

METAPLECTIC TORI OVER LOCAL FIELDS

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ABSTRACT. Smooth irreducible representations of tori over local fields have been parameterized by Langlands, using class field theory and Galois cohomology. This paper extends this parameterization to central extensions of such tori, which arise naturally in the setting of nonlinear covers of reductive groups.

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1. INTRODUCTION

1.1. Motivation. Let $\mathbf{T}$ be an algebraic torus over a local field F ; let $T = \mathbf{T}(F)$. Let L/F be a finite Galois extension over which $\mathbf{T}$ splits, with $\Gamma = \text{Gal}(L/F)$. Let $X(T)$ denote the group of continuous characters of T with values in $\mathbb{C}^\times$. In a preprint from 1968, now appearing as [9] (cf. the 1985 article by Labesse [8]), Langlands proves the following:

Theorem 1.1. *There is a natural isomorphism:*

$$X(T) \cong H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}}),$$

where $\mathcal{W}_{L/F}$ denotes the Weil group of L/F , and $\hat{\mathcal{T}}$ denotes the complex dual torus of $\mathbf{T}$.

We may consider $\mathbf{T}$ as a sheaf of groups, on the big Zariski site over F . In addition, we may consider $\mathbf{K}_2$ as such a sheaf, using Quillen's algebraic K-theory. Let $\mathbf{T}'$ be a central extension of $\mathbf{T}$ by $\mathbf{K}_2$, in the category of sheaves of groups on the big Zariski site over F . Such objects are introduced and studied extensively by Deligne and Brylinski in [3].

Let $T' = \mathbf{T}'(F)$ be the resulting extension of T by $K_2 = \mathbf{K}_2(F)$. If $F \not\cong \mathbb{C}$ and F has sufficiently many n^{th} roots of unity, one may push forward the central extension T' via the Hilbert symbol to obtain a central extension $\tilde{T}$:

$$1 \rightarrow \mu_n \rightarrow \tilde{T} \rightarrow T \rightarrow 1.$$

We are interested in the set $\text{Irr}_\epsilon(\tilde{T})$ of irreducible genuine representations of $\tilde{T}$, as defined in Section 3. Such representations arise frequently in the literature on “metaplectic groups”, especially when considering principal series representations of nonlinear covers of reductive groups (cf. [13], [6], [1], among others). It is the goal of this paper to parameterize the set $\text{Irr}_\epsilon(\tilde{T})$ in a way which naturally generalizes the aforementioned theorem of Langlands.

1.2. Main Results. Associated to the central extension $\mathbf{T}'$, Deligne and Brylinski associate two functorial invariants: an integer-valued quadratic form Q on the cocharacter lattice Y of $\mathbf{T}$, and a $\text{Gal}(L/F)$ -equivariant central extension $\hat{Y}$ of Y by $L^\times$. Associated to Q , one has a symmetric bilinear form $B_Q: Y \otimes_{\mathbb{Z}} Y \rightarrow \mathbb{Z}$.

Define:

$$Y^\# = \{y \in Y \text{ such that } B_Q(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y\}.$$

Associated to the inclusion $\iota: Y^\# \hookrightarrow Y$, there is an isogeny of complex tori: $\hat{\iota}: \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#$. This isogeny is also a morphism of $\mathcal{W}_{L/F}$ -modules. We prove the following result in this paper:

Theorem 1.2. *Suppose that one of the following conditions is satisfied:*

- (1) $\mathbf{T}$ is a split torus.
- (2) F is nonarchimedean, $\mathbf{T}$ splits over an unramified extension of F , and n is relatively prime to the residue characteristic of F .
- (3) F is archimedean.

Then, there exists a finite-to-one map:

$$\Phi: \text{Irr}_\epsilon(\tilde{T}) \rightarrow \frac{H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})}{\eta(H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#))},$$

such that the map intertwines canonical actions of $H^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})$.

We call the fibres of Φ “packets”, in analogy with the theory of L-packets. In each of the three cases of the above theorem, we explicitly describe the structure of the packets. When $\mathbf{T}$ is a split torus, all packets are singletons. In the other cases, the packets are naturally torsors for an explicitly determined finite abelian group, which we call $P^\dagger$. The definition of $P^\dagger$ is quite uniform; it may have a cohomological interpretation over the residue field, when considering nonarchimedean local fields.

The parameterization Φ of irreducible genuine representations is not unique; rather, it depends upon the choice of a base-point. The choice of this base-point is a significant problem. We identify a natural class of “pseudo-spherical” representations (following previous authors such as [13] and [1]). Moreover, we parameterize pseudo-spherical irreducible representations as a torsor for a complex algebraic torus in Section 6; perhaps more naturally, the category of pseudo-spherical representations can be identified with the category of modules over a “quantum dual torus”.

1.3. Acknowledgements. We would like to thank Jeffrey Adams and Gordan Savin, for providing some advice and insight related to this paper. In addition, we are thankful for the hospitality and excellent working environment provided by the Hausdorff Institute for Mathematics in Bonn, Germany, during the preparation of this paper. In addition, we thank the University of Michigan for their hospitality,

while this paper was being finished. We thank Brian Conrad and Stephen DeBacker for helpful conversations at Michigan during this time.

We are heavily indebted to Pierre Deligne, who read an early draft of this paper, and gave extensive helpful comments. His generosity has led to significant improvements in presentation and content.

2. BACKGROUND

2.1. Fields and sheaves. F will always denote a local field. F_{Zar} will denote the big Zariski site over F . By this, we mean that F_{Zar} is the full subcategory of the category of schemes over F , whose objects are schemes of finite type over F , endowed with the Zariski topology. $\mathbf{Set}_F$ will denote the topos of sheaves of sets over F_{Zar} , and $\mathbf{Sp}_F$ will denote the topos of sheaves of groups over F_{Zar} .

Any scheme or algebraic group over F will be identified with its functor of points, i.e., the associated object of $\mathbf{Set}_F$ or $\mathbf{Sp}_F$, respectively. Quillen's K-theory, defined in [12], yields sheaves $\mathbf{K}_n$ of abelian groups on F_{Zar} . We only work with $\mathbf{K}_1$ and $\mathbf{K}_2$, viewed as objects of $\mathbf{Sp}_F$.

For any field L , the group $\mathbf{K}_2(L)$ is identified as a quotient:

$$\mathbf{K}_2(L) = \frac{L^\times \otimes_{\mathbb{Z}} L^\times}{\langle x \otimes (1 - x) \rangle_{1 \neq x \in L^\times}}.$$

If $l_1, l_2 \in L$, then we write $\{l_1, l_2\}$ for the image of $l_1 \otimes l_2$ in $\mathbf{K}_2(L)$. The bilinear form $\{\cdot, \cdot\}$ is called the *universal symbol*; it is skew-symmetric. It is usually not alternating, but $\{x, x\} = \{x, -1\}$ for all $x \in L^\times$.

2.2. Local Nonarchimedean Fields. Suppose that F is a nonarchimedean local field. Then $\mathcal{O}_F$ will denote the valuation ring of F , and $\mathbf{f}$ the residue field of $\mathcal{O}_F$. We let p denote the characteristic of $\mathbf{f}$, and assume that the value group of F is $\mathbb{Z}$. We let q denote the cardinality of $\mathbf{f}$.

There is a canonical short exact sequence of abelian groups, given by inclusion and valuation:

$$1 \rightarrow \mathcal{O}_F^\times \rightarrow F^\times \rightarrow \mathbb{Z} \rightarrow 1.$$

It is sometimes convenient to split this sequence of abelian groups, by choosing a uniformizing element $\varpi \in F^\times$. However, none of our main results depend on the which uniformizing element is chosen.

Reduction yields another canonical short exact sequence:

$$1 \rightarrow \mathcal{O}_F^{\times 1} \rightarrow \mathcal{O}_F^\times \rightarrow \mathbf{f}^\times \rightarrow 1.$$

This sequence is split by the Teichmüller lifting:

$$\Theta: \mathbf{f}^\times \rightarrow \mathcal{O}_F^\times.$$

2.3. The Weil group. We let $\mathcal{W}_F$ denote a Weil group of F as discussed by Tate in [14]. In particular, we follow Tate's choices, and normalize the reciprocity isomorphism of nonarchimedean local class field theory, $rec: F^\times \rightarrow \mathcal{W}_F^{ab}$ in such a way that uniformizing elements of $F^\times$ act as the *geometric* Frobenius via rec .

When L is a finite Galois extension of F , we continue to follow Tate [14], and define:

$$\mathcal{W}_{L/F} = \mathcal{W}_F / [\overline{\mathcal{W}_L}, \mathcal{W}_L].$$

There is then a short exact sequence:

$$1 \rightarrow L^\times \rightarrow \mathcal{W}_{L/F} \rightarrow \text{Gal}(L/F) \rightarrow 1.$$

2.4. The Hilbert symbol. We say that F has *enough* n^{th} roots of unity if $\mu_n(F)$ has n elements. When F has enough n^{th} roots of unity, and $F \not\cong \mathbb{C}$, the Hilbert symbol provides a non-degenerate skew-symmetric bilinear map:

$$(\cdot, \cdot)_{F,n}: \frac{F^\times}{F^{\times n}} \otimes_{\mathbb{Z}} \frac{F^\times}{F^{\times n}} \rightarrow \mu_n(F).$$

In general, the Hilbert symbol is not alternating. The Hilbert symbol factors through $K_2(F)$, via the universal symbol.

The definition of the Hilbert symbol relies on a choice of reciprocity isomorphism in local class field theory – this choice has been made earlier, in sending a uniformizing element of $F^\times$ to a geometric Frobenius.

If F is nonarchimedean, and $(p, n) = 1$, then we say that the Hilbert symbol $(\cdot, \cdot)_{F,n}$ is tame. If p is odd, then in the tame case, $(\varpi, \varpi)_{F,n} = (-1)^{(q-1)/n}$, for every uniformizing element $\varpi \in F^\times$. When $p = 2$, in the tame case, $(\varpi, \varpi)_{F,n} = 1$. When $F \cong \mathbb{R}$, $(-1, -1)_{F,2} = -1$.

2.5. Tori. $\mathbf{T}$ will always denote an algebraic torus over F . Let L be a finite Galois extension of F , over which $\mathbf{T}$ splits, and define $\Gamma = \text{Gal}(L/F)$. We write $X = \text{Hom}(\mathbf{T}, \mathbf{G}_m)$ for the character group and Y for the cocharacter group $\text{Hom}(\mathbf{G}_m, \mathbf{T})$. We view X and Y as finite rank free $\mathbb{Z}$ -modules, endowed with actions of Γ . The groups X and Y are in canonical Γ invariant duality.

The dual torus $\hat{\mathbf{T}}$ is the split torus $\text{Spec}(\mathbb{Z}[Y])$ over $\mathbb{Z}$, with the resulting action of Γ . We write $\hat{\mathcal{T}} = \hat{\mathbf{T}}(\mathbb{C}) \equiv X \otimes_{\mathbb{Z}} \mathbb{C}^\times$ for the resulting $\mathbb{C}$ -torus, also endowed with the action of Γ .

2.6. Central Extensions of Tori by K_2 . Let $\mathcal{CExt}(\mathbf{T}, K_2)$ be the category of central extensions of $\mathbf{T}$ by K_2 in $\mathfrak{Sp}_F$. Let $\mathcal{CExt}_\Gamma(Y, L^\times)$ be the category of Γ -equivariant extensions of Y by $L^\times$.

In [3], Deligne and Brylinski study the following category, which we call $\mathfrak{DB}_\mathbf{T}$. Its objects are pairs $(Q, \tilde{Y})$, where:

- $Q: Y \rightarrow \mathbb{Z}$ is a Γ -invariant quadratic form.
- $\tilde{Y}$ is a Γ -equivariant central extension of Y by $L^\times$.
- The resulting commutator map $C: \bigwedge^2 Y \rightarrow L^\times$ satisfies:

$$C(y_1, y_2) = (-1)^{B_Q(y_1, y_2)}, \text{ for all } y_1, y_2 \in Y,$$

where B_Q is the symmetric bilinear form associated to Q .

If $(Q_1, \tilde{Y}_1)$, and $(Q_2, \tilde{Y}_2)$ are two objects of $\mathfrak{DB}_\mathbf{T}$, then a morphism from $(Q_1, \tilde{Y}_1)$ to $(Q_2, \tilde{Y}_2)$ exists only if $Q_1 = Q_2$, in which case the morphisms of $\mathfrak{DB}_\mathbf{T}$ are the just those from $\tilde{Y}_1$ to $\tilde{Y}_2$ in $\mathcal{CExt}_\Gamma(Y, L^\times)$.

In Section 3.10 of [3], Deligne and Brylinski construct an equivalence of categories, from $\mathcal{CExt}(\mathbf{T}, K_2)$ to the category $\mathfrak{DB}_\mathbf{T}$.

In particular, given a central extension $\mathbf{T}'$ of $\mathbf{T}$ by K_2 , the work of [3] (in part following Esnault [4]) yields a quadratic form $Q: Y \rightarrow \mathbb{Z}$, and a central extension $\tilde{Y}$ of Y by $L^\times$. Consider the central extension $\mathbf{T}'(L)$ below:

$$1 \rightarrow K_2(L) \rightarrow \mathbf{T}'(L) \rightarrow \mathbf{T}(L) \rightarrow 1.$$

In [3], it is shown that the resulting commutator $C_L: \bigwedge^2 \mathbf{T}(L) \rightarrow K_2(L)$ satisfies:

$$C_L(y_1(l_1), y_2(l_2)) = \{l_1, l_2\}^{B_Q(y_1, y_2)},$$

for all $y_1, y_2 \in Y$, $l_1, l_2 \in L^\times$.

2.7. Locally Compact Abelian Groups. An *LCA group* is a locally compact Hausdorff separable abelian topological group. There is a category $\mathfrak{LCAb}$ whose objects are LCA groups, and whose morphisms are continuous homomorphisms. Suppose that we are given a short exact sequence of LCA groups and continuous homomorphisms:

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Given a fourth LCA group D , the functor $\mathrm{Hom}(\bullet, D)$ is left-exact, yielding an exact sequence:

$$0 \rightarrow \mathrm{Hom}(C, D) \rightarrow \mathrm{Hom}(B, D) \rightarrow \mathrm{Hom}(A, D).$$

2.7.1. Continuous characters. When A is an LCA group, we write $X(A)$ for the group of continuous homomorphisms from A to the LCA group $\mathbb{C}^\times$, under pointwise multiplication. We call elements of $X(A)$ characters (or continuous characters) of A . If $\chi \in X(A)$, and $|\chi(a)| = 1$ for all $a \in A$, then we say that χ is a unitary character. We write $\hat{A}$ for the Pontrjagin dual of A , i.e., the set of unitary characters of A , with its natural topology as an LCA group.

We say that A is an elementary LCA group, if $A \cong \mathbb{R}^a \times \mathbb{Z}^b \times (\mathbb{R}/\mathbb{Z})^c \times F$, for some finite group F , and some non-negative integers a, b, c . When A is elementary, $X(A)$ has a natural structure as a complex algebraic group. In the case above, $X(A) \cong \mathbb{C}^a \times (\mathbb{C}^\times)^b \times \mathbb{Z}^c \times \hat{F}$.

If A is generated by a compact neighborhood of the identity, then A is canonically isomorphic to the inverse limit of its elementary quotients by compact subgroups. In this case, $X(A)$ is endowed with the (inductive limit) structure of a complex algebraic group. In this paper, all LCA groups will be generated by a compact neighborhood of the identity, and thus $X(A)$ will be viewed as a complex algebraic group.

2.7.2. Exactness Criteria. Given a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

there are two important cases, in which the induced map $X(B) \rightarrow X(A)$ is surjective, leading to an exact sequence:

$$0 \rightarrow X(C) \rightarrow X(B) \rightarrow X(A) \rightarrow 0.$$

Proposition 2.1. *Suppose that A is compact. Then $X(B) \rightarrow X(A)$ is surjective.*

Proof. When A is compact, every continuous character of A is unitary. The exactness of Pontrjagin duality implies that every unitary character of A extends to a unitary character of B . Hence $X(B)$ surjects onto $X(A)$. $\square$

Proposition 2.2. *Suppose that the map from A to B is an open embedding. Then $X(B) \rightarrow X(A)$ is surjective.*

Proof. The proof is not difficult – it follows directly from Proposition 3.3 of [5], for example. $\square$

2.7.3. Complex varieties and groups. We use a script letter, such as $\mathcal{M}$, to denote the (complex) points of a complex algebraic variety. It is unnecessary for us to distinguish between complex varieties and their complex points. If R is a commutative reduced finitely-generated $\mathbb{C}$ -algebra, then we write $\mathcal{M} = \mathcal{MSpec}(R)$ for

the maximal ideal spectrum of A , viewed as a complex variety. We view $\mathbb{C}^\times$ as a complex algebraic variety, identifying:

$$\mathbb{C}^\times \equiv \mathcal{MSpec}(\mathbb{C}[\mathbb{Z}]),$$

where $\mathbb{C}[\mathbb{Z}]$ denotes the group ring. We view $\mathbb{C}$ itself as an algebraic variety (the affine line over the field $\mathbb{C}$).

Suppose that $\mathcal{G}$ is a complex algebraic group, in other words, a group in the category of complex algebraic varieties. A $\mathcal{G}$ -variety is a complex algebraic variety $\mathcal{M}$, endowed with an action $\mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M}$ which is complex-algebraic. A $\mathcal{G}$ -torsor is a $\mathcal{G}$ -variety $\mathcal{M}$, such that the induced map $\mathcal{G} \times \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}$ sending (g, m) to $(g \cdot m, m)$, is an isomorphism of complex algebraic varieties.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{G}$ -varieties, then a morphism of $\mathcal{G}$ -varieties is a complex algebraic map from $\mathcal{M}_1$ to $\mathcal{M}_2$, which intertwines the action of $\mathcal{G}$. Morphisms of torsors are defined in the same way.

3. GENUINE REPRESENTATIONS OF METAPLECTIC TORI

In this section, the following will be fixed:

- $\mathbf{T}$ will be a torus over a local field F , which splits over a finite Galois extension L/F , with $\Gamma = \text{Gal}(L/F)$. X and Y will be the resulting character and cocharacter groups.
- $\mathbf{T}'$ will be an extension of $\mathbf{T}$ by $\mathbf{K}_2$ in $\mathfrak{Sp}_F$.
- $(Q, \tilde{Y})$ will be the Deligne-Brylinski invariants of $\mathbf{T}'$. B will be the symmetric bilinear form associated to Q .
- n will be a positive integer, such that F has enough n^{th} roots of unity.
- $\epsilon: \mu_n(F) \rightarrow \mathbb{C}^\times$ will be a fixed injective character.

3.1. Heisenberg Groups. Suppose that S is an LCA group, and A is a finite cyclic abelian group endowed with a faithful unitary character $\epsilon: A \rightarrow \mathbb{C}^\times$. Suppose that $\tilde{S}$ is a locally compact group, which is a central extension of S by A (in the category of locally compact groups and continuous homomorphisms:

$$1 \rightarrow A \rightarrow \tilde{S} \rightarrow S \rightarrow 1.$$

In this situation, the commutator on $\tilde{S}$ descends to a unique alternating form:

$$C: \bigwedge^2 S \rightarrow A.$$

Let $Z(\tilde{S})$ be the center of $\tilde{S}$. Then $Z(\tilde{S})$ is the preimage of a subgroup $Z^\dagger(S) \subset S$, where:

$$Z^\dagger(S) = \{s \in S \text{ such that } C(s, s') = 1 \text{ for all } s' \in S\}.$$

Throughout this paper, the following condition will be satisfied, and hence we assume that:

$Z^\dagger(S)$ is an open subgroup of finite index in S .

We define the following two sets:

- The set $\mathcal{X}_\epsilon(\tilde{S})$ of continuous genuine characters of $\tilde{S}$. These are elements of $\mathcal{X}(\tilde{S})$ whose restriction to A equals ϵ .

- The set $\text{Irr}_\epsilon(\tilde{S})$ of irreducible genuine representations of $\tilde{S}$. These are irreducible (algebraic) representations (π, V) of $\tilde{S}$ on a complex vector space, on which $Z(\tilde{S})$ acts via a continuous genuine character. In particular, since $Z(\tilde{S})$ will always have finite index in $\tilde{S}$, these are finite-dimensional representations.

We often use the following analogue of the Stone von-Neumann theorem:

Theorem 3.1. *Suppose that $\chi \in \mathcal{X}_\epsilon(Z(\tilde{S}))$ is a genuine continuous character. Let $\tilde{M}$ denote a maximal commutative subgroup of $\tilde{S}$. Then there exists an extension $\tilde{\chi} \in \mathcal{X}(\tilde{M})$ of χ to $\tilde{M}$. Define a representation of $\tilde{S}$ by:*

$$(\pi_\chi, V_\chi) = \text{Ind}_{\tilde{M}}^{\tilde{S}} \tilde{\chi}.$$

Algebraic induction suffices here, since we always assume that $Z(\tilde{S})$ has finite index in $\tilde{S}$. Then we have the following:

- (1) *The representation (π_χ, V_χ) is irreducible.*
- (2) *The representation (π_χ, V_χ) has central character χ .*
- (3) *The isomorphism class of (π_χ, V_χ) depends only upon χ , and not upon the choices of subgroup $\tilde{M}$ and extension $\tilde{\chi}$.*
- (4) *Every irreducible representation of $\tilde{S}$, on which $Z(\tilde{S})$ acts via χ , is isomorphic to (π_χ, V_χ) .*

Proof. Extension of χ to $\tilde{M}$ follows from Proposition 2.2. All but the last claim are proven in Section 0.3 of [6], and follow directly from Mackey theory. The last claim follows from the previous claims and Frobenius reciprocity. $\square$

3.2. Metaplectic Tori over Local Fields. The central extension of $\mathbf{T}$ by $\mathbf{K}_2$ yields a central extension of groups:

$$1 \rightarrow \mathbf{K}_2(F) \rightarrow \mathbf{T}'(F) \rightarrow \mathbf{T}(F) \rightarrow 1.$$

Since F is assumed to have enough n^{th} roots of unity, the Hilbert symbol allows us to push forward this extension to get:

$$1 \rightarrow \mu_n \rightarrow \tilde{T} \rightarrow T \rightarrow 1,$$

where $\mu_n = \boldsymbol{\mu}_n(F)$, and $T = \mathbf{T}(F)$. By results of Sections 10.2, 10.3 of [3], following Moore [10], this is a topological central extension, of the LCA group T by the LCA group μ_n .

In this case, the center $Z(\tilde{T})$ has finite index in $\tilde{T}$. Furthermore, Theorem 3.1 implies that:

Proposition 3.2. *There is a natural bijection between the set $\text{Irr}_\epsilon(\tilde{T})$ of irreducible genuine representations of $\tilde{T}$, and the set $\mathcal{X}_\epsilon(Z(\tilde{T}))$ of genuine characters of $Z(\tilde{T})$.*

There is a short exact sequence of LCA groups:

$$1 \rightarrow \mu_n \rightarrow Z(\tilde{T}) \rightarrow Z^\dagger(T) \rightarrow 1.$$

From Proposition 2.1, it follows that:

Proposition 3.3. *The space $\mathcal{X}_\epsilon(Z(\tilde{T}))$ of genuine continuous characters of $Z(\tilde{T})$ is a $\mathcal{X}(Z^\dagger(T))$ -torsor.*

As an immediate consequence, we find:

Corollary 3.4. *The set $\text{Irr}_\epsilon(\tilde{T})$ is a $\mathcal{X}(Z^\dagger(T))$ -torsor.*

In particular, we give $\text{Irr}_\epsilon(\tilde{T})$ the structure of a complex algebraic variety, in such a way that it is a complex algebraic $\mathcal{X}(Z^\dagger(T))$ -torsor.

Since $Z^\dagger(T)$ is a finite index subgroup of T , restriction of continuous characters yields a surjective homomorphism of complex algebraic groups:

$$\text{res}: \mathcal{X}(T) \rightarrow \mathcal{X}(Z^\dagger(T)).$$

As a result, the set $\text{Irr}_\epsilon(\tilde{T})$ is a homogeneous space for $\mathcal{X}(T)$, or equivalently (by Langlands theorem [9]), a homogeneous space for $H_c^1(W_{L/F}, \hat{\mathcal{T}})$.

4. SPLIT TORI

In this section, we carry on the assumptions of the previous section. In addition, we assume that $\mathbf{T}$ is a split torus of rank r over F . Thus, there is a canonical identification $\mathbf{T}(F) = Y \otimes_{\mathbb{Z}} F^\times$. We are interested in parameterizing $\text{Irr}_\epsilon(\tilde{T})$. By the results of the previous section, we may describe this set, up to a choice of base point, by describing the set $Z^\dagger(T)$.

4.1. An isogeny. Recall that $B: Y \otimes_{\mathbb{Z}} Y \rightarrow \mathbb{Z}$ is the symmetric bilinear form associated to Q . It allows us to construct a subgroup of finite index $Y^\# \subset Y$:

$$Y^\# = \{y \in Y \text{ such that } B(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y\}.$$

Note that we suppress mention of Q , B , and n , in our notation $Y^\#$.

The subgroup $Y^\#$ can be related to the ‘‘Smith normal form’’ of the bilinear form B . Namely, there exists a pair of group isomorphisms α, β :

$$\mathbb{Z}^r \xleftarrow{\beta} Y \xrightarrow{\alpha} \mathbb{Z}^r,$$

such that for all $y_1, y_2 \in Y$, one has:

$$B_Q(y_1, y_2) = D(\alpha(y_1), \beta(y_2)),$$

and D is a symmetric bilinear form on $\mathbb{Z}^r$ represented by a diagonal matrix with entries $(d_1, \dots, d_r)$ (the elementary divisors). Let e_j denote the smallest positive integer such that $d_j e_j \in n\mathbb{Z}$, for every $1 \leq j \leq r$. Then we find that:

$$Y^\# = \alpha^{-1}(e_1\mathbb{Z} \oplus e_2\mathbb{Z} \oplus \dots \oplus e_r\mathbb{Z}).$$

Let $\iota: Y^\# \rightarrow Y$ denote the inclusion of $\mathbb{Z}$ -modules. Since $Y^\#$ has finite index in Y , this corresponds to an F -isogeny of split tori:

$$\iota: \mathbf{T}^\# \rightarrow \mathbf{T},$$

where $\mathbf{T}^\#$ is the split algebraic torus with cocharacter lattice $Y^\#$. From the previous observations, we find that:

$$\iota(T^\#) = \iota(\mathbf{T}^\#(F)) = \alpha^{-1}(F^{\times e_1} \times \dots \times F^{\times e_r}).$$

4.2. Describing the Center. Recall that extension $\tilde{T}$ of T by μ_n yields a commutator:

$$C: \bigwedge^2 T \rightarrow \mu_n.$$

This commutator can be directly related (by [3]) to the bilinear form B . If $u_1, u_2 \in F^\times$, and $y_1, y_2 \in Y$, then one may directly compute:

$$C(y_1(u_1), y_2(u_2)) = (u_1, u_2)_n^{B(y_1, y_2)}.$$

The diagonalization of B , via group isomorphisms α, β , yields two isomorphisms of F -tori:

$$G_m^r \xleftarrow{\beta} T \xrightarrow{\alpha} G_m^r,$$

One arrives at a bilinear form on $(F^\times)^r$, given by:

$$\Delta(\vec{z}_1, \vec{z}_2) = \prod_{j=1}^r \left(z_1^{(j)}, z_2^{(j)} \right)_n^{d_j}.$$

This is related to the commutator C by:

$$C(t_1, t_2) = \Delta(\alpha(t_1), \beta(t_2)).$$

We can now characterize $Z^\dagger(T)$:

Proposition 4.1. *The subgroup $Z^\dagger(T)$ of T is equal to the image of the isogeny ι on the F -rational points:*

$$Z^\dagger(T) = \iota(T^\#).$$

Proof. We find that:

$$\begin{aligned} t_1 \in Z^\dagger(T) &\Leftrightarrow C(t_1, t_2) = 1 \text{ for all } t_2 \in T, \\ &\Leftrightarrow \Delta(\alpha(t_1), \beta(t_2)) = 1 \text{ for all } t_2 \in T, \\ &\Leftrightarrow \Delta(\alpha(t_1), \vec{z}_2) = 1 \text{ for all } \vec{z}_2 \in (F^\times)^r, \\ &\Leftrightarrow \alpha(t_1) \in (F^{\times e_1} \times \cdots \times F^{\times e_r}) \\ &\Leftrightarrow t_1 \in \iota(T^\#). \end{aligned}$$

The penultimate step above follows from the non-degeneracy of the Hilbert symbol. $\square$

We have now proven:

Theorem 4.2. *If F is a local field, then $\text{Irr}_e(\tilde{T})$ is a torsor for $X(\iota(T^\#))$.*

4.3. Character groups. The previous theorem motivates the further analysis of the group $X(\iota(T^\#))$. We write ι^* for the pullback homomorphism:

$$\iota^*: X(T) \rightarrow X(T^\#).$$

Proposition 4.3. *There is a natural identification:*

$$X(\iota(T^\#)) \cong \text{Im}(\iota^*).$$

Proof. There are short exact sequences of LCA groups:

$$1 \rightarrow \text{Ker}(\iota) \rightarrow T^\# \rightarrow \iota(T^\#) \rightarrow 1,$$

$$1 \rightarrow \iota(T^\#) \rightarrow T \rightarrow \text{Cok}(\iota) \rightarrow 1,$$

Using Propositions 2.1 and 2.2, we arrive at short exact sequences of character groups:

$$1 \rightarrow X(\iota(T^\#)) \rightarrow X(T^\#) \rightarrow X(\text{Ker}(\iota)) \rightarrow 1,$$

$$1 \rightarrow X(\text{Cok}(\iota)) \rightarrow X(T) \rightarrow X(\iota(T^\#)) \rightarrow 1,$$

Since $X(T)$ surjects onto $X(\iota(T^\#))$, we find that the image of $\iota^*: X(T) \rightarrow X(T^\#)$ is equal to the image of the injective map $X(\iota(T^\#)) \rightarrow X(T^\#)$. $\square$

4.4. The Dual Complex. The isogeny of split F -tori $\iota: T^\# \rightarrow T$ yields an isogeny of the complex dual tori:

$$\hat{\iota}: \hat{T} \rightarrow \hat{T}^\#.$$

One may pull back continuous characters:

$$\iota^*: \mathcal{X}(T) \rightarrow \mathcal{X}(T^\#).$$

The following result follows from local class field theory, and demonstrates the naturality of Langlands' classification [9]:

Proposition 4.4. *There is a commutative diagram of complex algebraic groups, whose rows are the reciprocity isomorphisms of local class field theory:*

$$\begin{array}{ccc} \mathcal{X}(T) & \longrightarrow & H_c^1(\mathcal{W}_F, \hat{T}) \\ \downarrow \iota^* & & \downarrow \hat{\iota} \\ \mathcal{X}(T^\#) & \longrightarrow & H_c^1(\mathcal{W}_F, \hat{T}^\#) \end{array}$$

Note that since T and $T^\#$ are split tori, the continuous cohomology groups are quite simple:

$$H_c^1(\mathcal{W}_F, \hat{T}) = \text{Hom}_c(\mathcal{W}_F, \hat{T}).$$

From this result, we now find:

Corollary 4.5. *There is a natural identification:*

$$\mathcal{X}(Z^1(T)) \equiv \text{Im} \left(\hat{\iota}: H^1(\mathcal{W}_F, \hat{T}) \rightarrow H^1(\mathcal{W}_F, \hat{T}^\#) \right).$$

4.5. Parameterization by Hypercohomology. We may parameterize representations, using the hypercohomology of the complex of tori:

$$\hat{T} \xrightarrow{\hat{\iota}} \hat{T}^\#$$

concentrated in degrees zero and one. We follow the treatment of Kottwitz-Shelstad in the Appendices of [7], when discussing hypercohomology of Weil groups, with coefficients in complexes of tori. In particular, we concentrate the complexes in degrees 0 and 1 following [7], and not in degrees -1 and 0 as in [2].

There is a long exact sequence in cohomology, which includes:

$$H_c^1(\mathcal{W}_F, \hat{T}) \longrightarrow \hat{T}^\# \xrightarrow{\eta} H_c^1(\mathcal{W}_F, \hat{T}) \longrightarrow H_c^1(\mathcal{W}_F, \hat{T}^\#).$$

Lemma 4.6. *The homomorphism η is injective.*

Proof. Extending the long exact sequence above, it suffices to prove the surjectivity of the preceding homomorphism:

$$H_c^0(\mathcal{W}_F, \hat{T}) \rightarrow H_c^0(\mathcal{W}_F, \hat{T}^\#).$$

But $\hat{T}$ and $\hat{T}^\#$ are complex tori, trivial as $\mathcal{W}_F$ -modules, and $\hat{\iota}$ is an isogeny. Therefore, the above map is surjective. $\square$

From this lemma, it makes sense to identify the hypercohomology group $H_c^1(\mathcal{W}_F, \hat{T} \rightarrow \hat{T}^\#)$ with a subgroup of $H_c^1(\mathcal{W}_F, \hat{T})$.

Lemma 4.7. *The group $H_c^1(\mathcal{W}_F, \hat{T} \rightarrow \hat{T}^\#)$ is finite.*

Proof. Since $\hat{i}$ is an isogeny, it has finite kernel and cokernel. There is a long exact sequence, which includes the following terms:

$$H_c^1(\mathcal{W}_F, \text{Ker}(\hat{i})) \rightarrow H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#) \rightarrow H_c^1(\mathcal{W}_F, \text{Cok}(\hat{i})).$$

The lemma follows. $\square$

This leads to the first main theorem:

Theorem 4.8. *There exists an isomorphism, in the category of varieties over $\mathbb{C}$ endowed with an action of $H_c^1(\mathcal{W}_F, \hat{\mathcal{T}})$:*

$$\text{Irr}_\epsilon(\tilde{T}) \cong \frac{H_c^1(\mathcal{W}_F, \hat{\mathcal{T}})}{H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#)}.$$

Remark 4.9. The global analogue of this result also seems to hold. Letting $\text{Irr}_\epsilon^{\text{aut}}(\tilde{T}_\mathbb{A})$ denote the appropriate set of genuine automorphic representations of $\tilde{T}_\mathbb{A}$, it seems likely that:

$$\text{Irr}_\epsilon^{\text{aut}}(\tilde{T}_\mathbb{A}) \cong \frac{H_c^1(\mathcal{W}_F, \hat{\mathcal{T}})}{H_c^1(\mathcal{W}_F, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#)},$$

when T is a split torus over a global field F . The proof follows the same techniques (together with the Hasse principle for isogenies of split tori), but requires some analytic care with extension of continuous characters and the appropriate Stone von-Neumann theorem. We hope to treat this global theorem in a future paper.

5. UNRAMIFIED TORI

For split tori, the cohomology groups that arise in Theorem 4.8 are quite simple, since $\mathcal{W}_F$ acts trivially on $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}^\#$. In fact, the statement of this theorem makes sense, even when T is a nonsplit torus. However, for general nonsplit tori, it seems that our explicit methods are insufficient to prove such a result. For “tame covers” of unramified tori over local nonarchimedean fields, such a parameterization is possible.

In this section, we fix the following:

- T will be a local nonarchimedean field F , which splits over a finite *unramified* Galois extension L/F , with $\Gamma = \text{Gal}(L/F)$. X and Y will be the resulting character and cocharacter groups.
- We define $d = [L : F]$, and write $\mathbf{l}$ for the residue field of $\mathcal{O}_L$ (note that $\mathbf{l}$ has cardinality q^d).
- We fix a uniformizer ϖ of $F^\times$ (and hence of $L^\times$ as well).
- Let γ be the generator of Γ , which acts upon $\mathbf{l}$ via $\gamma(x) = x^q$. Let $r = (q^d - 1)/(q - 1) = \#(\mathbf{l}^\times / \mathbf{f}^\times)$.
- T' will be an extension of T by K_2 in $\mathfrak{Sp}_F$.
- $(Q, \tilde{Y})$ will be the Deligne-Brylinski invariants of T' . B will be the symmetric bilinear form associated to Q .
- n will be a positive integer, such that F has enough n^{th} roots of unity. We also assume that $(p, n) = 1$.
- $\epsilon: \mu_n(F) \rightarrow \mathbb{C}^\times$ will be a fixed injective character.
- If W is a subgroup of Y , then we write W^Γ for the subgroup of Γ -fixed elements of W . We also define:

$$W^\# = \{y \in Y \text{ such that } B(y, w) \in n\mathbb{Z} \text{ for all } w \in W\}.$$

5.1. $\mathbb{Z}[\Gamma]$ -modules. Γ is a cyclic group, generated by γ , of order d . Let $\mathbb{Z}[\Gamma]$ denote the integral group ring of Γ . We define the following elements of $\mathbb{Z}[\Gamma]$:

- Let $Tr = \sum_{i=0}^{d-1} \gamma^i$, and let $Tr_q = \sum_{i=0}^{d-1} q^i \gamma^i$.
- Let $\delta = \gamma - 1$, and let $\delta_q = q\gamma - 1$.

Note that:

$$Tr \circ \delta = 0, \text{ and } Tr_q \circ \delta_q = q^d - 1.$$

When M is an $\mathbb{Z}[\Gamma]$ -module, we let $\bar{M} = M/(q^d - 1)M$. We write M^Γ for the Γ -invariant $\mathbb{Z}$ -submodule of M . Therefore,

$$M^\Gamma = \{m \in M \text{ such that } \delta m = 0\}.$$

We define $\bar{M}^{\Gamma, q}$ by:

$$\bar{M}^{\Gamma, q} = \{\bar{m} \in \bar{M} \text{ such that } \delta_q \bar{m} = 0\}.$$

Proposition 5.1. *Suppose that M is an $\mathbb{Z}[\Gamma]$ -module. Then,*

$$Tr(M) \subset M^\Gamma \text{ and } Tr_q(\bar{M}) \subset \bar{M}^{\Gamma, q}.$$

Proof. The first inclusion is obvious. For the second inclusion, suppose that $\bar{m} \in \bar{M}$. Then,

$$\delta_q Tr_q \bar{m} = Tr_q \delta_q \bar{m} = (q^d - 1)\bar{m} = 0.$$

□

Proposition 5.2. *Suppose that M is a $\mathbb{Z}[\Gamma]$ -module, which is free as an $\mathbb{Z}$ -module. Then, δ_q and Tr_q act as injective endomorphisms of M , and*

$$Im(\delta_q) = \{m \in M \text{ such that } Tr_q m \in (q^d - 1)M\}.$$

Proof. Since M is free as an $\mathbb{Z}$ -module, $\delta_q \circ Tr_q = Tr_q \circ \delta_q = q^d - 1$ acts as an injective endomorphism of M . Hence δ_q , and Tr_q must also act as injective endomorphisms of M , proving the first assertion.

Since $Tr_q \circ \delta_q = q^d - 1$, it follows that:

$$Im(\delta_q) \subset \{m \in M \text{ such that } Tr_q m \in (q^d - 1)M\}.$$

In the other direction, if $Tr_q m \in (q^d - 1)M$, then $Tr_q m = Tr_q \delta_q m'$, for some $m' \in M$. Since Tr_q acts via an injective endomorphism, it follows that $m = \delta_q m'$. □

5.2. Unramified Tori. Much of our treatment of *unramified tori* is inspired by Section 2 of Ono [11]. Recall that X and Y are naturally $\mathbb{Z}[\Gamma]$ -modules, in such a way that the pairing is Γ -invariant.

We fix a smooth model $\underline{T}$ of T over $\mathcal{O}_F$. We make the following identifications:

$$T_L = \underline{T}(L) \equiv Y \otimes_{\mathbb{Z}} L^\times, \text{ and } T_F = \underline{T}(F) \equiv (Y \otimes_{\mathbb{Z}} L^\times)^\Gamma.$$

Similarly, for the integral points, we identify:

$$T_L^\circ = \underline{T}(\mathcal{O}_L) \equiv Y \otimes_{\mathbb{Z}} \mathcal{O}_L^\times, \text{ and } T_F^\circ = \underline{T}(\mathcal{O}_F) \equiv (Y \otimes_{\mathbb{Z}} \mathcal{O}_L^\times)^\Gamma.$$

We write $\bar{T}$ for the special fibre of $\underline{T}$. Then, we also identify:

$$\bar{T}_1 = \bar{T}(\mathbf{1}) \equiv Y \otimes_{\mathbb{Z}} \mathbf{1}^\times, \text{ and } \bar{T}_{\mathbf{f}} = \bar{T}(\mathbf{f}) \equiv (Y \otimes_{\mathbb{Z}} \mathbf{1}^\times)^\Gamma.$$

There are natural reduction homomorphisms:

$$T_L^\circ \rightarrow \bar{T}_1, \text{ and } T_F^\circ \rightarrow \bar{T}_{\mathbf{f}}.$$

Let T_L^1 and T_F^1 denote the kernels of these reduction maps. The reduction morphisms are split by the Teichmüller lift, and we arrive at a decomposition of $\mathbb{Z}[\Gamma]$ -modules:

$$T_L^\circ \equiv T_L^1 \times \bar{T}_1.$$

Together with the valuation map, we arrive at a short exact sequence of $\mathbb{Z}[\Gamma]$ -modules:

$$1 \rightarrow T_L^1 \times \bar{T}_1 \rightarrow T_L \rightarrow Y \rightarrow 1.$$

The choice of (Γ -invariant) uniformizing element ϖ splits this exact sequence, leading to a decomposition of $\mathbb{Z}[\Gamma]$ -modules:

$$T_L \equiv Y \times \bar{T}_1 \times T_L^1.$$

We use this decomposition to “get our hands on” elements of T_L . First, every element of T_L can be expressed as $y(\varpi)t^\circ$, for uniquely determined $y \in Y$, $t^\circ \in T_L^\circ$. Let θ_1 denote a generator of the cyclic group $\mathbf{1}^\times$, and $\theta_{\mathbf{f}} = \theta_1^r$. Thus $\theta_{\mathbf{f}}$ is a generator of the cyclic group $\mathbf{f}^\times$. Let $\vartheta_L \in \mathcal{O}_L^\times$ and $\vartheta_F \in \mathcal{O}_F^\times$ denote the Teichmüller lifts of θ_1 and $\theta_{\mathbf{f}}$, respectively.

Let $\zeta_L = (\varpi, \vartheta_L)_{L, q^d-1}$. Let $\zeta_F = \zeta_L^r$. Note that ζ_L is a primitive $(q^d - 1)^{th}$ root of unity, and ζ_F is a primitive $(q - 1)^{th}$ root of unity.

Recall that $\bar{Y} = Y/(q^d - 1)Y$; thus, for $\bar{y} \in \bar{Y}$, it makes sense to write $\bar{y}(\vartheta_L)$ for an element of T_L° . According to the decomposition $T_L \equiv Y \times \bar{T}_1 \times T_L^1$, every element $t \in T_L$ has a unique expression:

$$t = y_1(\varpi)\bar{y}_2(\vartheta_L)t^1,$$

where $y_1 \in Y$, $\bar{y}_2 \in \bar{Y}$, and $t^1 \in T_L^1$. To determine when such an expression lies in T_F , we have the following characterization:

Proposition 5.3. *An element $y_1(\varpi)\bar{y}_2(\vartheta_L)t^1$ of T_L , with $y_1, \bar{y}_2, t^1$ as above, lies in T_F if and only if the following three conditions hold:*

- $y_1 \in Y^\Gamma$. In other words, $\delta(y_1) = 0$.
- $\bar{y}_2 \in \bar{Y}^{\Gamma, q}$. In other words, $\delta_q(\bar{y}_2) = 0$.
- $t^1 \in T_F^1$.

Proof. By the Γ -invariance of the decomposition $T_L \equiv Y \times \bar{T}_1 \times T_L^1$, we find that $y_1(\varpi)\bar{y}_2(\vartheta_L)t^1 \in T_F$ if and only if the three factors are fixed by Γ . The proposition follows from three observations:

- Since $\varpi \in F$, we have $y_1(\varpi) \in T_L^\Gamma$ if and only if $y_1 \in Y^\Gamma$.
- Since $\gamma(\vartheta_L) = \vartheta_L^q$, we find that $\bar{y}_2(\vartheta_L) \in T_L^\Gamma$ if and only if:

$$\bar{y}_2 = q\gamma(\bar{y}_2), \text{ in } \bar{Y}.$$

- Since the reduction map intertwines the action of Γ , we have $t^1 \in (T_L^1)^\Gamma$ if and only if $t^1 \in T_F^1$.

□

5.3. Tame Metaplectic Unramified Tori. The structure of $\mathbf{T}'(L)$ and $\mathbf{T}'(F)$ is based on Sections 12.8-12.12 of [3]. In particular, letting $T_L' = \mathbf{T}'(L)$, and

$T'_F = \mathbf{T}'(F)$, there is a natural commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{K}_2(F) & \longrightarrow & T'_F & \longrightarrow & T_F \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{K}_2(L) & \longrightarrow & T'_L & \longrightarrow & T_L \longrightarrow 1. \end{array}$$

There is a natural action of Γ on the bottom row, in such a way that $\mathbf{K}_2(F)$ maps to $\mathbf{K}_2(L)^\Gamma$, $T_F = T_L^\Gamma$, and T'_F maps to $(T'_L)^\Gamma$. The tame symbols yield a commutative diagram, where the downward arrows arise from the functoriality of $\mathbf{K}_2$ and $\mathbf{K}_1$:

$$\begin{array}{ccc} \mathbf{K}_2(F) & \xrightarrow{t} & \mathbf{f}^\times \\ \downarrow & & \downarrow \\ \mathbf{K}_2(L) & \xrightarrow{t} & \mathbf{l}^\times \end{array}$$

The bottom row is a morphism of $\mathbb{Z}[\Gamma]$ -modules. Pushing forward T'_F and T'_L via the tame symbols yields a commutative diagram of locally compact groups, with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{f}^\times & \longrightarrow & \tilde{T}_F^t & \longrightarrow & T_F \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{l}^\times & \longrightarrow & \tilde{T}_L^t & \longrightarrow & T_L \longrightarrow 1. \end{array}$$

The downward arrows arise from the inclusion of F in L , and of $\mathbf{f}$ in $\mathbf{l}$. In Section 12.8 of [3], Deligne and Brylinski note the following:

Proposition 5.4. *In the commutative diagram above, the groups in the top row are precisely the Γ -invariant subgroups of the bottom row. In other words, $\mathbf{f}^\times = (\mathbf{l}^\times)^\Gamma$, $T_F = T_L^\Gamma$, and $\tilde{T}_F^t = (\tilde{T}_L^t)^\Gamma$.*

We may push forward the covers further to obtain all *tame covers*. Recall that $(p, n) = 1$, and F has enough n^{th} roots of unity. Then, we find that $n|(q-1)$, and there is a natural surjective map:

$$\psi_F: \mathbf{f}^\times \rightarrow \mu_n(F),$$

obtained by first applying the Teichmüller map (from $\mathbf{f}^\times$ to μ_{q-1}), and then raising to the $m = (q-1)/n$ power. Recall that $r = (q^d - 1)/(q - 1)$. One obtains a similar map:

$$\psi_L: \mathbf{l}^\times \rightarrow \mu_{nr}(L),$$

obtained by applying the Teichmüller map (from $\mathbf{l}^\times$ to $\mu_{q^d-1}(L)$) and then raising to the $m = (q-1)/n$ power. The compatibility of these maps yields a new commutative diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_n(F) & \longrightarrow & \tilde{T}_F & \longrightarrow & T_F \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_{nr}(L) & \longrightarrow & \tilde{T}_L & \longrightarrow & T_L \longrightarrow 1. \end{array}$$

With this construction, we say that $\tilde{T}_F$ is a *tame metaplectic cover* of T_F , and $\tilde{T}_L$ is a tame metaplectic cover of T_L as well. $\tilde{T}_F$ is identified as a subgroup of $\tilde{T}_L$.

Note that the commutator map for $\tilde{T}_L$ satisfies:

$$C_L(y_1(u), y_2(v)) = (u, v)_{L, nr}^{B(y_1, y_2)} = (u, v)_{L, q^d-1}^{mB(y_1, y_2)},$$

where $(\cdot, \cdot)_{L, nr}$ and $(\cdot, \cdot)_{L, q^d-1}$ denotes the appropriate Hilbert symbols (in this case, norm residue symbols) on $L^\times$. The commutator on T_F is simply the restriction of C_L ; as a result, $Z^\dagger(T_F) \subset Z^\dagger(T_L) \cap T_F$, where the pre-image of $Z^\dagger(T_F)$ is the center of $\tilde{T}_F$, and the pre-image of $Z^\dagger(T_L)$ is the center of $\tilde{T}_L$.

5.4. Computation of the center. Recall that the set $\text{Irr}_\epsilon(\tilde{T}_F)$ is a torsor for $\mathcal{X}(Z^\dagger(T_F))$. Therefore, we wish to study the group $Z^\dagger(T_F)$ in more detail. To this end, we first observe:

Proposition 5.5. *The group T_L^1 is contained in $Z^\dagger(T_L)$. Similarly, T_F^1 is contained in $Z^\dagger(T_F)$.*

Proof. Since $(q^d-1, p) = 1$, the Hilbert symbol (in this case, a norm-residue symbol) is trivial, when one of its “inputs” is contained in $\mathcal{O}_L^1$. Hence the commutator $C_L(\cdot, \cdot)$ is trivial when one of its inputs is contained in T_L^1 . Hence $T_L^1 \subset Z^\dagger(T_L)$. Since $Z^\dagger(T_F) \subset Z^\dagger(T_L) \cap T_F$, we find that $T_F^1 \subset Z^\dagger(T_F)$ as well. $\square$

Since T_F^1 is contained in $Z^\dagger(T_F)$, $Z^\dagger(T_F)$ corresponds to a subgroup of T_F/T_F^1 . Our choice of uniformizing element, together with the previously mentioned splittings, yields a decomposition of $\mathbb{Z}[\Gamma]$ -modules:

$$T_L/T_L^1 \equiv Y \times \bar{T}_1.$$

Namely, every element t of T_L/T_L^1 can be represented by $y_1(\varpi)\bar{y}_2(\vartheta_L)$, for uniquely determined $y_1 \in Y$ and $\bar{y}_2 \in \bar{Y}$.

In order to describe $Z^\dagger(T_F)$, we work with a number of subgroups of Y . Recall that $Y^{\Gamma\#}$ is given by:

$$Y^{\Gamma\#} = \{y \in Y \text{ such that } B(y, y') \in n\mathbb{Z} \text{ for all } y' \in Y^\Gamma\}.$$

Note that $Y^{\Gamma\#} \supset Y^\#$. Also, it is important to distinguish between $Y^{\Gamma\#}$ and $Y^{\#\Gamma} = (Y^\#)^\Gamma$.

Lemma 5.6. *There are inclusions of $\mathbb{Z}[\Gamma]$ -modules, of finite index in Y :*

$$Y \supset Y^{\Gamma\#} \supset Y^\# \supset (q^d-1)Y.$$

Furthermore, $\delta_q(Y) \subset Y^{\Gamma\#}$, and $\text{Tr}_q(Y^{\Gamma\#}) \subset Y^\#$.

Proof. The inclusions are clear, since n divides q^d-1 . If $y \in Y$, and $y' \in Y^\Gamma$, then we find:

$$\begin{aligned} B(\delta_q y, y') &= B(q\gamma y - y, y') \\ &= qB(y, \gamma^{-1}y') - B(y, y') \\ &= (q-1)B(y, y') \in n\mathbb{Z}, \text{ since } q-1 = mn. \end{aligned}$$

Hence $\delta_q(Y) \subset Y^{\Gamma\#}$.

Now, suppose that $w \in Y^{\Gamma\#}$, and $y' \in Y$. Then, we find:

$$\begin{aligned} B(Tr_q(w), y') &= \sum_{i=0}^{d-1} B(q^i \gamma^i w, y') \\ &\equiv \sum_{i=0}^{d-1} B(\gamma^i w, y') \pmod{n} \text{ since } q-1 = mn \\ &\equiv B(w, Tr(y')) \in n\mathbb{Z}, \text{ since } Tr(y') \in Y^\Gamma. \end{aligned}$$

Hence $Tr_q(Y^{\Gamma\#}) \in Y^\#$. □

Now, we fully describe $Z^\dagger(T_F)$ with two results:

Theorem 5.7. *Suppose that $y_1 \in Y$, and $\bar{y}_2 \in \bar{Y}$. Then if the element $t = y_1(\varpi)\bar{y}_2(\vartheta_L)$ is contained in $Z^\dagger(T_F)$, then for every lift $y_2 \in Y$ of $\bar{y}_2$,*

$$y_1, y_2 \in Y^\#, \text{ and } \delta_q y_2 \in (q^d - 1)Y^{\Gamma\#}.$$

Proof. For reference during this proof, we recall that:

$$nm = q - 1, \text{ and } r = 1 + q + \cdots + q^{d-1}, \text{ and } nmr = q^d - 1.$$

Suppose furthermore that $y'_1, y'_2 \in Y$, and let $\bar{y}'_2 \in \bar{Y}$ be the reduction of y'_2 . Then, we find that $[Tr(y'_1)](\varpi)$ and $[Tr_q(\bar{y}'_2)](\vartheta_L)$ are elements of T_F . It follows that:

$$\begin{aligned} C_L(y_1(\varpi)\bar{y}_2(\vartheta_L), [Tr(y'_1)](\varpi)) &= 1, \text{ and} \\ C_L(y_1(\varpi)\bar{y}_2(\vartheta_L), [Tr_q(\bar{y}'_2)](\vartheta_L)) &= 1. \end{aligned}$$

The explicit formula for the commutator C_L yields:

$$\begin{aligned} 1 &= C_L(y_1(\varpi)y_2(\vartheta_L), [Tr_q(\bar{y}'_2)](\vartheta_L)), \\ &= \prod_{i=0}^{d-1} (\varpi, \vartheta_L)_{L, q^d-1}^{mq^i B(y_1, \gamma^i y'_2)}, \\ &= \zeta_L^{\sum_{i=0}^{d-1} mq^i B(y_1, \gamma^i y'_2)}, \text{ since } (\varpi, \vartheta_L)_{L, q^d-1} = \zeta_L, \\ &= \zeta_L^{\sum_{i=0}^{d-1} mq^i B(\gamma^{d-i} y_1, y'_2)}, \text{ by the } \Gamma\text{-invariance of } B, \\ &= \zeta_L^{\sum_{i=0}^{d-1} mq^i B(y_1, y'_2)}, \text{ by the } \Gamma\text{-invariance of } y_1, \\ &= \zeta_L^{mr B(y_1, y'_2)}, \text{ by summing a partial geometric series,} \\ &= \zeta_F^{mB(y_1, y'_2)}, \text{ since } \zeta_F = \zeta_L^r. \end{aligned}$$

Since $1 = \zeta_F^{mB(y_1, y'_2)}$ for all $y'_2 \in Y$, we find that:

$$y_1 \in Y^\#.$$

Carrying out a similar analysis, an explicit computation yields:

$$\begin{aligned} 1 &= C_L(y_1(\varpi)y_2(\vartheta_L), [Tr(y'_1)](\varpi)), \\ &= \prod_{i=0}^{d-1} (\varpi, \varpi)_{L, q^d-1}^{mB(y_1, \gamma^i y'_1)} (\varpi, \vartheta_L)_{L, q^d-1}^{mB(y_2, \gamma^i y'_1)}. \end{aligned}$$

Now, if p is odd, we find that $q-1$ is even. Since $(\varpi, \varpi)_{L, q^d-1} = \pm 1$, and $mB(y_1, \gamma^i y'_1) \in mn\mathbb{Z} = (q-1)\mathbb{Z} \subset 2\mathbb{Z}$ (since $y_1 \in Y^\#$), we find that:

$$(\varpi, \varpi)_{L, q^d-1}^{mB(y_1, \gamma^i y'_1)} = 1.$$

On the other hand, if $p = 2$, $(\varpi, \varpi)_{L, q^d-1} = 1$, and once again the above equality holds. Continuing our computations yields:

$$\begin{aligned}
1 &= \prod_{i=0}^{d-1} (\varpi, \varpi)_{L, q^d-1}^{mB(y_1, \gamma^i y'_1)} (\varpi, \vartheta_L)_{L, q^d-1}^{mB(y_2, \gamma^i y'_1)}, \\
&= \prod_{i=0}^{d-1} \zeta_L^{mB(y_2, \gamma^i y'_1)}, \text{ since } (\varpi, \varpi)_{L, q^d-1}^{mB(y_2, \gamma^i y'_1)} = 1 \text{ and } (\varpi, \vartheta_L)_{L, q^d-1} = \zeta_L, \\
&= \zeta_L^{\sum_{i=0}^{d-1} m q^i B(y_2, y'_1)}, \text{ since } q\gamma(\bar{y}_2) = \bar{y}_2, \\
&= \zeta_L^{mrB(y_2, y'_1)}, \text{ by summing a partial geometric series,} \\
&= \zeta_F^{mB(y_2, y'_1)}, \text{ since } \zeta_F = \zeta_L^r.
\end{aligned}$$

Hence, we find that $y_2 \in Y^\#$.

Finally, we prove that $\delta_q y_2 \in (q^d - 1)Y^{\Gamma\#}$. Note that $\delta_q y_2 \in (q^d - 1)Y$, since $\bar{y}_2 \in \bar{Y}^{\Gamma, q}$. Thus, $\delta_q y_2 = (q^d - 1)y_3$, for some $y_3 \in Y$. It suffices to prove that $y_3 \in Y^{\Gamma\#}$.

Now, to prove that $y_3 \in Y^{\Gamma\#}$, suppose that $y' \in Y^\Gamma$. It follows that:

$$\begin{aligned}
1 &= C_L(y_1(\varpi)\bar{y}_2(\vartheta_L), y'(\varpi)) \\
&= (\varpi, \varpi)_{q^d-1}^{mB(y_1, y')} (\vartheta_L, \varpi)_{q^d-1}^{mB(y_2, y')} \\
&= \zeta_L^{mB(y_2, y')}.
\end{aligned}$$

Hence $B(y_2, y') \in nr\mathbb{Z}$. It follows that:

$$\begin{aligned}
B(y_3, y') &= (q^d - 1)^{-1} B(\delta_q y_2, y') \\
&= (q^d - 1)^{-1} (B(q\gamma y_2, y') - B(y_2, y')) \\
&= r^{-1} B(y_2, y') \in n\mathbb{Z}.
\end{aligned}$$

Thus $y_3 \in Y^{\Gamma\#}$. □

Theorem 5.8. *Suppose that $y_1, y_2 \in Y^\#$. Also, suppose that $y_1 \in Y^\Gamma$, and $\bar{y}_2 \in \bar{Y}^{\Gamma, q}$. Furthermore, suppose that $\delta_q y_2 \in (q^d - 1)Y^{\Gamma\#}$. Then $y_1(\varpi)\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$.*

Proof. Since $y_1 \in Y^\Gamma$, and $\bar{y}_2 \in \bar{Y}^{\Gamma, q}$, it follows that $y_1(\varpi)\bar{y}_2(\vartheta_L) \in T_F$. Now, we may compute some commutators.

Suppose that $y'_1 \in Y^\Gamma$, and $y'_2 \in Y$, and $\bar{y}'_2 \in \bar{Y}^{\Gamma, q}$. Thus $y'_1(\varpi)$ and $\bar{y}'_2(\vartheta_L)$ are elements of T_F . We begin by computing:

$$C_L(y_1(\varpi), y'_1(\varpi)) = (\varpi, \varpi)_{L, q^d-1}^{mB(y_1, y'_1)}.$$

If p is odd, then $mn = q - 1$ is even, and thus $mB(y_1, y'_1)$ is even. Hence the commutator is trivial. If p is even, then $q^d - 1$ is odd, and hence $(\varpi, \varpi)_{L, q^d-1} = 1$. In either case, the commutator is trivial.

Now, consider the following commutator:

$$C_L(y_1(\varpi), \bar{y}'_2(\vartheta_L)) = \zeta_L^{mB(y_1, y'_2)}.$$

We claim that $mB(y_1, y'_2) \in (q^d - 1)\mathbb{Z}$. Indeed, we have:

$$B(y_1, y'_2) = B(\gamma y_1, y'_2) = B(y_1, \gamma^{-1} y'_2) = B(y_1, qy'_2 + (q^d - 1)y'_3),$$

for some $y'_3 \in Y$. Since $y_1 \in Y^\#$, we have:

$$B(y_1, (q^d - 1)y'_3) \in n(q^d - 1)\mathbb{Z}.$$

It follows that:

$$(q - 1)B(y_1, y'_2) \in n(q^d - 1)\mathbb{Z}.$$

From this, we find:

$$B(y_1, y'_2) \in nr\mathbb{Z}.$$

Hence $mB(y_1, y'_2) \in mn r\mathbb{Z} = (q^d - 1)\mathbb{Z}$. This proves our claim, and we have proven that:

$$C_L(y_1(\varpi), \bar{y}'_2(\vartheta_L)) = 1.$$

Next, consider the following commutator:

$$C_L(\bar{y}_2(\vartheta_L), y'_1(\varpi)) = \zeta_L^{-mB(y_2, y'_1)}.$$

We claim now that $mB(y_2, y'_1) \in (q^d - 1)\mathbb{Z}$. Indeed, we have:

$$B(y_2, y'_1) = B(q\gamma y_2 + (q^d - 1)y_3, y'_1) = qB(y_2, y'_1) + (q^d - 1)B(y_3, y'_1),$$

for some $y_3 \in Y^{\Gamma\#}$. In particular, $B(y_3, y'_1) \in n\mathbb{Z}$, since $y'_1 \in Y^\Gamma$. It follows that:

$$(q - 1)B(y_2, y'_1) \in n(q^d - 1)\mathbb{Z}.$$

From this we find that $B(y_2, y'_1) \in nr\mathbb{Z}$, from which the claim follows. We have proven that:

$$C_L(\bar{y}_2(\vartheta_L), y'_1(\varpi)) = 1.$$

Finally, note that $(\vartheta_L, \vartheta_L)_{L, q^d - 1} = 1$. Hence,

$$C_L(\bar{y}_2(\vartheta_L), \bar{y}'_2(\vartheta_L)) = 1.$$

We have proven that $y_1(\varpi)$, and $\bar{y}_2(\vartheta_L)$ commute with a set of generators for T_F/T_F^1 . Since $T_F^1 \in Z^\dagger(T_F)$, this suffices to prove that:

$$y_1(\varpi)\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F).$$

□

The previous two theorems fully characterize the subgroup $Z^\dagger(T_F)$.

Corollary 5.9. *Suppose that $y_1 \in Y$, $\bar{y}_2 \in \bar{Y}$, and $t^1 \in T_F^1$. Then $t = y_1(\varpi)\bar{y}_2(\vartheta_L)t^1 \in Z^\dagger(T_F)$ if and only if all of the following conditions hold:*

- $y_1 \in Y^{\#\Gamma}$.
- $y_2 \in Y^\#$, for any choice of representative y_2 of $\bar{y}_2$.
- $\delta_q y_2 \in (q^d - 1)Y^{\Gamma\#}$, for any choice of representative y_2 of $\bar{y}_2$.

Proof. This corollary follows directly from the previous two theorems. One important observation is the following: The latter two conditions do not depend upon the choice of representative $y_2 \in Y$ for a given $\bar{y}_2 \in \bar{Y}$.

Indeed, suppose that $y'_2 = y_2 + (q^d - 1)z$, for some $z \in Y$, so that y_2 and y'_2 are representatives for $\bar{y}_2$. Since $Y^\# \subset nY$, and n divides $(q^d - 1)$, we find that $y_2 \in Y^\#$ if and only if $y'_2 \in Y^\#$.

Similarly, we find that $\delta_q y'_2 = \delta_q y_2 + (q^d - 1)\delta_q z$. By Lemma 5.6, $\delta_q z \in Y^{\Gamma\#}$. It follows that $\delta_q y_2 \in (q^d - 1)Y^{\Gamma\#}$ if and only if $\delta_q y'_2 \in (q^d - 1)Y^{\Gamma\#}$. □

The above corollary implies that $y_1(\varpi) \in Z^\dagger(T_F)$, for a given $y_1 \in Y$, if and only if $y_1 \in Y^{\#\Gamma}$. It also implies the following:

Corollary 5.10. *Suppose that $\bar{y}_2 \in \bar{Y}$. Then $\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$ if and only if*

$$(5.1) \quad \bar{y}_2 \in \text{Im}(Tr_q(\bar{Y}^{\Gamma^\#}) \rightarrow \bar{Y}).$$

Proof. The previous corollary implies that $\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$ if and only if the following two conditions hold:

- (1) $y_2 \in Y^\#$ for some (equivalently, every) representative y_2 of $\bar{y}_2$.
- (2) $\delta_q y_2 \in (q^d - 1)Y^{\Gamma^\#}$ for some (equivalently, every) representative y_2 of $\bar{y}_2$.

Given these conditions, and a representative y_2 of $\bar{y}_2$, there exists $w \in Y^{\Gamma^\#}$ such that $\delta_q(y_2) = (q^d - 1)w$. Hence $\delta_q(y_2) = \delta_q Tr_q(w)$. The injectivity of δ_q implies that $y_2 = Tr_q(w)$. It follows that $\bar{y}_2$ is the image of $Tr_q(\bar{w})$ in $\bar{Y}$. Hence, the conditions (1) and (2) imply the one condition 5.1 of this corollary.

Conversely, suppose that Equation 5.1 is satisfied. Then we may choose $w \in Y^{\Gamma^\#}$, such that $\bar{y}_2$ equals the image of $Tr_q(\bar{w})$ in $\bar{Y}$. Thus $y_2 = Tr_q(w)$ is a representative for $\bar{y}_2$ in $\bar{Y}$. Since $Tr_q(Y^{\Gamma^\#}) \subset Y^\#$ by Lemma 5.6, the condition (1) is satisfied. Since $\delta_q y_2 = Tr_q \delta_q w = (q^d - 1)w$, condition (2) is satisfied as well. Therefore, $\bar{y}_2(\vartheta_L) \in Z^\dagger(T_F)$. $\square$

5.5. The image of an isogeny. For split metaplectic tori, we found a useful characterization of $Z^\dagger(T_F)$ as the image of an isogeny on F -rational points. The same isogeny makes sense for non-split tori, however there is a small but important difference between the image of the isogeny and $Z^\dagger(T_F)$. We view this difference as accounting for “packets” of representations of metaplectic tori, with the same parameter.

Consider the inclusion of $\mathbb{Z}[\Gamma]$ -modules $\iota: Y^\# \hookrightarrow Y$. Note that we use the fact that Q is a Γ -invariant quadratic form, so that $Y^\#$ is a $\mathbb{Z}[\Gamma]$ -submodule. This inclusion corresponds to an isogeny of algebraic tori over F :

$$\iota: \mathbf{T}^\# \rightarrow \mathbf{T}.$$

Our description of the F -rational and L -rational points for $\mathbf{T}$ is also valid, mutatis mutandis, for $\mathbf{T}^\#$. When $y \in Y^\#$, and $u \in L^\times$, we simply write $(y \otimes u)$ for the corresponding element of $\mathbf{T}^\#(L) \equiv Y^\# \otimes L^\times$. We choose this notation, rather than $y(u)$, since we do not wish to confuse cocharacters of $\mathbf{T}$ with cocharacters of $\mathbf{T}^\#$. Since $Y^\#$ is an $\mathbb{Z}[\Gamma]$ -module, we find that:

Proposition 5.11. *The torus $\mathbf{T}^\#$ splits over an unramified extension of F . Suppose that $y_1, y_2 \in Y^\#$. Then $(y_1 \otimes \varpi)(y_2 \otimes \vartheta_L) \in T^\# = \mathbf{T}^\#(F)$ if and only if*

$$y_1 \in Y^{\# \Gamma}, \text{ and } \bar{y}_2 \in (\bar{Y}^\#)^{\Gamma, q}.$$

The isogeny ι has the following effect on L -rational points:

$$\iota(y \otimes u) = y(u), \text{ for all } y \in Y^\#, u \in L^\times, (y \otimes u) \in \mathbf{T}^\#(L).$$

Hence, we find that:

Proposition 5.12. *Suppose that $y_1 \in Y$, $\bar{y}_2 \in \bar{Y}$, and $t^1 \in T_L^1$. Then $y_1(\varpi)\bar{y}_2(\vartheta)t^1$ is an element of the image of $\iota: \mathbf{T}^\#(F) \rightarrow \mathbf{T}(F)$ if and only if:*

- $y_1 \in Y^{\# \Gamma}$.
- There exists $y_2 \in Y^\#$ representing $\bar{y}_2$, such that $\delta_q y_2 \in (q^d - 1)Y^\#$.
- $t^1 \in T_F^1$.

Proof. Since $(n, p) = 1$, the image of ι contains T_F^1 . It suffices only to consider the images:

$$\iota((y_1 \otimes \varpi)(y_2 \otimes \vartheta_L)),$$

for all $y_1 \in Y^{\# \Gamma}$, and all $y_2 \in Y^{\#}$ such that $\bar{y}_2 \in (\bar{Y}^{\#})^{\Gamma, q}$. $\square$

Then, we find:

Corollary 5.13. *Suppose that $\bar{y}_2 \in \bar{Y}$. Then $\bar{y}_2(\vartheta_L) \in \iota(\mathbf{T}^{\#}(F))$ if and only if*

$$\bar{y}_2 \in \text{Im}(Tr_q(\bar{Y}^{\#}) \rightarrow \bar{Y}).$$

Proof. If $\bar{y}_2 \in \text{Im}(Tr_q(\bar{Y}^{\#}) \rightarrow \bar{Y})$, there exists an element $y_3 \in Y^{\#}$ such that $\bar{y}_2$ equals the image of $Tr_q(y_3)$ in $\bar{Y}$. If $y_2 = Tr_q(y_3)$, then y_2 is a representative for $\bar{y}_2$ in Y . Note that $y_2 \in Y^{\#}$, since $y_3 \in Y^{\#}$. Furthermore, $\delta_q y_2 = \delta_q Tr_q(y_3) = (q^d - 1)y_3 \in (q^d - 1)Y^{\#}$. Hence, $\bar{y}_2(\vartheta_L) \in \iota(\mathbf{T}^{\#}(F))$ by the previous proposition.

Conversely, suppose that $\bar{y}_2(\vartheta_L) \in \iota(\mathbf{T}^{\#}(F))$. By the previous proposition, there exists a representative y_2 of $\bar{y}_2$ in Y , such that $y_2 \in Y^{\#}$, and $\delta_q(y_2) \in (q^d - 1)Y^{\#}$. It follows that $\delta_q(y_2) = \delta_q Tr_q(y_3)$, for some $y_3 \in Y^{\#}$. Hence $y_2 = Tr_q(y_3)$. Hence, $\bar{y}_2$ is contained in the image of $Tr_q(\bar{Y}^{\#})$ in $\bar{Y}$. $\square$

5.6. Endoscopic Parameterization. From the previous two sections, we have described the groups $Z^{\dagger}(T_F)$ and $\iota(\mathbf{T}^{\#}(F))$. They are quite similar, with one exception. Given $\bar{y}_2 \in \bar{Y}$, we have:

- $\bar{y}_2(\vartheta_L) \in Z^{\dagger}(T_F)$ if and only if

$$\bar{y}_2 \in \text{Im}(Tr_q(\bar{Y}^{\Gamma \#}) \rightarrow \bar{Y}).$$

- $\bar{y}_2(\vartheta_L) \in \iota(\mathbf{T}^{\#}(F))$ if and only if

$$\bar{y}_2 \in \text{Im}(Tr_q(\bar{Y}^{\#}) \rightarrow \bar{Y}).$$

Define a finite group $P^{\dagger}$ by:

$$P^{\dagger} = \frac{\text{Im}(Tr_q(\bar{Y}^{\Gamma \#}) \rightarrow \bar{Y})}{\text{Im}(Tr_q(\bar{Y}^{\#}) \rightarrow \bar{Y})}$$

Since $Y^{\#} \subset Y^{\Gamma \#}$, one may give a more practical definition of $P^{\dagger}$:

$$P^{\dagger} = \frac{Tr_q(Y^{\Gamma \#})}{Tr_q(Y^{\#}) \cdot (q^d - 1)Y}.$$

Since Tr_q acts as an injective endomorphism of $Y^{\Gamma \#}$ and $Y^{\#}$, and $q^d - 1 = \delta_q \circ Tr_q$, we may further identify $P^{\dagger}$:

$$P^{\dagger} \equiv \frac{Y^{\Gamma \#}}{Y^{\#} \cdot \delta_q Y}.$$

The description of $P^{\dagger}$ given above depends upon the choice of root of unity θ_L . However, it can be defined canonically in terms of the $\mathbb{Z}[\Gamma]$ -module $\bar{T}_1$, and the bilinear form B .

It follows from Proposition 5.12, and Corollary 5.9, that there is a short exact sequence:

$$1 \rightarrow \iota(\mathbf{T}^{\#}(F)) \rightarrow Z^{\dagger}(T_F) \rightarrow P^{\dagger} \rightarrow 1.$$

The main theorem of Langlands in [9], parameterizing smooth characters of tori over local fields, determines isomorphisms:

$$\mathcal{X}(T_F) \cong H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}}), \text{ and } \mathcal{X}(T_F^{\#}) \cong H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}}^{\#}).$$

As before, the characters of the image of an isogeny can be parameterized cohomologically:

Proposition 5.14. *The Langlands parameterization yields a finite-to-one parameterization of the smooth characters of $Z^\dagger(T_F)$:*

$$1 \rightarrow \mathcal{X}(P^\dagger) \rightarrow \mathcal{X}(Z^\dagger(T_F)) \rightarrow \frac{H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})}{H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#)} \rightarrow 1.$$

Remark 5.15. In order to view $H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#)$ as a subgroup of $H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})$ as above, we must know that the map $H_c^0(\mathcal{W}_{L/F}, \hat{\mathcal{T}}) \rightarrow H_c^0(\mathcal{W}_{L/F}, \hat{\mathcal{T}}^\#)$ is surjective. This follows from the identifications:

$$H_c^0(\mathcal{W}_{L/F}, \hat{\mathcal{T}}) \equiv \text{Hom}_{\mathbb{Z}}(Y^\Gamma, \mathbb{C}^\times), \text{ and } H_c^0(\mathcal{W}_{L/F}, \hat{\mathcal{T}}^\#) \equiv \text{Hom}_{\mathbb{Z}}(Y^{\#\Gamma}, \mathbb{C}^\times),$$

and the fact that $Y^{\#\Gamma}$ has finite index in Y^Γ .

This leads directly, via a Stone-von-Neumann theorem, to a main theorem for tame covers of unramified tori:

Theorem 5.16. *Suppose that we are given a tame metaplectic cover of an unramified torus:*

$$1 \rightarrow \mu_n \rightarrow \tilde{T} \rightarrow T \rightarrow 1,$$

Then, with the sublattices $Y^\# \subset Y^{\Gamma\#} \subset Y$ defined as before, and the resulting isogeny $\iota: \mathbf{T}^\# \rightarrow \mathbf{T}$, we find:

There is a finite-to-one surjective map, intertwining the natural action of $H^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})$:

$$\Phi: \text{Irr}_\epsilon(\tilde{T}) \rightarrow \frac{H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})}{H_c^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}_{Q,n})}.$$

The fibres of this map are torsors for the finite group $\mathcal{X}(P^\dagger)$, where:

$$P^\dagger \equiv \frac{Y^{\Gamma\#}}{Y^\# \cdot \delta_q Y}.$$

Remark 5.17. We do not know if a parameterization, such as that above, holds for general metaplectic tori over local fields. Namely, we have not been able to describe the center of such metaplectic tori, when $\mathbf{T}$ is ramified, or when $\mathbf{T}$ is an unramified torus, but the cover is not tame. We hope that such a parameterization is possible, though the packets might be substantially different.

Remark 5.18. In the process of proving the previous theorem, we chose a uniformizing element $\varpi \in F^\times$ and a root of unity θ_L . However, this choice does not have any effect on the parameterization given above. The sublattices $Y^\#$ and $Y^{\Gamma\#}$ clearly do not depend upon such a choice. Moreover, the action of $\mathcal{X}(P^\dagger)$ on the fibres of Φ does not depend on such a choice.

6. PSEUDO-SPHERICAL AND PSEUDO-TRIVIAL REPRESENTATIONS

We maintain all of the conventions of the previous section. In particular, we have a *tame* metaplectic cover of an *unramified* torus:

$$1 \rightarrow \mu_n \rightarrow \tilde{T}_F \rightarrow T_F \rightarrow 1.$$

We have shown that the irreducible genuine representations of $\tilde{T}_F$ can be parameterized by the points of a homogeneous space on which $H^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})$ acts transitively. However, such a parameterization is *not* unique; one must choose a “base point” in the space of irreducible genuine representations of $\tilde{T}_F$, in order to choose a specific morphism of homogeneous spaces:

$$\Phi: \text{Irr}_\epsilon(\tilde{T}_F) \rightarrow \frac{H^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}})}{H^1(\mathcal{W}_{L/F}, \hat{\mathcal{T}} \rightarrow \hat{\mathcal{T}}^\#)}.$$

In this section, we discuss the data which determines such base points. Such choices arise frequently in treatments of metaplectic groups, often as choices of square roots of -1 in $\mathbb{C}$.

6.1. The Residual Extension. Recall that the unramified torus $\mathbf{T}$ has a smooth model $\underline{\mathbf{T}}$ over $\mathcal{O}_F$, and $T_F^\circ = \underline{\mathbf{T}}(\mathcal{O}_F)$. In this case, T_F° is the maximal compact subgroup of T , and we let $\tilde{T}_F^\circ$ be its preimage in $\tilde{T}_F$. Also, $\bar{\mathbf{T}}$ denotes the special fibre of $\underline{\mathbf{T}}$, which is a torus over $\mathbf{f}$. Recall that $\mathbf{T}'$ is a central extension of $\mathbf{T}$ by $\mathbf{K}_2$. Pushing forward via the tame symbol led to the tame central extension:

$$1 \rightarrow \mathbf{f}^\times \rightarrow T_F^t \rightarrow T_F \rightarrow 1.$$

We write $T_F^{t\circ}$ for the preimage of T_F in T_F^t .

In Section 12.11 of [3], Deligne and Brylinski construct an extension $\bar{\mathbf{T}}'$ of $\bar{\mathbf{T}}$ by $\mathbf{G}_m$ (in the category of groups over $\mathbf{f}$). We call $\bar{\mathbf{T}}'$ the *residual extension* associated to $\mathbf{T}'$. The residual extension fits into the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{f}^\times & \longrightarrow & T_F^{t\circ} & \longrightarrow & T_F^\circ \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{f}^\times & \longrightarrow & \bar{\mathbf{T}}'_\mathbf{f} & \longrightarrow & \bar{\mathbf{T}}_\mathbf{f} \longrightarrow 1. \end{array}$$

Here, the map from $\mathbf{f}^\times$ to itself is the identity, the map from T_F° to $\bar{\mathbf{T}}_\mathbf{f}$ is the reduction map, and the diagram identifies the top row with the pullback of the bottom row via reduction.

As an extension of $\bar{\mathbf{T}}$ by $\mathbf{G}_m$ over $\mathbf{f}$, the group $\bar{\mathbf{T}}'$ is an algebraic torus over $\mathbf{f}$. Note that the category of extensions of $\bar{\mathbf{T}}$ by $\mathbf{G}_m$, in the category of groups over $\mathbf{f}$, is equivalent to the category of extensions of Y by $\mathbb{Z}$, in the category of $\mathbb{Z}[\Gamma]$ -modules (where $\mathbb{Z}$ is given the trivial module structure). In this way, the construction of Section 12.11 of [3] associates an extension of Y by $\mathbb{Z}$, to any extension of an unramified torus $\mathbf{T}$ by $\mathbf{K}_2$.

Remark 6.1. Recall that $\tilde{Y}$ is a Γ -equivariant extension of Y by $L^\times$, constructed as a functorial invariant of the extension $\mathbf{T}'$ of $\mathbf{T}$ by $\mathbf{K}_2$. Let Y' be the extension of Y by $\mathbb{Z}$, obtained by pushing forward $\tilde{Y}$ via the valuation map $L^\times \rightarrow \mathbb{Z}$:

$$0 \rightarrow \mathbb{Z} \rightarrow Y' \rightarrow Y \rightarrow 0.$$

We do not know whether this extension is naturally isomorphic to the exact sequence of cocharacter groups of the residual extension of tori described above

Definition 6.2. Let $\text{Spl}(\bar{\mathbf{T}}')$ denote the set of splittings, in the category of algebraic groups over $\mathbf{f}$, of the short exact sequence:

$$1 \rightarrow \mathbf{G}_m \rightarrow \bar{\mathbf{T}}' \rightarrow \bar{\mathbf{T}} \rightarrow 1.$$

We say that the extension $\mathbf{T}'$ of the unramified torus $\mathbf{T}$ is a *residually split extension*, if $\text{Spl}(\bar{\mathbf{T}}')$ is non-empty.

Proposition 6.3. *Then, if $\text{Spl}(\bar{\mathbf{T}}')$ is non-empty, then $\text{Spl}(\bar{\mathbf{T}}')$ is a torsor for the abelian group X^Γ .*

Proof. Any two algebraic splittings are related by an element of $\text{Hom}_{\mathbf{F}}(\bar{\mathbf{T}}, \mathbf{G}_m)$. This group may be identified with the Γ -fixed characters of $\mathbf{T}$. $\square$

6.2. Pseudo-spherical representations. Suppose now that $\mathbf{T}'$ is a residually split extension of $\mathbf{T}$ by $\mathbf{K}_2$. Fix a splitting $s \in \text{Spl}(\bar{\mathbf{T}}')$. The splitting lifts to a splitting $\sigma: T_F^\circ \rightarrow T_F^{\text{to}}$. Pushing forward via the m^{th} power map, we may also view σ as a splitting $T_F^\circ \rightarrow \tilde{T}_F^\circ$. From such a splitting s , we let $\theta_s^\circ: \tilde{T}_F^\circ \rightarrow \mathbb{C}^\times$ denote the character obtained by projecting onto μ_n (via the splitting σ), and then applying the injective homomorphism $\epsilon: \mu_n \rightarrow \mathbb{C}^\times$.

Let $\tilde{T}_F^1$ be the centralizer of $\tilde{T}_F^\circ$ in $\tilde{T}_F$. Then, we find:

Proposition 6.4. *The group $\tilde{T}_F^1$ is the preimage of a subgroup $T_F^1 \subset T_F$. Considering the valuation map:*

$$\text{val}: T_F \rightarrow Y^\Gamma,$$

whose kernel is T_F° , T_F^1 is equal to the preimage of $Y^{\#\Gamma}$.

Proof. Since $T_F^1 \supset T_F^\circ$, it suffices to identify the set of $y \in Y^\Gamma$ such that:

$$C_L(y(\varpi), \bar{y}'(\vartheta_L)) = 1 \text{ for all } \bar{y}' \in \bar{Y}^{\Gamma, q}.$$

In fact, the set of such y has been identified in the proofs of Theorems 5.7 and 5.8. The above condition is satisfied if and only if $y \in Y^{\#\Gamma}$. $\square$

Corollary 6.5. *The group $\tilde{T}_F^1$ is abelian.*

Proof. As $\tilde{T}_F^1$ is the centralizer of the abelian group $\tilde{T}_F^\circ$, it suffices to prove that $C(y(\varpi), y'(\varpi)) = 1$ for all $y, y' \in Y^{\#\Gamma}$. This is proven in the beginning of the proof of Theorem 5.8. $\square$

Directly following Section 4 of Savin [13], we find

Proposition 6.6. *There is a natural bijection between the following two sets:*

- *The set $\text{Irr}_{s, \epsilon}^{\text{sph}}(\tilde{T}_F)$ of pseudo-spherical irreducible representations of $\tilde{T}_F$ (for the splitting s). These are the genuine irreducible representations of $\tilde{T}_F$, whose restriction to $\tilde{T}_F^\circ$ via the splitting s contains a nontrivial θ° -isotypic component.*
- *The set of extensions of θ° to the group $\tilde{T}_F^1$.*

Namely, if (π, V) is a pseudo-spherical irreducible representation, its θ° -isotypic subrepresentation is an extension of θ° to the group $\tilde{T}_F^1$. Conversely, given such an extension θ^1 of θ° to a character of $\tilde{T}_F^1$, the induced representation $\text{Ind}_{\tilde{T}_F^1}^{\tilde{T}_F} \theta^1$ is a pseudo-spherical irreducible representation.

One may rephrase the above bijection slightly; the splitting s yields an injective homomorphism from T_F° onto a normal subgroup of $\tilde{T}_F^1$. This fits into a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & 1 & \longrightarrow & T_F^\circ & \longrightarrow & T_F^\circ \longrightarrow 1 \\
 & & \downarrow & & \downarrow s & & \downarrow \\
 1 & \longrightarrow & \mu_n & \longrightarrow & \tilde{T}_F^1 & \longrightarrow & T_F^1 \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mu_n & \longrightarrow & \tilde{Y}^{\#\Gamma} & \longrightarrow & Y^{\#\Gamma} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

Hence, the splitting s determines an extension $\tilde{Y}^{\#\Gamma}$ of $Y^{\#\Gamma}$ by μ_n . A standard diagram chase now yields:

Proposition 6.7. *There is a natural bijection:*

$$\text{Irr}_{s,\epsilon}^{\text{sph}}(\tilde{T}_F) \leftrightarrow \mathcal{X}_\epsilon(\tilde{Y}^{\#\Gamma}).$$

Corollary 6.8. *The space $\text{Irr}_{s,\epsilon}^{\text{sph}}(\tilde{T}_F)$ is naturally a torsor for the complex algebraic torus $\mathcal{X}(Y^{\#\Gamma})$.*

Remark 6.9. One may view $\mathcal{X}_\epsilon(\tilde{Y}^{\#\Gamma})$ as the set of irreducible representations of a “quantum torus”. Indeed, consider the ring:

$$\mathbb{C}_\epsilon[\tilde{Y}^{\#\Gamma}] = \frac{\mathbb{C}[\tilde{Y}^{\#\Gamma}]}{\langle \zeta - \epsilon(\zeta) \rangle_{\zeta \in \mu_n}}.$$

The ring $\mathbb{C}_\epsilon[\tilde{Y}^{\#\Gamma}]$ can be viewed as (the coordinate ring of) a quantum complex torus, which we call $\hat{\mathcal{T}}_\epsilon^{\#\Gamma}$. $\hat{\mathcal{T}}_\epsilon^{\#\Gamma}$ is the quantization of a complex torus, at a root of unity. Quasi-coherent sheaves on this quantum torus (i.e. modules over its coordinate ring) correspond naturally to pseudo-spherical representations of $\tilde{T}_F$.

6.3. Pseudo-Trivial Representations. In many practical situations, the extension $\tilde{Y}^{\#\Gamma}$ of $Y^{\#\Gamma}$ by μ_n splits over a quite large submodule of $Y^{\#\Gamma}$. For example, in many cases, the extension splits over $Y^{\#\Gamma} \cap 2Y$.

Suppose that $V \subset Y^{\#\Gamma}$ is a finite index subgroup, endowed with a splitting v of the resulting exact sequence:

$$1 \longrightarrow \mu_n \longrightarrow \tilde{V} \xrightarrow{\quad v \quad} V \longrightarrow 1.$$

Let $U = Y^{\#\Gamma}/V$ denote the quotient. The splitting v yields an extension of finite abelian groups:

$$1 \rightarrow \mu_n \rightarrow \tilde{U} \rightarrow U \rightarrow 1.$$

Pulling back yields natural inclusions:

$$\mathcal{X}_\epsilon(U) \hookrightarrow \mathcal{X}_\epsilon(\tilde{Y}^{\#\Gamma}) \equiv \text{Irr}_{s,\epsilon}^{\text{sph}}(\tilde{T}_F).$$

Therefore, within the set of pseudo-spherical representations of $\tilde{T}_F$, we find a *finite* set of “pseudo-trivial” representations (relative to the choice of splitting subgroup (V, v) of $Y^\# \Gamma$):

Definition 6.10. The genuine pseudo-trivial representations of $\tilde{T}_F$, are those irreducible pseudo-spherical genuine representations, that are in the image of $\mathcal{X}_\epsilon(\tilde{U})$. This definition depends upon the following choices:

- The splitting s (to determine the pseudo-spherical representations).
- The splitting subgroup (V, v) (to determine the pseudo-trivial representations).

Remark 6.11. Most often, one chooses a pseudo-trivial “base point” in the space $\text{Irr}_\epsilon(\tilde{T}_F)$. Very often (cf. the examples of [13]) $\tilde{U}$ is a finite abelian group of exponent 4. It follows that pseudo-trivial representations may often be given by specifying certain characters of an abelian group of exponent 4. This explains the frequent need to choose fourth roots of unity, in the literature on metaplectic groups.

7. TORI OVER $\mathbb{R}$

In this section, the following will be fixed:

- $\mathbf{T}$ will be a torus over $\mathbb{R}$, which splits over $\mathbb{C}$, with $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$. X and Y will be the resulting character and cocharacter groups.
- $\mathbf{T}'$ will be an extension of $\mathbf{T}$ by $\mathbf{K}_2$ in $\mathfrak{Sp}_{\mathbb{R}}$.
- $(Q, \tilde{Y})$ will be the Deligne-Brylinski invariants of $\mathbf{T}'$. B will be the symmetric bilinear form associated to Q .
- We fix $n = 2$, so that $\mathbb{R}$ has enough n^{th} roots of unity.
- $\epsilon: \mu_2(\mathbb{R}) \rightarrow \mathbb{C}^\times$ will be the unique injective character.
- We view $T = \mathbf{T}(\mathbb{R})$ as a real Lie group. The extension $\mathbf{T}'$, and the quadratic Hilbert symbol, yields an extension of Lie groups:

$$1 \rightarrow \mu_2 \rightarrow \tilde{T} \rightarrow T \rightarrow 1.$$

We are interested in parameterizing the irreducible genuine representations of $\tilde{T}$, and the set of such representations is called $\text{Irr}_\epsilon(\tilde{T})$, as before.

7.1. Structure of metaplectic tori over $\mathbb{R}$. Let T° be the connected component of the identity in the Lie group T . Then, we find a canonically split short exact sequence of Lie groups:

$$1 \rightarrow T^\circ \rightarrow T \rightarrow (Y^\Gamma \otimes_{\mathbb{Z}} \mu_2) \rightarrow 1.$$

We define $\overline{Y^\Gamma} = Y^\Gamma \otimes_{\mathbb{Z}} \mu_2 \equiv Y^\Gamma / 2Y^\Gamma$. Every element t of T has a decomposition $t = t^\circ \bar{y}(-1)$, for uniquely determined $t^\circ \in T^\circ$ and $\bar{y} \in \overline{Y^\Gamma}$.

Now, we consider the metaplectic cover of T :

$$1 \rightarrow \mu_2 \rightarrow \tilde{T} \rightarrow T \rightarrow 1.$$

The commutator $C: T \times T \rightarrow \mu_2$ is bi-multiplicative and continuous. It follows that the commutator is trivial when either of its inputs is in T° . Hence we find:

Proposition 7.1. T° is a subgroup of $Z^\dagger(T)$.

7.2. Description of the center. It follows from the previous proposition that, to describe $Z^\dagger(T)$, it suffices to describe its image in T/T° . Hence, it suffices to determine when, given $\bar{y} \in \overline{Y^\Gamma}$, $\bar{y}(-1) \in Z^\dagger(T)$. We must be able to compute the commutator $C(\bar{y}(-1), \bar{y}'(-1))$ for arbitrary $\bar{y}, \bar{y}' \in \overline{Y^\Gamma}$.

Here, we note that such elements $\bar{y}(-1)$ and $\bar{y}'(-1)$ are contained in the maximal split subtorus $\mathbf{T}_s \subset \mathbf{T}$ over $\mathbb{R}$. Restricting the central extension of $\mathbf{T}$ by $\mathbf{K}_2$, to the subtorus $\mathbf{T}_s$, the formula of Corollary 3.14 of [3] is valid for computing commutators. We find that:

Proposition 7.2. *If $y, y' \in Y^\Gamma$, then $C(\bar{y}(-1), \bar{y}'(-1)) = (-1)^{B(y, y')}$.*

Proof. This follows directly from Corollary 3.14 of [3], and the Hilbert symbol over $\mathbb{R}$: $(-1, -1)_{\mathbb{R}, 2} = -1$. $\square$

Proposition 7.3. *Given $\bar{y} \in \overline{Y^\Gamma}$, $\bar{y}(-1) \in Z^\dagger(T)$ if and only if every representative y of $\bar{y}$ in Y satisfies:*

$$y \in Y^{\Gamma \# \Gamma}.$$

Proof. Suppose $\bar{y}, \bar{y}' \in \overline{Y^\Gamma}$. Let y be a representative of $\bar{y}$ in Y . The commutator has been computed:

$$C(\bar{y}(-1), \bar{y}'(-1)) = (-1)^{B(\bar{y}, \bar{y}')}.$$

Thus, we find that $C(\bar{y}(-1), \bar{y}'(-1)) = 1$ for all $\bar{y}' \in \overline{Y^\Gamma}$, if and only if $B(y, y') \in 2\mathbb{Z}$ for all representatives y' of all $\bar{y}' \in \overline{Y^\Gamma}$. This occurs if and only if $B(y, y') \in 2\mathbb{Z}$ for all $y' \in Y^\Gamma$, i.e., $y \in Y^{\Gamma \#}$.

Thus, we find that, given $\bar{y} \in \overline{Y^\Gamma}$, $\bar{y}(-1) \in Z^\dagger(T)$ if and only if $y \in Y^{\Gamma \#} \cap Y^\Gamma = Y^{\Gamma \# \Gamma}$. $\square$

7.3. The image of an isogeny. As in the nonarchimedean case, the inclusion $Y^\# \rightarrow Y$ of $\mathbb{Z}[\Gamma]$ -modules corresponds to an isogeny of tori over $\mathbb{R}$:

$$\iota: \mathbf{T}^\# \rightarrow \mathbf{T}.$$

We are interested in the resulting continuous homomorphism of real Lie groups:

$$\iota: T^\# \rightarrow T.$$

Since ι is an isogeny, we find that $\iota(T^\#) \supset T^\circ$. Thus, in order to fully describe $\iota(T^\#)$ it suffices to determine for which $\bar{y} \in \overline{Y^\Gamma}$, $\bar{y}(-1) \in \iota(T^\#)$.

Proposition 7.4. *Suppose that $\bar{y} \in \overline{Y^\Gamma}$. Then $\bar{y}(-1) \in \iota(T^\#)$ if and only if $\bar{y} \in \text{Im}(\overline{Y^{\Gamma \# \Gamma}} \rightarrow \overline{Y^\Gamma})$.*

Proof. We find that $\bar{y}(-1) \in \iota(T^\#)$, if and only if there exists $y \in Y^{\Gamma \# \Gamma}$ which represents $\bar{y}$. The proposition follows. $\square$

Comparing the image of the isogeny ι , to the group $Z^\dagger(T)$, yields a short exact sequence:

$$1 \rightarrow \iota(T^\#) \rightarrow Z^\dagger(T) \rightarrow P^\dagger \rightarrow 1,$$

where we may identify the finite group:

$$P^\dagger \equiv \frac{Y^{\Gamma \# \Gamma}}{Y^{\Gamma \# \Gamma} \cdot 2Y^\Gamma}.$$

7.4. Parameterization. As for the case of nonarchimedean fields, we choose to parameterize the genuine irreducible representations of $\tilde{T}$, through a finite-to-one map and a description of the fibres. Over $\mathbb{R}$, the previous two sections imply the following:

Theorem 7.5. *The space $\text{Irr}_\epsilon(\tilde{T})$ can be identified (via Theorem 3.1) with the complex variety of genuine characters $X_\epsilon(Z(\tilde{T}))$. This is a torsor for the complex algebraic group of characters $X(Z^\dagger(T))$. There is a short exact sequence:*

$$1 \rightarrow X(P^\dagger) \rightarrow X(Z^\dagger(T)) \rightarrow \frac{H_c^1(W_{\mathbb{R}}, \hat{T})}{H_c^1(W_{\mathbb{R}}, \hat{T} \rightarrow \hat{T}^\#)} \rightarrow 1.$$

Thus, there exists a finite-to-one parameterization:

$$\Phi: \text{Irr}_\epsilon(\tilde{T}) \rightarrow \frac{H_c^1(W_{\mathbb{R}}, \hat{T})}{H_c^1(W_{\mathbb{R}}, \hat{T} \rightarrow \hat{T}^\#)},$$

which intertwines the actions of $H_c^1(W_{\mathbb{R}}, \hat{T})$, and whose fibres are torsors for the finite group $X(P^\dagger)$.

Note that this theorem is quite similar to the parameterization of $\text{Irr}_\epsilon(\tilde{T})$ for tame covers of unramified tori over nonarchimedean local fields. The primary difference is that the “packet group” $P^\dagger$ is built from Γ -fixed subgroups. For example, the analogue of $\delta_q = q\gamma - 1$ (in the nonarchimedean case) is $-\gamma - 1$ (in the real case), which restricts to multiplication by -2 on the Γ -fixed subgroup.

On the other hand, the extension $\mathbb{C}/\mathbb{R}$ is more analogous to a ramified extension of nonarchimedean fields, so one should not expect a perfect analogy. Perhaps, for ramified extensions of nonarchimedean fields, the packet groups look similar to the groups $P^\dagger$ above.

REFERENCES

1. J. Adams, D. Barbasch, A. Paul, P. Trapa, and D. A. Vogan, Jr., *Unitary Shimura correspondences for split real groups*, J. Amer. Math. Soc. **20** (2007), no. 3, 701–751 (electronic). MR MR2291917
2. Mikhail Borovoi, *Abelian Galois cohomology of reductive groups*, Mem. Amer. Math. Soc. **132** (1998), no. 626, viii+50. MR MR1401491 (98j:20061)
3. Jean-Luc Brylinski and Pierre Deligne, *Central extensions of reductive groups by $\mathbf{K}_2$* , Publ. Math. Inst. Hautes Études Sci. (2001), no. 94, 5–85. MR MR1896177 (2004a:20049)
4. Hélène Esnault, Bruno Kahn, Marc Levine, and Eckart Viehweg, *The Arason invariant and mod 2 algebraic cycles*, J. Amer. Math. Soc. **11** (1998), no. 1, 73–118. MR MR1460391 (98d:14010)
5. Norbert Hoffmann and Markus Spitzweck, *Homological algebra with locally compact abelian groups*, Adv. Math. **212** (2007), no. 2, 504–524. MR MR2329311
6. D. A. Kazhdan and S. J. Patterson, *Metaplectic forms*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 59, 35–142. MR MR743816 (85g:22033)
7. Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190. MR MR1687096 (2000k:22024)
8. J.-P. Labesse, *Cohomologie, L -groupes et fonctorialité*, Compositio Math. **55** (1985), no. 2, 163–184. MR MR795713 (86j:11117)
9. R. P. Langlands, *Representations of abelian algebraic groups*, Pacific J. Math. (1997), no. Special Issue, 231–250, Olga Taussky-Todd: in memoriam. MR MR1610871 (99b:11125)
10. Calvin C. Moore, *Extensions and low dimensional cohomology theory of locally compact groups. I, II*, Trans. Amer. Math. Soc. **113** (1964), 40–63. MR MR0171880 (30 #2106)
11. Takashi Ono, *Arithmetic of algebraic tori*, Ann. of Math. (2) **74** (1961), 101–139. MR MR0124326 (23 #A1640)

12. Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR MR0338129 (49 #2895)
13. Gordan Savin, *On unramified representations of covering groups*, J. Reine Angew. Math. **566** (2004), 111–134. MR MR2039325 (2005a:22014)
14. J. Tate, *Number theoretic background*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26. MR MR546607 (80m:12009)

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