(\lambda)$. Set $\mathfrak{l}$ equal to $\mathfrak{g}(0)$, and let L denote the centralizer in G of $\mathfrak{l}$. Then L acts with finitely many orbits on each $\mathfrak{g}(i)$ [Vi].

Let

$$\mathfrak{u} = \bigoplus_{i>0} \mathfrak{g}(i) \qquad \text{and} \qquad \overline{\mathfrak{u}} = \bigoplus_{i<0} \mathfrak{g}(i);$$

and

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}.$$

Thus $\mathfrak{p}$ is a parabolic subalgebra containing $\mathfrak{b}$. Write $\mathcal{P}$ for the variety of conjugates of $\mathfrak{p}$. Then $\mathcal{P} \simeq G/P$ where $P = LU$ is the centralizer in G of $\mathfrak{p}$.

Let $y(\lambda)$ denote $\exp(i\pi\lambda)$. Since λ is integral, the square of $y(\lambda)$ is central. Conjugation by $y(\lambda)$ therefore defines an involution θ of G. Let K denote the identify component of its fixed points. Note that the Lie algebra of K is

$$\mathfrak{k} = \bigoplus_{i} \mathfrak{g}(2i)$$

and K contains L by definition.

Example 2.1. Let $G = \mathrm{GL}(n, \mathbb{C})$ and let $\mathfrak{t}$ denote the diagonal Cartan subalgebra. After a central shift, the integrality of λ implies that we may assume λ consists of integer entries. Then $K \simeq \mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})$ where p is the number of even entries of λ and q is the number of odd entries. The symmetric pair (G, K) corresponds to the real group $\mathrm{U}(p, q)$. This is the setting of [CiT].

Next let $G = \mathrm{Sp}(2n, \mathbb{C})$ and let $\mathfrak{t}$ denote the diagonal Cartan subalgebra in the standard realization. For λ to be integral, either all of its entires are integers, or else they are all integers shifted by $1/2$. If λ consists of all half-integers, then $K \simeq \mathrm{GL}(n, \mathbb{C})$, corresponding to the real group $\mathrm{Sp}(2n, \mathbb{R})$. If λ consists of all integers, then $K \simeq \mathrm{Sp}(p, \mathbb{C}) \times \mathrm{Sp}(q, \mathbb{C})$ where p is the number of even entries of λ and q is the number of odd entries. This case corresponds to the real group $\mathrm{Sp}(p, q)$. $\square$

Recall that our goal is to relate L orbits on $\mathfrak{g}(-1)$ and K orbits on $\mathcal{P}$. As mentioned in the introduction, we do this using a kind of truncated exponential map. In order to define the map, we need to introduce certain hypotheses which we now describe.

Let $\mathscr{P}$ denote a collection of parabolic subalgebras each of which properly contains $\mathfrak{p}$. Choose an order on the elements of $\mathscr{P}$ and write

$$\mathscr{P} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell\}.$$

For each i, write the Levi decomposition as $\mathfrak{p}_i = \mathfrak{l}_i \oplus \mathfrak{u}_i$. Since $\mathfrak{p}_i$ contains $\mathfrak{p}$,

$$\mathfrak{l}_i \cap \overline{\mathfrak{u}}$$

is the nilradical of a parabolic subgroup of $\mathfrak{l}_i$. We will be interested in imposing the following hypotheses on the collection $\mathscr{P}$: first, that

$$\mathfrak{l}_i \cap \mathfrak{g}(-1) \neq \{0\}; \tag{2.1}$$

and second that

$$\mathfrak{g}(-1) = \bigoplus_{i=1}^{\ell} \left[\mathfrak{l}_i \cap \overline{\mathfrak{u}}\right]. \tag{2.2}$$

Definition 2.2. Fix an ordered collection $\mathscr{P} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_\ell\}$ satisfying (2.1) and (2.2). (Such a collection $\mathscr{P}$ always exists if G is classical, but need not always exist in the exceptional cases; see Examples 2.3–2.7 below.) Define

$$\epsilon \ : \ \mathfrak{g}(-1) \longrightarrow \mathcal{P}$$

by writing $x \in \mathfrak{g}(-1)$ as $x_1 + \cdots + x_\ell$ according to (2.2) and setting

$$\epsilon(x) = \exp(x_1)\exp(x_2)\cdots\exp(x_\ell)\cdot\mathfrak{p}.$$

Fix an orbit $\mathcal{O}$ of L on $\mathfrak{g}(-1)$. Since ϵ is L equivariant and since $L \subset K$, $K\cdot\epsilon(\mathcal{O})$ consists of a single K orbit which we call $Q_{\mathcal{O}}$. The assignment

$$\mathcal{O} \mapsto Q_{\mathcal{O}} \tag{2.3}$$

defines an injection of L orbits on $\mathfrak{g}(-1)$ into K orbits on $\mathcal{P}$. Note that the definition of ϵ (and hence $Q_{\mathcal{O}}$) depends on the choice of ordering of the elements of $\mathscr{P}$. Let

$$Y = \bigcup_{\mathcal{O}} Q_{\mathcal{O}} \tag{2.4}$$

where $\mathcal{O}$ ranges over all orbit of L on $\mathfrak{g}(-1)$. Thus, by definition, Y is the K saturation of the image of ϵ. □

The next examples investigate some instances when (2.2) holds (or cannot hold).

Example 2.3. Suppose $\lambda = \rho^\vee$, the half-sum of the coroots corresponding to the roots of $\mathfrak{t}$ in $\mathfrak{n}$. As $\rho^\vee$ is regular, $\mathfrak{p} = \mathfrak{b}$. Let $\mathscr{P}$ denote the set of parabolic subalgebras are minimal among those that properly contain $\mathfrak{b}$. If we enumerate the simple roots of $\mathfrak{t}$ in $\mathfrak{n}$ as $\alpha_1, \dots, \alpha_\ell$, then we can enumerate $\mathscr{P}$ as $\mathfrak{p}_1, \dots \mathfrak{p}_\ell$ with

$$\mathfrak{p}_i = \mathfrak{g}_{-\alpha_i} \oplus \mathfrak{b},$$

where $\mathfrak{g}_{-\alpha_i}$ is the root space for $-\alpha_i$ in $\mathfrak{g}$. Thus

$$\mathfrak{l}_i \cap \overline{\mathfrak{u}} = \mathfrak{g}_{-\alpha_i}$$

Since $\mathfrak{g}(-1)$ is the span of the negative simple root spaces, $\mathscr{P}$ satisfies (2.1) and (2.2). This is the setting of [BT]. □

Example 2.4. Suppose G is a classical group, and λ is an arbitrary (possibly singular) integral element. Let $\mathscr{P}$ denote the set of parabolic subalgebras are minimal with respect to the properties of: (1) properly containing $\mathfrak{p}$; and (2) having a levi factor that meets $\mathfrak{g}(-1)$ in a nonzero subspace. Then $\mathscr{P}$ always satisfies (2.1) and (2.2).

To see this, first take the case of $G = \mathrm{GL}(n,\mathbb{C})$, let $\mathfrak{b}$ be upper-triangular matrices with $\mathfrak{t}$ the diagonal ones. Fix

$$\lambda = (\overbrace{a_1, \cdots, a_1}^{n_1}, \overbrace{a_2, \cdots, a_2}^{n_2}, \dots, \overbrace{a_k, \cdots, a_k}^{n_k}) \tag{2.5}$$

with each a_i an integer, $a_i > a_{i+1}$ and $n = n_1 + \cdots + n_k$. Then $\mathfrak{p}$ consists of block-upper triangular matrices with diagonal blocks of size $n_1, \dots, n_k$. Set

$$S = \{j \mid a_j - a_{j+1} = 1\}.$$

The space $\mathfrak{g}(-1)$ identifies with a subspace of block matrices just below the diagonal,

$$\mathfrak{g}(-1) \simeq \bigoplus_{j\in S} \mathrm{Mat}_{n_{j+1},n_j} \tag{2.6}$$

A minimal parabolic $\mathfrak{p}_j$containing $\mathfrak{p}$ is obtained by enlarging the adjacent Levi factors $\mathfrak{gl}(n_j)\oplus\mathfrak{gl}(n_{j+1})$ to $\mathfrak{gl}(n_j+n_{j+1})$, and $\mathfrak{p}_j$ has nonzero intersection with $\mathfrak{g}(-1)$ when $j \in S$. Thus $\mathscr{P} = \{\mathfrak{p}_j \mid j \in S\}$. The element $\mathfrak{p}_j$ has

$$\mathfrak{l}_j \cap \bar{\mathfrak{u}}$$

consisting of the block lower-triangular matrices in $\mathfrak{gl}(n_j+n_{j+1})$, namely $\mathrm{Mat}_{n_{j+1},n_j}$. Comparing with (2.6), one sees (2.2) holds. Note that $\lambda = \rho^\vee$ in the previous example is simply the case when all block sizes are 1.

The other classical cases are similar. For example, suppose $G = \mathrm{Sp}(2n, \mathbb{C})$ and λ is as in (2.5) in standard coordinates. Suppose all entries a_i are half-integers (but not integers). Then $\mathfrak{p}$ has Levi factor $\mathfrak{gl}(n_1)\oplus\cdots\oplus\mathfrak{gl}(n_k)$. Once again we obtain a minimal parabolic $\mathfrak{p}_j$ containing $\mathfrak{p}$ by expanding adjacent factors $\mathfrak{gl}(n_j)\oplus\mathfrak{gl}(n_{j+1})$ in the Levi factor for $\mathfrak{p}$ to $\mathfrak{gl}(n_j+n_{j+1})$, and $\mathfrak{p}_j$ meets $\mathfrak{g}(-1)$ nontrivially if $j \in S$ as defined above. Then $\mathscr{P}$ consists of the $\mathfrak{p}_j$ for $j \in S$ plus possibly one other element: if $a_k = 1/2$, then $\mathscr{P}$ contains $\mathfrak{p}_\circ$ whose Levi factor is $\mathfrak{gl}(n_1)\oplus\cdots\oplus\mathfrak{gl}(n_{k-1})\oplus\mathfrak{sp}(2n_k)$. If instead all entries of λ are nonnegative integers, then $\mathfrak{p}$ has Levi factor $\mathfrak{gl}(n_1)\oplus\cdots\oplus\mathfrak{gl}(n_k)$ if $a_k \neq 0$ and $\mathfrak{gl}(n_1)\oplus\cdots\oplus\mathfrak{gl}(n_{k-1})\oplus\mathfrak{sp}(2n_k)$ if $a_k = 0$. As before $\mathfrak{p}_j$ may be defined by collapsing adjacent Levi factors for $j \in S$, and this time if $k-1 \in S$ and $a_k = 0$, define $\mathfrak{p}_{k-1}$ to have Levi factor $\mathfrak{gl}(n_1)\oplus\cdots\oplus\mathfrak{gl}(n_{k-2})\oplus\mathfrak{sp}(2(n_{k-1}+n_k))$. Once again one can easily verify that $\mathscr{P} = \{\mathfrak{p}_j \mid j \in S\}$ satisfies (2.1) and (2.2). □

Example 2.5. Continuing the example of $G = \mathrm{GL}(n, \mathbb{C})$ in Example 2.4, if we order the set $\mathscr{P}$ as described there, then one may quickly verify that for $x \in \mathfrak{g}(-1)$,

$$\epsilon(x) = (\mathrm{Id}_n + x)\cdot\mathfrak{p}.$$

This is the map that is used in [CiT] to define $\mathcal{O} \mapsto Q_{\mathcal{O}}$. But note that we may take *any* ordering of the set $\mathscr{P}$ and thus define different maps $\mathcal{O} \mapsto Q_{\mathcal{O}}$ which will have the properties that we describe in Theorem 3.1. □

Example 2.6. Let G be simple of type F4. Suppose λ is one-half the middle element of a Jacobson-Morozov triple for the nilpotent orbit labeled in the Bala-Carter classification by F4(a3). The weighted Dynkin diagram of this orbit (in the standard Bourbaki order) is 0200. Thus $\mathfrak{p}$ is the maximal parabolic corresponding the long middle root on the Dynkin diagram of F4. The opposite nilradical $\bar{\mathfrak{u}}$ is a three-step nilpotent algebra whose first step is $\mathfrak{g}(-1)$. Since $\mathfrak{p}$ is maximal, the only possibility is for $\mathscr{P}$ to consist of a single algebra, namely all of $\mathfrak{g}$. But we have already remarked that $\mathfrak{g}\cap\bar{\mathfrak{u}} = \bar{\mathfrak{u}}$ properly contains $\mathfrak{g}(-1)$. Thus there is no choice of $\mathscr{P}$ that satisfies (2.2). □

Example 2.7. Let G be simple of type G2. Suppose λ is one-half the middle element of a Jacobson-Morozov triple for the nilpotent orbit labeled in the Bala-Carter classification by G2(a1). Thus $\mathfrak{p}$ is the maximal parabolic corresponding to the short simple root. Its nilradical properly contains $\mathfrak{g}(-1)$, so once again there is no choice of $\mathscr{P}$ that satisfies (2.2). However, $\bar{\mathfrak{u}}$ in this case is a two-step nilpotent algebra whose first step is $\mathfrak{g}(-1)$. We will give a related argument in Section 5 which handles this case. □

Remark 2.8. If λ is not integral, one can repeat the constructions above replacing G by $G(\lambda)$, the centralizer in G of $\exp(2i\pi\lambda)$. The Lie algebra of $G(\lambda)$ is the sum of the integral eigenspaces of $\mathrm{ad}(\lambda)$ and $y(\lambda)$ is still an element whose square is central in $G(\lambda)$. So the definitions above carry over without change.

Note also that above we restricted attention to the L orbits on $\mathfrak{g}(-1)$. One could instead consider the orbits of L on an general eigenspace $\mathfrak{g}(c)$ for $c \in \mathbb{C}^\times$. Since the -1 eigenspace of $\mathrm{ad}(\lambda)$ is the c-eigenspace of $\mathrm{ad}(\lambda')$ for $\lambda' = -\lambda/c$, the study of L orbits on $\mathfrak{g}(c)$ for λ is equivalent to the study of L orbits on $\mathfrak{g}(-1)$ for λ'. □

3. Statement of main results

The goal of this section is to state Theorem 3.1 describing how $\mathcal{O} \mapsto Q_{\mathcal{O}}$ in Definition 2.2 preserves the singularities of the closure of $\mathcal{O}$ in a precise sense. In order to do so, we need some notation.

3.1. **Notation.** Suppose X is a complex algebraic variety on which a complex algebraic group H acts with finitely many orbits. Let $\mathcal{C}(H,X)$ be the category of H-equivariant constructible sheaves on X. Write $\mathcal{P}(H,X)$ for the category of H-equivariant perverse sheaves on X.

Irreducible objects in both categories are parametrized by the set $\Xi(H,X)$ consisting of pairs $(Q,\mathcal{V})$ with Q an orbit of H on X and $\mathcal{V}$ an irreducible H-equivariant local system supported on Q. For $\gamma \in \Xi(H,X)$, we write $\mathrm{con}(\gamma)$ and $\mathrm{per}(\gamma)$ for the corresponding irreducible constructible and perverse sheaves.

By taking Euler characteristics, we identify the Grothendieck group of the categories $\mathcal{P}(H,X)$ and $\mathcal{C}(H,X)$. In this way, we can consider the change of basis matrix,

$$[\mathrm{per}(\gamma)] = \sum_{\psi \in \Xi(H,X)} (-1)^{d(\psi)}\, C^g_{\psi,\gamma}[\mathrm{con}(\psi)]; \tag{3.1}$$

here $\psi = (Q_\psi, \mathcal{V}_\psi)$ and $d(\psi) = \dim(Q_\psi)$. The matrix $(C^g(\psi,\gamma))$ is called the geometric multiplicity matrix. Let $P_{\psi,\gamma} \in \mathbb{Z}[q]$ denote the graded occurrence of $\mathrm{con}(\psi)$ in the cohomology sheaves of $\mathrm{per}(\gamma)$. More precisely, define the coefficient of q^i in $P_{\psi,\gamma}$ to be the multiplicity of $\mathrm{con}(\psi)$ in the ith cohomology sheaves of $\mathrm{per}(\gamma)$. (In our applications below we will have vanishing in odd degrees.) Thus, up to a sign,

$$P_{\psi,\gamma}(1) = C^g_{\psi,\gamma}. \tag{3.2}$$

Finally, given an H orbit Q on X, let m_Q^{mic} denote the $\mathbb{Z}$-valued linear functional on the Grothendieck group of $\mathcal{P}(H,X)$ that assigns to an irreducible perverse sheave the multiplicity of the conormal bundle to Q in its characteristic cycle. The notation is meant to indicate that m_Q^{mic} is a microlocal multiplicity.

3.2. **Statement of Main Results.** Let $A_L(x)$ denote the component group of the centralizer in L of $x \in \mathfrak{g}(-1)$, and let $A_K(\epsilon(x))$ denote the component group of the centralizer in K of $\epsilon(x)$. Since ϵ is L equivariant, there is a natural map $A_L(x) \to A_K(\epsilon(x))$. Since $A_K(\epsilon(x))$ is abelian (an elementary abelian 2-group) with only one-dimensional irreducible representations, composition defines a map on irreducible representations,

$$A_K(\epsilon(x))\widehat{\ } \longrightarrow A_L(x)\widehat{\ }. \tag{3.3}$$

Fix $\gamma = (Q_{\mathcal{O}}, \mathcal{L}) \in \Xi(K,Y)$, with $Q_{\mathcal{O}} = K\cdot\epsilon(x)$ for $x \in \mathfrak{g}(-1)$. Then $\mathcal{L}$ is parametrized by an irreducible representation of $A_K(\epsilon(x))$. By (3.3), this maps to an irreducible representation of $A_L(x)$, and hence an irreducible local system $\mathcal{L}'$ on $\mathcal{O} = L\cdot x$. Define

$$\varphi \;:\; \Xi(K,Y) \longrightarrow \Xi(L,\mathfrak{g}(-1)) \tag{3.4}$$

by

$$\varphi(Q_{\mathcal{O}},\mathcal{L}) = (\mathcal{O},\mathcal{L}').$$

More conceptually, this is just the pullback via ϵ of irreducible local systems from Y to $\mathfrak{g}(-1)$.

Theorem 3.1. *Recall the setting of* (2.3) *and the definition of* Y *in* (2.4)*. In particular, recall that we assume that there exists a fixed ordered set of parabolic subalgebras* $\mathscr{P}$ *satisfying* (2.1) *and* (2.2) *that is used to define* $\epsilon : \mathfrak{g}(-1) \to \mathcal{P}$*. Recall the definition of* φ *of* (3.4)*, and of the geometric multiplicity matrices and intersection homology polynomials of Section 3.1. Then for* $\psi, \gamma \in \Xi(K, Y)$*,*

$$P_{\psi,\gamma} = P_{\varphi(\psi),\varphi(\gamma)}; \tag{3.5}$$

in particular,

$$C^g_{\psi,\gamma} = C^g_{\varphi(\psi),\varphi(\gamma)}. \tag{3.6}$$

Finally, fix $\mathcal{O} = L \cdot x$*,* $Q_{\mathcal{O}} = K \cdot \epsilon(x)$*, and* $\gamma \in \Xi(K, Y)$*. The multiplicity of the conormal bundle to* $Q_{\mathcal{O}}$ *in the characteristic cycle of* $per(\gamma)$ *equals the multiplicity of the conormal bundle to* $\mathcal{O}$ *in* $per(\varphi(\gamma))$*,*

$$m^{mic}_{\mathcal{O}}(per(\varphi(\gamma))) = m^{mic}_{Q_{\mathcal{O}}}(per(\gamma)). \tag{3.7}$$

Before turning to the proof in Section 4, we make a few remarks.

Remark 3.2. Theorem 3.1 is most powerful when the map φ is surjective for some choice of $\mathscr{P}$. In this case, every polynomial arising from the L orbits on $\mathfrak{g}(-1)$ is matched with a Kazhdan-Lusztig-Vogan polynomial. In Example 4.8 (extending Example 2.4), we sketch that this is the case for some classical subgroups of $\mathrm{GL}(n, \mathbb{C})$. But note that there are some cases in spin groups and exceptional groups, for example, where $A_L(x)$ is nonabelian, and therefore has higher dimensional irreducible local systems as elements of $\Xi(L, \mathfrak{g}(-1))$. In these cases, φ can never be surjective (even when a choice of $\mathscr{P}$ satisfying (2.1) and (2.2) exists). $\square$

Remark 3.3. When $G = \mathrm{GL}(n, \mathbb{C})$ and the choice of $\mathscr{P}$ (and its ordering) is the one described in Example 2.5, then (3.6) in Theorem 3.1 is [CiT, Theorem 2.5]. $\square$

Remark 3.4. We describe a version of Theorem 3.1 that replaces K orbits with P orbits, and thus matches intersection homology polynomials for L orbits on $\mathfrak{g}(-1)$ with classical parabolic Kazhdan-Lusztig polynomials. Loosely speaking we replace every occurrence of K with P. In more detail, assuming the existence of $\mathscr{P}$, we can define a map that takes $\mathcal{O} = L \cdot x$ for $x \in \mathfrak{g}(-1)$ to $Q_{\mathcal{O}} = P \cdot \epsilon(\mathcal{O})$ where ϵ is defined as in Definition 2.2. If we once again set Y to be the union of all the various $Q_{\mathcal{O}}$, then ϵ sends $\mathfrak{g}(-1)$ to Y. Just as above, we obtain a map

$$\varphi \;:\; \Xi(P, Y) \longrightarrow \Xi(L, \mathfrak{g}(-1)).$$

The proof of Corollary 4.4 (see Remark 4.5) will apply to show $\mathcal{O} \simeq \epsilon(\mathcal{O})$ is open and dense in $Q_{\mathcal{O}}$. This allows one to directly conclude

$$P_{\psi,\gamma} = P_{\varphi(\psi),\varphi(\gamma)}, \tag{3.8}$$

and similarly match microlocal multiplicities. (See the argument and references in Section 4.2.) When $G = \mathrm{GL}(n, \mathbb{C})$ and the choice of ordering on $\mathscr{P}$ is as in Example 2.5 (so $\epsilon(x) = \mathrm{Id}_n + x$), (3.8) is exactly the main result of [Z].

Note, however, that there are no nontrivial local systems in $\Xi(P, Y)$. So there is no hope in matching polynomials that arise for nontrivial local systems for L orbits on $\mathfrak{g}(-1)$ with classical Kazhdan-Lusztig polynomials; cf. Remark 3.2. This is a reason for using K orbits in order to match more general polynomials for $\mathfrak{g}(-1)$. $\square$

Remark 3.5. Set $K' = G^\theta$ as in the introduction; so K' is potentially disconnected. In the setting of Theorem 3.1, we can copy the definitions above to define

$$\varphi' \;:\; \Xi(K', Y) \longrightarrow \Xi(L, \mathfrak{g}(-1)).$$

Fix $\psi, \gamma \in \Xi(K', Y)$ and write Q_γ for K' orbit that is the support of γ. If we further assume that Q_γ is irreducible (which of course is automatic if K' is connected), then the proof of Theorem 3.1 that we give below will show

$$P_{\psi,\gamma} = P_{\varphi'(\psi),\varphi'(\gamma)} \tag{3.9}$$

and that the analogous conclusion of (3.7) also holds. If we do not assume Q_γ is irreducible, then (3.9) can fail; see [BT, Remark 3.6] for a discussion of an example in SO(8). The one place in the argument below that we need Q_γ to be irreducible is to deduce the conclusion of Corollary 4.4.

4. Proof of Theorem 3.1

4.1. **Preliminary Results.** As a first step toward Theorem 3.1, we show $\mathcal{O} \mapsto Q_{\mathcal{O}}$ preserves dimensions; see Proposition 4.3 below. We start with some preliminaries, following the approach of [BT] closely.

Lemma 4.1. *In the setting of Theorem 3.1, if $\mathcal{O} = L \cdot x$ is an orbit of L on $\mathfrak{g}(-1)$, then*

$$\dim(\epsilon(\mathcal{O})) = \dim(\mathcal{O}).$$

Proof. Since ϵ is L-equivariant, $Z_L(x) \subset Z_L(\epsilon(x))$. So the result follows from the other containment

$$Z_L(x) \supset Z_L(\epsilon(x)). \tag{4.1}$$

Write $x = \sum_j x_i$ with possibly some of x_j's equal to zero. Because $[\mathfrak{g}(-1), \mathfrak{g}(-1)] \subset \mathfrak{g}(-2)$, there is a $z \in \bigoplus_{k \le -2} \mathfrak{g}(k)$ so that

$$\begin{aligned}
\epsilon(x) &:= \exp(x_1)\exp(x_2)\cdots\exp(x_\ell)\cdot\mathfrak{p}\\
&= \exp(x_1 + x_2 + \cdots + x_\ell + z)\cdot\mathfrak{p}\\
&= \exp(x + z)\cdot\mathfrak{p}.
\end{aligned}$$

If $l \in L$ centralizes $\epsilon(x)$, it thus centralizes $x+z$. Since L preserves the grading of $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$, if $l \in L$ centralizes $x + z$, it must centralize x, and so (4.1) follows. $\square$

Lemma 4.2. *In the setting of Theorem 3.1, write $\bar{P} = L\bar{U}$ for the opposite parabolic subgroup to $P = LU$. Then $\bar{U} \cap K$ acts freely on*

$$[\bar{U} \cap K] \cdot \epsilon(\mathfrak{g}(-1)).$$

Moreover, for all $x \in \mathfrak{g}(-1)$,

$$\big([\bar{U} \cap K] \cdot \epsilon(x)\big) \cap \epsilon(\mathfrak{g}(-1)) = \epsilon(x).$$

In particular, ϵ is injective.

Proof. Suppose $k \in \bar{U} \cap K$ and $x = \sum x_j \in \mathfrak{g}(-1)$ such that

$$k \cdot \exp(x_1)\exp(x_2)\cdots\exp(x_\ell)\cdot\mathfrak{p} = \exp(x_1)\exp(x_2)\cdots\exp(x_\ell)\cdot\mathfrak{p}. \tag{4.2}$$

The stabilizer in G of $\mathfrak{p}$ is P and $\bar{U} \cap P = 1$. Thus (4.2) implies

$$k \cdot \exp(x_1)\exp(x_2)\;\ldots\exp(x_\ell) = \exp(x_1)\exp(x_2)\ldots\exp(x_\ell),$$

from which we conclude that $k = 1$, verifying the first assertion of the lemma. The second assertion follows in a similar way. $\square$

Proposition 4.3. *In the setting of Theorem 3.1, let $\mathcal{O}$ be an orbit of L on $\mathfrak{g}(-1)$ and define $Q_{\mathcal{O}}$ as in* (2.3)*. Then,*

$$\dim(Q_{\mathcal{O}}) = \dim(Q_{\{0\}}) + \dim(\mathcal{O}). \tag{4.3}$$

Proof. We first show that

$$\dim(Q_{\mathcal{O}}) \leq \dim(Q_{\{0\}}) + \dim(\mathcal{O}). \tag{4.4}$$

To see this, we factor ϵ as follows. Write P_i for the centralizer in G of $\mathfrak{p}_i$. Define

$$\mathcal{X} = K \underset{K\cap P}{\times} P_1 \underset{P}{\times} P_2 \underset{P}{\times} \ldots \underset{P}{\times} P_\ell \tag{4.5}$$

to be the quotient of $K \times P_1 \times \ldots \times P_\ell$ by the action

$$(p_0, p_1, \ldots, p_\ell) \cdot (k_0, y_1, y_2, \ldots, y_\ell) = (k_0\, p_0, p_0^{-1}\, y_1\, p_1, p_1^{-1}\, y_2\, p_2, \ldots, p_\ell^{-1}\, y_\ell).$$

Let K act on $\mathcal{X}$ via

$$k \cdot [k_0, y_1, y_2, \ldots, y_\ell] = [k\, k_0, y_1, y_2, \ldots, y_\ell].$$

Then $\mathcal{X}$ comes equipped with a natural K-equivariant map

$$\tau \;:\; \mathcal{X} \longrightarrow \mathcal{P} \tag{4.6}$$

mapping

$$[k_0, y_1, y_2, \ldots, y_\ell] \mapsto k_0 y_1 \cdots y_\ell \cdot \mathfrak{p}.$$

Define an L equivariant map

$$\iota \;:\; \mathfrak{g}_{-1} \longrightarrow \mathcal{X}$$

mapping $x = x_1 + \cdots + x_\ell$ as

$$\iota(x) = [1, \exp(x_1), \exp(x_2), \ldots, \exp(x_\ell)].$$

Then, by definition, $\epsilon = \tau \circ \iota$ and

$$Q_{\mathcal{O}} = \tau(K \cdot \iota(\mathcal{O})).$$

Thus

$$\dim(Q_{\mathcal{O}}) \leq \dim(K \cdot \iota(\mathcal{O})) \leq \dim(K/K \cap P) + \dim(\mathcal{O}).$$

Once we note that $Q_{\{0\}} = K \cdot \mathfrak{p} \simeq K/K \cap P$, (4.4) follows.

We argue that the converse inequality holds. Since $\bar{U} \cap K \cdot \epsilon(\mathcal{O})$ is contained in $Q_{\mathcal{O}}$, Lemma 4.2 implies

$$\dim(Q_{\mathcal{O}}) \geq \dim(\bar{U} \cap K) + \dim(\epsilon(\mathcal{O})). \tag{4.7}$$

By Lemma 4.1, we know that $\dim(\epsilon(\mathcal{O})) = \dim(\mathcal{O})$. Since $Q_{\{0\}} = K \cdot \mathfrak{p} \simeq K/(K \cap P)$,

$$\dim(Q_{\{0\}}) = \dim(\bar{U} \cap K).$$

Thus, (4.7) becomes

$$\dim(Q_{\mathcal{O}}) \geq \dim(Q_{\{0\}}) + \dim(\mathcal{O}),$$

as we wished to show. $\square$

Corollary 4.4. $[K \cap \bar{P}] \cdot \epsilon(x)$ *is open and dense in* $K \cdot \epsilon(x)$.

Proof. Since $\dim(Q_{\{0\}}) = \dim(K/(K \cap P)) = \dim(\bar{U} \cap K)$, Lemma 4.2 and the dimension count of Proposition 4.3 implies the result. $\square$

Remark 4.5. We can repeat the analysis above with K replace by P (as in Remark 3.4). In this case $Q_{\{0\}}$ is just the point $\mathfrak{p}$ in $\mathcal{P}$. Replacing K with P in the proofs of Proposition 4.3 and Corollary 4.4 implies that

$$\dim(Q_{\mathcal{O}}) = \dim(\mathcal{O}),$$

and $\epsilon(\mathcal{O})$ is open and dense in $Q_{\mathcal{O}}$. $\square$

4.2. **Consequences of Corollary 4.4.** Recall that the component group of the centralizer in K of an element of $\mathcal{P}$ is an elementary abelian 2-group. Thus, every irreducible K-equivariant local system on $\mathcal{P}$ is one-dimensional. In the setting of Corollary 4.4 and notation for Y in (2.4), we thus obtain a restriction map,

$$\varphi_1 \;:\; \Xi(K, Y) \longrightarrow \Xi\left(K \cap \bar{P}, [K \cap \bar{P}] \cdot \epsilon(\mathfrak{g}(-1))\right). \tag{4.8}$$

This is once again dual to the corresponding restriction of A-group representations as in (3.3). By definition, if $\gamma \in \Xi(K, Y)$, then the constructible sheaf $\mathrm{con}(\gamma)$ restricts to $\mathrm{con}(\varphi_1(\gamma))$. By the density statement, the intersection homology polynomials for $K \cap \bar{P}$ orbit closures on $[K \cap \bar{P}] \cdot \epsilon(\mathfrak{g}(-1))$ match those for K orbits on Y. (This is part of the unicity of perverse extensions. A discussion of such a statement can be found in the proof of Proposition 7.14(c) and around Equation (7.16)(e) in [ABV].) More precisely, for $\psi, \gamma \in \Xi(K, Y)$,

$$P_{\psi,\gamma} = P_{\varphi_1(\psi),\varphi_1(\gamma)};$$

in particular,

$$C^g_{\psi,\gamma} = C^g_{\varphi_1(\psi),\varphi_1(\gamma)}.$$

Finally, using the method of calculating of characteristic cycles via normal slices (sketched, for example, at the bottom of page 186 and top of page 187 in [ABV]), one sees that the microlocal multiplicities match: if $\mathcal{O} = K \cdot \epsilon(x)$, $\mathcal{O}' = [K \cap \bar{P}] \cdot \epsilon(x)$ and $\gamma \in \Xi(K, Y)$,

$$m_{\mathcal{O}'}(\mathrm{per}(\gamma)) = m_{\mathcal{O}}(\mathrm{per}(\varphi_1(\gamma)).$$

4.3. **Induced bundles.** The previous section relates the geometry of orbits of K on Y to the orbits of $K \cap \bar{P}$ on $[K \cap \bar{P}] \cdot \epsilon(\mathfrak{g}(-1))$. In this section, we use an induced bundle construction to relate these $K \cap \bar{P}$ orbits to L orbits on $\mathfrak{g}(-1)$.

To begin, recall a general construction. Suppose H acts on a variety X with finitely many orbits. Suppose $H \subset H'$. The induced bundle

$$H' \times_H X$$

is defined by quotienting $H' \times X$ by $(h'h, x) \sim (h', hx)$ for all $h \in H$. See [ABV, Chapter 7], for example.

Proposition 4.6. *In the setting of Section 3, the map*

$$A : [K \cap \bar{P}] \times_L \mathfrak{g}(-1) \longrightarrow [K \cap \bar{P}] \cdot \epsilon(\mathfrak{g}(-1))$$

defined by

$$A(k, x) \mapsto k\, \epsilon(x)$$

is a $K \cap \bar{P}$ equivariant isomorphism.

Proof. Clearly A is surjective and $K\cap\bar{P}$ equivariant. We prove that A is injective. Suppose $A(k_1,x_1)=A(k_2,x_2)$. Write $k_1=\bar{n}_1\,t_1$ with $\bar{n}_1\in[\bar{U}\cap K]$ and $t_1\in L\cap K$. Similarly, write $k_2=\bar{n}_2\,t_2$. Thus

$$\bar{n}_1t_1\epsilon(x_1)=\bar{n}_2t_2\epsilon(x_2).$$

Since ϵ is L equivariant, this implies

$$\bar{n}_1\epsilon(\mathrm{Ad}(t_1)x_1)=\bar{n}_2\epsilon(\mathrm{Ad}(t_2)x_2). \tag{4.9}$$

By Lemma 4.2, $\bar{n}_1=\bar{n}_2$ and

$$\epsilon(\mathrm{Ad}(t_1)x_1)=\epsilon(\mathrm{Ad}(t_2)x_2). \tag{4.10}$$

Since ϵ is injective (Lemma 4.2), $\mathrm{Ad}(t_1)x_1=\mathrm{Ad}(t_2)x_2$. We use this in the third equality below to conclude,

$$\begin{aligned}(k_2,x_2)&=(\bar{n}_2t_2,x_2)=(\bar{n}_2t_2,\mathrm{Ad}(t_2)^{-1}\,\mathrm{Ad}(t_2)x_2)=(\bar{n}_2t_2,\mathrm{Ad}(t_2)^{-1}\,\mathrm{Ad}(t_1)x_1)\\&\sim(\bar{n}_2,\mathrm{Ad}(t_1)x_1)=(\bar{n}_1,\mathrm{Ad}(t_1)x_1)\\&=(k_1t_1^{-1},\mathrm{Ad}(t_1)x_1)\sim(k_1,x_1).\end{aligned}$$

Thus $A(k_1,x_1)=A(k_2,x_2)$ implies $(k_1,x_1)\sim(k_2,x_2)$, and so A is injective as we wished to show. ◻

Corollary 4.7. *In the setting of Proposition 4.6, there is a is a natural correspondence of L orbits on $\mathfrak{g}(-1)$ and $K\cap\bar{P}$ orbits on $[K\cap\bar{P}]\cdot\epsilon(\mathfrak{g}(-1))$,*

$$\mathcal{O}=L\cdot x\mapsto\mathcal{O}'=[K\cap\bar{P}]\cdot\epsilon(x).$$

Write $A_L(x)$ for the component group of the centralizer of x in L, and similarly for $A_{K\cap\bar{P}}(\epsilon(x))$. Then the map $L\to[K\cap\bar{P}]$ induces an isomorphism

$$A_L(x)\simeq A_{K\cap\bar{P}}(\epsilon(x)).$$

The resulting bijection

$$\varphi_2\;:\;\Xi(K\cap\bar{P},[K\cap\bar{P}]\cdot\epsilon(\mathfrak{g}(-1)))\to\Xi(L,\mathfrak{g}(-1)) \tag{4.11}$$

implements an identification of the geometric mulitplicity matrices and intersection homology polynomials of Section 3.1. More precisely, for ψ,γ in $\Xi(K\cap\bar{P},[K\cap\bar{P}]\cdot\epsilon(\mathfrak{g}(-1)))$

$$P_{\psi,\gamma}=P_{\varphi_2(\psi),\varphi_2(\gamma)} \tag{4.12}$$

and, in particular,

$$C^g_{\psi,\gamma}=C^g_{\varphi_2(\psi),\varphi_2(\gamma)}. \tag{4.13}$$

Finally, the microlocal multiplicities match,

$$m_{\mathcal{O}'}(per(\gamma))=m_{\mathcal{O}}(per(\varphi_2(\gamma))). \tag{4.14}$$

Proof. According to [ABV, Proposition 7.14], there is a bijective correspondence of L orbits on $\mathfrak{g}(-1)$ and $K\cap\bar{P}$ orbits on the induced bundle $[K\cap\bar{P}]\times_L\mathfrak{g}(-1)$ with properties as listed in (4.11)-(4.13), while [ABV, Proposition 20.1(e)] implies (4.14). Composing with the isomorphism of Proposition 4.6 completes the proof. ◻

4.4. **Proof of Theorem 3.1.** Once one observes that φ in Theorem 3.1 is simply the composition of φ_2 in (4.11) and φ_1 in (4.8), the theorem follows by combining the results of Section 4.2 and Corollary 4.7. $\square$

Remark 4.8. Determining when φ is surjective requires case-by-case analysis, and we simply record the results here for classical groups. If $G = \mathrm{GL}(n)$, then all local systems are trivial, so φ is automatically surjective for any ordering of the set $\mathscr{P}$ defined in Example 2.4. As [BT, Example 3.5] already indicates, for some choices of ordering of $\mathscr{P}$, φ can fail to be surjective. But there is always some choice for which φ *is* surjective. If $G = \mathrm{Sp}(2n)$, and λ consists of all integers, there are again no nontrivial local systems, so surjectivity is automatic. If, however, λ consists of half-integers, $\mathscr{P}$ in Example 2.4 sometimes contains a distinguished element $\mathfrak{p}_\circ$. This element must appear first in the ordering on $\mathscr{P}$ in order for φ to be surjective. The situation is similar for $\mathrm{SO}(n)$ where there is sometimes a distinguished element in the set $\mathscr{P}$ defined in Example 2.4 which has a Levi factor component of the form $\mathfrak{so}(n)$. Again, this element must be taken first in the order on $\mathscr{P}$ in order for φ to be surjective. $\square$

5. ABELIAN AND TWO-STEP CASE

In this section, we study a special class of examples that includes some cases (like Example 2.7) where there does not exist a set $\mathscr{P}$ satisfying (2.1) and (2.2). In these cases, we do not need to truncate the exponential map. Alternatively, one can think of this section as a kind of basic case, and the definition of ϵ in Definition 2.2 and some of the proofs of Section 4 as an induction using the bundle constructed in (4.5). This can be made precise, but isn't necessary for our purposes here.

Assume that the i-eigenspace $\mathfrak{g}(i)$ is zero if $|i| > 2$. In other words, the nilradical of $\mathfrak{p}$ is either abelian or a two-step nilpotent Lie algebra. Under this hypothesis, we simply define

$$\epsilon' : \mathfrak{g}(-1) \longrightarrow \mathcal{P} \tag{5.1}$$

by

$$\epsilon'(x) = \exp(x) \cdot \mathfrak{p}.$$

Using ϵ' instead of ϵ in (2.3), for an L orbits $\mathcal{O}$ on $\mathfrak{g}(-1)$ we define

$$Q'_{\mathcal{O}} = K \cdot \epsilon'(\mathcal{O}),$$

and Y' to be the union of the various $Q'_{\mathcal{O}}$. As in (3.3), for $x \in \mathfrak{g}(-1)$ we have a natural map

$$A_K(\epsilon'(x))\widehat{\ } \longrightarrow A_L(x)\widehat{\ }, \tag{5.2}$$

and hence a map

$$\varphi' \ : \ \Xi(K, Y') \longrightarrow \Xi(L, \mathfrak{g}(-1)). \tag{5.3}$$

We then have the following analog of Theorem 3.1.

Theorem 5.1. *For $\psi, \gamma \in \Xi(K, Y')$,*

$$P_{\psi,\gamma} = P_{\varphi'(\psi),\varphi'(\gamma)}. \tag{5.4}$$

Fix $\mathcal{O} = L \cdot x$, $Q'_{\mathcal{O}} = K \cdot \epsilon'(x)$, and $\gamma \in \Xi(K, Y')$. Then

$$m^{mic}_{\mathcal{O}}(per(\varphi'(\gamma))) = m^{mic}_{Q'_{\mathcal{O}}}(per(\gamma)).$$

This will follow in exactly the same way as Theorem 3.1 once we prove analogs of Lemma 4.2, Proposition 4.3 and Corollary 4.4.

Lemma 5.2. *The group $\bar{U} \cap K$ acts freely on*

$$[\bar{U} \cap K] \cdot \epsilon'(\mathfrak{g}(-1)).$$

Moreover, for all $x \in \mathfrak{g}(-1)$,

$$\big([\bar{U} \cap K] \cdot \epsilon'(x)\big) \cap \epsilon'(\mathfrak{g}_{-1}) = \epsilon'(x).$$

Proof. This follows as in the proof of Lemma 4.2. □

Proposition 5.3. *Let $\mathcal{O}$ be the L orbit of $x \in \mathfrak{g}(-1)$. Then,*

$$\dim(Q'_{\mathcal{O}}) = \dim(Q'_{\{0\}}) + \dim(\mathcal{O}). \tag{5.5}$$

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ denote the Cartan decomposition. Because of our two-step assumption, $\mathfrak{s} = \mathfrak{g}(-1) \oplus \mathfrak{g}(1)$, and $\mathfrak{k} = \mathfrak{g}(-2) \oplus \mathfrak{l} \oplus \mathfrak{g}(2)$. The tangent space to $Q'_{\mathcal{O}}$ at $\mathfrak{q} := \epsilon'(x) = \exp(x) \cdot \mathfrak{p}$ identifies with $\mathfrak{g}/(\mathfrak{k} + \mathfrak{q})$. If we fix a nondegenerate invariant form to identify $\mathfrak{g}$ and $\mathfrak{g}^*$, then then conormal space to $Q'_{\mathcal{O}}$ at $\mathfrak{q}$ identifies with

$$[\mathfrak{g}/(\mathfrak{k} + \mathfrak{q})]^{*,\perp} \simeq \mathrm{Ad}(\exp(x))\bar{\mathfrak{u}} \cap \mathfrak{s}.$$

Since $\bar{\mathfrak{u}} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1)$, $x \in \mathfrak{g}(-1)$, and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$,

$$\mathrm{Ad}(\exp(x))\bar{\mathfrak{u}} \cap \mathfrak{s} = \ker(\mathrm{ad}(x)|_{\mathfrak{g}(-1)}).$$

The dimension of $\mathcal{P}$ is simply $\dim(\bar{\mathfrak{u}})$. On the other hand, $\dim(\mathcal{P})$ equals the the sum of the dimension of $Q'_{\mathcal{O}}$ and the dimension of the conormal space to $Q'_{\mathcal{O}}$ at $\mathfrak{q}$. Thus

$$\dim(\bar{\mathfrak{u}}) = \dim(Q'_{\mathcal{O}}) + \ker(\mathrm{ad}(x)|_{\mathfrak{g}(-1)}).$$

So

$$\begin{aligned} \dim(Q'_{\mathcal{O}}) &= \dim(\bar{\mathfrak{u}}) - \ker(\mathrm{ad}(x)|_{\mathfrak{g}(-1)}) \\ &= \dim(\mathfrak{g}(-2)) + \big[\dim(\mathfrak{g}(-1)) - \ker(\mathrm{ad}(x)|_{\mathfrak{g}(-1)})\big] \\ &= \dim(\mathfrak{g}(-2)) + \dim(\mathcal{O}). \end{aligned}$$

To complete the proof, note that $Q_{\{0\}} = K \cdot \mathfrak{p}$ has dimension equal to that of $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{p}) \simeq \mathfrak{g}(-2)$. □

Corollary 5.4. *$[K \cap \bar{B}] \cdot \epsilon'(x)$ is open and dense in $K \cdot \epsilon'(x)$.*

Proof. This now follows in the same way as Corollary 4.4. □

With Corollary 5.4 in hand, we can argue as in the proof of Theorem 3.1 to establish Theorem 5.1. We omit the details.

6. Remarks on ABV micro-packets

To conclude, we sketch some consequences related to micro-packets of representations. Part of our motivation is to explain the appearance of the Kashiwara-Saito singularity discovered by Cunningham-Fiora-Kitt [CFK].

Fix λ integral and semisimple as above. According to the [Vo4] (reinterpreted in [ABV, Theorem 1.24]), $\Xi_0(K,Y)$ parametrizes a subset of irreducible representations with infinitesimal character λ of various real forms $G'_{\mathbb{R}}$ of the Langlands dual group $G^\vee$. (In order to think of λ as an infinitesimal character for $G'_{\mathbb{R}}$, we view $\lambda \in \mathfrak{h}$ as an element of $(\mathfrak{h}^\vee)^*$ for the dual Cartan subalgebra $\mathfrak{h}^\vee \simeq \mathfrak{h}^*$.) Write $\pi_{\mathbb{R}}(\gamma)$ for the irreducible representation corresponding to $\gamma \in \Xi(K,Y)$, and let Π_Y denote all of the representations of the form $\pi(\gamma)$.

Fix a K orbit Q on Y. Following [ABV, Definition 19.15] define a subset $\Pi^g_{\mathbb{R}}(Q)$ of Π_Y consisting of those $\pi_{\mathbb{R}}(\gamma)$ such that the conormal bundle to Q occurs in the characteristic cycle of $\mathrm{per}(\gamma)$. The subset $\Pi^g_{\mathbb{R}}(Q)$ is called a micro-packet. Arthur packets are defined in [ABV, Definition 22.6] as certain special kinds of micro-packets arising from Arthur parameters.

Meanwhile, if we assume further that λ is hyperbolic, then according to Lusztig's classification of unipotent representation of graded affine Hecke algebras ([Lu1, Lu2, Lu3]), the set $\Xi(L,\mathfrak{g}(-1))$ parameterizes certain unipotent representations of the split $F = \mathbb{Q}_p$ form G'_F of the Langlands dual group $G^\vee$. (The subset $\Xi_0(L,\mathfrak{g}(-1))$ consisting of irreducible local systems of Springer type parametrizes irreducible unramified representations of G'_F.) Write $\pi_F(\gamma)$ for the irreducible unipotent representation of G'_F corresponding to $\gamma \in \Xi(L,\mathfrak{g}(-1))$. Fix an L orbit $\mathcal{O}$ on $\mathfrak{g}(-1)$. Just as above, we can define a micro-packet $\Pi^g_F(\mathcal{O})$ consisting of those $\pi_F(\gamma)$ such that the conormal bundle to $\mathcal{O}$ occurs in the characteristic cycle of $\mathrm{per}(\gamma)$. Vogan has proposed a definition of Arthur packets as a certain special kind of micro-packets in the p-adic case. See [Vo] and [CFMMX].

Because φ preserves microlocal multiplicities according to Theorem 3.1, we conclude that φ of (3.4) maps micro-packets for real groups into micro-packets for p-adic groups. More precisely, fix $\Pi^g_{\mathbb{R}}(Q_{\mathcal{O}}) \subset \Pi_Y$. Then

$$\left\{\pi_F(\varphi(\gamma)) \mid \pi_{\mathbb{R}}(\gamma) \in \Pi^g_{\mathbb{R}}(Q_{\mathcal{O}})\right\} \tag{6.1}$$

is contained in micro-packet for G'_F parametrized by $\mathcal{O}$; if φ is surjective, then this will be the entire packet.

For example, consider $G = \mathrm{GL}(n,\mathbb{C})$. In [BST], we found occurrences of the Kashiwara-Saito singularity in K orbits on $\mathcal{P}$, and deduced the existence of reducible characteristic cycles. In the notation above, this gives examples of micro-packets of the form $\Pi^g_{\mathbb{R}}(Q)$ with more than one element for $\mathrm{GL}(n,\mathbb{R})$. From the proof of Theorem 3.1, one immediately deduces the occurrence of the Kashiwara-Saito singularity in the closure of an L orbit on $\mathfrak{g}(-1)$, and the existence of micro-packets for $\mathrm{GL}(n,F)$ with more than one element, as discovered by [CFK]. (It seems plausible that all of the interesting examples in [CFK] are accounted for by matching with the real case as the choice of ordering on $\mathscr{P}$ varies.) In any event, since the theory in the real case is better developed, it is interesting to study other classical p-adic cases from this viewpoint. We would like to return to this elsewhere.

Finally, one especially interesting class of micro-packets for real groups are the special unipotent Arthur packets of [ABV, Chapter 27]. The construction of (6.1) applied to special unipotent packets for real classical groups should give rise to what one might call special unipotent packets for split p-adic classical groups. It would be interesting to compare this notion with the definition recently given by Ciubotaru, Mason-Brown, and Okada in [CiMBO].

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