arXiv:math/0310140v1 [math.RT] 9 Oct 2003

Generalized Harish-Chandra Modules: A New Direction in the Structure Theory of Representations

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Abstract. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbf{C}$. We say that a $\mathfrak{g}$-module M is a generalized Harish-Chandra module if, for some subalgebra $\mathfrak{k} \subset \mathfrak{g}$, M is locally $\mathfrak{k}$-finite and has finite $\mathfrak{k}$-multiplicities. We believe that the problem of classifying all irreducible generalized Harish-Chandra modules could be tractable. In this paper, we review the recent success with the case when $\mathfrak{k}$ is a Cartan subalgebra. We also review the recent determination of which reductive in $\mathfrak{g}$ subalgebras $\mathfrak{k}$ are essential to a classification. Finally, we present in detail the emerging picture for the case when $\mathfrak{k}$ is a principal 3-dimensional subalgebra.

INTRODUCTION

Deep mathematical theories are usually rooted in a combination of ideas. It is also true that the core of mathematics, whatever it may be, consists of theories which, due to their complexity, have taken a long time to mature. Representation theory is a perfect illustration of both of these statements. By the end of the 20^{th} century, representation theory has grown to an enormous subject, with many different aspects and with complex relations to theoretical physics, and with flavors ranging from combinatorics, through abstract algebra, algebraic geometry, homological algebra, to harmonic analysis and mathematical physics.

A central part of the foundation of representation theory is the Cartan-Killing classification of finite dimensional complex simple Lie algebras, or equivalently of all connected reduced Dynkin diagrams. Moreover, many other fundamental results which have shaped the face of representation theory are also classifications. Strictly speaking, the representation theory of Lie groups or Lie algebras starts with Cartan's classification of irreducible finite dimensional representations, i.e. with the classification of integral dominant weights. One may say that the skeleton of representation theory consists of several explicit results and classifications such as H. Weyl's character formula, the formula for the multiplicity of one Verma module in another (commonly referred to as the Kazhdan-Lusztig conjecture) the classification of Harish-Chandra modules, the classification of simple Lie superalgebras, the constructions of Kac-Moody algebras and

Math. Subject Classification 2000: Primary 17B10, Secondary 22E46, 22E47

quantum groups, etc. Some classification problems have been agreed upon to be unrealistic: a classical example is the problem of classifying all irreducible representations of a simple Lie algebra of rank greater than 1.

We believe that in the last decade a combination of conceptual developments has led to a possibility to restate this latter problem, in a more restrictive but still enormously general way, and to turn it into a tractable problem. More precisely, let $\mathfrak{g}$ be a fixed complex reductive Lie algebra. We are interested in the problem of classifying all irreducible $\mathfrak{g}$-modules M which have finite multiplicities as $\mathfrak{g}[M]$-modules, where $\mathfrak{g}[M] \subset \mathfrak{g}$ is the subalgebra of all elements of $\mathfrak{g}$ acting locally finitely on M, i.e. $g \in \mathfrak{g}[M]$, if, for any $m \in M$, the span $\langle m, g \cdot m, g^2 \cdot m, \dots \rangle_{\mathbf{C}}$ is a finite dimensional subspace of M.

The purpose of the present paper is to provide a brief review of interrelated concepts and results which have led to this problem, to describe the status quo, and to present some recent new results. Clearly, Harish-Chandra modules play a central role in the subject, but we do not present them in this paper. Two fundamental references on Harish-Chandra modules are [V] and [KV]. We also omit all proofs given elsewhere in the literature.

Acknowledgement

We thank Professors G. van Dijk and V. Molchanov for giving us the opportunity to present our results at the conference "Representations of Lie Groups, Harmonic Analysis on Homogeneous Spaces and Quantization" in Leiden, December 2002. We are also grateful to Professor V. Serganova for some very interesting and stimulating discussions of the subject. Finally, we thank Professor J. Willenbring for the permission to state his unpublished joint result with the second named author.

Notational conventions

The ground field is $\mathbf{C}$. We set $\mathbf{Z}_+ := \{n \geq 0 \mid n \in \mathbf{Z}\}$. By $\langle \quad \rangle_{\mathbf{Z}_+}$, $\langle \quad \rangle_{\mathbf{R}_+}$ and $\langle \quad \rangle_{\mathbf{C}}$ we denote linear span over $\mathbf{Z}_+, \mathbf{R}_+$, or $\mathbf{C}$. The superscript $*$ indicates dual space. If $\mathfrak{l}$ is a Lie algebra, $Z(\mathfrak{l})$ stands for the center of $\mathfrak{l}$, and if $\mathfrak{l}$ is a Lie subalgebra in a fixed Lie algebra $\mathfrak{g}$, $C(\mathfrak{l})$ and $N(\mathfrak{l})$ denote respectively the centralizer and the normalizer of $\mathfrak{l}$ in $\mathfrak{g}$. If $\mathfrak{l}$ is reductive in $\mathfrak{g}, C(\mathfrak{l})$ is also reductive in $\mathfrak{g}$. The signs $\subset\!\!\!+$ and $\supset\!\!\!\!+$ stand for semidirect sum of Lie algebras. If $\mathfrak{l}$ is a reductive Lie algebra, we set $\mathfrak{l}_{ss} := [\mathfrak{l}, \mathfrak{l}]$.

1. General discussion and statement of the problem

Let $\mathfrak{g}$ be a reductive Lie algebra.

Theorem 1. *Let M be any $\mathfrak{g}$-module. Then the set*
$\mathfrak{g}[M] := \{g \in \mathfrak{g} \mid g$ acts locally finitely on $M\}$ is a Lie subalgebra of $\mathfrak{g}$.

This result has been proved independently and by different methods by V. Kac in [K] and by S. Fernando in [F]. We call $\mathfrak{g}[M]$ the *Fernando-Kac* subalgebra of M, cf. [PSZ].

Let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra. We define a $(\mathfrak{g}, \mathfrak{k})$-*module* M as a $\mathfrak{g}$-module M such that $\mathfrak{k} \subset \mathfrak{g}[M]$. A $(\mathfrak{g}, \mathfrak{k})$-module M is *strict* if $\mathfrak{k} = \mathfrak{g}[M]$. If M is a $(\mathfrak{g}, \mathfrak{k})$-module and N is a finite dimensional irreducible $\mathfrak{k}$-module (or a $\mathfrak{k}$-*type* for short), we define the multiplicity $[M : N]$ as the supremum of the multiplicities $[M' : N]$ for all finite dimensional $\mathfrak{k}$-submodules $M' \subset N$. We say that M is of *finite type over* $\mathfrak{k}$ if $[M : N] < \infty$ for any N, and we say that M is of *infinite type over* $\mathfrak{k}$ if $[M : N] \neq 0$ implies $[M : N] = \infty$ for all N. In [PS2] the following important technical result is proved.

Proposition 1. *Let M be an irreducible $\mathfrak{g}$-module and $\mathfrak{k}$ be a reductive in $\mathfrak{g}$ subalgebra with $\mathfrak{k} \subset \mathfrak{g}[M]$. Then $\mathfrak{k}$ acts semisimply on M, and M has either finite or infinite type over $\mathfrak{k}$. Therefore, there is a canonical isomorphism of $\mathfrak{k}$-modules*

$$M \cong \bigoplus_{W \in \hat{\mathfrak{k}}} Hom_{\mathfrak{k}}(W, M) \otimes W,$$

where $\hat{\mathfrak{k}}$ is the set of equivalence classes of $\mathfrak{k}$-types, and the isotypic components $Hom_{\mathfrak{k}}(W, M) \otimes W$ are either all finite dimensional or all infinite dimensional.

We say that a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is of *finite type* if there exists an irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module M of finite type over $\mathfrak{l}$.

A classical theorem of Harish-Chandra implies that if a subalgebra $\mathfrak{k}$ equals the fixed points of an involution on $\mathfrak{g}$ (in this, the so-called symmetric case, $\mathfrak{k}$ is necessarily reductive in $\mathfrak{g}$), then any irreducible $(\mathfrak{g}, \mathfrak{k})$-module has finite type over $\mathfrak{k}$. Assume $\mathfrak{g}$ is simple and $\mathfrak{k}$ is symmetric. In this case the only possible subalgebras $\mathfrak{l}$ with $\mathfrak{k} \subset \mathfrak{l} \subset \mathfrak{g}$ are parabolic, and it is easy to see that an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M is either a highest weight module or a strict $(\mathfrak{g}, \mathfrak{k})$-module, i.e. in the latter case $\mathfrak{g}[M] = \mathfrak{k}$. For the purpose of this paper, we define a *Harish-Chandra module* as a $(\mathfrak{g}, \mathfrak{k})$-module of finite type over a symmetric subalgebra $\mathfrak{k} \subset \mathfrak{g}$.

Classically Harish-Chandra modules are defined as $(\mathfrak{g}, K)$-modules, where K is a subgroup of an algebraic group G and the Lie algebra $\mathfrak{k}$ of K is a symmetric subalgebra of the Lie algebra $\mathfrak{g}$ of G. This definition is closely related to the above definition, but is not equivalent to it. In the original definition, Harish-Chandra modules have been classified by a monumental effort of several groups of mathematicians including R. Langlands, D. Vogan, A. Beilinson, J. Bernstein and others. The literature on the subject is enormous; see the texts [V] and [KV] (and the references therein) for a presentation and discussion of the classification.

Harish-Chandra modules have their roots in physics and have been recognized to be of fundamental importance because of their specific properties, in particular their relation to unitarizability of representations, and not because of a clear understanding of the place of Harish-Chandra modules among general $\mathfrak{g}$-modules. In our opinion, such an understanding can be based on the notion of the Fernando-Kac subalgebra, which was not part of the original theory. Moreover, this notion enables us to consider Harish-Chandra modules and weight modules, two seemingly unrelated subjects, from a single point of view.

We define a *generalized Harish-Chandra* $\mathfrak{g}$*-module* to be a $\mathfrak{g}$-module M which has finite type over a subalgebra of $\mathfrak{g}[M]$. An example of a generalized Harish-Chandra module is a weight module (the definition is recalled in subsection 2.1 below) with finite weight multiplicities. The theory of weight modules has developed practically independently with that of Harish-Chandra modules and has culminated in O. Mathieu's classification of irreducible weight modules with finite weight multiplicities, [M]. We present a summary of Mathieu's result in subsection 2.2.

We believe that the classification of Harish-Chandra modules, together with Mathieu's classification suggest that the problem of classifying all irreducible generalized Harish-Chandra modules (posed in slightly different terms in [PSZ]) could be tractable. If that is the case, this classification problem could be the ultimate substitute for the unrealistic problem of classifying all irreducible $\mathfrak{g}$-modules. Closely related problems are as follows.

A. For a given subalgebra $\mathfrak{l} \subset \mathfrak{g}$ of finite type, classify all irreducible $(\mathfrak{g}, \mathfrak{l})$-modules of finite type over $\mathfrak{l}$.

B. For a given subalgebra $\mathfrak{k} \subset \mathfrak{g}$, reductive in $\mathfrak{g}$ and of finite type, classify all irreducible $(\mathfrak{g}, \mathfrak{k})$-modules M of finite type over $\mathfrak{k}$ with $\mathfrak{g}[M]_{\mathrm{red}} = \mathfrak{k}$.

C. For a given Fernando-Kac subalgebra $\mathfrak{l} \subset \mathfrak{g}$ of finite type, classify all irreducible strict $(\mathfrak{g}, \mathfrak{l})$-modules of finite type over $\mathfrak{g}$.

No systematic solution of any of the above problems is known except in the cases of Harish-Chandra modules and weight modules. In this paper we give a brief account of some of the developments which have led naturally to those problems. In section 2 below, we present a summary of results on weight modules. In section 3, we summarize some recent results of [PS2] and [PSZ] on the description of Fernando-Kac subalgebras of finite type and related subjects. In particular we give an explicit description of any reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ which is the reductive part of a Fernando-Kac subalgebra $\mathfrak{l} \subset \mathfrak{g}$ of finite type.

In section 4, we discuss the case when $\mathfrak{g}$ is simple and $\mathfrak{k} = \mathfrak{sl}(2)$ is a principal $\mathfrak{sl}(2)$-subalgebra. As the principal $\mathfrak{sl}(2)$-subalgebra is never symmetric, unless $\mathfrak{g} \cong \mathfrak{sl}(2)$ or $\mathfrak{sl}(3)$, the corresponding $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$ are interesting objects of study in the new theory of generalized Harish-Chandra modules. Their existence is ensured by a result of [PSZ], see Theorem 4 below. A careful study of this result in the case when $\mathfrak{k}$ is a principal $\mathfrak{sl}(2)$-subalgebra, shows that the irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules M constructed in [PSZ] are in some sense generic, and not every given $\mathfrak{k}$-type occurs in some module M.

In the present paper, we prove two new results: that any infinite dimensional irreducible $(\mathfrak{g}, \mathfrak{k})$-module of finite type over $\mathfrak{k}$ is strict, and that any fixed $\mathfrak{k}$-type Z occurs in some irreducible $(\mathfrak{g}, \mathfrak{k})$-module as the $\mathfrak{k}$-type of minimal dimension. The second result suggests that some aspects of Vogan's theory of minimal $\mathfrak{k}$-types may carry over to generalized Harish-Chandra modules.

2. The case when $\mathfrak{h} \subset \mathfrak{g}[M]$: irreducible weight modules

The case when $\mathfrak{h} \subset \mathfrak{g}[M]$ is an important case in which much more is known than in the general case. Below we present a very brief survey of known results.

2.1. General invariants. In this subsection M is an arbitrary irreducible $\mathfrak{g}$-module such that $\mathfrak{h} \subset \mathfrak{g}[M]$. We do not assume that M is of finite type over $\mathfrak{g}[M]$ or over $\mathfrak{h}$. Let Δ denote the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $\Delta_{\mathfrak{g}[M]}$ denote the roots of $\mathfrak{g}[M]$ with respect to $\mathfrak{h}$.

Note that if $\mathfrak{h} \subset \mathfrak{g}[M]$, then $\mathfrak{g}[M]$ has a unique reductive part $\mathfrak{g}[M]_{\mathrm{red}}$ (which is the unique maximal reductive subalgebra of $\mathfrak{g}[M]$ and is automatically reductive in $\mathfrak{g}$), and by Proposition 1, $\mathfrak{g}[M]_{\mathrm{red}}$ acts semisimply on M. Consequently, $\mathfrak{h}$ acts also semisimply on M and as an $\mathfrak{h}$-module M has the decomposition

$$M = \bigoplus_{\nu \in \mathfrak{h}^*} M^\nu, \tag{1}$$

where the weight spaces $M^\nu := \{m \in M \mid h \cdot m = \nu(h) \cdot m, \forall h \in \mathfrak{h}\}$ are all either finite dimensional or infinite dimensional. A $\mathfrak{g}$-module which admits a decomposition (1) is by definition a *weight* $\mathfrak{g}$*-module*. We set $\mathrm{supp} M := \{\nu \in \mathfrak{h}^* \mid M^\nu \neq 0\}$. Let Γ_M denote the submonoid of $\langle \Delta \rangle_{\mathbf{Z}}$ generated by $\Delta \backslash \Delta_{\mathfrak{g}[M]}$. We define the M*-decomposition* of Δ, or *the shadow decomposition of* Δ *corresponding to* M, to be the decomposition

$$\Delta = \Delta_M^I \cup \Delta_M^F \cup \Delta_M^+ \cup \Delta_M^-, \tag{2}$$

where

$$\begin{aligned}
\Delta_M^I &:= \{\alpha \in \Delta \mid \alpha \in \langle \Gamma_M \rangle_{\mathbf{R}_+}, -\alpha \in \langle \Gamma_M \rangle_{\mathbf{R}_+}\},\\
\Delta_M^F &:= \{\alpha \in \Delta \mid \alpha \notin \langle \Gamma_M \rangle_{\mathbf{R}_+}, -\alpha \notin \langle \Gamma_M \rangle_{\mathbf{R}_+}\},\\
\Delta_M^+ &:= \{\alpha \in \Delta \mid \alpha \notin \langle \Gamma_M \rangle_{\mathbf{R}_+}, -\alpha \in \langle \Gamma_M \rangle_{\mathbf{R}_+}\},\\
\Delta_M^- &:= \{\alpha \in \Delta \mid \alpha \in \langle \Gamma_M \rangle_{\mathbf{R}_+}, -\alpha \notin \langle \Gamma_M \rangle_{\mathbf{R}_+}\}.
\end{aligned}$$

In particular, the M-decomposition of Δ is determined only by $\mathfrak{g}[M]$. The decomposition (2) induces a decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = (\mathfrak{g}_M^I + \mathfrak{g}_M^F) \oplus \mathfrak{g}_M^+ \oplus \mathfrak{g}_M^-,$$

where

$$\mathfrak{g}_M^I := \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_M^I} \mathfrak{g}^\alpha), \quad \mathfrak{g}_M^F := \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_M^F} \mathfrak{g}^\alpha),$$

$$\mathfrak{g}_M^\pm := \bigoplus_{\alpha \in \Delta_M^\pm} \mathfrak{g}^\alpha.$$

It follows from the main result of [DMP] that $\mathfrak{p}_M := (\mathfrak{g}_M^I + \mathfrak{g}_M^F) \oplus \mathfrak{g}_M^+$ is a parabolic subalgebra whose semisimple part is the direct sum $[\mathfrak{g}_M^I, \mathfrak{g}_M^I] \oplus [\mathfrak{g}_M^F, \mathfrak{g}_M^F]$.

Note that the shadow decomposition of Δ corresponding to M depends only on $\mathfrak{g}[M]$. In general it is not true that the shadow decomposition reconstructs $\mathfrak{g}[M]$. One can only show that

$$\mathfrak{g}[M] = (\mathfrak{g}_M^F + (\mathfrak{g}_M^I \cap \mathfrak{g}[M])) \oplus \mathfrak{g}_M^+,$$

and the results of [PS2] and [PSZ] imply that there are irreducible weight modules with the same shadow decomposition and different Fernando-Kac subalgebras, see subsection 3.3 below.

The shadow decomposition reconstructs "the shape" of supp$M$, more precisely it reconstructs supp M up to adding the support of an arbitrary irreducible (finite dimensional) $\mathfrak{g}_M^F \oplus \mathfrak{g}_M^+$-submodule of M. Indeed, a direct argument, see [PS1], shows that

$$\mathrm{supp} M = \mathrm{supp} M^F + \Gamma_M,$$

where M^F is any irreducible $\mathfrak{g}_M^F \oplus \mathfrak{g}_M^+$-submodule of M.

In general, irreducible weight modules M are not generalized Harish-Chandra modules (as they are not necessarily of finite type over $\mathfrak{g}[M]$) and no classification of irreducible weight modules is available. The following theorem provides a reduction, which can be extended to a classification in the case when M has finite $\mathfrak{h}$-type.

Theorem 2. *M has a unique irreducible $\mathfrak{p}_M$-submodule M^{IF} on which $\mathfrak{g}_M^+$ acts trivially. Furthermore, there is an isomoprhism of $(\mathfrak{g}_M^I + \mathfrak{g}_M^F)$-modules $M^{IF} \simeq M^I \otimes M^F$, where M^I is an irreducible $\mathfrak{g}_M^I$-module with $\mathfrak{g}_M^I = (\mathfrak{g}_M^I)_{M^I}^I$ and M^F is an irreducible finite dimensional $\mathfrak{g}_M^F$-module. Finally, M is the unique irreducible quotient of the induced $\mathfrak{g}$-module $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{p}_M)} M^{IF}$.*

The proof of Theorem 2 see in [DMP].

Theorem 2 reduces the study of an arbitrary irreducible $\mathfrak{g}$-module M with $\mathfrak{h} \subset \mathfrak{g}[M]$ to the study of the irreducible $\mathfrak{g}_M^I$-module M^I. The latter module is a *cuspidal* $\mathfrak{g}_M^I$-module, which by definition means that its shadow decomposition is trivial in the sense that $\Delta^I = (\Delta^I)_{M^I}^I$. Cuspidal modules arise in Harish-Chandra module theory, for if $\mathfrak{g}$ is simple, any Harish-Chandra module M for which $\mathfrak{h} \subset \mathfrak{g}[M]$ and such that M is not a highest (or a lowest) weight module, is necessarily a cuspidal $\mathfrak{g}$-module.

Mathieu's classification result, which we describe in the next subsection, provides an explicit classification of all cuspidal $\mathfrak{g}$-modules with finite dimensional weight spaces (i.e. of finite type over $\mathfrak{h}$), and via Theorem 2 this yields a complete classification of all irreducible generalized Harish-Chandra modules M such that $\mathfrak{h} \subset \mathfrak{g}[M]$ and M has finite type over $\mathfrak{h}$. No classification of irreducible generalized Harish-Chandra modules which have infinite type over $\mathfrak{h}$ is known, except when they are Harish-Chandra modules.

2.2 Fernando's theorem and Mathieu's classification. In [F], S. Fernando constructed the shadow decomposition (2) of any irreducible weight module M of finite type over $\mathfrak{h}$, proved Theorem 2 in that case, and showed that the Fernando-Kac subalgebra of M is determined by the shadow decomposition via the formula:

$$\mathfrak{g}[M] = \mathfrak{g}_M^F \oplus \mathfrak{g}_M^+.$$

In particular, if M is cuspidal, and of finite type over $\mathfrak{h}$, we have $\mathfrak{g}[M] = \mathfrak{h}$. Moreover, in this case all weight spaces M^ν are immediately seen to be of the same dimension d; by definition, M is called then a cuspidal module of *degree* d. In [F] Fernando established also the following key result.

Theorem 3. *The reductive Lie algebra* $\mathfrak{g}$ *admits an irreducible cuspidal weight module of finite type over* $\mathfrak{h}$ *if and only if all simple components of* $\mathfrak{g}_{ss}$ *are of type* A *and* C.

The further trivial observation that any irreducible cuspidal module over $\mathfrak{g}$ is isomorphic to a tensor product of cuspidal irreducible modules over the simple components of $\mathfrak{g}_{ss}$ with a 1-dimensional module over the center of $\mathfrak{g}$, reduces the problem of classifying all cuspidal irreducible $\mathfrak{g}$-modules of finite type over $\mathfrak{h}$ to the same problem for the Lie algebras $\mathfrak{sl}(n+1)$ and $\mathfrak{sp}(2n)$. This latter problem was solved completely in Mathieu's breakthrough paper [M].

Mathieu's main idea is that irreducible cuspidal weight modules come in coherent families, each family being determined by a finite dimensional, irreducible representation of a corresponding maximal reductive root subalgebra. The classification then reduces to describing a continuous parameter (the position of the module within the family) and a mixed (partly continuous, partly discrete) parameter (the highest weight of an irreducible representation of a reductive Lie algebra).

In the rest of this section $\mathfrak{g} = \mathfrak{sl}(n+1)$, $\mathfrak{sp}(2n)$. Following Mathieu we call a (reducible) weight $\mathfrak{g}$-module $\mathcal{M} = \bigoplus_{\nu \in \mathfrak{h}^*} \mathcal{M}^\nu$ a *coherent family of degree* d if $\mathrm{supp}\, \mathcal{M} = \mathfrak{h}^*$, $\dim \mathcal{M}^\nu = \dim \mathcal{M}^\mu = d$, and for any u in the centralizer of $\mathfrak{h}$ in $U(\mathfrak{g})$, the function $\lambda \in \mathfrak{h}^* \mapsto tr\ u\,|_{\mathcal{M}^\lambda}$ is a polynomial in λ. $\mathcal{M}$ is called *semisimple* if it is semisimple as a $\mathfrak{g}$-module.

A *fiber* of the family $\mathcal{M}$ is a $\mathfrak{g}$-submodule $\mathcal{M}'$ of the form $\bigoplus_{\gamma \in \langle \Delta \rangle_{\mathbf{Z}}} \mathcal{M}^{x+\gamma}$ for some $x \in \mathfrak{h}^*$. Clearly $\mathcal{M}$ is the direct sum of all its fibers. By definition, $\mathcal{M}$ is an *irreducible coherent* family if at least one fiber of $\mathcal{M}$ is irreducible (then necessarily almost any fiber of $\mathcal{M}$ is irreducible, see Lemma 4.7 in [M]).

Mathieu's result is that for each irreducible cuspidal weight module M of degree d, there is a unique (up to isomorphism) semisimple coherent family $\tilde{M}$ of degree d for which M is a fiber of $\tilde{M}$. Furthermore, $\tilde{M}$ has a (non-unique) irreducible infinite dimensional highest weight submodule $L(\lambda_M)$ (with respect to a fixed Borel subalgebra of $\mathfrak{g}$) which necessarily has the property that the dimensions of its weight spaces are bounded. More generally we will say that a weight module M has *bounded multiplicities* if, for some ℓ, $\dim M^\nu < \ell$ for all $\nu \in \mathrm{supp}\ M$.[1]

[1] Mathieu uses the term admissible weight module but, since this term is a synonym for a $(\mathfrak{g}, \mathfrak{k})$-module of finite type in Harish-Chandra module theory, we prefer the term weight module with bounded multiplicities.

The classification is based on three main facts:

- that $L(\lambda_M)$ determines $\tilde{M}$ up to isomorphism; in what follows we will write also $\widetilde{L(\lambda_M)}$ for $\tilde{M}$;

- that the correspondence $L(x) \mapsto \widetilde{L(x)}$ is defined for any infinite dimensional highest weight module $L(x)$ with bounded multiplicities;

- that one can describe explicitly all weights x for which $\dim L(x) = \infty$ and $L(x)$ has bounded multiplicities.

Here is, for instance, an explicit description of all such weights x for $\mathfrak{g} = \mathfrak{sp}(2n)$. Fix a basis $\varepsilon_1, \dots, \varepsilon_n$ of $\mathfrak{h}^*$ such that $\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$ is a system of simple roots.

Proposition 2. *([M], Lemma 9.1) Let $\mathfrak{g} = \mathfrak{sp}(2n)$. If $x = \sum_i x_i \varepsilon_i$ and $\dim L(x) = \infty$, then $L(x)$ has bounded multiplicities if and only if $x_i \in \mathbf{Z} + \frac{1}{2}$ and $x_1 > x_2 > \dots > x_{n-1} > |x_n|$.*

For $\mathfrak{g} = \mathfrak{sp}(2n)$ all weights x with the above property form a discrete set. For $\mathfrak{g} = \mathfrak{sl}(n+1)$ this is no longer true and the description is slightly more complicated, see [M].

Mathieu's classification can now be stated from the opposite end as follows.

1. Determine the highest weights x of all infinite dimensional irreducible modules $L(x)$ with bounded multiplicities. Define two such weights x and x' to be equivalent if $\widetilde{L(x)} \simeq \widetilde{L(x')}$. Then the set of equivalence classes parametrizes all semisimple coherent families. In particular, for $\mathfrak{g} = \mathfrak{sp}(2n)$ two weights $x = \Sigma x_i \varepsilon_i$ and $x' = \Sigma x'_i \varepsilon_i$ as above are equivalent if and only if $x'_n = \pm x_n$.

2. Let $\tilde{x}$ be the equivalence class of a weight x as in Step 1, and let $\mathcal{M} = \widetilde{L(x)}$. Describe the subset in $\mathfrak{h}^*/\langle \Delta \rangle_{\mathbf{Z}}$ for which the corresponding fibers on $\mathcal{M}$ are irreducible. Mathieu gives an explicit combinatorial description of this set in terms of $\tilde{x}$ and shows in particular that its complement (corresponding to reducible fibers) is always the union of precisely $n+1$ codimension 1 subsets in $\mathfrak{h}^*/\langle \Delta \rangle_{\mathbf{Z}}$ for $\mathfrak{g} = \mathfrak{sl}(n+1)$, and respectively n codimension 1 subsets for $\mathfrak{g} = \mathfrak{sp}(2n)$. In the latter case, the corresponding conditions on $\eta = (\eta_1, \dots, \eta_n) \in \mathfrak{h}^*/\langle \Delta \rangle_{\mathbf{Z}}$ are simply $\eta_i \notin \mathbf{Z} + \frac{1}{2}$.

3. Finally, in each case a weight x as above determines a natural maximal reductive root subalgebra $\mathfrak{g}'_x \subset \mathfrak{g}$ for which the weight x is dominant. If $\mathfrak{g} = \mathfrak{sl}(n+1)$, then $\mathfrak{g}'_x$ is always isomorphic to $\mathfrak{gl}(n)$, while for $\mathfrak{g} = \mathfrak{sp}(2n)$, we have $\mathfrak{g}'_x = \mathfrak{o}(2n)$. Furthermore, the degree d of the coherent family $\widetilde{L(x)}$ is computed in terms of x and $\mathfrak{g}'_x$ as follows:

- for $\mathfrak{g} = \mathfrak{sp}(2n)$, $d = \frac{1}{2^{n-1}} \dim L_{\mathfrak{g}'_x}(x + \varepsilon)$, where $\varepsilon = \sum_i \varepsilon_i$;

- for $\mathfrak{g} = \mathfrak{sl}(n+1)$, $d = \dim L_{\mathfrak{g}'_x}(x)$ unless the infinitesimal character of $L(x)$ is regular integral; in the latter case d is an alternating sum of dimensions of finite dimensional $\mathfrak{g}'_x$-modules.

The details are in [M].

Mathieu's classification has been generalized by D. Grantcharov [G] to the case of the Lie superalgebra $\mathfrak{sl}(n/1)$, and by I. Dimitrov [Di] to the case of the infinite dimensional Lie algebra $\mathfrak{gl}(\infty)$.

3. Fernando-Kac subalgebras of finite type

A part of a future classification of irreducible generalized Harish-Chandra modules should be a good understanding of their respective Fernando-Kac subalgebras. The problem of classifying all Fernando-Kac subalgebras and all Fernando-Kac subalgebras of finite type of a given reductive Lie algebra $\mathfrak{g}$ has been addressed in the recent papers [PS2] and [PSZ]. Below we present a summary of the results.

3.1. General Fernando-Kac subalgebras. Little is known about a general description of Fernando-Kac subalgebras of irreducible $\mathfrak{g}$-modules which are not generalized Harish-Chandra modules. In [PS2] the following two results are established.

1. An example of a subalgebra of $\mathfrak{sl}(n)$ which is not a Fernando-Kac subalgebra is constructed.

2. It is proved that any subalgebra $\mathfrak{l} \subset \mathfrak{g}$ which contains a Cartan subalgebra of $\mathfrak{g}$ is a Fernando-Kac subalgebra. The proof is an explicit $\mathcal{D}$ -module construction which provides a strict irreducible $(\mathfrak{g}, \mathfrak{l})$-module L. This result, together with Theorem 8 below, implies that not every Fernando-Kac subalgebra is of finite type.

3.2. A description of primal subalgebras. In [PSZ] a *primal subalgebra* of $\mathfrak{g}$ is defined as a reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ for which there exists an irreducible generalized Harish-Chandra module M such that $\mathfrak{k}$ is a maximal reductive subalgebra of $\mathfrak{g}[M]$. A main result of [PSZ] is the following theorem.

Theorem 4. *A reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ is primal if and only if $\mathfrak{k} \cap C(\mathfrak{k}_{ss})$ is a Cartan subalgebra of $C(\mathfrak{k}_{ss})$, or equivalently if $C(\mathfrak{k}) = Z(\mathfrak{k})$.*

The proof of Theorem 4 is also an explicit $\mathcal{D}$-module construction which yields an irreducible generalized Harish-Chandra module M for which $\mathfrak{k}$ is a maximal reductive subalgebra in $\mathfrak{g}[M]$; see Theorems 4.3 and 4.4 in [PSZ].

Theorem 4 gives an explicit description of all primal subalgebras of $\mathfrak{g}$. Indeed, recall that all semisimple subalgebras of a semisimple, or equivalently reductive, Lie algebra have been classified in the fundamental papers [D1] and [D2] of E. Dynkin. Then, Theorem 4 implies that for any semisimple subalgebra $\mathfrak{k}' \subset \mathfrak{g}$, the primal subalgebras $\mathfrak{k}$ with $\mathfrak{k}_{ss} = \mathfrak{k}'$ are precisely all direct sums $\mathfrak{k}' \oplus \mathfrak{h}_{C(\mathfrak{k}')}$, where $\mathfrak{h}_{C(\mathfrak{k}')}$ is any Cartan subalgebra of $C(\mathfrak{k}')$. In particular, Theorem 4 implies that any semisimple subalgebra of $\mathfrak{g}$ is the semisimple part of a Fernando-Kac subalgebra of finite type. Another corollary of Theorem 4 is that every maximal proper subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is a Fernando-Kac subalgebra of finite type. For the proof, as well as for other related corollaries of Theorem 4, see [PSZ].

3.3. Fernando-Kac subalgebras of finite type. In general, the problem of describing all Fernando-Kac subalgebras of finite type of a given reductive (or simple) Lie algebra $\mathfrak{g}$ is open. A complete description of all Fernando-Kac subalgebras of $\mathfrak{g}$ is known under various additional conditions. The following general theorem is proved in [PSZ].

Theorem 5. *Let $\mathfrak{l} \subset \mathfrak{g}$ be a Fernando-Kac subalgebra of finite type.*

1. $N(\mathfrak{l}) = \mathfrak{l}$; hence $\mathfrak{l}$ is an algebraic subalgebra of $\mathfrak{g}$.

2. There is a decomposition $\mathfrak{l} = \mathfrak{n}_{\mathfrak{l}} \supsetplus \mathfrak{l}_{\mathrm{red}}$, unique up to an inner automorphism of $\mathfrak{l}$, where $\mathfrak{l}_{\mathrm{red}}$ is a (maximal) subalgebra of $\mathfrak{l}$ reductive in $\mathfrak{g}$.

3. Any irreducible $(\mathfrak{g}, \mathfrak{l})$-module M of finite type over $\mathfrak{l}$ has finite type over $\mathfrak{l}_{\mathrm{red}}$, and $\mathfrak{l}_{\mathrm{red}}$ acts semisimply on M.

4. $C(\mathfrak{l}_{\mathrm{red}}) = Z(\mathfrak{l}_{\mathrm{red}})$ and $Z(\mathfrak{l}_{\mathrm{red}})$ is a Cartan subalgebra of $C(\mathfrak{l}_{ss})$.

5. $\mathfrak{l} \cap C(\mathfrak{l}_{ss})$ is a solvable Fernando-Kac subalgebra of finite type of $C(\mathfrak{l}_{ss})$.

The following theorem provides a complete description of all solvable Fernando-Kac subalgebras of finite type.

Theorem 6. *A solvable subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is a Fernando-Kac subalgebra of finite type if and only if $\mathfrak{s}$ contains a Cartan subalgebra of $\mathfrak{g}$ and its nilradical $\mathfrak{n}_{\mathfrak{s}}$ is the nilradical of a parabolic subalgebra of $\mathfrak{g}$ whose simple components are all of type A and C.*

For the proof see Proposition 3.2 in [PSZ].

For $\mathfrak{g} = \mathfrak{gl}(n)$, $\mathfrak{sl}(n)$ the following theorem provides a complete description of all reductive Fernando-Kac subalgebras of finite type.

Theorem 7. *(Theorem 5.1 in [PSZ]) A reductive in $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n)$ subalgebra $\mathfrak{k}$ is a Fernando-Kac subalgebra if and only if it is primal. (An explicit description of primal subalgebras is provided by Theorem 4).*

For simple Lie algebras other than of type A, it is not known whether every primal subalgebra is a Fernando-Kac subalgebra of finite type. The following proposition is proved in [PS2] and provides a partial answer to this question.

Proposition 3. *Let $\mathfrak{g}$ be simple of type other than $B_n, n \geq 3$, and F_4. Then any reductive root subalgebra (which is automatically primal by Theorem 4) is a Fernando-Kac subalgebra of finite type.*

We complete this section by a description of all Fernando-Kac subalgebras of finite type of $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n)$ which are root subalgebras. Let $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n)$, $\mathfrak{h}$ be a fixed Cartan subalgebra, and $\mathfrak{l} \supset \mathfrak{h}$ be a root subalgebra of $\mathfrak{g}$. Then $\mathfrak{l}$ is determined by its subset of roots $\Delta(\mathfrak{l}) \subset \Delta$. Let $\mathfrak{k} = \mathfrak{l}_{\mathrm{red}}$ and $\mathfrak{l} = \mathfrak{k} \ni \mathfrak{n}$. Fix an arbitrary Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$ and let N be any finite dimensional semisimple $\mathfrak{k}$-module. Denote by

$\mathcal{S}_{\mathfrak{k}}(N)$ the set of weights of all $\mathfrak{k}\cap\mathfrak{b}$-singular vectors in N, and put $\mathcal{C}_{\mathfrak{k}}(N) := \langle \mathcal{S}_{\mathfrak{k}}(N)\rangle_{\mathbf{Z}_+}$. The following theorem is proved in [PSZ] (Theorem 5.8).

Theorem 8. *A root subalgebra $\mathfrak{l} = \mathfrak{k} \ni \mathfrak{n} = \mathfrak{h}$ of $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n)$ is a Fernando-Kac subalgebra of finite type if and only if $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$.*

It is impossible not to notice that the condition in Theorem 8 is a generalization of a parabolic or "triangular" decomposition: if $\mathfrak{k} = \mathfrak{h}$, it means precisely that $\mathfrak{n}$ is the complement of a parabolic subalgebra, cf. Theorem 6 above. It would be very interesting to find the analog of Theorem 8 for a general simple Lie algebra.

Finally, note that Theorem 8 implies that any reductive root subalgebra $\mathfrak{k}$ of $\mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n)$ is a Fernando-Kac subalgebra of finite type. Furthermore, it is easy to check that any strict $(\mathfrak{g}, \mathfrak{k})$-module M is cuspidal and is necessarily of infinite type over $\mathfrak{h}$ unless $\mathfrak{k} = \mathfrak{h}$. In particular this shows, as claimed in subsection 2.1 above, that in general the shadow decomposition of an irreducible $\mathfrak{g}$-module M does not determine the subalgebra $\mathfrak{g}[M] \subset \mathfrak{g}$.

4. A case when $\mathfrak{h} \not\subset \mathfrak{g}[M]$: a principal $\mathfrak{sl}(2)$-subalgebra

In this section, we consider a specific, yet broad enough, class of generalized Harish-Chandra modules which has not been discussed in the literature.

According to Theorem 4, if $\mathfrak{g}$ is simple and $\mathfrak{k} \subset \mathfrak{g}$ is a simple subalgebra with $C(\mathfrak{k}) = 0$, then $\mathfrak{k}$ is primal. If, in addition, we know that any intermediate subalgebra $\mathfrak{l}, \mathfrak{k} \subset \mathfrak{l} \subset \mathfrak{g}$, is reductive, Theorem 4 implies that $\mathfrak{k}$ is itself a Fernando-Kac subalgebra of finite type. An important example of such a situation is when $\mathfrak{k}$ is a principal 3-dimensional subalgebra. More precisely, for any simple $\mathfrak{g}$, an injective homomorphism $\mathfrak{sl}(2) \to \mathfrak{g}$ is called *principal* if its image contains a regular nilpotent element. Principal $\mathfrak{sl}(2)$-subalgebras of simple Lie algebras have been studied in detail by [D2] and [Ko]. In particular, it is true that every intermediate subalgebra $\mathfrak{k} \subset \mathfrak{l} \subset \mathfrak{g}$ (where $\mathfrak{k}$ is a principal $\mathfrak{sl}(2)$-subalgebra) is semisimple, see Proposition 5 below. Hence $\mathfrak{k}$ is a Fernando-Kac subalgebra of finite type, and certain irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules of finite type have been constructed in [PSZ]. In the present paper, we prove the following more detailed result.

Theorem 9. *Let $\mathfrak{g}$ be simple, of rank greater or equal to 2, and $\mathfrak{k} \subset \mathfrak{g}$ be a principal $\mathfrak{sl}(2)$-subalgebra.*

(i) *Any infinite dimensional irreducible $(\mathfrak{g}, \mathfrak{k})$-module of finite type over $\mathfrak{k}$ is strict.*

(ii) *Let Z be a fixed $\mathfrak{k}$-type. There exists an irreducible infinite dimensional (and thus strict) $(\mathfrak{g}, \mathfrak{k})$-module X of finite type over $\mathfrak{k}$ such that Z occurs in X with multiplicity 1 and for any other $\mathfrak{k}$-type W which occurs in X, $\dim W > \dim Z$.*

In the next subsection we prove (i). To prove (ii) we need to recall some basic facts about a cohomological method of constructing $(\mathfrak{g}, \mathfrak{k})$-modules. We do this in subsection 4.2. Finally, we present the proof of (ii) in subsection 4.3.

4.1. A result about finite dimensional modules and its application to generalized Harish-Chandra modules. J. Willenbring and the second named author recently proved the following fact.

Proposition 4. *Let $\mathfrak{k}$ be an $\mathfrak{sl}(2)$-subalgebra of a semisimple Lie algebra $\mathfrak{s}$, none of whose simple factors is isomorphic to $\mathfrak{sl}(2)$. Then there exists a positive integer $b(\mathfrak{s})$, such that for every irreducible finite dimensional $\mathfrak{s}$-module V, there exists an injection of $\mathfrak{k}$-modules $W \to V$, where W is an irreducible $\mathfrak{k}$-module of dimension less than $b(\mathfrak{s})$.*

A more general statement will be proved in [WZ]. The proof uses invariant theory.

To relate Proposition 4 with the statement of Theorem 9 (i), we recall the notion of a stem subalgebra. In a slight modification of the original terminology of [D2], a *stem subalgebra* is defined in [PSZ] as a subalgebra $\mathfrak{s}$ of a semisimple Lie algebra $\mathfrak{r}$ which is not contained in a root subalgebra of $\mathfrak{r}$. The following result is established in [D2] (see p. 160).

Proposition 5. *Let $\mathfrak{s} \subset \mathfrak{r}$ be a stem subalgebra. Then any intermediate subalgebra $\mathfrak{s}'$, $\mathfrak{s} \subset \mathfrak{s}' \subset \mathfrak{r}$, is a semisimple stem subalgebra. If $\mathfrak{r}$ is simple and $\mathfrak{l}$ is a principal $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{r}$, then $\mathfrak{k}$ is a stem subalgebra of $\mathfrak{r}$, and for any intermediate subalgebra $\mathfrak{s}'$, no simple factor of $\mathfrak{s}'$ is isomorphic to $\mathfrak{sl}(2)$.*

Propositions 4 and 5 imply Theorem 9 (i). Indeed, consider any infinite dimensional irreducible $(\mathfrak{g}, \mathfrak{k})$-module M of finite type, where $\mathfrak{g}$ is simple and $\mathfrak{k}$ is a principal $\mathfrak{sl}(2)$-subalgebra. By Proposition 5, $\mathfrak{g}[M]$ is a semisimple proper subalgebra of $\mathfrak{g}$, and no simple factor of $g[M]$ is isomorphic to $\mathfrak{sl}(2)$. Theorem 9 (i) is then equivalent to the claim that $\mathfrak{k} = \mathfrak{g}[M]$. Assume, to the contrary, that the inclusion $\mathfrak{k} \subset \mathfrak{g}[M]$ is proper. Then M has finite type over $\mathfrak{g}[M]$, and as $\dim M = \infty$, infinitely many $\mathfrak{g}[M]$-types occur in M. Now Proposition 4 implies that the multiplicity of some $\mathfrak{k}$-type W_0 with $\dim W_0 < b(\mathfrak{g}[M])$ is infinite in M. This proves Theorem 9 (i).

4.2. Generalities on cohomological induction. Let $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ be a triple of reductive Lie algebras such that $\mathfrak{t}$ is reductive in both $\mathfrak{k}$ and $\mathfrak{g}$ and $\mathfrak{k}$ is reductive in $\mathfrak{g}$. Let $C(\mathfrak{g}, \mathfrak{k})$ (respectively, $C(\mathfrak{g}, \mathfrak{t})$) be the category of $(\mathfrak{g}, \mathfrak{k})$-modules (resp. $(\mathfrak{g}, \mathfrak{t})$-modules) which are semisimple as $\mathfrak{k}$-modules (resp. $\mathfrak{t}$-modules). Note that, by Proposition 1, every irreducible $(\mathfrak{g}, \mathfrak{k})$-module (resp. $(\mathfrak{g}, \mathfrak{t})$-module) is an object of $C(\mathfrak{g}, \mathfrak{k})$ (resp. $C(\mathfrak{g}, \mathfrak{t})$). Furthermore, it is well known that

$$\begin{aligned} &\Gamma_{\mathfrak{k},\mathfrak{t}} : C(\mathfrak{g}, \mathfrak{t}) \rightsquigarrow C(\mathfrak{g}, \mathfrak{k}) \\ &V \rightsquigarrow \Sigma_{W \subset V, \dim W = 1, \dim U(\mathfrak{k}) \cdot W < \infty} \; W \end{aligned}$$

is a well-defined left exact functor. We denote by $R^i\Gamma_{\mathfrak{k},\mathfrak{t}}$ its i-th right derived functor. The following theorems summarize some important properties of the functors $R^i\Gamma_{\mathfrak{k},\mathfrak{t}}$.

Theorem 10. [EW] *Let M be an object of $C(\mathfrak{g}, \mathfrak{t})$.*

(i) $R^i\Gamma_{\mathfrak{k},\mathfrak{t}}(M) = 0$ *for* $i > \dim \mathfrak{k} - \dim \mathfrak{t}$.
(ii) *If M has finite type over $\mathfrak{t}$, then $R^i\Gamma_{\mathfrak{k},\mathfrak{t}}(M)$ is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type over $\mathfrak{k}$ for every $i \geq 0$.*

(iii) *If M has finite type over $\mathfrak{t}$, then for each $i \leq \dim\mathfrak{k} - \dim\mathfrak{t}$, there is a natural isomorphism of $(\mathfrak{g},\mathfrak{k})$-modules*

$$R^i\Gamma_{\mathfrak{k},\mathfrak{t}}(M) \cong (R^{\dim\mathfrak{k}-\dim\mathfrak{t}-i}\Gamma_{\mathfrak{k},\mathfrak{t}}(M^*_{\mathfrak{t}}))^*_{\mathfrak{k}},$$

where N^* denotes the $\mathfrak{g}$-module dual to a $\mathfrak{g}$-module N, and $N^*_{\mathfrak{t}}$ (respectively, $N^*_{\mathfrak{k}}$) stands for the maximal submodule of N^* which is an object of $C(\mathfrak{g},\mathfrak{t})$ (resp., $C(\mathfrak{g},\mathfrak{k})$).

Theorem 11. [V] *Suppose M is an object of finite type in $C(\mathfrak{g},\mathfrak{t})$ and W is an irreducible finite dimensional $\mathfrak{k}$-module. Then*

$$\sum_i (-1)^i \dim Hom_{\mathfrak{k}}(W, R^i\Gamma_{\mathfrak{k},\mathfrak{t}}(M)) =$$
$$\sum_i (-1)^i \dim Hom_{\mathfrak{t}}(W \otimes \Lambda^i(\mathfrak{k}/\mathfrak{t}), M),$$

where Λ^i stands for i-th exterior power.

Finally, assume that $\mathfrak{t}$ is abelian and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \supset \mathfrak{t}$. Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{b}$ and let $M(\lambda)$ denote the Verma module $U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$, where $\mathbf{C}_\mu$ is the 1-dimensional $\mathfrak{h}$-module on which $\mathfrak{h}$ acts via $\mu : \mathfrak{h} \to \mathbf{C}$. Then, for each $i \geq 0$, we define the family of $(\mathfrak{g},\mathfrak{k})$-modules $A^i(\lambda)$ by setting

$$A^i(\lambda) := R^i\Gamma_{\mathfrak{k},\mathfrak{t}}(M(\lambda)).$$

In general, $A^i(\lambda)$ need not be of finite type over $\mathfrak{k}$ as $M(\lambda)$ may not be of finite type over $\mathfrak{t}$. However, to ensure this latter condition, it suffices to assume that $\mathfrak{t}$ contains an element with strictly positive real eigenvalues in $\mathfrak{n} := [\mathfrak{b},\mathfrak{b}]$. Under this assumption, the construction of $A^i(\lambda)$ is a close analog of a construction in Harish-Chandra module theory known as cohomological induction, cf. [KV].

4.3. Proof of Theorem 9 (ii). Now let $\mathfrak{k} \subset \mathfrak{g}$ be a principal $\mathfrak{sl}(2)$-subalgebra, and $\{e,h,f\}$ be a standard basis for $\mathfrak{k}$. Then h is a regular semisimple element in $\mathfrak{g}$, and $ad_h : \mathfrak{g} \to \mathfrak{g}$ has even integral eigenvalues, since as a $\mathfrak{k}$-module $\mathfrak{g}$ is isomorphic to a direct sum of irreducible odd dimensional modules, see [D2]. We set $\rho_{\mathfrak{k}} := \frac{\alpha}{2}$, where α is the root of $\mathfrak{k}$ with root space $\langle e\rangle_{\mathbf{C}}$.

Let $\mathfrak{t} := \langle h\rangle_{\mathbf{C}}$. Then $\mathfrak{h} := C(\mathfrak{t})$ is a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \supset \mathfrak{t}$. Define the Borel subalgebra $\bar{\mathfrak{b}} \supset \mathfrak{h}$ as the sum of all nonnegative eigenspaces of ad_h, and put $\bar{\mathfrak{n}} := [\bar{\mathfrak{b}},\bar{\mathfrak{b}}]$. Then $\bar{\mathfrak{n}} = \langle e\rangle_{\mathbf{C}} + \bar{\mathfrak{n}} \cap \mathfrak{k}^\perp$, where $\perp$ stands for orthogonal complement with respect to the Killing form on $\mathfrak{g}$. Furthermore, for any weight x of $\bar{\mathfrak{n}} \cap \mathfrak{k}^\perp$ as a module over $\langle h\rangle_{\mathbf{C}}$, $x(h)$ is a positive even integer. Finally, let $\mathfrak{b}$ be the Borel subalgebra opposite to $\mathfrak{b}$, i.e. $\mathfrak{b} \cap \bar{\mathfrak{b}} = \mathfrak{h}$.

Consider the $(\mathfrak{g}, \mathfrak{k})$-modules $A^0(\lambda)$, $A^1(\lambda)$ and $A^2(\lambda)$ defined as above (by generalized cohomological induction) for the fixed $\mathfrak{t}, \mathfrak{k}, \mathfrak{b}$ and $\mathfrak{h}$. As the eigenvalues of ad_{-h} on $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ are positive, the $\mathfrak{g}$-modules $A^0(\lambda), A^1(\lambda), A^2(\lambda)$ have finite type over $\mathfrak{k}$. We claim first that $A^0(\lambda) = 0$ for any λ. Indeed, by definition $A^0(\lambda) = \Gamma_{\mathfrak{k},\mathfrak{t}}(M(\lambda))$ and thus $\mathfrak{k}, \mathfrak{b} \subset \mathfrak{g}[\Gamma_{\mathfrak{k},\mathfrak{t}}(M(\lambda))]$. Since $\mathfrak{k}$ is a stem subalgebra, $\mathfrak{k}$ and $\mathfrak{b}$ generate $\mathfrak{g}$, i.e. $\mathfrak{g}[A^0(\lambda)] = \mathfrak{g}$. Hence $A^0(\lambda) = 0$ as $M(\lambda)$ has no nontrivial integrable $\mathfrak{g}$-submodules.

We claim next that $A^2(\lambda) = 0$ for any nonintegral λ. This follows from Theorem 10 (iii), once we check that $\Gamma_{\mathfrak{k},\mathfrak{t}}(M(\lambda)^*_{\mathfrak{t}}) = 0$. We easily see that $M(\lambda)^*_{\mathfrak{t}} = M(\lambda)^*_{\mathfrak{h}}$. Since λ is nonintegral, $M(\lambda)$ has no finite dimensional nontrivial quotient $\mathfrak{g}$-module and thus $M(\lambda)^*_{\mathfrak{h}}$ has no nontrivial integrable $\mathfrak{g}$-submodule.

The following proposition is a statement about the structure of $A^1(\lambda)$ as a $\mathfrak{k}$-module and is proved by a non-difficult computation based on Theorem 11. If $\mu \in \mathfrak{t}^*$ is a dominant integral weight for $\mathfrak{k}$, let $W(\mu)$ be the irreducible finite dimensional $\mathfrak{k}$-module with highest weight μ. For $\nu \in \mathfrak{t}^*$, let $P_{\bar{\mathfrak{n}}\cap\mathfrak{k}^\perp}(\nu)$ be the multiplicity of ν in the symmetric algebra on $\bar{\mathfrak{n}} \cap \mathfrak{k}^\perp$.

Proposition 6. *For any dominant integral weight μ for $\mathfrak{k}$ and for any non $\mathfrak{g}$-integral $\lambda \in \mathfrak{h}^*$, we have*

$$\dim Hom_{\mathfrak{k}}(W(\mu), A^1(\lambda)) = P_{\bar{\mathfrak{n}}\cap\mathfrak{k}^\perp}(\mu - \lambda\,|_t + 2\rho_{\mathfrak{k}}) - P_{\bar{\mathfrak{n}}\cap\mathfrak{k}^\perp}(-\mu - \lambda\,|_{\mathfrak{t}}). \tag{3}$$

Note that if $0 \neq \lambda(h) \in \mathbf{Z}_+$, then the second term on the right hand side of (3) vanishes. This leads to the following corollary.

Corollary 1. *Suppose $\lambda(h) - 2 = n \in \mathbf{Z}_+$. Then $W(n\rho_{\mathfrak{k}})$ occurs with multiplicity 1 in $A^1(\lambda)$, and $m \geq n$ whenever $W(m\rho_{\mathfrak{k}})$ occurs in $A^1(\lambda)$.*

The proof of Theorem 9(ii) is now immediate. Let the highest weight of Z be $n\rho_{\mathfrak{k}}$ for $n \in \mathbf{Z}_+$. Choose $\lambda \in \mathfrak{h}^*$ so that λ is not $\mathfrak{g}$-integral but such that $\lambda(h) - 2 = n$. By Corollary 1, $Z \cong W(n\rho_{\mathfrak{k}})$ occurs with multiplicity 1 in $A^1(\lambda)$ and $m > n$ whenever $W(m\rho_{\mathfrak{k}})$ occurs in $A^1(\lambda)$. Define X as an irreducible quotient of the $\mathfrak{g}$-submodule of $A^1(\lambda)$ generated by Z. Then Z occurs with multiplicity 1 in X and $m > n$ whenever $W(m\rho_{\mathfrak{k}})$ occurs in X.

As λ is not $\mathfrak{g}$-integral, the infinitesimal parameter of $M(\lambda)$, and hence (see [V]) of $A^1(\lambda)$ and X, is not integral. Thus X is infinite dimensional. $\square$

Remark. If we fix n and set $\lambda(h) - 2 = n$, then λ still has $\ell - 1$ complex parameters ($\ell = \dim \mathfrak{h}$). It will be quite interesting to determine the $\mathfrak{g}$-module structure of $A^1(\lambda)$ as λ varies continuously with $\lambda(h)$ fixed. Will $A^1(\lambda)$ have finite length for all λ ? This is known (see [V]) to be true if $\mathfrak{g} = \mathfrak{sl}(3)$.

References

[DMP] I. Dimitrov, O. Mathieu, I. Penkov, *On the structure of weight modules*, Trans. Amer. Math. Soc. **352** (2000), 2857-2869.

[Di] I, Dimitrov, Work in progress.

[D1] E. Dynkin, *Maximal subgroups of the classical groups*, (Russian) Trudy Moskov, Mat Obšč **1** (1952) 36-166, English: AMS Translations **6** (1957), 245-389.

[D2] E. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, (Russian) Mat. Sbornik N.S. **30** (**72**) (1952) 349-462, English: AMS Translations **6** (1957), 111-244.

[EW] T.J. Enright, N.R. Wallach, *Notes on homological algebra and representations of Lie algebras*, Duke Math. J. **47** (1980), 1-15.

[F] S. Fernando, *Lie algebra modules with finite dimensional weight spaces I*, Trans. Amer. Math. Soc. **322** (1990), 757-781.

[G] D. Grantcharov, *Coherent families of weight modules of Lie superalgebras and an explicit description of the simple admissible* $\mathfrak{sl}(n/1)$*-modules*, J. Algebra **265** (2003), 711-733.

[K] V. Kac, *Constructing groups associated to infinite dimensional Lie algebras, Infinite dimensional groups with applications*, (Berkeley, 1984), Math. Sci. Res. Inst. Publ. **4**, 167-216.

[KV] A. Knapp, D. Vogan, *Cohomological induction and unitary representations*, Princeton Mathematical Series **45**, Princeton University Press, Princeton (1995).

[Ko] B. Kostant, *The principal three dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973-1032.

[M] O. Mathieu, *Classification of irreducible weight modules*, Ann. Inst. Fourier **50** (2000), 537-592.

[PS1] I. Penkov, V. Serganova, *The support of an irreducible Lie algebra representation*, J. Algebra **209** (1998), 129-142.

[PS2] I. Penkov, V. Serganova, *Generalized Harish-Chandra modules*, Moscow Math. J. **2** (2003), 753-767.

[PSZ] I. Penkov, V. Serganova, G. Zuckerman, *On the existence of* $(\mathfrak{g}, \mathfrak{k})$*-modules of finite type*, Max-Planck Institute preprint (2003).

[V] D. Vogan, *Representations of real reductive Lie groups*, Progress in Mathematics **15**, Birkhäuser, Boston (1981).

[WZ] J. Willenbring, G. Zuckerman, Work in progress.

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