Relations Between Unitary Representations of Real and p-adic Groups

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MIT

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OUTLINE

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(2) G (split) p-adic. Review parallel (less well-developed) theory for the graded affine Hecke algebra $\mathbb{H}$ (Barbasch, Ciubotaru, Moy): toward determining if $\pi \in \widehat{G}_{ur}$ is unitary.

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(3) Idea: find relations between spaces of ~~Langlands~~ Adams-Barbasch-Vogan parameters for $G_{\mathbb{R}}$ and Langlands parameters for $G_{\mathbb{Q}_p}$ to develop theory in (2).

(4) A precise theorem in the context of $\mathrm{GL}(n)$.

(ALMOST) NOTHING ABOUT THE ORBIT METHOD

Dubrovnik, 2002

Canyonlands, 2004

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$K_{\mathbb{R}}$ maximal compact subgroup
$\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $K = (K_{\mathbb{R}})_{\mathbb{C}} \subset G$
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Theorem (Harish Chandra)

Passing to $K_{\mathbb{R}}$-finite vectors induces a bijection from (equivalence classes of) irreducible unitary representations of $G_{\mathbb{R}}$ and (equivalence classes of) irreducible $(\mathfrak{g}, K)$ modules admitting a positive definite σ-invariant form.

STRATEGY

Locate unitary dual as a subset of irreducible $(\mathfrak{g}, K)$ modules admitting a positive definite invariant Hermitian form.
Notation:

$$\widehat{G}^u_{\mathbb{R}} \subset \widehat{G}^h_{\mathbb{R}} \subset \widehat{G}_{\mathbb{R}} := \text{irred } (\mathfrak{g}, K) \text{ modules.}$$

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Recall that Langlands, Knapp, Zuckerman, Vogan classified $\widehat{G}_{\mathbb{R}}$: to each "parameter" $\gamma \in \mathcal{P}$, there corresponds a standard object $\mathrm{std}(\gamma)$ and canonical irreducible subquotient $\mathrm{irr}(\gamma)$ so that the correspondence

$$\begin{aligned} \mathcal{P} &\longrightarrow \widehat{G}_{\mathbb{R}} \\ \gamma &\longrightarrow \mathrm{irr}(\gamma) \end{aligned}$$

is bijective. Easy to determine if $\mathrm{irr}(\gamma) \in \widehat{G}^h_{\mathbb{R}}$.

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(e.g. if $\nu = 0$, this is again tempered)

Unitarizability of certain series of representations

By David A. Vogan, Jr.

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ON THE EXISTENCE AND IRREDUCIBILITY OF CERTAIN SERIES OF REPRESENTATIONS[1]

BY BERTRAM KOSTANT

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be the function that assign $\mu \in \widehat{K}$ the multiplicity of the μ-isotypic space of X:

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When X is irreducible m_X is computable as a consequence of Vogan's proof of the KL conjectures for real groups: In appropriate Grothendieck group, X is a computable sum of standard modules whose m functions are computable.

REFINEMENT: SIGNATURE CHARACTER

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Notice: If $X \in G^h_{\mathbb{R}}$, then $X \in G^u_{\mathbb{R}}$ if $q \equiv 0$.

CONVENIENT REPACKAGING

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Then

$$s \cdot \langle\ ,\ \rangle_X$$

is the "opposite" signature character: if

$$\langle\ ,\ \rangle_X(\mu) = (p, q)$$

then

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Let X be an irreducible $(\mathfrak{g}, K)$ module with real infinitesimal character and invariant Hermitian form $\langle\ ,\ \rangle$. Then there exist finitely many irreducible tempered modules with real infinitesimal character $Z_1, \cdots, Z_k$ and unique nonzero elements $a_i \in \mathbb{W}$ such that

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i}$$

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If we equip each Z_i with its positive form, then $\langle\ ,\ \rangle_X$ is positive iff all $a_i \in \mathbb{Z}$.

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First prove a refinement of the KL conjectures and write

$$\langle\ ,\ \rangle_X = \sum_j b_j \cdot \langle\ ,\ \rangle_{\mathrm{std}(\gamma_j)}$$

as a $\mathbb{W}$ combination of invariant forms of standard modules with the same (real) infinitesimal character.

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In terms of computable Jantzen filtrations, Vogan then explicitly described the behavior of

$$\langle\ ,\ \rangle_{\mathrm{std}(\gamma)}$$

as the continuous parameter ν deformed to 0 (where the standard module becomes tempered).

VOGAN'S STRATEGY TO COMPUTE a_i

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Overcome in Adams-van Leeuwen-T-Vogan and Yee.

Black box for today: foundational work of Barbasch and Moy (building on the Borel-Casselman equivalence and the Kazhdan-Lusztig classification and further work of Lusztig) gives algebraic reduction of the unitary dual of split reductive p-adic groups to the graded affine Hecke algebra.

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Has natural conjugate linear antiautomorphism τ:

$$\tau(t_w) = t_{w^{-1}}, \quad w \in W,$$
$$\tau(\omega) = -\overline{t_{w_0} \omega t_{w_0}}, \quad \omega \in V^\vee.$$

A Hermitian form $\langle\ ,\ \rangle$ on a $\mathbb{H}$ module X is <u>τ-invariant</u> if

$$\langle Y \cdot u, v\rangle = \langle u, -\tau(Y) \cdot v\rangle \qquad Y \in \mathbb{H}$$

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A LITTLE MORE ABOUT STANDARD MODULES

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Standard modules are of the form

$$\mathbb{H} \otimes_{\mathbb{H}_P} (Z \otimes \mathbb{C}_\nu)$$

where Z is tempered and $\nu \in V$.

If $\langle\ ,\ \rangle$ is an invariant form on an irreducible $\mathbb{H}$ module, restriction to W isotypic spaces defines a function

$$\langle\ ,\ \rangle_X : \widehat{W} \longrightarrow \mathbb{W} = \mathbb{Z} \oplus s\mathbb{Z}.$$

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▸ go to real case

No known way to compute the a_i. Main result today: deduce some computations of a_i from the real case. ▸ main result

If $G_{\mathbb{R}}$ and $\mathbb{H}$ correspond to split forms arising from the same root data, one might ask if there is a correspondence

$$X \leftrightarrow X' \qquad Z_i \leftrightarrow Z_i'$$

so that the two kinds of formulas have matching coefficients:

$G_{\mathbb{R}}$:

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Main result: this is exactly true in the case of $\mathrm{GL}(n)$ and suggests generalizations.

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Set $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$, and let $\mathbb{H}$ correspond to the root system of $\mathfrak{gl}(n)$.

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Ciubotaru-T defined exact functors

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More on this in a moment.

Theorem

In the setting of $\mathrm{GL}(n)$*, suppose* X' *is an irreducible* $\mathbb{H}$ *module with real central character admitting an invariant form. Then there exists an irreducible* $(\mathfrak{g}, K)$ *module* X *admitting an invariant form such that* $F(X) = X'$*. Moreover if*

$$\langle \ , \ \rangle_X = \sum_i a_i \cdot \langle \ , \ \rangle_{Z_i} \ : \ \widehat{K} \to \mathbb{W}$$

then

$$\langle \ , \ \rangle_{F(X)} = \sum_i a_i \cdot \langle \ , \ \rangle_{F(Z_i)} \ : \ \widehat{W} \to \mathbb{W}.$$

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$$e_j \mapsto \sum_j [E_j \text{ acting on } X] \otimes [E_j \text{ acting in } ith \text{ component}]$$

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Ciubotaru-T: Natural relationship exists.

From geometry conclude

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(This is a real version of a result of Lusztig and Zelvinsky for category $\mathcal{O}$.) Since

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sends standard modules to the "right" standard modules (or zero), conclude F sends irreducibles to irreducibles (or zero).

IC4

IRREDUCIBLE CHARACTERS OF SEMISIMPLE LIE GROUPS IV. CHARACTER-MULTIPLICITY DUALITY

DAVID A. VOGAN, JR.

IRREDUCIBLE CHARACTERS OF SEMISIMPLE LIE GROUPS IV. CHARACTER-MULTIPLICITY DUALITY

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THEOREM 13.13. *Let G be a real reductive group, and $B = \{cl(\gamma_1), \ldots, cl(\gamma_r)\}$ a block of regular characters of G having nonsingular infinitesimal character. Let $\check{G}$ be a second reductive group, and $\check{B} = \{cl(\check{\gamma}_1), \ldots, cl(\check{\gamma}_r)\}$ a block for $\check{G}$; and assume that the bijection $\gamma_i \to \check{\gamma}_i$ satisfies conditions* (a)–(d) *of Theorem* 11.9. *Then the inverse of the Kazhdan–Lusztig matrix* $(P_{\phi\gamma})$ (*Lemma* 12.15) *is the matrix*

(a) $(P_{\phi\gamma})^{-1} = ((-1)^{l'(\phi) - l'(\gamma)} P_{\check{\gamma}\check{\phi}}$.

more IC4 trivia: famous last words

$$G^{\mathbb{C}} = \check{G}^0 AN. \tag{16.6c}$$

This verifies a conjecture of P. Sally that $\check{G}^0 AN$ may be regarded as a complex simple Lie group; I am grateful to him for supplying this example.

In context of $\mathrm{GL}(n)$ and real infinitesimal char, trying to prove

THEOREM

Suppose X' is an irreducible $\mathbb{H}$ module admitting an invariant form. Then there exists an irreducible $(\mathfrak{g}, K)$ module X admitting an invariant form such that $F(X) = X'$. Moreover if

$$\langle\ ,\ \rangle_X = \sum_i a_i \cdot \langle\ ,\ \rangle_{Z_i} \;:\; \widehat{K} \to \mathbb{W}$$

then

$$\langle\ ,\ \rangle_{F(X)} = \sum_i a_i \cdot \langle\ ,\ \rangle_{F(Z_i)} \;:\; \widehat{W} \to \mathbb{W}.$$

In particular if X is unitary, then so it $F(X)$.

using

$$F(X) := \mathrm{Hom}_{K_{\mathbb{R}}}\big(1, X \otimes \overbrace{(\mathbb{C}^n \otimes \cdots \mathbb{C}^n)}^{n}\big)$$

FINISHING THE PROOF SHOULD BE EASY...

Given $\mathrm{GL}(n,\mathbb{R})$ invariant form $\langle\ ,\ \rangle$ on X, take invariant form $(\ ,\ )$ on $\mathbb{C}^n$ and build invariant form

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Tiny problem: No $\mathrm{GL}(n,\mathbb{R})$ invariant form exists on $\mathbb{C}^n$.

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Definition

A Hermitian form $\langle\ ,\ \rangle$ on a $(\mathfrak{g}, K)$ module X is <u>c-invariant</u> if

$$\langle Y \cdot u, v\rangle = \langle u, -\sigma_c(Y) \cdot v\rangle \qquad Y \in \mathfrak{g};$$

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Minor miracle: c-invariant forms exist and can be canonically normalized. This is the starting point of ALTV to complete Vogan's strategy voganstrategy

$\mathbb{H}$ has natural conjugate linear antiautomorphism τ_c:

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Again they exist and are canonical (Barbasch-Ciubotaru).

The proof should be easy...

Given canonical σ_c-invariant form $\langle \ , \ \rangle^c$ on $(\mathfrak{g}, K)$ module X and $(\ , \ )^c$ on $\mathbb{C}^n$, build σ_c-invariant form

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Now we can follow ALTV to prove a version of the main result for c-invariant forms.

Theorem

In the setting of $\mathrm{GL(n)}$*, suppose* X' *is an irreducible* $\mathbb{H}$ *module with real infinitesimal character. Then there exists an irreducible* $(\mathfrak{g}, K)$ *module* X *such that* $F(X) = X'$*. Moreover the signature characters of the canonical c-invariant forms on* X *and* $F(X)$ *are related as follows: if*

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Useless unless we can say something about invariant forms.

Back to the real case. Recall we had σ_c (defining compact real form) and σ (defining $G_{\mathbb{R}}$). Differ by the Cartan involution for $G_{\mathbb{R}}$

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Trivial to deduce the σ invariance from the σ_c invariance.

FROM c-FORMS TO σ-INVARIANT FORMS

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Trivial to deduce the σ invariance from the σ_c invariance. So really should be keeping track of this information everywhere.

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Same formalism shows that action of δ relates c-invariant forms on $\mathbb{H}^\delta$ modules to honest invariant forms.

BACK TO GL(N) TO FINISH THE PROOF

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Redo the functors for groups and algebras extended by fundamental automorphism δ:

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So the functors once again have good properties, preserve Jantzen filtrations, etc.

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$$\text{decomposition numbers for } \mathbb{H}^\delta \text{ modules}$$

are related to a subset of

$$\text{decomposition numbers for } (\mathfrak{g}, K^\delta) \text{ modules.}$$

So the functors once again have good properties, preserve Jantzen filtrations, etc.

The rest is just following your nose....

BUT STILL NOTHING ABOUT THE ORBIT METHOD...

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Happy Birthday!