ON THE EXISTENCE OF $(\mathfrak{g}, \mathfrak{k})$-MODULES OF FINITE TYPE

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To Israel Moiseevich Gelfand on his 90th birthday

Abstract

Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field of characteristic zero, and let $\mathfrak{k}$ be a subalgebra reductive in $\mathfrak{g}$. We prove that $\mathfrak{g}$ admits an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M which has finite $\mathfrak{k}$-multiplicities and which is not a $(\mathfrak{g}, \mathfrak{k}')$-module for any proper inclusion of reductive subalgebras $\mathfrak{k} \subset \mathfrak{k}' \subset \mathfrak{g}$ if and only if $\mathfrak{k}$ contains its centralizer in $\mathfrak{g}$. The main point of the proof is a geometric construction of $(\mathfrak{g}, \mathfrak{k})$-modules which is analogous to cohomological induction. For $\mathfrak{g} = \mathfrak{gl}(n)$ we show that whenever $\mathfrak{k}$ contains its centralizer, there is an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M of finite type over $\mathfrak{k}$ such that $\mathfrak{k}$ coincides with the subalgebra of all $g \in \mathfrak{g}$ which act locally finitely on M. Finally, for a root subalgebra $\mathfrak{k} \subset \mathfrak{gl}(n)$, we describe all possibilities for the subalgebra $\mathfrak{l} \supset \mathfrak{k}$ of all elements acting locally finitely on some M.

1. Introduction

Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field of characteristic zero, and let $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra reductive in $\mathfrak{g}$. In his program talk [G], I. Gelfand introduced the notion of a $(\mathfrak{g}, \mathfrak{k})$-module with finite $\mathfrak{k}$-multiplicities. This paper focuses on a new notion relevant to Gelfand's program: we call $\mathfrak{k}$ *primal* if $\mathfrak{g}$ admits an irreducible $(\mathfrak{g}, \mathfrak{k})$-module with finite $\mathfrak{k}$-multiplicities which is not a $(\mathfrak{g}, \mathfrak{k}')$-module for any proper inclusion of reductive subalgebras $\mathfrak{k} \subset \mathfrak{k}' \subset \mathfrak{g}$. Our central result is that $\mathfrak{k}$ is primal if and only if $\mathfrak{k}$ contains its centralizer in $\mathfrak{g}$ or, equivalently, if and only if $\mathfrak{k}$ is a direct sum of a semisimple subalgebra $\mathfrak{k}'$ in $\mathfrak{g}$ and a Cartan subalgebra of the centralizer $C(\mathfrak{k}')$ in $\mathfrak{g}$. This provides a complete description of all primal subalgebras, as the

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semisimple subalgebras of a reductive Lie algebra have been classified by E. Dynkin [D].

Here is a brief account of our motivation. It is common wisdom that classifying all irreducible representations of a reductive Lie algebra $\mathfrak{g}$ is not a well-posed problem. In contrast with that, classifying irreducible representations with natural finiteness properties has remained a core problem in representation theory since the work of E. Cartan and H. Weyl. A landmark success has been the celebrated classification of irreducible Harish-Chandra modules (see [V, Chapter 6], [KV, Chapter 11]). The case of $(G_2, \mathfrak{sl}(3))$-modules has been considered by P. Kekäläinen in [Ke] and by G. Savin in [S]. In 1998 O. Mathieu [M] (following up on work of S. Fernando and others) obtained a very important different classification: that of irreducible weight modules with finite-dimensional weight spaces.

In [PS] it was noticed that both classifications, those of irreducible Harish-Chandra modules and those of irreducible weight modules, are particular cases of the problem of classifying irreducible $\mathfrak{g}$-modules that have finite type over their Fernando-Kac subalgebra. The *Fernando-Kac subalgebra* $\mathfrak{g}[M]$ associated to an irreducible $\mathfrak{g}$-module M is by definition the set of all elements in $\mathfrak{g}$ which act locally finitely on M. The fact that $\mathfrak{g}[M]$ is a Lie subalgebra of $\mathfrak{g}$ was discovered independently by S. Fernando [F] and V. Kac [K]. Furthermore, M is of *finite type* over a given subalgebra $\mathfrak{l} \subset \mathfrak{g}[M]$ if the multiplicity of an arbitrary fixed irreducible $\mathfrak{l}$-module in any (varying) finite-dimensional $\mathfrak{l}$-submodule of M is bounded. The subalgebra $\mathfrak{l}$ is called a *Fernando-Kac subalgebra of finite type* if $\mathfrak{g}$ admits an irreducible $\mathfrak{g}$-module M with $\mathfrak{g}[M] = \mathfrak{l}$ which is of finite type over $\mathfrak{l}$. The problem of classifying all, not necessarily reductive, Fernando-Kac subalgebras of finite type is of fundamental importance for the structure theory of $\mathfrak{g}$-modules. In this article we classify the reductive parts of Fernando-Kac subalgebras of finite type, as a subalgebra is primal if and only if it is a reductive part of a Fernando-Kac subalgebra of finite type.

A short outline of the paper is as follows. In Section 3 we establish some necessary (but, in general, not sufficient) conditions for a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ to be a Fernando-Kac subalgebra of finite type. We show in particular that a Fernando-Kac subalgebra of finite type $\mathfrak{l}$ is algebraic and admits a natural decomposition $\mathfrak{l} = \mathfrak{l}_{\mathrm{red}} \supsetplus \mathfrak{n}_{\mathfrak{l}}$, where $\mathfrak{l}_{\mathrm{red}}$ is a reductive in $\mathfrak{g}$ subalgebra that contains its centralizer and $\mathfrak{n}_{\mathfrak{l}}$ is a nilpotent ideal in $\mathfrak{l}$. We also characterize completely all solvable Fernando-Kac subalgebras of finite type in $\mathfrak{g}$. In Section 4 we fix an arbitrary algebraic subalgebra $\mathfrak{k}$, reductive in $\mathfrak{g}$, and we construct irreducible $(\mathfrak{g}, \mathfrak{k})$-modules M of finite type over $\mathfrak{k}$. The construction of M is a $\mathscr{D}$-module version of cohomological induction: M equals the global sections of a $\mathscr{D}^{\mu}$-module supported on the preimage in G/B of $K \cdot P \subset G/P$ for a suitable parabolic subgroup $P \subset G$. Here G is a connected algebraic group with Lie algebra $\mathfrak{g}$, and K is a connected subgroup with Lie algebra $\mathfrak{k}$. We next show that if $\mathfrak{k}$ contains its centralizer in $\mathfrak{g}$, then $\mathfrak{g}[M]_{\mathrm{red}} = \mathfrak{k}$ for some M. Therefore $\mathfrak{k}$ is primal if and only if it contains its centralizer. Furthermore, as a corollary we obtain that any semisimple subalgebra of $\mathfrak{g}$ is the derived subalgebra of a primal subalgebra and that any subalgebra that is not a proper subalgebra of a maximal root subalgebra is a Fernando-Kac subalgebra of finite type. In Section 5 we consider in more detail the case $\mathfrak{g} = \mathfrak{gl}(n)$. We prove that here any primal subalgebra $\mathfrak{k}$ is itself a reductive Fernando-Kac subalgebra of finite type, and we also give an explicit description of all Fernando-Kac subalgebras of finite type which contain a Cartan subalgebra.

In conclusion, for an arbitrary reductive Lie algebra $\mathfrak{g}$, we give a complete description of all primal subalgebras $\mathfrak{k} \subset \mathfrak{g}$, and for each primal subalgebra $\mathfrak{k}$, we construct certain "series" of irreducible $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$. A direct comparison with known results in the case of a symmetric pair $(\mathfrak{g}, \mathfrak{k})$ shows that the $(\mathfrak{g}, \mathfrak{k})$-modules obtained by our construction are only a part of all irreducible $(\mathfrak{g}, \mathfrak{k})$-modules. Consequently, the problem of classifying all irreducible $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over an arbitrary primal subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is still open.

2. General preliminaries

The ground field F is algebraically closed of characteristic zero. If X is a topological space and $\mathscr{F}$ is a sheaf of abelian groups on X, then $\Gamma(\mathscr{F})$ denotes the global sections of $\mathscr{F}$ on X. If $f: X \to Y$ is a continuous map of topological spaces, f^{-1} denotes the topological inverse image functor from sheaves on Y to sheaves on X. If X is an algebraic variety, $\mathscr{O}_X$ stands for the structure sheaf of X, and if $f: X \to Y$ is a morphism of algebraic varieties, f^* (resp., f_*) denotes the inverse image (resp., direct image) functor of $\mathscr{O}$-modules. A *multiset* is defined as a map from a set Y into $\mathbb{Z}_+ \cup \infty$, where $\mathbb{Z}_+ := \{0, 1, 2, 3, \ldots\}$, or, more informally, as a set whose elements have finite or infinite multiplicities.

Throughout this paper, $\mathfrak{g}$ is a fixed reductive Lie algebra and G stands for a connected algebraic group with Lie algebra $\mathfrak{g}$. Denote by $C(\mathfrak{l})$ (resp., $N(\mathfrak{l})$) the centralizer (resp., normalizer) of a subalgebra $\mathfrak{l} \subset \mathfrak{g}$. Furthermore, $U(\mathfrak{l})$ stands for the universal enveloping algebra of $\mathfrak{l}$, $Z(\mathfrak{l})$ stands for the center of $\mathfrak{l}$, $\mathfrak{r}_\mathfrak{l}$ stands for the solvable radical of $\mathfrak{l}$, and $\mathfrak{n}_\mathfrak{l}$ stands for the maximal ideal in $\mathfrak{l}$ which acts nilpotently on $\mathfrak{g}$. The sign $\in\!\!\!\!+$ denotes the semidirect sum of Lie algebras, and $\mathfrak{l}_{ss}$ is a Levi component of $\mathfrak{l}$. If $\mathfrak{l}$ is reductive, then $\mathfrak{l}_{ss}$ simply equals the derived subalgebra $[\mathfrak{l}, \mathfrak{l}]$. For a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ which contains a Cartan subalgebra $\mathfrak{h}$, $\rho_\mathfrak{b}$ denotes as usual the half-sum of the roots of $\mathfrak{b}$. In what follows, a *root subalgebra* $\mathfrak{l} \subset \mathfrak{g}$ means a subalgebra containing a Cartan subalgebra of $\mathfrak{g}$.

By definition, a $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{l})$-*module* if $\mathfrak{l} \subset \mathfrak{g}[M]$. M is a *strict* $(\mathfrak{g}, \mathfrak{l})$-module if $\mathfrak{l} = \mathfrak{g}[M]$. We also need the following definition from [PS]: M is an *isotropic* $(\mathfrak{g}, \mathfrak{l})$-module if, for each $0 \neq m \in M$, the set of elements $g \in \mathfrak{g}$ acting finitely on m coincides with $\mathfrak{l}$. An irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module is automatically isotropic.

The following statement is a reformulation of [PS, Lemma 1].

LEMMA 2.1
Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$, let $\mathfrak{l} \supset \mathfrak{h}$ be a solvable subalgebra, and let M be an isotropic strict $(\mathfrak{g}, \mathfrak{l})$-module of finite type over $\mathfrak{h}$. Then there exists a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$, $\mathfrak{q} \cap \mathfrak{l} = \mathfrak{h}$, such that the semisimple part of $\mathfrak{q}$ is a direct sum of simple Lie algebras of types A and C.

3. Necessary conditions for $\mathfrak{l}$ to be a Fernando-Kac subalgebra of finite type

THEOREM 3.1
Let $\mathfrak{l} \subset \mathfrak{g}$ be a Fernando-Kac subalgebra of finite type.

(1) *The Lie algebra $\mathfrak{l}$ equals its normalizer $N(\mathfrak{l})$; hence $\mathfrak{l}$ is an algebraic subalgebra of $\mathfrak{g}$.*
(2) *There is a decomposition $\mathfrak{l} = \mathfrak{n}_{\mathfrak{l}} \in\!\!\!\!+ \mathfrak{l}_{\mathrm{red}}$, unique up to an inner automorphism of $\mathfrak{l}$, where $\mathfrak{l}_{\mathrm{red}}$ is a (maximal) subalgebra of $\mathfrak{l}$ reductive in $\mathfrak{g}$.*
(3) *Any irreducible $(\mathfrak{g}, \mathfrak{l})$-module M of finite type over $\mathfrak{l}$ has finite type over $\mathfrak{l}_{\mathrm{red}}$, and $\mathfrak{l}_{\mathrm{red}}$ acts semisimply on M.*
(4) *The equality $C(\mathfrak{l}_{\mathrm{red}}) = Z(\mathfrak{l}_{\mathrm{red}})$ holds, and $Z(\mathfrak{l}_{\mathrm{red}})$ is a Cartan subalgebra of $C(\mathfrak{l}_{\mathrm{ss}})$.*
(5) *The subalgebra $\mathfrak{l} \cap C(\mathfrak{l}_{\mathrm{ss}})$ is a solvable Fernando-Kac subalgebra of finite type of $C(\mathfrak{l}_{\mathrm{ss}})$.*

Proof
Let M be an irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module, and let $M_0 \subset M$ be an irreducible finite-dimensional $\mathfrak{l}$-submodule. To prove statement (1), assume that $N(\mathfrak{l}) \neq \mathfrak{l}$. Then one can choose $x \in N(\mathfrak{l}) \backslash \mathfrak{l}$ such that $[x, \mathfrak{l}_{\mathrm{ss}}] = 0$ for a fixed Levi decomposition $\mathfrak{l} = \mathfrak{l}_{\mathrm{ss}} \ni\!\!\!\!+ \mathfrak{r}_{\mathfrak{l}}$. Since $x \notin \mathfrak{l}$, x acts freely on any nonzero vector in M. Set

$$M_n := M_0 + x \cdot M_0 + x^2 \cdot M_0 + \cdots + x^n \cdot M_0.$$

A simple calculation, using $[x, \mathfrak{l}_{\mathrm{ss}}] = 0$ and $[x, \mathfrak{r}_{\mathfrak{l}}] \subset \mathfrak{r}_{\mathfrak{l}}$, shows that M_n is $\mathfrak{l}$-invariant and M_n / M_{n-1} is isomorphic to M_0 as an $\mathfrak{l}$-module. Therefore the multiplicity of M_0 in M is infinite. This is a contradiction. To show the algebraicity of $\mathfrak{l}$, consider the normalizer J of $\mathfrak{l}$ in G. The Lie subalgebra of $\mathfrak{g}$ corresponding to J is $N(\mathfrak{l})$. Hence $N(\mathfrak{l}) = \mathfrak{l}$ is an algebraic subalgebra of $\mathfrak{g}$.

Statement (2) follows from (1) via some well-known facts. For instance, [B, §5, Cor. 1] implies that a self-normalizing subalgebra $\mathfrak{l}$ is splittable; that is, for $y \in \mathfrak{l}$, the semisimple and nilpotent parts of y are contained in $\mathfrak{l}$. Proposition 7 in [B, Section 5] claims that any splittable subalgebra has a decomposition as required in statement (2).

To prove statement (3), note first that M is a quotient of the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} M_0$. As the adjoint action of $\mathfrak{l}_{\text{red}}$ on $U(\mathfrak{g})$ is semisimple, $\mathfrak{l}_{\text{red}}$ acts semisimply on $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} M_0$ and therefore also on M. Now note that there exists $\nu \in \mathfrak{n}_{\mathfrak{l}}^*$ such that

$$x \cdot m = \nu(x)m$$

for any $m \in M_0$ and $x \in \mathfrak{n}_{\mathfrak{l}}$. Since the adjoint action of $\mathfrak{n}_{\mathfrak{l}}$ on $U(\mathfrak{g})$ is locally nilpotent, we obtain that for any $x \in \mathfrak{n}_{\mathfrak{l}}$, $x - \nu(x)$ acts locally nilpotently on $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} M_0$ and hence on M. Therefore $\mathfrak{n}_{\mathfrak{l}}$ acts via the character ν on any irreducible $\mathfrak{l}$-subquotient of M, and consequently two irreducible $\mathfrak{l}$-subquotients of M are isomorphic if and only if they are isomorphic as $\mathfrak{l}_{\text{red}}$-modules. This implies that M also has finite type over $\mathfrak{l}_{\text{red}}$, and statement (3) is proven.

To prove statement (4), we note that, by statement (2), any irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module M has an $\mathfrak{l}_{\text{red}}$-module decomposition

$$M = \bigoplus_i M_i'$$

for finite-dimensional isotypic components M_i'. Clearly, each M_i' is $(C(\mathfrak{l}_{\text{red}}))$-invariant, and as it is finite-dimensional, $C(\mathfrak{l}_{\text{red}}) \subset \mathfrak{g}[M] = \mathfrak{l}$. Note that $C(\mathfrak{l}_{\text{red}}) \cap \mathfrak{l}$ is solvable. Consequently, since $C(\mathfrak{l}_{\text{red}}) = C(\mathfrak{l}_{\text{ss}}) \cap C(Z(\mathfrak{l}_{\text{red}})) \subset \mathfrak{l}$, the centralizer of $Z(\mathfrak{l}_{\text{red}})$ in $C(\mathfrak{l}_{\text{ss}})$ is solvable. On the other hand, as $C(\mathfrak{l}_{\text{ss}})$ is reductive and $Z(\mathfrak{l}_{\text{red}})$ is reductive in $C(\mathfrak{l}_{\text{ss}})$, the centralizer of $Z(\mathfrak{l}_{\text{red}})$ in $C(\mathfrak{l}_{\text{ss}})$ is reductive. Therefore $Z(\mathfrak{l}_{\text{red}})$ coincides with its centralizer in $C(\mathfrak{l}_{\text{ss}})$. This implies that $C(\mathfrak{l}_{\text{red}}) = C(\mathfrak{l}_{\text{ss}}) \cap C(Z(\mathfrak{l}_{\text{red}})) = Z(\mathfrak{l}_{\text{red}})$ and that $Z(\mathfrak{l}_{\text{red}})$ is a Cartan subalgebra of $C(\mathfrak{l}_{\text{ss}})$.

To show statement (5), decompose M as

$$M = \bigoplus_i (M_i \otimes V_i),$$

where M_i are pairwise nonisomorphic irreducible $\mathfrak{l}_{\text{ss}}$-modules and V_i are $(C(\mathfrak{l}_{\text{ss}}))$-modules. Then each V_i is a strict isotropic $(C(\mathfrak{l}_{\text{ss}}), \mathfrak{l} \cap C(\mathfrak{l}_{\text{ss}}))$-module of finite type over $\mathfrak{l} \cap C(\mathfrak{l}_{\text{ss}})$. Furthermore, $\mathfrak{l} \cap C(\mathfrak{l}_{\text{ss}})$ is solvable, and statement (5) follows from Lemma 2.1. □

Statements (1) – (5) in Theorem 3.1 are necessary but not sufficient conditions for a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ to be a Fernando-Kac subalgebra of finite type (see the example in Section 5.3). In general, the problem of a complete characterization of a Fernando-Kac subalgebra of finite type is open. However, for a solvable $\mathfrak{l}$ we have the answer.

PROPOSITION 3.2
A solvable subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is a Fernando-Kac subalgebra of finite type if and only if $\mathfrak{l} = \mathfrak{h} \supsetplus \mathfrak{n}_{\mathfrak{l}}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_{\mathfrak{l}}$ is the nilradical of a parabolic subalgebra of $\mathfrak{g}$ whose simple components are all of types A and C.

Proof
Here $\mathfrak{l}_{ss} = 0$, $C(\mathfrak{l}_{ss}) = \mathfrak{g}$, and Theorem 3.1(4) implies that $\mathfrak{h} := \mathfrak{l}_{red}$ is a Cartan subalgebra of $\mathfrak{g}$. The claim of the proposition follows now immediately from [PS, Section 3], where a criterion for $\mathfrak{l}$ to be a Fernando-Kac subalgebra of finite type is established under the assumption that $\mathfrak{l} \supset \mathfrak{h}$. □

Note that Theorem 3.1(3) and Proposition 3.2, applied to a solvable $\mathfrak{l}$, yield that any strict irreducible $(\mathfrak{g}, \mathfrak{l})$-module of finite type over $\mathfrak{l}$ is a weight module with finite-dimensional weight spaces. Such modules are classified by O. Mathieu in [M]. More precisely, any irreducible weight module M with finite-dimensional weight spaces is the unique irreducible quotient of an induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{n}_{\mathfrak{p}}}$, where $\mathfrak{p}$ is a parabolic subalgebra and $M^{\mathfrak{n}_{\mathfrak{p}}}$ is the $\mathfrak{p}$-submodule of $\mathfrak{n}_{\mathfrak{p}}$-invariants in M. The Fernando-Kac subalgebra $\mathfrak{g}[M]$ of M equals $(\mathfrak{g}[M] \cap \mathfrak{p}_{red}) \supsetplus \mathfrak{n}_{\mathfrak{p}}$, and it is solvable if and only if $\mathfrak{g}[M] \cap \mathfrak{p}_{red}$ is a Cartan subalgebra of $\mathfrak{g}$. (In general, $\mathfrak{g}[M] \cap \mathfrak{p}_{red}$ is the sum of a Cartan subalgebra and an ideal in $\mathfrak{p}_{ss}$.)

4. A construction of irreducible $(\mathfrak{g}, \mathfrak{k})$-modules of finite type

4.1. A geometric setup
Let $\mathfrak{k} \subset \mathfrak{g}$ be an algebraic subalgebra, reductive in $\mathfrak{g}$ and such that $\mathfrak{k}_{ss}$ is proper in $\mathfrak{g}_{ss}$. Denote by K the connected subgroup of G with Lie algebra $\mathfrak{k}$, and let K_{ss} be the connected subgroup corresponding to $\mathfrak{k}_{ss}$. By H_K we denote a fixed Cartan subgroup of K, with Lie algebra $\mathfrak{h}_{\mathfrak{k}}$. Fix an element $h \in \mathfrak{h}_{\mathfrak{k}}$ such that $C(Fh) \subset C(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{ss})$ and for which the operator $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$ has rational eigenvalues. The element h defines the parabolic subalgebra

$$\mathfrak{p} := \bigoplus_{\gamma \geq 0} \mathfrak{g}_h^{\gamma}, \tag{4.1}$$

where $\mathfrak{g}_h^{\gamma}$ is the γ-eigenspace of $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$. Clearly, $\mathfrak{b}_{\mathfrak{k}} := \mathfrak{p} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}_{\mathfrak{k}}$. Notice also that $\mathfrak{p}_{red} := \mathfrak{g}_{\mathfrak{h}}^0$ is a maximal reductive in $\mathfrak{g}$ subalgebra of $\mathfrak{p}$. Let P be the subgroup of G corresponding to $\mathfrak{p}$, and let $P_{ss} \subset P$ be the connected subgroup corresponding to a fixed Levi component $\mathfrak{p}_{ss}$ of $\mathfrak{p}$. Furthermore, let $B \subset P$ be a Borel subgroup of G such that $B_K = B \cap K$ has Lie algebra $\mathfrak{b}_{\mathfrak{k}}$. Set $X := G/B$, $Y := G/P$, and let $\pi : X \to Y$ be the natural projection. Denote by S the K-orbit of the closed point in Y corresponding to P, and put $V := \pi^{-1}(S)$.

LEMMA 4.1
There is a canonical isomorphism of K_{ss}-varieties $V \cong S \times T$, where $T := P/B$ and the action of K_{ss} on T is trivial.

Proof
V is a relative flag variety over S with fiber $T = P/B \cong P_{\mathrm{ss}}/(P_{\mathrm{ss}} \cap B)$. Moreover, $V = K_{\mathrm{ss}} \times_{K_{\mathrm{ss}} \cap P} T$. To be able to conclude that the bundle $V \to S$ is canonically trivial, it suffices to check that the action of $K_{\mathrm{ss}} \cap P$ on T is trivial. The solvable radical of P lies in B; hence the action of $K_{\mathrm{ss}} \cap P$ on T factors through the action of $K_{\mathrm{ss}} \cap P_{\mathrm{ss}}$ on T. The fact that $K_{\mathrm{ss}} \cap P_{\mathrm{ss}}$ acts trivially on T follows from the inclusion

$$K_{\mathrm{ss}} \cap P_{\mathrm{ss}} \subset Z(P_{\mathrm{ss}}), \tag{4.2}$$

where $Z(G')$ stands for the center of an algebraic group G'. In the rest of the proof, we establish (4.2).

We show first that $\mathfrak{p}_{\mathrm{ss}} \cap \mathfrak{k}_{\mathrm{ss}} = 0$. By the definition of $\mathfrak{p}$,

$$\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}} \subset \mathfrak{p}_{\mathrm{red}} \subset C(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}}).$$

Therefore

$$\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}} \subset Z(\mathfrak{p}_{\mathrm{red}}) \tag{4.3}$$

and

$$\mathfrak{p}_{\mathrm{ss}} \cap \mathfrak{k}_{\mathrm{ss}} \subset C(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}}) \cap \mathfrak{k}_{\mathrm{ss}}. \tag{4.4}$$

Furthermore, $\mathfrak{p}_{\mathrm{ss}} \cap \mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}} = 0$ as

$$\mathfrak{p}_{\mathrm{red}} = \mathfrak{p}_{\mathrm{ss}} \oplus Z(\mathfrak{p}_{\mathrm{red}}). \tag{4.5}$$

The observation that $C(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}}) \cap \mathfrak{k}_{\mathrm{ss}}$ equals the centralizer in $\mathfrak{k}_{\mathrm{ss}}$ of $\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}}$, together with (4.4), yields

$$\mathfrak{p}_{\mathrm{ss}} \cap \mathfrak{k}_{\mathrm{ss}} \subset C(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}}) \cap \mathfrak{k}_{\mathrm{ss}} = \mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}} \subset Z(\mathfrak{p}_{\mathrm{red}}).$$

This implies $\mathfrak{p}_{\mathrm{ss}} \cap (\mathfrak{p}_{\mathrm{ss}} \cap \mathfrak{k}_{\mathrm{ss}}) = 0$ or, equivalently, $\mathfrak{p}_{\mathrm{ss}} \cap \mathfrak{k}_{\mathrm{ss}} = 0$.

On the group level, (4.5) implies $P_{\mathrm{ss}} \cap Z(P_{\mathrm{red}}) \subset Z(P_{\mathrm{ss}})$. Similarly, (4.3) yields $H_K \cap K_{\mathrm{ss}} \subset Z(P_{\mathrm{red}})$. Hence $P_{\mathrm{ss}} \cap H_K \cap K_{\mathrm{ss}} \subseteq Z(P_{\mathrm{ss}})$. By (4.4),

$$P_{\mathrm{ss}} \cap K_{\mathrm{ss}} \subset C(H_K \cap K_{\mathrm{ss}}) \cap K_{\mathrm{ss}}, \tag{4.6}$$

where $C(G')$ now stands for the centralizer in G of a closed subgroup $G' \subset G$. Since $C(H_K \cap K_{\mathrm{ss}}) \cap K_{\mathrm{ss}}$ is the centralizer of $H_K \cap K_{\mathrm{ss}}$ in K_{ss}, the fact that a Cartan subgroup of K_{ss} is self-centralizing yields, via (4.6),

$$P_{\mathrm{ss}} \cap K_{\mathrm{ss}} \subset C(H_K \cap K_{\mathrm{ss}}) \cap K_{\mathrm{ss}} = H_K \cap K_{\mathrm{ss}} \subset Z(P_{\mathrm{red}}).$$

Therefore

$$P_{ss} \cap (P_{ss} \cap K_{ss}) \subset P_{ss} \cap Z(P_{red}) = Z(P_{ss})$$

or, equivalently, $P_{ss} \cap K_{ss} \subset Z(P_{ss})$. This completes the proof. □

4.2. $\mathscr{D}$-module preliminaries

For any $\mu \in \mathfrak{h}^*$, let $\mathscr{D}^\mu$ denote the twisted sheaf of differential operators on X defined in [BB]. A *$\mathscr{D}^\mu$-module* is by convention a sheaf $\mathscr{F}$ of $\mathscr{D}^\mu$-modules on X which is quasi-coherent as a sheaf of $\mathscr{O}_X$-modules. The support of $\mathscr{F}$ is the closure of the subvariety of all closed points for which the sheaf-theoretic fiber of $\mathscr{F}$ is nonzero. A weight $\mu \in \mathfrak{h}^*$ defines the character θ^μ of the center of $U(\mathfrak{g})$ via the Harish-Chandra map (see [B, Section 6]).

When the ground field F is not **C**, by a *dominant* weight we mean an element $\mu \in \mathfrak{h}^*$ whose value on all B-positive coroots is a nonnegative rational number. For $F = \mathbb{C}$ it suffices that the value have nonnegative real part. The Beilinson-Bernstein localization theorem claims that for a regular dominant μ, the functor of global sections

$$\Gamma\colon \mathscr{D}^\mu\text{-mod} \to U(\mathfrak{g}) / \left(\ker \theta^\mu\right)\text{-mod}$$

is an equivalence between the category of $(\mathscr{D}^\mu)$-modules and the category of $(U(\mathfrak{g})/(\ker \theta^\mu))$-modules, where $(\ker \theta^\mu)$ stands for the two-sided ideal in $U(\mathfrak{g})$ generated by the kernel of the central character θ^μ. The inverse equivalence (usually referred to as localization) is given by the functor

$$R \mapsto \mathscr{D}^\mu \otimes_{\Gamma(\mathscr{D}^\mu)} R,$$

where the $(U(\mathfrak{g})/(\ker \theta^\mu))$-module R is endowed with a $\Gamma(\mathscr{D}^\mu)$-module structure via the natural isomorphism $U(\mathfrak{g})/(\ker \theta^\mu) \to \Gamma(\mathscr{D}^\mu)$ (see [BB]).

Let $i : W \to X$ define a nonsingular locally closed subvariety of X; we denote by $\mathscr{D}_W^\mu$ the sheaf of right $i^*\mathscr{D}^\mu$-module endomorphisms of the inverse image sheaf $i^*\mathscr{D}^\mu$ which are left $\mathscr{O}_W$-module differential operators. Furthermore, the inverse image functor i^* of $\mathscr{O}$-modules yields a functor

$$i^\bigstar\colon \mathscr{D}^\mu\text{-mod} \to \mathscr{D}_W^\mu\text{-mod}.$$

If W is a closed subvariety, we also consider the direct image functor

$$i_\bigstar : \mathscr{D}_W^\mu\text{-mod} \to \mathscr{D}^\mu\text{-mod},$$

$$\mathscr{F} \mapsto \mathscr{D}^\mu_{\leftarrow W} \otimes_{\mathscr{D}_W^\mu} \mathscr{F},$$

where $\mathscr{D}^\mu_{\leftarrow W} := i^\bigstar(\mathscr{D}^\mu \otimes_{\mathscr{O}_X} \Omega_X^*) \otimes_{\mathscr{O}_W} \Omega_W$ and Ω stands for volume forms. Kashiwara's theorem claims that $i_\bigstar$ is an equivalence between the category of $\mathscr{D}_W^\mu$-modules and the category of $\mathscr{D}^{\mu}$-modules supported in W. It also is important for us that the sheaf $i^{-1}i_{\bigstar}\mathscr{F}$ has a natural $\mathscr{O}_W$-module filtration with successive quotients

$$\Lambda^{\max}(\mathscr{N}_{W|X}) \otimes_{\mathscr{O}_W} S^i(\mathscr{N}_{W|X}) \otimes_{\mathscr{O}_W} \mathscr{F}, \tag{4.7}$$

where $i \in \mathbb{Z}_+$, $\mathscr{N}_{W|X}$ denotes the normal bundle of W in X, S^i stands for ith symmetric power, and $\Lambda^{\max}$ stands for maximal exterior power.

In [PS] the following lemma is proven.

LEMMA 4.2

If Q is the support of a $\mathscr{D}^{\mu}$-module $\mathscr{F}$, then $\mathfrak{g}[\Gamma(\mathscr{F})] \subset \mathrm{Stab}_{\mathfrak{g}}\, Q$, where $\mathrm{Stab}_{\mathfrak{g}}\, Q$ is the Lie algebra of the subgroup of G which stabilizes Q.

4.3. The construction

Let L be an irreducible $(\mathfrak{p}, \mathfrak{h}_{\mathfrak{k}})$-module of finite type over $\mathfrak{h}_{\mathfrak{k}}$ with trivial action of $\mathfrak{n}_{\mathfrak{p}} + (Z(\mathfrak{p}_{\mathrm{red}}) \cap \mathfrak{k}_{\mathrm{ss}})$ and with $\mathfrak{p}_{\mathrm{red}}$-central character $\theta^{\nu}_{\mathfrak{p}_{\mathrm{red}}}$ for some $P_{\mathrm{ss}} \cap B$-dominant weight $\nu \in \mathfrak{h}^*$. Consider $T = P/B$ a (nonsingular) closed subvariety of $X = G/B$. Set $\mathscr{L} := \mathscr{D}^{\eta}_T \otimes_{\Gamma(\mathscr{D}^{\eta}_T)} L$, where $\eta = \nu + \rho_{\mathfrak{b}\cap\mathfrak{p}_{\mathrm{red}}} - \rho_{\mathfrak{b}}$. Furthermore, let $\mathscr{O}_S(\zeta)$ be the invertible K_{ss}-sheaf of local sections on S of the line bundle $K \times_{K\cap P} (F_{w(\zeta)})$, where w is the longest element in the Weyl group of $\mathfrak{k}_{\mathrm{ss}}$, ζ is a $\mathfrak{k}_{\mathrm{ss}}$-integral weight in $\mathfrak{h}^*$, and F_{ξ} stands for the one-dimensional $\mathfrak{h}$-module of weight ξ. Then Lemma 4.1 enables us to consider $\mathscr{F} := \mathscr{O}_S(\zeta) \boxtimes \mathscr{L}$ as a $\mathscr{D}^{\mu}_V$-module for $\mu = \zeta + \eta$, and $\mathscr{M} = i_{\bigstar}\mathscr{F}$ is a $\mathscr{D}^{\mu}$-module. Finally, set $M = \Gamma(\mathscr{M})$.

THEOREM 4.3

Assume that ζ is dominant and that μ is regular and dominant. Then

(1) *M is an infinite-dimensional irreducible $\mathfrak{g}$-module;*

(2) *$\mathfrak{g}[M] = \mathfrak{k}_{\mathrm{ss}} \supsetplus \mathfrak{m}_L$, where $\mathfrak{m}_L$ is the maximal $\mathfrak{k}_{\mathrm{ss}}$-invariant subspace in $\mathfrak{p}[L]$; moreover, $\mathfrak{g}[M]$ is the unique maximal subalgebra in $\mathfrak{p}[L] + \mathfrak{k}$ which contains $\mathfrak{k}$;*

(3) *M is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type over $\mathfrak{k}$.*

Proof

The sheaf $\mathscr{D}^{\eta}_T$ is a sheaf of twisted differential operators on the flag variety T. By the Beilinson-Bernstein theorem applied to T, $\mathscr{L}$ is an irreducible $\mathscr{D}^{\eta}_T$-module. Furthermore, $\mathscr{F}$ is an irreducible $\mathscr{D}^{\mu}_V$-module. Since V is a nonsingular closed subvariety, $\mathscr{M}$ is an irreducible $\mathscr{D}^{\mu}$-module by Kashiwara's theorem. Finally, by the Beilinson-Bernstein theorem applied to X, $M = \Gamma(\mathscr{M})$ is an irreducible $\mathfrak{g}$-module. Statement (1) is proven.

To prove statement (2), consider the subalgebra $\mathrm{Stab}_{\mathfrak{g}}\, Q$, where Q is the support of the $\mathscr{D}^{\mu}$-module $\mathscr{M}$. Note that $Q \subset V$ and that $V = \pi^{-1}(\pi(Q))$. Hence $\mathrm{Stab}_{\mathfrak{g}}\, Q$

is a subalgebra of $\mathfrak{st} := \mathrm{Stab}_{\mathfrak{g}} V$. One can check easily that

$$\mathfrak{st} = \mathfrak{k}_{ss} \supsetplus \mathfrak{m}, \tag{4.8}$$

where $\mathfrak{m}$ is the maximal $\mathfrak{k}_{ss}$-invariant subspace in $\mathfrak{p}$. Thus $\mathfrak{st}$ is a maximal subalgebra in $\mathfrak{k} + \mathfrak{p}$ containing $\mathfrak{k}$. By Lemma 4.2, $\mathfrak{g}[M] \subset \mathrm{Stab}_{\mathfrak{g}} Q \subset \mathfrak{st}$, and therefore $\mathfrak{g}[M] = \mathfrak{st}[M]$.

Recall now that by (4.7), $i^{-1}\mathscr{M} = i^{-1}i_{\bigstar}\mathscr{F}$ (considered as an $\mathfrak{st}$-sheaf) has a natural $\mathfrak{st}$-sheaf filtration with successive quotients

$$\Lambda^{\max}(\mathscr{N}_{V|X}) \otimes_{\mathscr{O}_V} S^i(\mathscr{N}_{V|X}) \otimes_{\mathscr{O}_V} \mathscr{F}.$$

In particular, $\mathscr{M}_0 := \Lambda^{\max}(\mathscr{N}_{V|X}) \otimes_{\mathscr{O}_V} \mathscr{F}$ is a subsheaf of $i^{-1}\mathscr{M}$. As $\mathscr{N}_{V|X} \cong \mathscr{N}_{S|Y} \boxtimes \mathscr{O}_Z$, $\Lambda^{\max}(\mathscr{N}_{V|X}) \cong \mathscr{O}_S(\tau) \boxtimes \mathscr{O}_Z$, where $\tau = -w\big(\sum_{\alpha\in\Delta(\mathfrak{n}_{\mathfrak{p}})} \alpha\big) - 2\rho_{\mathfrak{b}\cap\mathfrak{k}_{ss}}$. Therefore $\mathscr{M}_0 \cong \mathscr{O}_S(\tau+\zeta) \boxtimes \mathscr{L}$ and

$$M_0 := \Gamma(\mathscr{M}_0) \cong \Gamma(\pi_*\mathscr{M}_0) \cong \Gamma\big(\mathscr{O}_S(\tau+\zeta)\big) \otimes L.$$

The weights τ and ζ are both dominant. Hence $\tau + \zeta$ is $\mathfrak{k}_{ss}$-dominant, $M_0 \neq 0$, and by the irreducibility of M,

$$\mathfrak{g}[M] = \mathfrak{st}[M] = \mathfrak{st}[M_0]. \tag{4.9}$$

To calculate $\mathfrak{st}[M_0]$ we use the fact that $\Gamma(\mathscr{M}_0) \cong \Gamma(\pi_*\mathscr{M}_0)$. Observe that $\pi_*\mathscr{M}_0$ is the sheaf of sections of the induced vector bundle $K_{ss} \times_{K_{ss}\cap P} (F_{w(\zeta+\tau)} \otimes L)$. The latter is a K_{ss}-sheaf; hence $\mathfrak{k}_{ss} \subset \mathfrak{st}[M_0]$. By (4.8) and (4.9), $\mathfrak{g}[M] = \mathfrak{k}_{ss} \supsetplus \mathfrak{m}_L$, where $\mathfrak{m}_L = \mathfrak{g}[M] \cap \mathfrak{m}$. To calculate $\mathfrak{m}_L$, let us write down the action of $\mathfrak{m}$ on $\Gamma(\pi_*\mathscr{M}_0)$. An element of $\Gamma(\pi_*\mathscr{M}_0)$ is a function $\phi : K_{ss} \to F_{w(\zeta+\tau)} \otimes L$ satisfying the condition $\phi(ab) = b^{-1}\phi(a)$ for all $a \in K_{ss}$, $b \in K_{ss} \cap P$. For $x \in \mathfrak{m}$ and $a \in K_{ss}$, we have

$$(L_x\phi)(a) = \mathrm{Ad}_a^{-1}(x)\big(\phi(a)\big), \tag{4.10}$$

where $L_x\phi$ stands for the action of x on ϕ. This formula immediately implies that

$$\mathfrak{m}_L \subset \big\{x \in \mathfrak{m} \,\big|\, \mathrm{Ad}_{K_{ss}}(x) \subset \mathfrak{m}[F_{w(\zeta+\tau)} \otimes L] = \mathfrak{m}[L]\big\}.$$

To see that $\mathfrak{m}_L$ is equal to the right-hand side, let U be a unipotent subgroup of K_{ss} complementary to $K_{ss} \cap P$. The group U acts simply transitively on an open dense subset of S. Consider a U-invariant function $f : K_{ss} \to L$. For any $a \in U$, we have $f(a) = f(1)$. Let x be in $\mathfrak{m}[L]$, and assume that x is $\mathrm{Ad}\, U$-invariant. Then by (4.10), x acts locally finitely on f, and by the irreducibility of M, x acts locally finitely on M. Finally, any y obtained from x by the action of K_{ss} also acts locally finitely on M. Hence

$$\mathfrak{m}_L = \big\{x \in \mathfrak{m} \,\big|\, \mathrm{Ad}_{K_{ss}}(x) \subset \mathfrak{m}[F_{w(\zeta+\tau)} \otimes L] = \mathfrak{m}[L]\big\}. \tag{4.11}$$

In other words, $\mathfrak{m}_L$ is the maximal $\mathfrak{k}_{ss}$-invariant subspace in $\mathfrak{m}[L]$ or, equivalently, in $\mathfrak{p}[L]$. Consequently, $\mathfrak{k}_{ss} \supsetplus \mathfrak{m}_L$ is the maximal subalgebra in $\mathfrak{k}+\mathfrak{p}[L]$ containing $\mathfrak{k}$, and statement (2) is proven.

It remains to prove statement (3). Let $j : S \to Y$ be the natural embedding. Observe that the isomorphism $\mathcal{N}_{V|X} \cong \mathcal{N}_{S|Y} \boxtimes \mathcal{O}_Z$ yields isomorphisms of $\mathfrak{k}$-sheaves

$$j^{-1} j_\bigstar \mathcal{O}_S(\zeta) \boxtimes \mathcal{L} \cong i^{-1} i_\bigstar\big(\mathcal{O}_S(\zeta) \boxtimes \mathcal{L}\big) \cong i^{-1}\mathcal{M}.$$

Therefore we have an isomorphism of $\mathfrak{k}$-modules

$$\Gamma(\mathcal{M}) \cong \Gamma(\pi_*\mathcal{M}) \cong \Gamma\big(j_\bigstar \mathcal{O}_S(\zeta)\big) \otimes L,$$

where the action of $\mathfrak{k}_{ss}$ on L is trivial and the action of $Z(\mathfrak{k})$ is induced by the embedding $Z(\mathfrak{k}) \subset \mathfrak{p}_{\mathrm{red}}$. By (4.7), $j^{-1} j_\bigstar \mathcal{O}_S(\zeta)$ has a filtration by $\mathfrak{k}$-sheaves with successive quotients

$$S^i(\mathcal{N}_{S|Y}) \otimes_{\mathcal{O}_S} \mathcal{O}_S(\zeta+\tau).$$

Consequently, M has a $\mathfrak{k}$-module filtration whose associated graded $\mathfrak{k}$-module is a submodule of

$$\Gamma\big(S^{\cdot}(\mathcal{N}_{S|Y}) \otimes_{\mathcal{O}_S} \mathcal{O}_S(\zeta+\tau)\big) \otimes L.$$

The sheaf $S^{\cdot}(\mathcal{N}_{S|Y}) \otimes_{\mathcal{O}_S} \mathcal{O}_S(\zeta+\tau)$ is locally free on S and has a filtration with invertible successive quotients $\mathcal{O}_S(\kappa)$, where κ runs over the multiset Θ of weights in $\mathfrak{h}_{\mathfrak{k}}^*$:

$$\Theta = \Big\{\zeta+\tau+\sum_{n_\alpha \in \mathbb{N}} n_\alpha \alpha \;\Big|\, n_\alpha \in \mathbb{Z}_+\Big\}.$$

Here we take the summation over all weights α of the $\mathfrak{h}_{\mathfrak{k}}$-module $\mathfrak{n}_{\mathfrak{p}}/(\mathfrak{n}_{\mathfrak{p}} \cap \mathfrak{k})$. Thus the multiplicity of the irreducible $\mathfrak{k}$-module with the highest weight κ in M is majorized by the multiplicity of κ in $\Theta+\Theta_L$, where Θ_L is the multiset of $\mathfrak{h}_{\mathfrak{k}}$-weights of L. Our goal is to show that the multiset $\Theta+\Theta_L$ has finite multiplicities. For any multiset $C \subset \mathfrak{h}_{\mathfrak{k}}^*$ and any element $t \in F$, put $C^t := \{\kappa \in C \mid \kappa(h) = t\}$. Then $\Theta_L = \Theta_L^{t_0}$ for some $t_0 \in F$, and $(\Theta+\Theta_L)^t = \Theta^{t-t_0} + \Theta_L$. As L has finite type over $\mathfrak{h}_{\mathfrak{k}}$, Θ_L has finite multiplicities. Furthermore, Θ^{t-t_0} is a finite multiset as $\alpha(h)$ are all positive. Therefore $(\Theta+\Theta_L)^t$ has finite multiplicities, and thus $\Theta+\Theta_L$ also has finite multiplicities. Theorem 4.3 is proven. □

The construction in Theorem 4.3 does not provide all irreducible $(\mathfrak{g},\mathfrak{k})$-modules of finite type over $\mathfrak{k}$. Consider, for instance, the case when $\mathfrak{k}$ is symmetric, that is, when $\mathfrak{k}$ is stable under an involution of $\mathfrak{g}$. Here irreducible $(\mathfrak{g},\mathfrak{k})$-modules of finite type over $\mathfrak{k}$ are nothing but irreducible Harish-Chandra modules. The Beilinson-Bernstein classification of irreducible Harish-Chandra modules implies that the supports of their corresponding localizations (the latter are $\mathcal{D}^\mu$-modules on $X = G/B$) run over the closures of all K-orbits in X. In particular, there are infinite-dimensional irreducible Harish-Chandra modules whose localizations are supported on the closure X of the open orbit of K in G/B. These latter modules do not appear among the modules constructed in Theorem 4.3 as all $\mathscr{D}^{\mu}$-modules $\mathscr{M}$ considered above are supported on a closed proper subvariety of X.

4.4. Description of primal subalgebras

THEOREM 4.4

Let $\mathfrak{k}$ be a reductive in $\mathfrak{g}$ subalgebra with $C(\mathfrak{k}) = Z(\mathfrak{k})$. Then $\mathfrak{k}$ is primal; that is, there exists a Fernando-Kac subalgebra of finite type $\mathfrak{l} \subset \mathfrak{g}$ such that $\mathfrak{l}_{\mathrm{red}} = \mathfrak{k}$. In addition, $\mathfrak{l}$ can be chosen so that $\mathfrak{n}_{\mathfrak{l}}$ is the nilradical of a Borel subalgebra of $C(\mathfrak{k}_{\mathrm{ss}})$.

Proof

The assumption $C(\mathfrak{k}) = Z(\mathfrak{k})$ implies that $Z(\mathfrak{k})$ is a Cartan subalgebra of $C(\mathfrak{k}_{\mathrm{ss}})$. Let h' be a semisimple element in $\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}}$ such that $C(Fh') = C(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{\mathrm{ss}})$ and $\mathrm{ad}_{h'} : \mathfrak{g} \to \mathfrak{g}$ has rational eigenvalues γ_i'. Furthermore, let $h'' \in \mathfrak{h}_{\mathfrak{k}} \cap Z(\mathfrak{k})$ be a regular element in $C(\mathfrak{k}_{\mathrm{ss}})$ for which $\mathrm{ad}_{h''} : \mathfrak{g} \to \mathfrak{g}$ has rational eigenvalues γ_j'', such that

$$|\gamma_j''| < \min_{\gamma_i' \neq 0} |\gamma_i'| \tag{4.12}$$

for all j. Denote by $\mathfrak{p}$ the parabolic subalgebra of $\mathfrak{g}$ defined by the element $h := h' + h''$, and let L be a one-dimensional $\mathfrak{p}$-module. Theorem 4.3 applies to the triple $(\mathfrak{k}, \mathfrak{p}, L)$ (as $\mathfrak{k}$ is automatically algebraic) and hence yields an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M of finite type over $\mathfrak{k}$. Put $\mathfrak{l} := \mathfrak{g}[M]$. Then $\mathfrak{l} = \mathfrak{k}_{\mathrm{ss}} \supsetplus \mathfrak{m}$, where, as in the proof of Theorem 4.3, $\mathfrak{m}$ is the maximal $\mathfrak{k}_{\mathrm{ss}}$-invariant subspace in $\mathfrak{p}$. Let κ be the $\mathfrak{p} \cap \mathfrak{k}_{\mathrm{ss}}$-lowest weight of an irreducible $\mathfrak{k}_{\mathrm{ss}}$-submodule in $\mathfrak{m}$. We have $\kappa(h') = \gamma_i' \leq 0$ for some i. On the other hand, as $\mathfrak{m} \subset \mathfrak{p}$, $\kappa(h' + h'') \geq 0$. Condition (4.12) gives $\kappa(h') = 0$; that is, $\mathfrak{m} = C(\mathfrak{k}_{\mathrm{ss}}) \cap \mathfrak{p}$. As h'' is regular in $C(\mathfrak{k}_{\mathrm{ss}})$, $\mathfrak{m}$ is a Borel subalgebra in $C(\mathfrak{k}_{\mathrm{ss}})$; hence $\mathfrak{m}$ is solvable and $\mathfrak{m} = Z(\mathfrak{k}_{\mathrm{ss}}) + [\mathfrak{m}, \mathfrak{m}]$. Therefore $\mathfrak{l}_{\mathrm{red}} = \mathfrak{k}$, $\mathfrak{n}_{\mathfrak{l}} = [\mathfrak{m}, \mathfrak{m}]$, and $[\mathfrak{n}_{\mathfrak{l}}, \mathfrak{k}_{\mathrm{ss}}] = 0$. □

COROLLARY 4.5

A reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ is primal if and only if $C(\mathfrak{k}) = Z(\mathfrak{k})$.

Proof

The statement follows directly from Theorems 4.4 and 3.1(4). □

Corollary 4.5, together with the remark that $\mathfrak{k}$ is primal if and only if $\mathfrak{k} = \mathfrak{l}_{\mathrm{red}}$ for a Fernando-Kac subalgebra of finite type, reduces the problem of classifying all Fernando-Kac subalgebras of finite type to the problem of describing all nilpotent subalgebras $\mathfrak{n}$ such that $\mathfrak{k} \supsetplus \mathfrak{n}$ is a Fernando-Kac subalgebra of finite type, $\mathfrak{k}$ being a fixed primal subalgebra of $\mathfrak{g}$. The latter problem is open. In Section 5, we solve this problem in the case when $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{k}$ is a root subalgebra, and we show also that every primal subalgebra of $\mathfrak{gl}(n)$ is a Fernando-Kac subalgebra of finite type.

For simple Lie algebras not of type A, it is not true that any primal subalgebra is itself a Fernando-Kac subalgebra of finite type. Indeed, Proposition 3.2 implies that a Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$ (which is always primal) is a Fernando-Kac subalgebra of finite type if and only if $\mathfrak{g}$ is of type A or C. (This was proven first by S. Fernando in [F].) An important particular case of the above open problem is the problem of characterizing all primal subalgebras that are Fernando-Kac subalgebras of finite type.

We conclude this section with an application to the classical theory of subalgebras of a semisimple Lie algebra. In the fundamental paper [D], an important role is played by subalgebras $\mathfrak{s} \subset \mathfrak{g}$ which are not contained in a proper root subalgebra. We propose the term *stem subalgebra*. By [D, Theorems 7.3, 7.4], a stem subalgebra is necessarily semisimple with zero centralizer. Here are some well-known examples of stem subalgebras.

(1) A principal $\mathfrak{sl}(2)$-subalgebra is a stem subalgebra.
(2) If $\mathfrak{g} = \mathfrak{sl}(n)$, a proper subalgebra $\mathfrak{s} \in \mathfrak{g}$ is a stem subalgebra if and only if the defining representation of $\mathfrak{g}$ is irreducible over $\mathfrak{s}$.
(3) In general, any semisimple maximal subalgebra is either a stem subalgebra or a root subalgebra. For instance, if $n \geq 3$ is odd, $\mathfrak{o}(n) \oplus \mathfrak{o}(n)$ is a stem subalgebra of $\mathfrak{o}(2n)$. (This is, moreover, a symmetric pair.) Another well-known example of a stem subalgebra is $G_2 \oplus F_4$ in E_8.
(4) If $\mathfrak{g}$ is an exceptional simple Lie algebra over $\mathbb{C}$, [D, Table 39] gives a complete catalog of the stem subalgebras of $\mathfrak{g}$.

Theorem 4.4 combined with [D, Theorems 7.3, 7.4] implies the following.

COROLLARY 4.6
If $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}}$, any stem subalgebra is a Fernando-Kac subalgebra of finite type.

Finally, we have the following.

COROLLARY 4.7
If $\mathfrak{g} = \mathfrak{g}_{\mathrm{ss}}$, every maximal proper subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is a Fernando-Kac subalgebra of finite type.

Proof
By a theorem of F. Karpelevič [Ka], $\mathfrak{l}$ is a parabolic subalgebra or a semisimple

subalgebra. If $\mathfrak{l}$ is parabolic, the statement is obvious as any module induced from a finite-dimensional $\mathfrak{l}$-module has finite $\mathfrak{l}$-multiplicities. Let $\mathfrak{l}$ be semisimple. Then $C(\mathfrak{l}) = 0$, and thus $\mathfrak{l}$ is primal by Corollary 4.5. But as $\mathfrak{l}$ is maximal, any irreducible infinite-dimensional $(\mathfrak{g}, \mathfrak{l})$-module of finite type is strict; that is, $\mathfrak{l}$ is a Fernando-Kac subalgebra of finite type. $\square$

5. The case $\mathfrak{g} = \mathfrak{gl}(n)$

5.1. Description of reductive Fernando-Kac subalgebras of finite type

THEOREM 5.1
A reductive in $\mathfrak{g} = \mathfrak{gl}(n)$ subalgebra $\mathfrak{k}$ is a Fernando-Kac subalgebra of finite type if and only if it is primal or, equivalently, if and only if $C(\mathfrak{k}) = Z(\mathfrak{k})$.

Proof
By Theorem 3.1, it suffices to prove that if $C(\mathfrak{k}) = Z(\mathfrak{k})$, then $\mathfrak{k}$ is a Fernando-Kac subalgebra of finite type. We modify the argument in the proof of Theorem 4.4 under the assumption that $\mathfrak{g} = \mathfrak{gl}(n)$.

Let $h = h' + h''$, and let $\mathfrak{p}$ and $\mathfrak{m}$ be as in the proof of Theorem 4.4. In particular, $\mathfrak{m} = C(\mathfrak{k}_{ss}) \cap \mathfrak{p}$. We claim that h'' can be chosen so that, in addition, there is a decomposition $\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{a}$ and an isomorphism $p : C(\mathfrak{k}_{ss}) \to \mathfrak{a}$. Here is how this claim implies the theorem. Note that $C(\mathfrak{k}_{ss})$ is a direct sum of an abelian ideal and simple ideals of type A. Now choose L to be a strict irreducible $(C(\mathfrak{k}_{ss}), Z(\mathfrak{k}))$-module of finite type over $Z(\mathfrak{k})$. Define a $\mathfrak{p}$-module structure on L by putting $\mathfrak{a}' \cdot L = 0$ and letting $\mathfrak{a}$ act on L via the isomorphism p. One can see immediately that L is an irreducible $(\mathfrak{p}, \mathfrak{h}_{\mathfrak{k}})$-module of finite type over $\mathfrak{h}_{\mathfrak{k}}$ with $\mathfrak{p}[L] = \mathfrak{a}' + \mathfrak{h}$. Apply the construction in Theorem 4.3 to the triple $(\mathfrak{k}, \mathfrak{p}, L)$ to obtain a $(\mathfrak{g}, \mathfrak{k})$-module M of finite type over $\mathfrak{k}$. As $\mathfrak{m} \subset C(\mathfrak{k}_{ss})$, we have $\mathfrak{m}_L = Z(\mathfrak{k})$ and, consequently, $\mathfrak{g}[M] = \mathfrak{k}$.

It remains to prove our claim about the choice of h''. We consider the parabolic subalgebra $\mathfrak{p}'$ defined via (4.1) by the fixed element h', and then we choose h'' such that $\mathfrak{p}$ is a certain subalgebra of $\mathfrak{p}'$. Let E be the defining (n-dimensional) $\mathfrak{g}$-module. There is an isomorphism of $(\mathfrak{k}_{ss} \oplus C(\mathfrak{k}_{ss}))$-modules

$$E \cong \bigoplus_i (E_i \otimes V_i), \tag{5.1}$$

where the E_i's are pairwise nonisomorphic irreducible $\mathfrak{k}_{ss}$-modules and the V_i's are irreducible $C(\mathfrak{k}_{ss})$-modules. We have

$$C(\mathfrak{k}_{ss}) \cong \bigoplus_i \mathrm{End}(V_i). \tag{5.2}$$

One can check that

$$\mathfrak{p}'_{\mathrm{red}} = C(\mathfrak{h}_{\mathfrak{k}_{\mathrm{ss}}}) \cong \bigoplus_{\lambda \in \mathfrak{h}^*_{\mathfrak{k}_{\mathrm{ss}}}} \mathrm{End}(E^\lambda), \tag{5.3}$$

where E^λ denotes the $\mathfrak{h}_{\mathfrak{k}_{\mathrm{ss}}}$-weight space of weight λ. Furthermore, by (5.1),

$$E^\lambda \cong \bigoplus_i (E_i^\lambda \otimes V_i). \tag{5.4}$$

Put $\mathscr{E}_{ij}^\lambda := \mathrm{Hom}(E_i^\lambda, E_j^\lambda) \otimes \mathrm{Hom}(V_i, V_j)$ and $\mathscr{E}^\lambda := \oplus_{i,j} \mathscr{E}_{ij}^\lambda$. Then combining (5.3) and (5.4), one obtains $\mathfrak{p}'_{\mathrm{red}} \cong \bigoplus_\lambda \mathscr{E}^\lambda$. Note that $\mathscr{E}_+^\lambda := \bigoplus_{i \le j} \mathscr{E}_{ij}^\lambda$ is a parabolic subalgebra of $\mathscr{E}^\lambda$.

We now choose $h'' \in Z(\mathfrak{k})$ such that the parabolic subalgebra $\mathfrak{p}$ associated to $h' + h''$ by (4.1) is precisely $\left(\bigoplus_\lambda \mathscr{E}_+^\lambda\right) \ni\!\!\!\!\supset \mathfrak{n}_{\mathfrak{p}'}$. Note that $\mathfrak{p}_{\mathrm{red}} = \bigoplus_{i,\lambda} \mathscr{E}_{i,i}^\lambda$. For each $\mathfrak{p} \cap \mathfrak{k}_{\mathrm{ss}}$-singular weight λ of $\mathfrak{k}$ in E, there is a unique index i_λ such that the $\mathfrak{p} \cap \mathfrak{k}_{\mathrm{ss}}$-highest weight of E_{i_λ} equals λ. Let $\mathfrak{a} := \bigoplus_\lambda \mathscr{E}_{i_\lambda i_\lambda}^\lambda$, and let $\mathfrak{a}'$ be the ideal complementary to $\mathfrak{a}$. Since $E_{i_\lambda}^\lambda$ is one-dimensional and $\mathscr{E}_{i_\lambda i_\lambda}^\lambda \cong \mathrm{End}(V_{i_\lambda})$, the isomorphism (5.2) enables us to conclude that $C(\mathfrak{k}_{\mathrm{ss}})$ is isomorphic to $\mathfrak{a}$. □

COROLLARY 5.2

A reductive in $\mathfrak{g} = \mathfrak{gl}(n)$ subalgebra $\mathfrak{k}$ is a Fernando-Kac subalgebra of finite type if and only if the defining $\mathfrak{g}$-module is multiplicity free as a $\mathfrak{k}$-module.

5.2. A combinatorial setup

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g} = \mathfrak{gl}(n)$, and let $\mathfrak{l} \supset \mathfrak{h}$ be a root subalgebra of $\mathfrak{g}$. The subalgebra $\mathfrak{l}$ is defined by its subset of roots $\Delta(\mathfrak{l}) \subset \Delta$, where $\Delta \subset \mathfrak{h}^*$ is the root system of $\mathfrak{g}$. Recall that $\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$ for an orthonormal basis $\varepsilon_1, \dots, \varepsilon_n$ of $\mathfrak{h}^*$. Set $\mathfrak{k} := \mathfrak{l}_{\mathrm{red}}$ and $\mathfrak{n} := \mathfrak{n}_{\mathfrak{l}}$. Then $\mathfrak{l} = \mathfrak{k} \ni\!\!\!\!\supset \mathfrak{n}$. Fix an arbitrary Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$, and let $\mathscr{S}_{\mathfrak{k}}(\mathfrak{g}) \subset \Delta$ be the set of weights of all $\mathfrak{k} \cap \mathfrak{b}$-singular vectors in $\mathfrak{g}$. For any $\alpha \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g})$, denote by $\mathfrak{g}(\alpha)$ the irreducible $\mathfrak{k}$-submodule in $\mathfrak{g}$ with highest weight α. Obviously, any $\alpha, \beta \in \Delta$ satisfy the condition

$$\alpha + \beta \in \Delta \ \text{ for } \alpha, \beta \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}) \Rightarrow \alpha + \beta \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}). \tag{5.5}$$

More generally, for any $\mathfrak{k}$-submodule $\mathfrak{f}$ of $\mathfrak{g}$, let $\mathscr{S}_{\mathfrak{k}}(\mathfrak{f})$ denote the set of all weights of $\mathfrak{k} \cap \mathfrak{b}$-singular vectors in $\mathfrak{f}$. As $\mathfrak{k}$ and $\mathfrak{n}$ are subalgebras, $\mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$ and $\mathscr{S}_{\mathfrak{k}}(\mathfrak{k})$ satisfy the analog of condition (5.5).

The following lemma is an easy consequence of the description of root subalgebras in $\mathfrak{gl}(n)$, and we leave its proof to the reader.

LEMMA 5.3

There exist pairwise nonintersecting subsets $I, J, K \subset \{1, \dots, n\}$ such that $|I| = |J|$

and

$$\mathscr{S}_{\mathfrak{k}}(\mathfrak{g}) = \left\{\varepsilon_i - \varepsilon_j \mid i \in I \cup K, j \in J \cup K\right\}.$$

Let $\mathscr{C}_{\mathfrak{k}}(\mathfrak{f})$ denote the set of all linear combinations of vectors from $\mathscr{S}_{\mathfrak{k}}(\mathfrak{f})$ with coefficients in $\mathbb{Z}_+$.

LEMMA 5.4
Let $\mathfrak{g} = \mathfrak{gl}(n)$. If $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$, one of the following relations holds:
(1) $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$,
(2) $\alpha_1 + \alpha_2 = \beta$
for some $\alpha_1, \alpha_2 \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta_1, \beta_2, \beta \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$, where $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = 0$ in the case of (1).

Proof
If $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$, there is a nontrivial relation

$$\alpha_1 + \cdots + \alpha_k = \beta_1 + \cdots + \beta_l \tag{5.6}$$

for $\alpha_i \in S_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$ and $\beta_i \in S_{\mathfrak{k}}(\mathfrak{n})$. Among all such relations we fix one with minimal k and minimal l for the fixed k. Consider first the case when $\alpha_1 + \alpha_p \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g})$ for some $p \leq k$. We claim that then $\alpha_1 + \alpha_p \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$. For if $\alpha_1 + \alpha_p \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$, one can reduce k in (5.6) by the substitution $\beta = \alpha_1 + \alpha_p$, which contradicts our assumption. Thus $\beta := \alpha_1 + \alpha_p \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$, and to show that $\alpha_1 + \alpha_p = \beta$ is a relation of type (2), we need only verify that $\alpha_1, \alpha_p \notin \mathscr{S}_{\mathfrak{k}}(\mathfrak{k})$. But the assumption $\alpha_1 \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{k})$ (and similarly $\alpha_p \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{k})$) is obviously contradictory as then $-\alpha_1 \in \Delta(\mathfrak{k})$ and $\alpha_p = \alpha_1 + \alpha_p - \alpha_1 = \beta - \alpha_1 \in \Delta(\mathfrak{n})$. Therefore $\alpha_1, \alpha_p \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$, and $\alpha_1 + \alpha_p = \beta$.

In the remainder of the proof we assume that $\alpha_1 + \alpha_p \notin \mathscr{S}_{\mathfrak{k}}(\mathfrak{g})$ for all $p \leq k$. If $\alpha_1 = \varepsilon_i - \varepsilon_j$, then ε_i and $-\varepsilon_j$ appear in $\alpha_1 + \cdots + \alpha_k$ with positive coefficients. Therefore there exist a and b such that $\beta_a = \varepsilon_i - \varepsilon_r$ and $\beta_b = \varepsilon_s - \varepsilon_j$, $s \neq r$, by minimality. By Lemma 5.3, $\gamma := \varepsilon_s - \varepsilon_r \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g})$. We claim that $\gamma \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$. Indeed, assume to the contrary that $\gamma \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$. Then one can modify (5.6) by removing α_1 and replacing $\beta_a + \beta_b$ by γ. Since (5.6) is minimal, the new relation must be trivial. Thus $\alpha_1 = \beta_1 + \cdots + \beta_l$. Since $\beta_1 + \cdots + \beta_l \in \Delta$, $\beta := \beta_1 + \cdots + \beta_l \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$, and hence $\alpha = \beta \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$. This is a contradiction. Therefore indeed $\gamma \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$, and we have a relation $\alpha_1 + \gamma = \beta_a + \beta_b$, where $\alpha_1, \gamma \in \mathscr{S}(\mathfrak{g}/\mathfrak{n})$, $\beta_a, \beta_b \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$. Obviously, $(\alpha_1, \gamma) = (\beta_a, \beta_b) = 0$. To complete the proof we need to show that $\alpha_1, \gamma \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$. But the assumption $\alpha_1 \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{k})$ (and similarly $\gamma \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{k})$) is contradictory as it implies that $\beta_b - \alpha_1 \in \Delta(\mathfrak{n})$. Hence $\gamma = \beta_a + (\beta_b - \alpha_1) \in \Delta(\mathfrak{n})$. □

COROLLARY 5.5
The equalities $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$ and $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$ are equivalent.

Proof

As $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \subset \mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$, $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$ implies $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$. To prove the converse, assume that $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$ but $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$. Then one has a relation of type (1) or (2) as in Lemma 5.4 with $\alpha_1, \alpha_2 \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$. Hence $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$. This is a contradiction. □

LEMMA 5.6

Let $\mathfrak{s} = \mathfrak{gl}(m)$, let $\mathfrak{q} \subset \mathfrak{s}$ be a maximal parabolic subalgebra, and let $\mathfrak{k} := \mathfrak{q}_{\mathrm{red}}$. Let V_κ be the irreducible $\mathfrak{s}$-module with highest weight κ, and let $V_\mu(\mathfrak{k})$ be the irreducible $\mathfrak{k}$-module with highest weight μ. If λ is a dominant $\mathfrak{k}$-integral weight and β is the highest root of $\mathfrak{s}$, then, for large enough $q \in \mathbb{Z}_+$, the multiplicity of $V_{\lambda+q\beta}(\mathfrak{k})$ in $V_{\lambda+p\beta}$ is one for any $p \in \mathbb{Z}_+$, $p \geq q$.

Proof

Set $\mu := \lambda + p\beta$, $\nu := \lambda + q\beta$ for fixed $p \geq q \in \mathbb{Z}_+$. Note that μ and ν are $\mathfrak{k}$-dominant and hence that $V_\nu(\mathfrak{k})$ is finite-dimensional. If M_μ denotes the Verma module over $\mathfrak{s}$ and $M_\mu(\mathfrak{k})$ denotes the Verma module over $\mathfrak{k}$, then M_μ is isomorphic to $M_\mu(\mathfrak{k}) \otimes S(\mathfrak{q}/\mathfrak{k})^*$ as a $\mathfrak{k}$-module. Thus M_μ admits a filtration by $\mathfrak{k}$-submodules such that the associated graded $\mathfrak{k}$-module is a direct sum of Verma modules over $\mathfrak{k}$, each appearing with finite multiplicity. As the multiplicity of the weight $(q-p)\beta$ in $S(\mathfrak{q}/\mathfrak{k})^*$ is one, the multiplicity of $M_\nu(\mathfrak{k})$ in M_μ is one. Therefore the multiplicity of $V_\nu(\mathfrak{k})$ in M_μ is also one.

Now let $N \neq V_\mu$ be an irreducible subquotient of M_μ. We show that for q large, the multiplicity of $V_\nu(\mathfrak{k})$ in N is zero. It is known (see, e.g., [Di, Theorem 7.6.23]) that N is a subquotient of $M_{w_\alpha(\mu+\rho)-\rho}$ for some positive root α such that $(\mu, \alpha) \in \mathbb{Z}_+$. Therefore it suffices to prove that the multiplicity of $M_\nu(\mathfrak{k})$ in $M_{w_\alpha(\mu+\rho)-\rho}$ is zero. This is equivalent to showing that $w_\alpha(\mu+\rho) - \rho - \nu$ is not a weight of $S(\mathfrak{q}/\mathfrak{k})$, that is, that $w_\alpha(\mu+\rho) - \rho - \nu$ does not belong to the convex hull $\mathscr{C}$ of $\Delta(\mathfrak{q}/\mathfrak{k})$.

Choose q such that $(\nu, \alpha) > 0$ for any positive α satisfying $(\alpha, \beta) = 1$, and assume $p \geq q$. First, consider the case when $(\alpha, \beta) = 0$. Here $w_\alpha(\mu+\rho) - \rho - \nu = w_\alpha(\nu+\rho) - \rho - \nu + (p-q)\beta$. But $w_\alpha(\nu+\rho) - \rho - \nu = a\alpha$ for some negative a, which implies that $w_\alpha(\mu+\rho) - \rho - \nu = (p-q)\beta + a\alpha$ does not belong to $\mathscr{C}$. Next, consider the case when $(\alpha, \beta) = 1$. Here

$$\begin{aligned} w_\alpha(\mu+\rho) - \rho - \nu &= w_\alpha(\nu+\rho) - \rho - \nu + (p-q)w_\alpha(\beta) \\ &= -(b+1+p-q)\alpha + (p-q)\beta, \end{aligned}$$

where $b = (\nu, \alpha)$ is positive by our choice of q. One can see that $-(b+1+p-q)\alpha + (p-q)\beta$ is not in $\mathscr{C}$. Finally, the case $\alpha = \beta$ is obvious. □

COROLLARY 5.7

Let $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_j$, where each $\mathfrak{s}_i$ is isomorphic to $\mathfrak{gl}(m_i)$, let $\mathfrak{q}_i$ be a maximal parabolic subalgebra of $\mathfrak{s}_i$, and let β_i be the highest root of $\mathfrak{s}_i$. Let $\mathfrak{q} = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_l \oplus \mathfrak{s}_{l+1} \oplus \cdots \oplus \mathfrak{s}_j$ for $l \leq j$, let $\mathfrak{k}$ be the reductive part of $\mathfrak{q}$, and let $\beta = \beta_1 + \cdots + \beta_l$. If λ is a dominant $\mathfrak{k}$-integral weight, then there is a positive integer q such that the multiplicity of $V_{\lambda+q\beta}(\mathfrak{k})$ in $V_{\lambda+p\beta}$ is one for any $p \geq q$.

Proof

The statement follows easily from Lemma 5.6 as an irreducible highest weight module over a direct sum of reductive Lie algebras is isomorphic to the tensor product of irreducible modules over the components. □

5.3. Description of Fernando-Kac root subalgebras of finite type

THEOREM 5.8

A root subalgebra $\mathfrak{l} = (\mathfrak{k} \supset\!\!\!\!+ \mathfrak{n}) \subset \mathfrak{g} = \mathfrak{gl}(n)$ is a Fernando-Kac subalgebra of finite type if and only if $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$.

Proof

First, we show that if $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$, then $\mathfrak{l}$ is not a Fernando-Kac subalgebra of finite type. If $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) \neq 0$, Lemma 5.4 provides us with a relation of type (1) or (2). Assume that the relation is of type (2), that is, that $\alpha_1 + \alpha_2 = \beta$ for some $\alpha_1, \alpha_2 \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n})$. Let $\mathfrak{s}$ be the subalgebra generated by $\mathfrak{k}$ and $\mathfrak{g}^{\pm\beta}$, and let $\mathfrak{q}$ be the subalgebra generated by $\mathfrak{k}$ and $\mathfrak{g}^{\beta}$. Then the triple $(\mathfrak{s}, \mathfrak{q}, \beta)$ satisfies the hypothesis of Corollary 5.7 with $l = 1$. Moreover, $\mathfrak{g}(\beta)$ commutes with $\mathfrak{g}^{\alpha_i}$.

Let M be an irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module. There exists a $(\mathfrak{b} \cap \mathfrak{k})$-singular vector $v \in M$ such that $\mathfrak{g}(\beta) \cdot v = 0$. Let λ denote the weight of v. For any positive integer t, set $v_t = (g^{\alpha_1})^t (g^{\alpha_2})^t \cdot v$ for $0 \neq g^{\alpha_i} \in \mathfrak{g}^{\alpha_i}$. As g^{α_i} acts freely on M, we have $v_t \neq 0$. Furthermore, v_t is $(\mathfrak{b} \cap \mathfrak{k})$-singular and $\mathfrak{g}(\beta) v_t = 0$. Hence v_t generates an $\mathfrak{s}$-submodule $M(v_t) \subset M$ of highest weight $\lambda + t\beta$. By Corollary 5.7, for a fixed large $r \in \mathbb{Z}_+$, the multiplicity of $V_{\lambda+r\beta}$ in $M(v_t)$ is not zero for any $t \geq r$. Therefore the multiplicity of $V_{\lambda+r\beta}$ in M is infinite. This is a contradiction.

In the case of a relation of type (1), $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$, let $\mathfrak{s} \subset \mathfrak{g}$ be the subalgebra generated by $\mathfrak{k}$, $\mathfrak{g}^{\pm\beta_1}$, and $\mathfrak{g}^{\pm\beta_2}$, and let $\mathfrak{q} \subset \mathfrak{s}$ be the subalgebra generated by $\mathfrak{k}$, $\mathfrak{g}^{\beta_1}$, and $\mathfrak{g}^{\beta_2}$. The reader can check that the triple $(\mathfrak{s}, \mathfrak{q}, \beta)$ satisfies the conditions of Corollary 5.7 with $l = 2$ and, moreover, that $\mathfrak{g}(\beta_1) \oplus \mathfrak{g}(\beta_2)$ commutes with $\mathfrak{g}^{\alpha_i}$. Therefore an argument similar to that in the case of a relation of type (2) leads to a contradiction.

It remains to prove that $\mathfrak{l}$ is a Fernando-Kac subalgebra of finite type whenever $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$. Using Theorem 4.3, we construct an irreducible strict $(\mathfrak{g}, \mathfrak{l})$-

module M of finite type over $\mathfrak{l}$.

Note first that $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g})$ consist of $\mathfrak{k}$-dominant roots; therefore $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}) \cap -\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}) = \mathscr{C}_{\mathfrak{k}}(C(\mathfrak{k}_{\mathrm{ss}}))$ and $\mathscr{S}_{\mathfrak{k}}(\mathfrak{g}) \cap -\mathscr{S}_{\mathfrak{k}}(\mathfrak{g}) = \mathscr{S}_{\mathfrak{k}}(C(\mathfrak{k}_{\mathrm{ss}}))$. Furthermore, as $\mathfrak{n}$ is nilpotent, $\mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) \cap -\mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$. Let $\mathscr{C}_0 = \mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap -\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$, and let $\Delta_0 = \mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap -\mathscr{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$. The above implies immediately that $\Delta_0 \subset \mathscr{S}_{\mathfrak{k}}(C(\mathfrak{k}_{\mathrm{ss}}))$ and that Δ_0 generates $\mathscr{C}_0$. By Corollary 5.5, $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$. Therefore one can find $h \in \mathfrak{h}$ such that all eigenvalues of $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$ are rational and

$$\begin{aligned} \alpha(h) &> 0 \quad \text{for } \alpha \in \mathscr{S}_{\mathfrak{k}}(\mathfrak{n}), \\ \alpha(h) &= 0 \quad \text{for } \alpha \in \Delta_0, \\ \alpha(h) &< 0 \quad \text{for } \alpha \in \mathscr{S}_{\mathfrak{k}}\big(\mathfrak{g}/(\mathfrak{n} + C(\mathfrak{k}_{\mathrm{ss}}))\big). \end{aligned} \tag{5.7}$$

One can easily verify that, in additon, h can be chosen so that

$$\alpha(h) < 0 \quad \text{for all } \alpha \in \Delta(\mathfrak{b} \cap \mathfrak{k}_{\mathrm{ss}}). \tag{5.8}$$

Let $\mathfrak{p}$ be defined by (4.1). Then $\Delta(\mathfrak{p}_{\mathrm{red}}) = \Delta_0$, and $\mathfrak{n} \subset \mathfrak{n}_{\mathfrak{p}}$.

Let L be an irreducible $(\mathfrak{p},\mathfrak{h})$-module of finite type over $\mathfrak{h}$ with trivial action of $\mathfrak{n}_{\mathfrak{p}}$ and such that $\mathfrak{p}_{\mathrm{red}}[L] = \mathfrak{h}$. Such an L exists as $\mathfrak{p}_{\mathrm{ss}}$ is a sum of ideals of type A. Let M be as in Section 4.3. Then, by Theorem 4.3, M is an irreducible $(\mathfrak{g},\mathfrak{k})$-module of finite type over $\mathfrak{k}$. Let $\mathfrak{g}[M] = \mathfrak{k} \supplus \mathfrak{n}'$. We claim that $\mathfrak{n}' = \mathfrak{n}$. Indeed, $\mathfrak{g}(\alpha) \subset \mathfrak{n}'$ if and only if $\mathfrak{g}(\alpha) \subset \mathfrak{p}[L]$. In particular, $\alpha(h) \geq 0$. If $\alpha(h) > 0$, then by (5.7) and (5.8), $\mathfrak{g}(\alpha) \subset \mathfrak{n} \subset \mathfrak{n}_{\mathfrak{p}} \subset \mathfrak{p}[L]$. If $\alpha(h) = 0$, then $\alpha \in \Delta_0$. As $\mathfrak{p}_{\mathrm{red}}(L) = \mathfrak{h}$, we have $\mathfrak{g}(\alpha) \not\subset \mathfrak{p}[L]$. Thus $\mathfrak{n} = \mathfrak{n}'$. Theorem 5.8 is proven. □

COROLLARY 5.9
A root subalgebra $\mathfrak{l} = (\mathfrak{k} \supplus \mathfrak{n}) \subset \mathfrak{gl}(n)$ *with* $\mathfrak{n} \subset C(\mathfrak{k}_{\mathrm{ss}})$ *is a Fernando-Kac subalgebra of finite type if and only if* $\mathfrak{n}$ *is the nilradical of a parabolic subalgebra in* $C(\mathfrak{k}_{\mathrm{ss}})$.

Proof
For the necessity, see Theorem 3.1(5). For the sufficiency, we use Theorem 5.8. By hypothesis, $\mathfrak{n}$ is the nilradical of a parabolic subalgebra in $C(\mathfrak{k}_{\mathrm{ss}})$. We show that $\mathscr{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$. Suppose not. Then there exist roots $\alpha_1, \ldots, \alpha_k \in \mathscr{C}_{\mathfrak{k}}(\mathfrak{n}/\mathfrak{l})$ and roots $\beta_1, \ldots, \beta_l \in \mathscr{C}_{\mathfrak{k}}(\mathfrak{n})$ such that (5.6) holds. Restrict both sides of (5.6) to $\mathfrak{h}_{\mathfrak{k}_{\mathrm{ss}}}$, and write $\widetilde{\gamma}$ for the restriction of a weight γ to $\mathfrak{h}_{\mathfrak{k}_{\mathrm{ss}}}$. Because $\mathfrak{n} \subset C(\mathfrak{k}_{\mathrm{ss}})$, $\widetilde{\beta}_i = 0$ for all i, and hence $\widetilde{\alpha}_1 + \cdots + \widetilde{\alpha}_l = 0$. But the $\widetilde{\alpha}_j$'s are dominant weights for $\mathfrak{k}_{\mathrm{ss}}$. Therefore $\widetilde{\alpha}_j = 0$ for all j, and each $\alpha_j \in \mathscr{C}_{\mathfrak{k}}(C(\mathfrak{k}_{\mathrm{ss}})) = \Delta(C(\mathfrak{k}_{\mathrm{ss}}))$. Equation (5.6) becomes a nontrivial relation among roots in $\Delta(\mathfrak{n})$ and $\Delta(C(\mathfrak{k}_{\mathrm{ss}})) \setminus \Delta(\mathfrak{n})$. This is a contradiction. □

Example. Let $\mathfrak{g} = \mathfrak{gl}(4)$, let $\mathfrak{h}$ be the diagonal subalgebra, and let $\mathfrak{l} \supset \mathfrak{h}$ be a root subalgebra of $\mathfrak{g}$. The rank of $\mathfrak{l}_{\mathrm{ss}}$ can be 0, 1, or 2. In the first case, $\mathfrak{l}$ is solvable, and by Proposition 3.2, $\mathfrak{l}$ is of finite type if and only if $\mathfrak{n}_{\mathfrak{l}}$ is the nilradical of a parabolic subalgebra. In the third case, $\mathfrak{l}_{\mathrm{red}}$ equals the fixed points of an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{l}$ is always a Fernando subalgebra of finite type; the corresponding strict $(\mathfrak{g}, \mathfrak{l})$-modules are Harish-Chandra modules.

In the case when $\mathfrak{l}_{ss} \cong \mathfrak{sl}(2)$, we can fix the roots of $\mathfrak{l}_{ss}$ to be $\pm(\varepsilon_1 - \varepsilon_2)$. To determine $\mathfrak{l}$ we need to specify the roots of $\mathfrak{n}_{\mathfrak{l}}$. Up to automorphisms of $\mathfrak{g}$ that stabilize $\mathfrak{l}_{ss}$, there are eight choices for $\mathfrak{n}_{\mathfrak{l}}$ (including the possibility $\mathfrak{n}_{\mathfrak{l}} = 0$). A direct checking based on Theorem 5.8 and Corollary 5.9 shows that there is a single choice of $\mathfrak{n}_{\mathfrak{l}}$ for which $\mathfrak{l}$ is not a Fernando-Kac subalgebra of finite type. We may normalize this $\mathfrak{l}$ so that the roots in $\mathfrak{n}_{\mathfrak{l}}$ are $\varepsilon_1 - \varepsilon_3$ and $\varepsilon_2 - \varepsilon_3$. Furthermore, for the so-defined $\mathfrak{l} = \mathfrak{l}_{\mathrm{red}} \supsetplus \mathfrak{n}_{\mathfrak{l}}$, statements (1) – (5) in Theorem 3.1 hold. This shows, in particular, that the statements in Theorem 3.1 are not sufficient conditions for a subalgebra $\mathfrak{l}$ to be a Fernando-Kac subalgebra of finite type.

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