Annals of Mathematics, 120 (1984), 141–187

Unitarizability of certain series of representations

By David A. Vogan, Jr.

1. Introduction

Let G be a reductive Lie group in Harish-Chandra's class, K a maximal compact subgroup, and θ the corresponding Cartan involution. Write $\mathfrak{g}_0 =$ Lie(G), and $\mathfrak{g}$ for its complexification; analogous notation is used for other groups. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ such that

(1.1)(a) $$\theta \mathfrak{q} = \mathfrak{q},$$

(b) $\mathfrak{q}$ and its complex conjugate $\bar{\mathfrak{q}}$ are opposite; that is, they intersect in a Levi factor $\mathfrak{l} = \mathfrak{q} \cap \bar{\mathfrak{q}}$ of $\mathfrak{q}$.

If $\mathfrak{u}$ is the nil radical of $\mathfrak{q}$, then

(1.1)(c) $$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$$

is a θ-stable Levi decomposition of $\mathfrak{q}$, and

(1.1)(d) $$\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}$$

is a triangular decomposition of $\mathfrak{g}$. Unless $\mathfrak{q} = \mathfrak{g}$, $\mathfrak{q}$ is *not* the complexification of a subalgebra of $\mathfrak{g}_0$; but $\mathfrak{l} = (\mathfrak{l}_0)_{\mathbf{C}}$, with $\mathfrak{l}_0 \subseteq \mathfrak{g}_0$. Define

(1.1)(e) $$L = \text{normalizer of } \mathfrak{q} \text{ in } G.$$

Then L is again a reductive group in Harish-Chandra's class, with maximal compact subgroup $L \cap K$. We will summarize these assumptions by saying "$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ-stable parabolic subalgebra".

In [S-V], a more or less formal correspondence was established from certain $(\mathfrak{l}, L \cap K)$ modules to $(\mathfrak{g}, K)$ modules. Here is one version of it (except for a definition of $\mathscr{R}$, which will be given in Section 4).

THEOREM 1.2 [S-V]. *Let* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *be a* θ*-stable parabolic subalgebra of* $\mathfrak{g}$. *Fix a Cartan subalgebra* $\mathfrak{h} \subseteq \mathfrak{l}$, *and a weight* $\lambda \in \mathfrak{h}^*$. *Define*

$$\rho(\mathfrak{u}) = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} \alpha \in \mathfrak{h}^*.$$

Let Y *be an irreducible* $(\mathfrak{l}, L \cap K)$ *module of infinitesimal character* $\lambda - \rho(\mathfrak{u})$. *Assume that*

$$(1.2)(a) \qquad \operatorname{Re}\langle \alpha, \lambda \rangle \geq 0, \quad \textit{for all } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Then there is a $(\mathfrak{g}, K)$ *module* $\mathscr{R}Y$ *of infinitesimal character* λ *naturally attached to* Y; $\mathscr{R}Y$ *is irreducible or zero. If (in addition to* (a))

$$(1.2)(b) \qquad \langle \alpha, \lambda \rangle \neq 0, \quad \textit{for all } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}),$$

then $\mathscr{R}Y$ *is non-zero.*

More important, [S-V] shows how to compute the global character of $\mathscr{R}Y$ from that of Y (Corollary 4.21 below). If Y is tempered, then $\mathscr{R}Y$ is as well. In conjunction with other results of [S-V], this immediately suggests the conjecture that $\mathscr{R}Y$ is unitary whenever Y is. That conjecture is proved here:

THEOREM 1.3. *Fix notation as in Theorem* 1.2.

a) *Assume that* Y *is a unitarizable* $(\mathfrak{l}, L \cap K)$ *module of infinitesimal character* $\lambda - \rho(\mathfrak{u})$, *and that* λ *satisfies* (1.2)(a). *Then* $\mathscr{R}Y$ *is a unitarizable* $(\mathfrak{g}, K)$ *module.*

b) *Assume that* λ *satisfies* (1.2)(a) *and* (b), *and that* $\mathscr{R}Y$ *is a unitarizable* $(\mathfrak{g}, K)$ *module. Then* Y *is a unitarizable* $(\mathfrak{l}, L \cap K)$ *module.*

After [S-V] was written, Zuckerman independently constructed (among other things) the representations $\mathscr{R}Y$ for Y one dimensional, and conjectured their unitarity. (These representations have a special importance, discussed in [V-Z].) At about the same time, Parthasarathy in [P1] gave yet another construction (still for Y one dimensional); he was undoubtedly also aware that they ought to be unitary. Since that time, the problem has been widely studied; some of the successes may be found in [A], [P3], [R-S-W], and [E-P-W-W]. (Most of these prove substantially *more* than Theorem 1.3 in special cases.)

The central idea in the proof of Theorem 1.3 is to exploit the existence of a large family of well understood unitary representations, the tempered ones. Any representation admitting an invariant Hermitian form may be obtained from a tempered one by analytic continuation of a tempered representation through Hermitian ones. An almost trivial analysis of this process leads to some control on the possible signatures of Hermitian forms. The point is that the signature must be known *a priori* somewhere; the tempered representations are the starting point. To give a precise result, we need a definition.

Definition 1.4. Suppose X is a $(\mathfrak{g}, K)$ module of finite length endowed with a non-degenerate $\mathfrak{g}$-invariant Hermitian form $\langle\,,\rangle$. Fix a representation $\delta \in \hat{K}$, and a positive-definite Hermitian form on the space V_δ of δ. Then the finite dimensional space

$$X^\delta = \mathrm{Hom}_K(V_\delta, X)$$

acquires a non-degenerate Hermitian form; write $(p(\delta), q(\delta))$ for its signature. The *signature of* $\langle\,,\rangle$ is the pair of formal sums

$$\left(\sum_{\delta\in\hat{K}} p(\delta)\delta,\ \sum_{\delta\in\hat{K}} q(\delta)\delta\right).$$

(For notational convenience, we may sometimes regard the signature as the pair (p, q) of functions from $\hat{K}$ to $\mathbf{N}$.) Similarly, we define

$$m(\delta) = p(\delta) + q(\delta),$$

the *multiplicity of* δ. The *formal K-character of* X is

$$\Theta_K(X) = \sum_{\delta\in\hat{K}} m(\delta)\delta.$$

(or sometimes just the function m).

THEOREM 1.5. *Suppose X is a $(\mathfrak{g}, K)$ module of finite length endowed with a non-degenerate Hermitian form $\langle\,,\rangle$. Then there are finitely many tempered irreducible $(\mathfrak{g}, K)$ modules*

$$Z_1, \ldots, Z_p$$

and integers

$$r_1^+, \ldots, r_p^+, r_1^-, \ldots, r_p^-,$$

such that the signature of $\langle\,,\rangle$ is

$$\left(\sum_{i=1}^{p} r_i^+\Theta_K(Z_i),\ \sum_{j=1}^{p} r_j^-\,\Theta_K(Z_j)\right).$$

We may assume that all the Z_i have real infinitesimal character; in this case they, and the integers $r_i^\pm$, are unique (except for terms with $r_i^\pm = 0$).

If this result were proved by an effective computation, it would be an (unsatisfactory) determination of the unitary dual of G. However, it uses two non-computable invariants. The first is a Jantzen-type filtration of certain induced representations; we need to know the character of each subquotient. There is an obvious conjecture for this (at least if G is linear), and it may be provable by the methods used by Beilinson-Bernstein in category $\mathcal{O}$. The second invariants are some signs, associated with the fact that an indefinite form is not intrinsically preferable to its negative. I have no idea how to calculate these.

Here is an outline of the proof of Theorem 1.5. The basic idea is to understand how the signature of a one parameter analytic family $\langle\, ,\rangle_t$ of Hermitian forms can vary with the parameter. Obviously the signature does not change at all over intervals where the form is non-degenerate. At each point t_0 where $\langle\, ,\rangle_{t_0}$ is singular, Jantzen has defined in [J] a family of "residual" forms $\langle\, ,\rangle^k$ which make precise the idea of "signature on the subspace where $\langle\, ,\rangle_t$ vanishes to order k at t_0". Theorem 3.8 says that the change in signature of $\langle\, ,\rangle_t$ as one crosses t_0 is given by the sum of all the signatures of odd order residual forms. (This generalizes the fact that a real analytic function changes sign exactly at zeros of odd order. The generalization is not really much harder than that fact, at least in the presence of Jantzen's results.) Theorem 1.5 is deduced from Theorem 3.8 by a standard inductive argument in terms of the Langlands-Knapp-Zuckerman classification of irreducible representations.

In the context of Theorem 1.3, the proof of Theorem 1.5 and the results of [S-V] show that the tempered characters involved in the signature of (say) $\mathscr{R}Y$ must have their lowest K-types in the "bottom layer" of $\mathscr{R}Y$ (see [S-V] or Section 6). Consequently, $\mathscr{R}Y$ is unitary if and only if its form is definite on the bottom layer. If Y is one dimensional, there is only one K-type in the bottom layer, and we are done. In general, the ideas of [E-W] make possible a comparison of forms for Y and $\mathscr{R}Y$ on the bottom layer; and Theorem 1.3 follows.

As explained in [V3], Theorem 1.3 reduces the classification of unitary representations of G to a finite set F_0 of lowest K-types. (The proof of Theorem 1.5 gives an effective finite set $F_1 \subseteq \hat{K}$ such that any representation having a lowest K-type in F_0, and an invariant Hermitian form positive inside F_1, must be unitary. For fixed G, the determination of the unitary dual is therefore reduced to a finite computation.)

The paper is organized as follows. Section 2 recalls the general theory of Hermitian forms and irreducible $(\mathfrak{g}, K)$ modules. Section 3 is devoted to the proof of Theorem 1.5. Section 4 translates [S-V] into the language of Zuckerman's functor. Section 5 describes the Enright-Wallach construction of Hermitian forms on the representations in Theorem 1.3, and the proof of that theorem is in Section 6. An extension of Theorem 1.3, which is important for some applications, is given in Sections 7 and 8.

This paper grew out of joint work with Dan Barbasch, and I would like to thank him for many fruitful discussions. The results in Section 8 also owe a great deal to suggestions of Tony Knapp.

2. Preliminary results

Recall that G is a reductive group in Harish-Chandra's class. In particular, G may be non-linear; its Cartan subgroups may be nonabelian; lowest K-types may not have multiplicity one; and so on. Many of the results we need have published proofs only under additional hypotheses, but we will avoid most of those which cannot easily be extended to the present setting. All references to the bibliography should be interpreted with this in mind. General notation follows [V2]. In particular,

$$Z(\mathfrak{g}) = \text{center of } U(\mathfrak{g}). \tag{2.1)(a}$$

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then we write

$$\xi\colon Z(\mathfrak{g}) \to S(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})} \tag{2.1)(b}$$

for the Harish-Chandra isomorphism. The "infinitesimal character λ", for $\lambda \in \mathfrak{h}^*$, refers to the map

$$\xi_\lambda\colon Z(\mathfrak{g}) \to \mathbf{C} \tag{2.1)(c}$$

obtained by composing ξ with evaluation at λ.

Definition 2.2. Suppose $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ is a θ-stable Cartan subalgebra, and H its centralizer in $\mathfrak{g}$. Write

$$\begin{aligned} \mathfrak{t}_0 &= 1 \text{ eigenspace of } \theta \text{ on } \mathfrak{h}_0, \\ \mathfrak{a}_0 &= -1 \text{ eigenspace}, \\ T &= H \cap K, \\ A &= \exp(\mathfrak{a}_0), \end{aligned}$$

so that $H = T \times A$, a direct product. A *regular pseudocharacter* (or simply *regular character*) of H is a pair

$$\gamma = (\Gamma, \bar{\gamma})$$

subject to the following conditions:

R-1) $\Gamma \in \hat{H}$, and $\bar{\gamma} \in \mathfrak{h}^*$;

R-2) If α is an imaginary root of $\mathfrak{h}$ in $\mathfrak{g}$, then $\langle \alpha, \bar{\gamma} \rangle$ is a non-zero real number. Write Ψ for the unique system of positive imaginary roots such that

$$\langle \alpha, \bar{\gamma} \rangle > 0, \quad \text{for all } \alpha \in \Psi.$$

R-3) Write $\rho(\Psi)$ for half the sum of the roots in Ψ, and $\rho_c(\Psi)$ for half the sum of the compact ones. Then

$$d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi).$$

We write $\mathscr{P}(H)$ or $\mathscr{P}^G(H)$ for the set of regular characters of H. (In [S-V] and

[V2], the set $\mathscr{P}(H)$ was called $\hat{H}'$. That notation is inconvenient when several Cartan subgroups must be considered simultaneously.)

Definition 2.3. In the setting of Definition 2.2, write

$$M = \text{centralizer of } A \text{ in } G.$$

There is a unique (relative) discrete series $(\mathfrak{m}, M \cap K)$ module

$$X^M = X^M(\gamma)$$

satisfying

$$\mathfrak{a} \text{ acts in } X^M \text{ by } \bar{\gamma}|_{\mathfrak{a}},$$

$$X^M \text{ has a lowest } M \cap K\text{-type of highest weight } \Gamma|_T.$$

(We require X^M to be square integrable only on the commutator subgroup of M. On M itself, X^M need not even be unitary.)

Choose a real parabolic subgroup

$$P = MN$$

so that

$$\mathrm{Re}\langle \alpha, \bar{\gamma} \rangle \leq 0, \quad \text{for all } \alpha \in \Delta(\mathfrak{n}, \mathfrak{a}).$$

The *standard representation with parameter* γ is the $(\mathfrak{g}, K)$ module

$$X(\gamma) = X^G(\gamma) = \mathrm{Ind}_{MN}^G(X^M \times 1).$$

We use normalized induction, and consider only the K-finite vectors in the induced representation.

The $(\mathfrak{g}, K)$ module $X(\gamma)$ has finite length, and infinitesimal character $\bar{\gamma}$ (cf. (2.1)).

Definition 2.4. Suppose $H = TA$ is a θ-stable Cartan subgroup of G. A *limit pseudocharacter* (or *limit character*) of H is a triple

$$\gamma = (\Psi, \Gamma, \bar{\gamma})$$

with the following properties.

L-1) Ψ is a positive system for the imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\Gamma \in \hat{H}$, and $\bar{\gamma} \in \mathfrak{h}^*$.

L-2) If $\alpha \in \Psi$, the $\langle \alpha, \bar{\gamma} \rangle \geq 0$.

L-3) $d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi)$.

Write $\mathscr{P}_{\lim}(H)$ or $\mathscr{P}^G_{\lim}(H)$ for the set of limit characters of H; of course we may regard $\mathscr{P}(H)$ as a subset of $\mathscr{P}_{\lim}(H)$. A limit character is called *final* if it also satisfies the following conditions.

F-1) Suppose α is a simple root of Ψ, and $\langle \alpha, \bar{\gamma} \rangle = 0$. Then α is noncompact.

F-2) Suppose α is a real root of $\mathfrak{h}$ in $\mathfrak{g}$, and $\langle \alpha, \bar{\gamma} \rangle = 0$. Then α does not satisfy the parity condition ([V2], Definition 8.3.11).

Write $\mathscr{P}_f(H)$ or $\mathscr{P}^G_f(H)$ for the set of final limit characters on H.

Attached to every limit character γ is a *standard limit representation*

$$X(\gamma) = X^G(\gamma) \tag{2.5}$$

given by a generalization of Definition 2.3; it may be zero. It is induced from a limit of discrete series representations on $M = G^A$; a discussion may be found in [S-V]. The conditions (F-1) and (F-2) are chosen because of the following result.

PROPOSITION 2.6. *Suppose* $\gamma \in \mathscr{P}_{\text{lim}}(H)$ (*Definition* 2.4).

a) γ *satisfies* (F-1) $\Leftrightarrow X(\gamma) \neq 0$.

b) *If* γ *does not satisfy* (F-2), *then* $X(\gamma)$ *is a direct sum of standard limit representations attached to a more compact Cartan subgroup.*

c) *If* γ *is final, then* $X(\gamma)$ *has a unique irreducible submodule.*

This result (due in part to Hecht-Schmid and Milićič) is proved for connected linear G in [K-Z2]. The general case is similar. Whenever γ satisfies (F-1) of Definition 2.4, set

$$\begin{aligned}\overline{X}^G(\gamma) = \overline{X}(\gamma) &= \operatorname{soc}(X(\gamma)) \\ &= \text{largest completely reducible } (\mathfrak{g}, K) \text{ submodule of } X(\gamma).\end{aligned} \tag{2.7)(a}$$

Whenever γ does not satisfy (F-1), define

$$\overline{X}(\gamma) = 0. \tag{2.7)(b}$$

Definition 2.8. Suppose $\gamma \in \mathscr{P}_{\text{lim}}(H)$. The *Langlands subrepresentation* of $X(\gamma)$ is $\overline{X}(\gamma)$ (cf. (2.7)).

By Proposition 2.6(c), $\overline{X}(\gamma)$ is irreducible whenever γ is final.

THEOREM 2.9 (Langlands, Knapp-Zuckerman [L], [K-Z2]).

a) *Suppose* $\gamma_i \in \mathscr{P}_f(H_i)$ $(i = 1, 2)$. *Then* $\overline{X}(\gamma_1)$ *is equivalent to* $\overline{X}(\gamma_2)$ *if and only if* (H_1, γ_1) *is conjugate by K to* (H_2, γ_2).

b) *Suppose X is any irreducible* $(\mathfrak{g}, K)$ *module. Then there are a θ-stable Cartan subgroup H, and a* $\gamma \in \mathscr{P}_f(H)$, *such that X is equivalent to* $\overline{X}(\gamma)$.

The Knapp-Zuckerman result was formulated differently, and proved only for connected linear groups. This version is a consequence of the results on translation functors in [S-V] and [V2].

We need some generalities on Hermitian forms, all of which are entirely standard.

Definition 2.10. Suppose X is a $(\mathfrak{g}, K)$ module. The *Hermitian dual of* X, denoted X^h, is the $(\mathfrak{g}, K)$ module defined as follows. The underlying vector space of X^h is, as an abelian group, equal to the K-finite dual X^c of X ([V], Definition 8.5.1). For the purpose of the rest of the definition only, write

$$\phi: X^c \to X^h$$

for this identification. The scalar multiplication of $\mathbf{C}$ on X^h is

$$z \cdot \phi(x) = \phi(\bar{z} \cdot x) \qquad (x \in X^c, z \in \mathbf{C}).$$

The action of $\mathfrak{g}$ is

$$(U + iV) \cdot \phi(x) = \phi((U - iV) \cdot x) \qquad (U, V \in \mathfrak{g}_0, x \in X^c).$$

Finally, K acts exactly as in X^c, via the isomorphism ϕ. Write

$$\langle\, ,\rangle: X \times X^h \to \mathbf{C}$$

for the pairing induced by ϕ and that on $X \times X^c$. It satisfies

a) $\langle\, ,\rangle$ is linear in the first variable, and conjugate linear in the second:

$$\langle x, ay + bz\rangle = \bar{a}\langle x, y\rangle + \bar{b}\langle x, z\rangle$$

for $a, b \in \mathbf{C}$, $x \in X$, and $y, z \in X^h$.

b) For $U, V \in \mathfrak{g}_0$, $x \in X$, and $y \in X^h$,

$$\langle (U + iV)x, y\rangle = \langle x, (-U + iV)y\rangle$$

c) For $k \in K$, $x \in X$, and $y \in X^h$,

$$\langle k \cdot x, y\rangle = \langle x, k^{-1}y\rangle.$$

Definition 2.11. Suppose X is a $(\mathfrak{g}, K)$ module. An (*invariant*) *Hermitian form* on X is a pairing

$$\langle\, ,\rangle: X \times X \to \mathbf{C}$$

satisfying (a)–(c) of Definition 2.10, and also

d) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

The *radical of* $\langle\, ,\rangle$ is

$$\mathrm{Rad}(\langle\, ,\rangle) = \{x \in X \mid \langle x, X\rangle = 0\}.$$

We say $\langle\, ,\rangle$ is *non-degenerate* if its radical is $\{0\}$.

PROPOSITION 2.12. *Suppose X is an irreducible $(\mathfrak{g}, K)$ module. Then X admits a non-zero invariant Hermitian form if and only if X is isomorphic to X^h. In this case any such form is non-degenerate, and any two differ by multiplication by a real constant.*

This is standard.

Definition 2.13. Suppose $\gamma = (\Psi, \Gamma, \bar{\gamma}) \in \mathscr{P}_{\mathrm{lim}}(H)$. Define a new limit character

$$\gamma^h = (\Psi^h, \Gamma^h, \bar{\gamma}^h)$$

by

$$\Psi^h = \Psi,$$
$$\Gamma^h = \text{Hermitian dual of the representation } \Gamma \text{ of } H,$$
$$\bar{\gamma}^h = -\ (\text{complex conjugate of } \bar{\gamma}).$$

(Recall that the bar here is part of the notation, and does *not* mean complex conjugate.)

PROPOSITION 2.14. *Suppose $\gamma \in \mathscr{P}_{\text{lim}}(H)$. Choose $P = MN$ as in Definition 2.3. Then the opposite parabolic $P^{\text{op}} = MN^{\text{op}}$ satisfies the same conditions for γ^h. We have*

$$X(\gamma) = \operatorname{Ind}_P^G(X^M(\gamma) \otimes 1),$$
$$X(\gamma)^h = \operatorname{Ind}_P^G(X^M(\gamma^h) \otimes 1),$$
$$X(\gamma^h) = \operatorname{Ind}_{P^{\text{op}}}^G(X^M(\gamma^h) \otimes 1).$$

Consequently $X(\gamma^h)$ and $X(\gamma)^h$ have the same irreducible composition factors and multiplicities.

Again this is standard and easy.

COROLLARY 2.15. *Suppose $\gamma \in \mathscr{P}_f(H)$ (Definition 2.4). Then $\overline{X}(\gamma)$ admits a non-degenerate Hermitian form if and only if γ is conjugate to γ^h by an element of $W(G, H)$.*

The proof of Theorem 1.5 depends on having some control on how the Hermitian form on $\overline{X}(\gamma)$ varies as γ varies continuously.

Definition 2.16. Suppose $H = TA$ is a θ-stable Cartan subgroup, and

$$\gamma = (\Psi, \Gamma, \bar{\gamma}) \in \mathscr{P}_{\text{lim}}(H).$$

For every real number t, define

$$\gamma_t = (\Psi, \Gamma_t, \bar{\gamma}_t)$$

with

$$\Gamma_t|_T = \Gamma|_T,$$
$$\Gamma_t|_A = (\Gamma|_A)^t,$$
$$\bar{\gamma}_t|_{\mathfrak{t}} = \bar{\gamma}|_{\mathfrak{t}},$$
$$\bar{\gamma}_t|_{\mathfrak{a}} = t\bar{\gamma}|_{\mathfrak{a}}.$$

THEOREM 2.17. *Suppose $\gamma \in \mathscr{P}_f(H)$ (Definition 2.4), and that $\overline{X}(\gamma)$ admits a non-zero invariant Hermitian form. Choose P as in Definition 2.3, and define*

$$I_P(\gamma_t) = \operatorname{Ind}_P^G(X^M(\gamma_t) \otimes 1).$$

(Thus $I_P(\gamma_t) = X(\gamma_t)$ for $t \geq 0$.) Regard all the $I_P(\gamma_t)$ as realized on a fixed space V; so all the Hermitian duals $I_P(\gamma_t)^h$ are realized on a fixed space V^h. Then there is an analytic family

$$\{A(t) | t \in \mathbf{R}\}$$

of maps from V^h to V, with the following properties:
a) *For fixed t, $A(t)$ is a $(\mathfrak{g}, K)$ module map from $I_P(\gamma_t)^h$ to $I_P(\gamma_t)$;*
b) *$A(t)$ is not zero for any t;*
c) *$A(0)$ is an isomorphism;*
d) *There is a non-zero Hermitian form $\langle\,,\rangle_t$ defined on $I_P(\gamma_t)^h$ by the formula*

$$\langle x, y\rangle_t = \langle x, A(t)y\rangle;$$

e) *The radical of $\langle\,,\rangle_t$ is the kernel of $A(t)$; and*
f) *For $t > 0$, the image of $A(t)$ is precisely $\overline{X}(\gamma_t)$.*
Consequently, $\langle\,,\rangle_1$ may be identified (after dividing $X(\gamma)^h$ by the radical) with a non-zero invariant Hermitian form on $\overline{X}(\gamma)$.

This is a consequence of the results in [K-S]; see also [K-Z1]. Usually $A(t)$ is proved only to be meromorphic. There are two ways around this. First, we will need analyticity only for $t \geq 0$, where it is standard. Alternatively, one can remember that $A(t)$ is usually taken meromorphic in order to have a nice functional equation. Since we do not need such an equation, we can easily eliminate poles and zeroes by multiplying $A(t)$ by an appropriate meromorphic function of t.

A few remarks about the statement are in order. The space of $I_P(\gamma_t)$ consists of the K-finite maps from K to $X^M(\gamma_t)$ transforming properly under $M \cap K$. All the $X^M(\gamma_t)$ may be taken on a fixed space, with a fixed action of $M \cap K$; only the action of A changes. This is why V exists. Notice also that the action of K is independent of t. Thus

$$V = \bigoplus_{\delta \in \hat{K}} V(\delta), \tag{2.18}$$

$$A(t)\colon V(\delta)^h \to V(\delta).$$

So $A(t)$ is just a direct sum of maps between finite dimensional vector spaces, and it is these which we are requiring to be analytic: There is no functional analysis involved.

Although we make no use of the fact, it is worth observing that $A(t)$ is unique up to multiplication by a nowhere zero real valued analytic function of t. This follows from the fact that $X(\gamma_t)$ is irreducible for t outside some discrete set.

3. The Jantzen filtration and the proof of Theorem 1.5

Definition 3.1. Let E be a finite dimensional complex vector space, and $\langle\, ,\rangle_t$ an analytic family of Hermitian forms on E defined for small real t. Assume that $\langle\, ,\rangle_t$ is non-degenerate for sufficiently small non-zero t. The *Jantzen filtration* of E is the sequence of subspaces

$$E = E_0 \supset E_1 \supset \cdots \supset E_N = \{0\},$$

defined as follows. Fix $n \geq 0$. Then an element $e \in E$ belongs to E_n if and only if for some $\varepsilon > 0$ there is an analytic function

$$f_e: (-\varepsilon, \varepsilon) \to E,$$

with the following properties:
a) $f_e(0) = e$,
b) $\langle f_e(t), e'\rangle_t$ vanishes at least to order n at $t = 0$, for any $e' \in E$.
(It is equivalent to consider only polynomial functions f_e.) Suppose e, e' are in E_n; choose $f_e, f_{e'}$ accordingly. Set

$$\langle e, e'\rangle^n = \lim_{t\to 0} \frac{1}{t^n}\langle f_e(t), f_{e'}(t)\rangle_t$$

(which is easily seen to be independent of the choices of f_e and $f_{e'}$).

THEOREM 3.2 (Jantzen [J], 5.1). *In the setting of Definition* 3.1, $\langle\, ,\rangle^n$ *is a Hermitian form on* E_n *with radical exactly equal to* E_{n+1}. *In particular,*
a) $\mathrm{Rad}\langle\, ,\rangle_0 = E_1$; *and*
b) $\langle\, ,\rangle^n$ *is a non-degenerate Hermitian form on* E_n/E_{n+1}.

Proof. Let A be the ring of germs of analytic functions at $t = 0$. Germs of analytic functions from $(-\varepsilon, \varepsilon)$ to E may be regarded as elements of the A module

$$M = A \underset{\mathbf{C}}{\otimes} E.$$

The family of Hermitian forms on E may be regarded as a single pairing

$$\langle\, ,\rangle: M \times M \to A,$$
$$\langle f, g\rangle = \langle f(t), g(t)\rangle_t.$$

With these definitions, we are essentially in the setting of [J]. This pairing is Hermitian instead of symmetric, but Jantzen's arguments still apply. Q.E.D.

PROPOSITION 3.3. *In the setting of Theorem* 3.2, *let* (p_n, q_n) *be the signature of* $\langle\, ,\rangle^n$.

a) *For small positive* t, $\langle\,,\rangle_t$ *has signature*

$$\left(\sum_{n=0}^{N} p_n,\ \sum_{n=0}^{N} q_n\right)$$

b) *For small negative* t, $\langle\,,\rangle_t$ *has signature*

$$\left(\sum_{n \text{ even}} p_n + \sum_{n \text{ odd}} q_n,\ \sum_{n \text{ odd}} p_n + \sum_{n \text{ even}} q_n\right)$$

Proof. For (a), it suffices to produce (for small positive t) a subspace F_t of E of dimension $\sum p_n$ on which $\langle\,,\rangle_t$ is positive definite. (For if we replace $\langle\,,\rangle_t$ by its negative, we accomplish the same thing for the negative part of the signature. These two things together give (a).) For each n, choose

$$\{e_1^n, \ldots, e_{p_n}^n\} \subseteq E_n \tag{3.4)(a}$$

in such a way that

$$\langle e_i^n, e_j^n \rangle^n = \delta_{ij}. \tag{3.4)(b}$$

Write

$$\{f_i^n\colon (-\varepsilon, \varepsilon) \to E\} \tag{3.4)(c}$$

for the corresponding functions from Definition 3.1. By that definition and (3.4)(b),

$$\langle f_i^n(t), f_j^m(t) \rangle_t = \begin{cases} t^n(\delta_{ij} + t(\ldots)) & \text{if } m = n \\ t^{\max(m,n)}(\ldots) & \text{always.} \end{cases} \tag{3.5}$$

In each case, "$\ldots$" represents an analytic function of t. Set

$$h_i^n(t) = t^{-n/2} f_i(t) \qquad (t > 0).$$

Then (3.5) becomes

$$\langle h_i^n(t), h_j^m(t) \rangle_t = \begin{cases} \delta_{ij} + t(\ldots) & \text{if } m = n \\ t^{|m-n|/2}(\ldots) & \text{always.} \end{cases}$$

Now define F_t to be the span of the various $h_i^n(t)$. Then the matrix of $\langle\,,\rangle_t$ on F_t is

$$I + t^{\frac{1}{2}}(\ldots).$$

It follows that for small t, the $h_i^n(t)$ are linearly independent, and $\langle\,,\rangle_t$ is positive definite on F_t.

For (b), apply (a) to the family of forms

$$\langle\,,\rangle_t' = \langle\,,\rangle_{-t}. \qquad \text{Q.E.D.}$$

COROLLARY 3.6. *In the setting of Proposition* 3.3, *write* (p, q) *for the signature of* $\langle\,,\rangle$ *for small positive* t, *and* (p', q') *for the signature for small negative* t. *Then*

$$p = p' + \sum_{n \text{ odd}} p_n - \sum_{n \text{ odd}} q_n,$$

$$q = q' + \sum_{n \text{ odd}} q_n - \sum_{n \text{ odd}} p_n.$$

We express this by saying that $\langle\,,\rangle_t$ changes sign on the odd levels of the Jantzen filtration as one passes through the singularity at zero.

Definition 3.7. Suppose $\gamma \in \mathscr{P}_f(H)$ (Definition 2.4), and that $\overline{X}(\gamma)$ admits a non-zero invariant Hermitian form. Define $I_P(\gamma_t)$ as in Theorem 2.17, so that there is an analytic family $\langle\,,\rangle_t$ of Hermitian forms on V^h (notation as in Theorem 2.17). The *Jantzen filtration of* $X(\gamma)^h$ (the space of which is V^h) is

$$X(\gamma)_0^h = X(\gamma)^h \supset X(\gamma)_1^h \supset \cdots \supset X(\gamma)_N^h = \{0\},$$

defined using $\langle\,,\rangle_t$ at $t = 1$ (instead of $t = 0$) as in Definition 3.1. Explicitly, $X(\gamma)_n^h$ consists of those elements $x \in X(\gamma)^h$ with the following property: There is an analytic function

$$f_x\colon (1 - \varepsilon, 1 + \varepsilon) \to V^h$$

satisfying three conditions:
a) f_x takes values in a fixed finite dimensional subspace of V^h;
b) $f_x(1) = x$;
c) For all $y \in V^h$, the function $\langle f_x(t), y\rangle_t$ vanishes at least to order n at $t = 1$. (Again, we could consider only polynomial functions f_x.) On each subquotient $X(\gamma)_n^h/X(\gamma)_{n+1}^h$, we have the nondegenerate Hermitian form $\langle\,,\rangle^n$ defined as in Definition 3.1 (cf. Theorem 3.2). The *Jantzen filtration of* $X(\gamma)$ is

$$X(\gamma)^N = X(\gamma) \supset X(\gamma)^{N-1} \supset \cdots \supset X(\gamma)^0 = \{0\};$$

here

$$\begin{aligned} X(\gamma)^n &= \left\{x \in X(\gamma) | \langle x, X(\gamma)_n^h\rangle = 0\right\} \\ &= \left(X(\gamma)_n^h\right)^{\perp}. \end{aligned}$$

(The pairing $\langle\,,\rangle$ is that between $X(\gamma)$ and its Hermitian dual $X(\gamma)^h$.) The form $\langle\,,\rangle$ provides an identification

$$X(\gamma)_n^h/X(\gamma)_{n+1}^h \cong \left[X(\gamma)^{n+1}/X(\gamma)^n\right]^h$$

(see Definition 2.10)). Therefore $\langle\,,\rangle_n$ may be transferred to a non-degenerate

Hermitian form on $X(\gamma)^{n+1}/X(\gamma)^n$; we call it $\langle\, ,\rangle^n$ as well.

THEOREM 3.8. *Suppose $\gamma \in \mathscr{P}_f(H)$ (Definition 2.4), and $\overline{X}(\gamma)$ admits a non-zero invariant Hermitian form. Then the Jantzen filtration*

$$X(\gamma) = X(\gamma)^N \supset X(\gamma)^{N-1} \supset \cdots \supset X(\gamma)^0 = \{0\}$$

is a filtration by $(\mathfrak{g}, K)$ submodules.

a) *$X(\gamma)^1$ is the Langlands subrepresentation $\overline{X}(\gamma)$.*

b) *The non-degenerate Hermitian form $\langle\, ,\rangle_n$ on $X(\gamma)^{n+1}/X(\gamma)^n$ is invariant (Definition 2.11). Write (p_n, q_n) for its signature (Definition 1.4; here p_n and q_n are functions from $\hat{K}$ to $\mathbf{N}$.)*

c) *For ε sufficiently small, the non-degenerate invariant Hermitian form $\langle\, ,\rangle_{1+\varepsilon}$ on $X(\gamma_{1+\varepsilon})$ (Theorem 2.17) has signature*

$$\left(\sum_{n=0}^{N-1} p_n,\ \sum_{n=0}^{N-1} q_n\right).$$

d) *For ε sufficiently small, the non-degenerate invariant Hermitian form $\langle\, ,\rangle_{1-\varepsilon}$ on $X(\gamma_{1-\varepsilon})$ has signature*

$$\left(\sum_m (p_{2m} + q_{2m+1}),\ \sum_m (p_{2m+1} + q_{2m})\right).$$

Proof. To show that $X(\gamma)^n$ is $\mathfrak{g}$-stable, it suffices to show that $X(\gamma)^h_n$ is. So fix $x \in X(\gamma)^h_n$, and $U \in \mathfrak{g}$. Let f_x be as in Definition 3.7, and define

$$f_{U\cdot x}(t) = U \cdot (f_x(t));$$

the action on the right is in $X(\gamma_t)^h$. It is a triviality to check that $f_{U\cdot x}$ satisfies (a)–(c) of Definition 3.7 for $U \cdot x$ instead of x. Therefore $U \cdot x$ belongs to $X(\gamma)^h_n$. The K invariance is similar. Part (a) follows from Theorem 2.17(e) and (f), with Theorem 3.2(a). Part (b) may be proved in the same way as the first claim of the theorem. Parts (c) and (d) follow from Proposition 3.3. Q.E.D.

The proof of Theorem 1.5 from Theorem 3.8 is entirely standard bookkeeping, which experts will probably wish to omit; they should proceed directly to Section 4.

LEMMA 3.9. *Suppose X is a $(\mathfrak{g}, K)$ module of finite length admitting a non-degenerate invariant Hermitian form of signature $(\sum p(\delta)\delta, \sum q(\delta)\delta)$ (Definition 1.4). Then we can find irreducible $(\mathfrak{g}, K)$ modules $\{X_i\}_{i=1}^r$, $\{Y_j\}_{j=1}^s$ (not necessarily distinct), with the following properties:*

a) *In the Grothendieck group of finite length $(\mathfrak{g}, K)$ modules,*

$$X = \sum_{i=1}^r X_i + \sum_{j=1}^s \left(Y_j + Y_j^h\right).$$

b) *Each* X_i *admits a non-degenerate invariant Hermitian form* $\langle\, ,\rangle_i$. *Write* (p_i, q_i) *for its signature; and write* m_j *for the K-character of* Y_j (*Definition* 1.4).

c)
$$p(\delta) = \sum_{i=1}^{r} p_i(\delta) + \sum_{j=1}^{s} m_j(\delta),$$
$$q(\delta) = \sum_{i=1}^{r} q_i(\delta) + \sum_{j=1}^{s} m_j(\delta).$$

Proof. We proceed by induction on the length of X. If $X = \{0\}$, we take $r = s = 0$; so suppose $X \neq \{0\}$. Then we can find an irreducible submodule $Z \subset X$. There are two possibilities.

Case I. *The restriction of* $\langle\, ,\rangle$ *to* Z *is non-zero*. By Proposition 2.12, $\langle\, ,\rangle|_Z$ is non-degenerate. Set

(3.10)(a) $$Z^\perp = \{x \in X | \langle x, Z\rangle = 0\}.$$

It follows that

(3.10)(b) $$\langle\, ,\rangle|_{Z^\perp} \quad \text{is nondegenerate},$$
$$Z \cap Z^\perp = \{0\},$$
$$X = Z \oplus Z^\perp.$$

Choose $\{X_i\}_{i=1}^{r'}$, $\{Y_j\}_{j=1}^{s'}$ for $Z^\perp$ (by inductive hypothesis). Let $s = s'$, $r = r' + 1$, and
$$X_r = Z, \langle\, ,\rangle_r = \langle\, ,\rangle|_Z.$$
The various conclusions of the lemma follow from (3.10) and the conclusions for $Z^\perp$.

Case II. *The restriction of* $\langle\, ,\rangle$ *to* Z *is zero*. The inclusion

(3.11)(a) $$Z \to X$$

induces a surjection

(3.11)(b) $$X^h \twoheadrightarrow Z^h.$$

The form $\langle\, ,\rangle$ induces an isomorphism

(3.12)(a) $$\phi\colon X \to X^h,$$

defined by

(3.12)(b) $$\langle x, \phi(y)\rangle = \langle x, y\rangle \qquad (x, y \in X);$$

the pairing on the left is that of Definition 2.10, and that on the right is the Hermitian form on X. Composing ϕ with the map of 3.11(b), we get a surjection

(3.13) $$\psi\colon X \twoheadrightarrow Z^h,$$
$$\langle z, \psi x\rangle = \langle z, x\rangle \qquad (z \in Z, x \in X).$$

The pairing on the left is that of Z and Z^h, and the one on the right is the Hermitian form on X. By the hypothesis of Case II,

$$Z^{\perp} = \ker \psi \supset Z. \tag{3.14}$$

Define

$$V = (\ker \psi)/Z. \tag{3.15}$$

Essentially by definition,

$$\mathrm{Rad}(\langle\,,\rangle|_{\ker\psi}) = Z; \tag{3.16}$$

so $\langle\,,\rangle$ induces a non-degenerate form $\langle\,,\rangle_V$ on V. Because of (3.13) and (3.15),

$$X = V + Z + Z^h \tag{3.17}$$

in the Grothendieck group. Choose $\{X_i\}_{i=1}^{r'}$, $\{Y_j\}_{j=1}^{s'}$ for V and $\langle\,,\rangle_V$ by inductive hypothesis. Put $r = r'$, $s = s' + 1$, and $Y_s = Z$. Parts (a) and (b) of the lemma follow from (3.17) and the inductive hypothesis. For (c), we use:

SUBLEMMA 3.18. *Let E be a finite dimensional vector space carrying a non-degenerate Hermitian form $\langle\,,\rangle$ of signature (p,q). Let S be a totally isotropic subspace of E (that is, $\langle\,,\rangle|_S$ is zero), and set*

$$S^{\perp} = \{e \in E \mid \langle e, S\rangle = 0\} \supseteq S.$$

a) *The radical of $\langle\,,\rangle|_{S^{\perp}}$ is S; so $\langle\,,\rangle$ induces a non-degenerate Hermitian form $\langle\,,\rangle_F$ on $F = S^{\perp}/S$.*

b) *Write (p',q') for the signature of $\langle\,,\rangle_F$, and m for the dimension of S. Then*

$$p = p' + m, \qquad q = q' + m.$$

This is elementary. Part (c) of the lemma follows, if we apply the sublemma to each K-isotypic subspace $X(\delta)$. Q.E.D.

LEMMA 3.1. *Suppose X is a $(\mathfrak{g}, K)$ module of finite length having a non-degenerate Hermitian form $\langle\,,\rangle$. Assume that the conclusion of Theorem 1.5 holds for every irreducible composition factor of X which admits a non-zero Hermitian form. Then Theorem 1.5 is true for X itself.*

Proof. By Lemma 3.9, it suffices to show that the K-character of any irreducible $(\mathfrak{g}, K)$ module is an integral combination of K-characters of tempered representations. This is well known ([V2], Proposition 6.6.7). Q.E.D.

To set up the inductive argument proving Theorem 1.5, we need a few more standard ideas.

Definition 3.20 ([V2], 5.4.11). Suppose $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ is a θ-stable Cartan subalgebra of $\mathfrak{g}_0$. Then

(a) $$\mathfrak{h}^* = [\mathfrak{a}_0^* \oplus (i\mathfrak{t}_0)^*] \oplus [i\mathfrak{a}_0^* \oplus \mathfrak{t}_0^*].$$

If $\mu \in \mathfrak{h}^*$, we define

(b) $\qquad \mathrm{RE}\,\mu = \text{projection of } \mu \text{ on first factor of (a)},$

an element of $\mathfrak{a}_0^* + i\mathfrak{t}_0^*$; it is called the *canonical real part of* μ.

LEMMA 3.21 ([V2], Lemma 5.4.12). *In the setting of Definition* 3.20, *the infinitesimal character* $\xi_{\mathrm{RE}(\mu)}$ (*cf.* (2.1)) *depends only on* ξ_μ. *The standard bilinear form* $\langle\, ,\rangle$ *is positive definite on* $\mathfrak{a}_0^* \oplus i\mathfrak{t}_0^*$.

Definition 3.22. Suppose $\gamma \in \mathscr{P}_{\mathrm{lim}}(H)$. Write

$$\begin{aligned}\bar{\gamma} &= (\lambda, \nu) \in \mathfrak{t}^* + \mathfrak{a}^* \\ &= (\lambda, (\mathrm{Re}\,\nu, \mathrm{Im}\,\nu)) \in i\mathfrak{t}_0^* + (\mathfrak{a}_0^* + i\mathfrak{a}_0^*).\end{aligned}$$

The *lambda norm of* γ is

$$\|\gamma\|_{\mathrm{lambda}} = \langle \lambda, \lambda \rangle^{\frac{1}{2}}.$$

Notice that

$$\|\gamma\|_{\mathrm{lambda}} \leq \|\mathrm{RE}\,\bar{\gamma}\|.$$

Definition 3.23 ([V2], Definition 5.4.1). Suppose $\delta \in \hat{K}$. Choose a highest weight μ of δ, and construct from it a parameter λ by [V1], Proposition 4.1 (cf. [V2], Proposition 5.33). The *lambda norm of* δ is

$$\|\delta\|_{\mathrm{lambda}} = \langle \lambda, \lambda \rangle^{\frac{1}{2}}.$$

If X is any $(\mathfrak{g}, K)$ module, the representations of K having minimal lambda norm among those occurring in X are called the *lambda-lowest K-types of* X. If $\gamma \in \mathscr{P}_{\mathrm{lim}}(H)$, define $A^G(\gamma) = A(\gamma) = $ set of lambda-lowest K-types of $X(\gamma) \subseteq \hat{K}$.

THEOREM 3.24 ([V1] or [V2], Chapter 6). *Suppose* $\gamma \in \mathscr{P}_{\mathrm{lim}}(H)$, *and* $\delta \in A(\gamma)$ (*Definition* 3.23).

a) $\|\gamma\|_{\mathit{lambda}} = \|\delta\|_{\mathit{lambda}}$.

b) *The δ-primary subspace* $X(\gamma)(\delta)$ *is contained in the Langlands subrepresentation* $\bar{X}(\gamma)$ (*cf.* (2.7)).

c) $\sum_{\delta \in A(\gamma)} X(\gamma)(\delta)$ *generates* $\bar{X}(\gamma)$.

COROLLARY 3.25. *Suppose* $\gamma \in \mathscr{P}_{\mathrm{lim}}(H)$, $\gamma' \in \mathscr{P}_f(H')$, *and* $\bar{X}(\gamma')$ *is a composition factor of* $X(\gamma)$. *Then*

$$\|\gamma'\|_{\mathit{lambda}} \geq \|\gamma\|_{\mathit{lambda}}.$$

If equality holds, then $\bar{X}(\gamma') \subseteq \bar{X}(\gamma)$.

COROLLARY 3.26. *Suppose X is an irreducible* $(\mathfrak{g}, K)$ *module of infinitesimal character* μ, *and δ is a lambda-lowest K-type of X. Then*

$$\|\mathrm{RE}\,\mu\| \geq \|\delta\|_{\mathit{lambda}}.$$

Of course, this uses the remark after Definition 3.22.

Proof of Theorem 1.5. We postpone for a moment the uniqueness assertion. Fix some large real number T. We will prove Theorem 1.5 for all representations whose composition factors have infinitesimal characters μ satisfying

$$\|\mathrm{RE}\,\mu\| \leq T. \tag{3.27}$$

Since T is arbitrary, this will suffice. We proceed by *downward* induction on the lambda norm of a lambda-lowest K-type of X; this is permitted by (3.27) and Corollary 3.26. By Lemma 3.19, we may as well suppose X is irreducible; say

$$X = \overline{X}(\gamma), \qquad \gamma \in \mathscr{P}_f(H). \tag{3.28}$$

We work with the tools of Theorem 2.17. Define

$$\{t_1 < \cdots < t_{r-1}\} = \{t \in (0,1) | A(t) \text{ is not an isomorphism}\}, \qquad t_0 = 0, \quad t_r = 1. \tag{3.29}$$

For $1 \leq l \leq r$, define

$$(p^l, q^l) = \text{signature of } \langle\, ,\rangle_s,\ s \in (t_{l-1}, t_l). \tag{3.30}$$

For each $l, 1 \leq l \leq r$, the representation $X(\gamma_{t_l})$ has a Jantzen filtration

$$\{0\} = X(\gamma_{t_l})^0 \subseteq X(\gamma_{t_l})^1 \subseteq \cdots. \tag{3.31}$$

The nth subquotient $X(\gamma_{t_l})^{n+1}/(X(\gamma_{t_l})^n$ carries a non-degenerate Hermitian form $\langle\, ,\rangle_n^l$, of signature

$$(p_n^l, q_n^l). \tag{3.32}$$

By Theorem 3.8, we have the following facts:

$$\overline{X}(\gamma_{t_l}) = X(\gamma_{t_l})^1 \qquad (l = 1, \ldots, r), \tag{3.33}$$

$$\text{(3.34)(a)} \quad p^{l+1} = \sum_n p_n^l, \qquad \text{(3.34)(b)} \quad q^{l+1} = \sum_n q_n^l, \qquad (l = 1, \ldots, r-1)$$

$$\text{(3.35)(a)} \quad p^l = \sum_m \left(p_{2m}^l + q_{2m+1}^l\right), \qquad \text{(3.35)(b)} \quad q^l = \sum_m \left(p_{2m+1}^l + q_{2m}^l\right). \qquad (l = 1, \ldots, r)$$

Combining (3.34) and (3.35), we get recursion formulas

$$\begin{aligned} p^{l+1} &= p^l + \sum_m \left(p_{2m+1}^l - q_{2m+1}^l\right), \\ q^{l+1} &= q^l + \sum_m \left(q_{2m+1}^l - p_{2m+1}^l\right). \end{aligned} \qquad (l = 1, \ldots, r-1) \tag{3.36}$$

Finally, (3.35) gives

$$p_0^r = p^r - \sum_m q_{2m+1}^r - \sum_{m>0} p_{2m}^r, \tag{3.37}$$

$$q_0^r = q^r - \sum_m p_{2m+1}^r - \sum_{m>0} p_{2m}^r.$$

Now (3.36) and (3.37) together give

$$\begin{aligned}\text{signature of form on } \overline{X}(\gamma) &= (p_0^r, q_0^r)\\ &= (p_1, q_1) + \sum_{l=1}^{r-1} \sum_m (p_{2m+1}^l, q_{2m+1}^l)\\ &\quad - \sum_{l=1}^{r} \sum_m (q_{2m+1}^l, p_{2m+1}^l) - \sum_{m>0} (p_{2m}^r, q_{2m}^r).\end{aligned} \tag{3.38}$$

To prove Theorem 1.5, it suffices to show that each term on the right in this formula has the desired form. Because

$$\|\mathrm{RE}\,\bar{\gamma}_t\| \leq \|\mathrm{RE}\,\bar{\gamma}\| \leq T \qquad (|t| \leq 1),$$

the inductive hypothesis tells us that Theorem 1.5 holds for all the various

$$\overline{X}(\gamma_{t_l})^{2m+2}/\overline{X}(\gamma_{t_l})^{2m+1} \qquad (l = 1, \ldots, r)$$

and

$$\overline{X}(\gamma)^{2m+1}/X(\gamma)^{2m} \qquad (m > 0).$$

This takes care of all terms but (p_1, q_1) in (3.38). That term is the signature of $\langle\,,\rangle_s$ for small positive s. By Theorem 2.17(c), it is also the signature of $\langle\,,\rangle_0$, a nondegenerate Hermitian form on

$$Y = \mathrm{Ind}_P^G\big(X^L(\gamma_0^h) \otimes 1\big).$$

Since γ_0^h is trivial on A (Definition 2.16), Y is a direct sum of irreducible unitary tempered representations $Z_1 \ldots Z_l$. The signature of $\langle\,,\rangle_0$ is therefore of the form

$$\Big(\sum_{i \in S} \Theta_K(Z_i),\ \sum_{j \notin S} \Theta_K(Z_j)\Big)$$

for some subset S of $\{1, \ldots, l\}$ (cf. Lemma 3.9). So every term on the right of (3.38) has the desired form. Q.E.D.

This proof actually produces tempered representations with infinitesimal character μ satisfying $\mu = \mathrm{RE}\,\mu$; this is what is meant by "real infinitesimal character" in the statement.

We turn now to the uniqueness statement.

LEMMA 3.39. *Suppose* $\gamma \in \mathscr{P}_f(H)$; *write*

$$\bar{\gamma} = (\lambda, \nu) \in \mathfrak{t}^* + \mathfrak{a}^*.$$

Then

a) $\overline{X}(\gamma)$ *is tempered if and only if* $\operatorname{Re}\nu = 0$,

b) $\overline{X}(\gamma)$ *has real infinitesimal character if and only if* $\nu = \operatorname{Re}\nu$,

c) $\overline{X}(\gamma)$ *is tempered and has real infinitesimal character if and only if* $\nu = 0$.

This is obvious.

THEOREM 3.40. *Suppose* $\gamma \in \mathscr{P}_f(H)$, *and* $\overline{X}(\gamma)$ *is tempered with real infinitesimal character. Then* $\overline{X}(\gamma)$ *has a unique lambda-lowest K-type* $\delta(\gamma)$:

a) $$A(\gamma) = \{\delta(\gamma)\}.$$

This K-type occurs with multiplicity one in $X(\gamma)$ *and* $\overline{X}(\gamma)$. *If* $\gamma' \in \mathscr{P}_f(H')$ *is of the same type, then*

b) $$\delta(\gamma) = \delta(\gamma') \Leftrightarrow (H,\gamma) \textit{ is conjugate to } (H',\gamma').$$

Finally,

c) $$X(\gamma) = \overline{X}(\gamma).$$

Sketch of proof. Suppose first that G is linear, with abelian Cartan subgroups. Then the result is a consequence of Theorem 2.9 and the results of Sections 6.5 and 6.6 of [V2] (notably Corollary 6.5.14). Next, suppose G is connected, but possibly non-linear. Then the result follows from Theorem 2.9 and [V1], Theorem 1.4. In general, one combines these two arguments. The main point for (a) and the multiplicity-one statement is that the assumptions on γ mean that all the real roots are good ([V2], Section 6). Consequently the R-group is trivial, and the delicate part of the multiplicity one proof disappears. Part (c) is immediate from (a) and Corollary 3.25. For (b), let $\gamma^0 \in \mathscr{P}(H^0)$ be the unique regular character such that $\overline{X}(\gamma) \subseteq \overline{X}(\gamma^0)$. If we write

$$\overline{\gamma^0} = (\lambda^0, \nu^0),$$

then γ^0 is determined by the two properties

$$\nu^0 = 0, \qquad \delta(\gamma) \in A(\gamma^0)$$

(Definition 3.23). Therefore

$$X(\gamma^0) \supseteq \overline{X}(\gamma), \overline{X}(\gamma'). \tag{3.41}$$

Let 2^n be the order of the R-group for γ^0. The method of [V1], Section 6, shows that we can write

(3.42)(a) $$n = p + 2q,$$

in such a way that

(3.42)(b) $A(\gamma^0)$ has 2^p elements, each occurring 2^q times in $X(\gamma^0)$.

Because of (3.41), part (b) of the theorem will follow if we can show that the 2^q composition factors of $X(\gamma^0)$ containing $\delta(\gamma)$ are all equivalent. By (3.42), this amounts to

$$\dim \operatorname{Hom}_{\mathfrak{g},K}\big(X(\gamma^0), X(\gamma^0)\big) \geq 2^n. \tag{3.43}$$

So it suffices to show that the standard intertwining operators on $X(\gamma^0)$ (attached to elements of the R-group) are linearly independent. For this, we may as well restrict to the identity component of G (which can only enlarge the R group). Then the linear independence was proved by Knapp-Stein in the linear case. For non-linear connected groups, one can either imitate their argument or invoke Theorem 1.4 of [V1] and Harish-Chandra's completeness theorem. (Alternatively, one can prove (3.43) by a similar algebraic argument: One first expresses the Hom as the dimension of a weight space in a cohomology group of $X(\gamma^0)$, then restricts to the identity component of G to compute it.) Q.E.D.

Corollary 3.44. *Suppose Z_1 and Z_2 are tempered irreducible $(\mathfrak{g}, K)$ modules having real infinitesimal character and the same lambda-lowest K-type. Then $Z_1 \cong Z_2$.*

Corollary 3.45. *Suppose $Z_1, \ldots, Z_m$ are distinct tempered irreducible $(\mathfrak{g}, K)$ modules having real infinitesimal character, and $r_1, \ldots, r_m$ are integers. Then there is an i such that the lambda-lowest K-type of Z_i has multiplicity r_i in $\sum r_i \Theta_K(Z_i)$.*

The uniqueness statement of Theorem 1.5 follows.

4. An up-to-date look at [S-V]

We fix throughout this section a θ-stable parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g} \tag{4.1}$$

as in (1.1); and we use the other notation of (1.1) freely. If $\mathfrak{h} \subseteq \mathfrak{l}$ is any Cartan subalgebra, we will often need

$$\rho(\mathfrak{u}) = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})} \alpha \in \mathfrak{h}^*; \tag{4.2}$$

this is the weight of a one-dimensional representation of $\mathfrak{l}$ (but not necessarily of L). Recall from [V2], Chapter 6, Zuckerman's functors ([V2], Definition 6.2.11):

$$\Gamma^i\colon \mathcal{M}(\mathfrak{g}, L \cap K) \to \mathcal{M}(\mathfrak{g}, K), \tag{4.3}$$

and the functor ([V2], Definition 6.1.21)

$$\begin{gathered} \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}\colon \mathcal{M}(\mathfrak{l}, L \cap K) \to \mathcal{M}(\mathfrak{g}, L \cap K), \\ \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}} Y = \operatorname{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), Y)_{L\cap K\text{-finite}}; \end{gathered} \tag{4.4}$$

we regard Y as a $(\mathfrak{q}, L \cap K)$ module by making $\mathfrak{u}$ act trivially. We will also need

$$\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}\colon \mathscr{M}(\mathfrak{l}, L \cap K) \to \mathscr{M}(\mathfrak{g}, L \cap K), \tag{4.5}$$
$$\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}} Y = U(\mathfrak{g}) \underset{\bar{\mathfrak{q}}}{\otimes} Y$$

([V2], Definition 6.1.5). The basic construction is cohomological parabolic induction:

$$\mathscr{R}^i\colon \mathscr{M}(\mathfrak{l}, L \cap K) \to \mathscr{M}(\mathfrak{g}, K). \tag{4.6}$$
$$\mathscr{R}^i Y = \Gamma^i(\operatorname{pro}(Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u})).$$

LEMMA 4.7 ([V2]). *Suppose Y is an $(\mathfrak{l}, L \cap K)$ module of finite length.*
a) *For all i, $\mathscr{R}^i Y$ is a $(\mathfrak{g}, K)$ module of finite length.*
b) *For $i > S = \frac{1}{2}\dim(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})$,*

$$\mathscr{R}^i Y = 0.$$

c) *Suppose $\mathfrak{h} \subseteq \mathfrak{l}$ is a Cartan subalgebra, $\lambda \in \mathfrak{h}^*$, and Y has infinitesimal character $\lambda - \rho(\mathfrak{u})$. Then $\mathscr{R}^i Y$ has infinitesimal character λ.*

Because it will recur constantly, we emphasize

$$S = \tfrac{1}{2}\dim(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k}) = \dim(\mathfrak{u} \cap \mathfrak{k}). \tag{4.8}$$

Here is the formal correspondence between representations of L and of G.

Definition 4.9. Suppose H is a θ-stable Cartan subgroup of L, and

$$\gamma_{\mathfrak{q}} = (\Psi_{\mathfrak{q}}, \Gamma_{\mathfrak{q}}, \bar{\gamma}_{\mathfrak{q}}) \in \mathscr{P}^L_{\mathrm{lim}}(H).$$

Assume that

$$\text{(a)} \qquad \operatorname{Re}\langle \bar{\gamma}_{\mathfrak{q}} + \rho(\mathfrak{u}), \alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Define

$$\gamma = (\Psi, \Gamma, \bar{\gamma}) \in \mathscr{P}^G_{\mathrm{lim}}(H)$$

by

$$\Psi = \Psi_{\mathfrak{q}} \cup \{\text{imaginary roots of } \mathfrak{h} \text{ in } \mathfrak{u}\},$$
$$\Gamma|_A = \Gamma_{\mathfrak{q}}|_A,$$
$$\Gamma|_T = (\Gamma_{\mathfrak{q}}|_T) \otimes (\Lambda^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{p})|_T),$$
$$\bar{\gamma} = \bar{\gamma}_{\mathfrak{q}} + \rho(\mathfrak{u}),$$

the limit character of H for G associated to $\gamma_{\mathfrak{q}}$. Thus

$$\text{(b1)} \qquad \operatorname{Re}\langle \bar{\gamma}, \alpha\rangle \geq 0, \quad \text{for all } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

$$\text{(b2)} \qquad \Psi \supseteq \{\text{imaginary roots of } \mathfrak{h} \text{ in } \mathfrak{u}\}.$$

LEMMA 4.10. *In the setting of Definition* 4.9, γ *is really a limit character for G. The correspondence* $\gamma_{\mathfrak{q}} \leftrightarrow \gamma$ *is a bijection between limit characters for L satisfying* (4.9)(a), *and those for G satisfying* (4.9)(b).

Proof. The only non-formal part of this is the fact that

$$\rho(\Psi) - 2\rho_c(\Psi) = \rho(\Psi_{\mathfrak{q}}) - 2\rho_c(\Psi_{\mathfrak{q}}) + 2\rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u}),$$

both sides regarded as elements of $\mathfrak{t}^*$. This is a consequence of Lemma 5.3.29 of [V2]; or one can argue more directly as in [V2], proof of Lemma 8.1.1. Q.E.D.

PROPOSITION 4.11 ([V2], Proposition 8.2.15). *Suppose* $H \subseteq L$ *is a* θ*-stable Cartan subgroup, and* $\gamma_{\mathfrak{q}} \in \mathscr{P}^L_{\lim}(H)$ *satisfies* (4.9)(a). *Define* $\gamma \in \mathscr{P}^G_{\lim}(H)$ *by Definition* 4.9. *Then*

$$\mathscr{R}^i X^L(\gamma_{\mathfrak{q}}) = \begin{cases} 0, & i \neq S \\ X^G(\gamma), & i = S. \end{cases}$$

The result in [V] is for regular characters only, but the general case follows immediately by the translation principle.

Definition 4.12. For $\lambda \in \mathfrak{h}^*$, write

$$\mathscr{M}(\mathfrak{g}, K)_\lambda$$

for the full subcategory of $\mathscr{M}(\mathfrak{g}, K)$ consisting of modules of generalized infinitesimal character λ.

PROPOSITION 4.13. *Assume* (1.2)(a). *The functor*

$$\mathscr{R}^S \colon \mathscr{M}(\mathfrak{l}, L \cap K)_{\lambda - \rho(\mathfrak{u})} \to \mathscr{M}(\mathfrak{g}, K)_\lambda$$

is exact.

Proof. We need only show that $\mathscr{R}^i$ vanishes on $\mathscr{M}(\mathfrak{l}, L \cap K)_{\lambda - \rho(\mathfrak{u})}$ for $i \neq S$. For standard limit representations, this is Proposition 4.11. The general case follows by a straightforward induction (compare the proof of Theorem 1.5 in Section 3). Q.E.D.

LEMMA 4.14. *Suppose* $\gamma_{\mathfrak{q}} \in \mathscr{P}^L_f(H)$ *satisfies* (4.9)(a). *Then the corresponding element* $\gamma \in \mathscr{P}^G_{\lim}(H)$ *satisfies* (F-2) *of Definition* 2.4, *so* $\overline{X}^G(\gamma)$ *is irreducible or zero.*

LEMMA 4.15. *Suppose* $\gamma_{\mathfrak{q}} \in \mathscr{P}^L_f(H)$ *satisfies* (4.9)(a). *Then*

$$\mathscr{R}^S\left[\overline{X}^L(\gamma_{\mathfrak{q}})\right] \supseteq \overline{X}^G(\gamma).$$

Proof. If $X(\gamma) = 0$, there is nothing to prove; so assume not. By Proposition 4.11, there is an irreducible composition factor

$$Y = \overline{X}^L(\gamma'_{\mathfrak{q}})$$

of $X^L(\gamma_{\mathfrak{q}})$, such that

$$\mathscr{R}^S Y \supseteq \overline{X}^G(\gamma).$$

Suppose $Y \neq \overline{X}^L(\gamma_{\mathfrak{q}})$. By Corollary 3.25,

$$\|\gamma\|_{\text{lambda}} < \|\gamma'\|_{\text{lambda}}. \tag{4.16}$$

By Propositions 4.11 and 4.13,

$$\overline{X}^G(\gamma) \subseteq \mathscr{R}^S\left(X^L(\gamma'_{\mathfrak{q}})\right) = X(\gamma'). \tag{4.17}$$

Now (4.16) and (4.17) contradict Corollary 3.25. Q.E.D.

PROPOSITION 4.18. *Suppose* $\gamma_{\mathfrak{q}} \in \mathscr{P}_f^L(H)$ *satisfies* (4.9)(a). *Then*

$$\mathscr{R}^S \overline{X}^L(\gamma_{\mathfrak{q}}) = \overline{X}^G(\gamma).$$

Proof. Enumerate the composition factors of $X^L(\gamma_{\mathfrak{q}})$ (with multiplicities) as $\{\overline{X}^L(\gamma^i_{\mathfrak{q}})\}$, with

$$\left(\gamma^i_{\mathfrak{q}} \in \mathscr{P}_f^L(H_i)\right).$$

Because (4.9)(a) is just a condition on infinitesimal character, all the $\gamma^i_{\mathfrak{q}}$ satisfy it; so by Lemma 4.15,

$$\mathscr{R}^S\left(\overline{X}^L\left(\gamma^i_{\mathfrak{q}}\right)\right) \supseteq \overline{X}^G(\gamma^i). \tag{4.19}$$

By Proposition 4.13, the formal difference

$$X^G(\gamma) - \sum_i \overline{X}^G(\gamma^i) \tag{4.20}$$

(in the Grothendieck group) is equal to

$$\begin{aligned}\mathscr{R}^S\left(X^L(\gamma_{\mathfrak{q}})\right) - \sum \overline{X}^G(\gamma^i) &= \mathscr{R}^S\left(\sum_i \overline{X}^L\left(\gamma^i_{\mathfrak{q}}\right)\right) - \sum \overline{X}^G(\gamma^i) \\ &= \sum_i \left[\mathscr{R}^S\left(\overline{X}^L\left(\gamma^i_{\mathfrak{q}}\right)\right) - \overline{X}^G(\gamma^i)\right].\end{aligned}$$

By (4.19), each term is the character of a representation. By [S-V], Theorem 4.23, (4.20) is zero (see Corollary 4.21 below). Consequently equality holds in (4.19). Q.E.D.

COROLLARY 4.21 ([S-V], Theorem 4.23). *Suppose* $\gamma_{\mathfrak{q}} \in \mathscr{P}^L_{\text{lim}}(H)$ *satisfies* (4.9)(a). *Write*

$$X^L(\gamma_{\mathfrak{q}}) = \sum_i \overline{X}^L\left(\gamma^i_{\mathfrak{q}}\right)$$

in the Grothendieck group, with $\gamma_{\mathfrak{q}}^i \in \mathscr{P}_f^L(H_i)$. *Then*

$$X^G(\gamma) = \sum_i \overline{X}^G(\gamma^i)$$

in the Grothendieck group, with $\gamma^i \in \mathscr{P}_{\lim}^G(H_i)$ *associated to* $\gamma_{\mathfrak{q}}^i$ *by Definition* 4.9. *Each term on the right is irreducible or zero; it is zero only if* $\bar{\gamma}^i$ *is orthogonal to some compact imaginary root of* $\mathfrak{h}_i$ *in* $\mathfrak{u}$.

Proposition 4.18 shows that $\mathscr{R}^S$ realizes the correspondence of [S-V] described in the introduction.

5. Duality for derived functor representations

In order to study Hermitian forms, it is convenient to introduce another family of functors:

(5.1)
$$\mathscr{L}^j: \mathscr{M}(\mathfrak{l}, L \cap K) \to \mathscr{M}(\mathfrak{g}, K),$$
$$\mathscr{L}^j Y = \Gamma^j(\mathrm{ind}(Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u}))$$

(compare 4.6). These have properties analogous to those of $\mathscr{R}^i$. For example,

LEMMA 5.2. *Suppose* Y *is an* $(\mathfrak{l}, L \cap K)$ *module of finite length.*

a) *For all* j, $\mathscr{L}^j Y$ *is a* $(\mathfrak{g}, K)$ *module of finite length.*

b) *For* $j < S = \frac{1}{2}\dim(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})$,

$$\mathscr{L}^j Y = 0.$$

c) *If* Y *has infinitesimal character* $\lambda - \rho(\mathfrak{u})$, *then* $\mathscr{L}^j Y$ *has infinitesimal character* λ.

The proof is identical to that of Lemma 4.7, and the rest of the results in Section 4 carry over equally well.

THEOREM 5.3 (Enright-Wallach). *Suppose* Y *is an* $(\mathfrak{l}, L \cap K)$ *module of finite length. Then there is a natural isomorphism*

$$[\mathscr{L}^i Y]^h \cong \mathscr{R}^{2S-i}(Y^h)$$

(*notation as in* (2.10), (4.6), *and* (5.1).

We will prove this in a moment.

PROPOSITION 5.4. *Suppose* Y *is an* $(\mathfrak{l}, L \cap K)$ *module of finite length. Then there is a natural map*

$$\mathscr{L}^S Y \xrightarrow{\Phi} \mathscr{R}^S Y.$$

If Y *has generalized infinitesimal character* $\lambda - \rho(\mathfrak{u})$, *and* λ *satisfies* (1.2)(a), *then* Φ *is an isomorphism.*

This will also be proved in a moment.

COROLLARY 5.5. *Suppose Y is an $(\mathfrak{l}, L \cap K)$ module admitting a nondegenerate invariant Hermitian form $\langle\, ,\rangle^L$. Then there is a natural invariant Hermitian form $\langle\, ,\rangle^G$ on $\mathscr{L}^S Y$; its radical is the kernel of the map Φ of Proposition* 5.4.

Proof. Since $Y \cong Y^h$ by hypothesis, Theorem 5.3 and Proposition 5.4 provide a map Ψ from $\mathscr{L}^S Y$ to its Hermitian dual $\mathscr{R}^S Y$. We may therefore define

$$(5.6) \qquad \langle u, v\rangle^G = \langle u, \Psi v\rangle \qquad (u, v \in \mathscr{L}^S Y).$$

Here the pairing on the right is the one between $\mathscr{L}^S Y$ and its Hermitian dual. Conditions (a)–(c) of Definition 2.10 are obvious, and (d) of Definition 2.11 follows by inspection of the proofs of Theorem 5.3 and Proposition 5.4 (see the proof of Theorem 5.7 and Lemma 5.19 below). The last assertion of the corollary is clear from (5.6). Q.E.D.

We begin now the proof of Theorem 5.3. The only difficult part is the following result. Recall that the functors Γ^i of (4.3) are defined even on $(\mathfrak{k}, L \cap K)$ modules.

THEOREM 5.7 ([E-W]). *Suppose X is a $(\mathfrak{k}, L \cap K)$ module. There is a natural isomorphism*

$$\left[\Gamma^i(X)\right]^h \cong \Gamma^{2S-i}(X^h).$$

Here h is Hermitian dual (Definition 2.10), and S is defined by (4.8). In particular, a non-degenerate Hermitian form on X induces one on $\Gamma^S X$. If X is actually a $(\mathfrak{g}, L \cap K)$ module, then all these constructions are $\mathfrak{g}$-invariant.

We will not give the proof, but it will be important to know how the pairing is defined. There is a natural isomorphism

$$(5.8) \qquad \Gamma^i X \cong \sum_{\delta \in \hat{K}} H^i(\mathfrak{k}, L \cap K; V_\delta^* \otimes X) \otimes V_\delta$$

as $(\mathfrak{k}, K)$ modules; the action on the right is on V_δ only. (This follows from [V2], Corollary 6.2.15.) The relative Lie algebra cohomology is computed by the complex

$$(5.9) \quad \operatorname{Hom}_{L \cap K}\left(\Lambda^i(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k}), V_\delta^* \otimes X\right) \cong \left[\Lambda^i(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})\right]^* \otimes V_\delta^* \otimes X.$$

Fix a non-zero linear functional

$$(5.10)(\mathrm{a}) \qquad \phi\colon \left[\Lambda^{2S}(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})\right]^* \to \mathbf{C}$$

such that

$$(5.10)(\mathrm{b}) \qquad \phi\left(\left[\Lambda^{2S}(\mathfrak{k}_0/\mathfrak{l}_0 \cap \mathfrak{k}_0)\right]^*\right) \subseteq a\mathbf{R};$$

here a is a constant to be chosen later. Now suppose

$$u \in \Gamma^i X, \quad v \in \Gamma^{2S-i}(X^h);$$

we may as well assume they belong to the δ-isotypic part. Choose representatives

$$\tilde{u} = \sum_j \omega_j \otimes v_j^* \otimes x_j \otimes v_j,$$
$$\tilde{v} = \sum_k \tau_k \otimes w_k^* \otimes x_k^h \otimes w_k.$$

Here for each j

$$\omega_j \in \left[\Lambda^i(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})\right]^*, \quad v_j^* \in V_\delta^*, \quad x_j \in X, \quad v_j \in V_\delta,$$

and similarly for v. Then the pairing in the theorem is

$$\langle u, v\rangle = \sum_{j,k} \phi(\omega_j \wedge \bar{\tau}_k)\langle v_j^*, w_k^*\rangle\langle x_j, x_k^h\rangle\langle v_j, w_k\rangle. \tag{5.11}$$

Here the second and fourth factors are fixed Hermitian pairings on V_δ and V_δ^*, assumed to be dual to each other. (This cancels the ambiguity of a constant multiple in their choice.) If $i = S$ and X carries a Hermitian form, then exactly the same formula gives the form on $\Gamma^S X$. Notice in that case that

$$\begin{aligned}\omega_j \wedge \bar{\tau}_k &= \left(\overline{\bar{\omega}_j \wedge \tau_k}\right)\\ &= (-1)^S\left(\overline{\tau_k \wedge \bar{\omega}_j}\right),\end{aligned}$$

and therefore that

$$\langle u, v\rangle = (-1)^S(\bar{a}/a)\langle\overline{v, u}\rangle. \tag{5.12)(a}$$

In order to get a Hermitian form, we can therefore choose

$$a = \pm(i^S); \tag{5.12)(b}$$

the sign will be determined later, by Proposition 6.8. We continue now with the proof of Theorem 5.3. Because of the twists in (4.6) and (5.1), we really need to study the $(\mathfrak{l}, L \cap K)$ module

$$\tilde{Y} = Y \otimes \Lambda^{\text{top}}\mathfrak{u}.$$

If Y has infinitesimal character $\lambda - \rho(\mathfrak{u})$, then $\tilde{Y}$ has infinitesimal character $\lambda + \rho(\mathfrak{u})$. Since $\Lambda^{\text{top}}\mathfrak{u}$ carries a positive $(\mathfrak{l}, L \cap K)$-invariant Hermitian form, all questions about forms transfer perfectly.

Lemma 5.13. *In the setting of* (4.4) *and* (4.5), *suppose $\tilde{Y}$ is an* $(\mathfrak{l}, L \cap K)$ *module. Then there is a natural isomorphism*

$$\left[\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\tilde{Y})\right]^h \cong \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}(\tilde{Y}^h).$$

Proof. The pairing is defined by

$$\langle u \otimes y, f \rangle = \langle y, f(\sigma(u)) \rangle \tag{5.14}$$

for $y \in \tilde{Y}$, $u \in U(\mathfrak{g})$, and $f \in \mathrm{pro}(\tilde{Y}^h)$. Here

$$u \to \sigma(u)$$

is the anti-automorphism of $U(\mathfrak{g})$ induced by

$$\sigma(U + iV) = -U + iV \qquad (U, V \in \mathfrak{g}_0),$$

and the pairing on the right in (5.14) is between $\tilde{Y}$ and $\tilde{Y}^h$. We leave the verifications to the reader. Q.E.D.

Theorem 5.3 follows from Theorem 5.7 and Lemma 5.13. We begin now the proof of Proposition 5.4.

LEMMA 5.15. *Suppose $\tilde{Y}$ is an $(\mathfrak{l}, L \cap K)$ module.*

a) *There are $(\mathfrak{l}, L \cap K)$ module maps*

$$\tilde{Y} \hookrightarrow \mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} \tilde{Y} \twoheadrightarrow \tilde{Y}$$

$$\tilde{Y} \hookrightarrow \mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}} \tilde{Y} \twoheadrightarrow \tilde{Y}$$

such that the composition is the identity.

b) *There is a $(\mathfrak{g}, L \cap K)$ module map ϕ such that*

$$\begin{tikzcd} \mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} \tilde{Y} \arrow[rr, "\phi"] \arrow[dr, shift left] & & \mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}} \tilde{Y} \arrow[dl, shift left] \\ & \tilde{Y} \arrow[ul, shift left, hook] \arrow[ur, shift left, hook] & \end{tikzcd}$$

commutes.

Proof. Write

$$U(\mathfrak{g}) = \mathfrak{u} U(\mathfrak{g}) \oplus U(\bar{\mathfrak{q}}).$$

This is stable under $\mathrm{ad}(\mathfrak{l})$ and $\mathrm{Ad}(L \cap K)$, so that

$$U(\mathfrak{g}) \underset{\bar{\mathfrak{q}}}{\otimes} \tilde{Y} = \left(\mathfrak{u} U(\mathfrak{g}) \underset{\bar{\mathfrak{q}}}{\otimes} \tilde{Y} \right) \oplus \tilde{Y}.$$

This gives the first half of (a); the second is similar. For (b),

$$\mathrm{Hom}_{\mathfrak{g}, L \cap K}(X, \mathrm{pro}\, \tilde{Y}) \cong \mathrm{Hom}_{\mathfrak{q}, L \cap K}(X, \tilde{Y}). \tag{5.16}$$

The kernel of the map

$$\mathrm{ind}\, \tilde{Y} \xrightarrow{\pi} \tilde{Y}$$

is $\mathfrak{u}(\mathrm{ind}\, \tilde{Y})$; so π is a $\mathfrak{q}$ module map. We define ϕ to be the map corresponding to π in (5.16) (with $X = \mathrm{ind}\, \tilde{Y}$). Then (b) is immediate. Q.E.D.

LEMMA 5.17. *Suppose* $\tilde{Y}$ *is an* $(\mathfrak{l}, L \cap K)$ *module of generalized infinitesimal character* $\lambda + \rho(\mathfrak{u}) = \lambda - \rho(\bar{\mathfrak{u}})$, *and that* λ *satisfies* (1.2)(a). *Then*

a) *Every non-zero submodule* S *of* $\mathrm{ind}_{\mathfrak{q}}^{\mathfrak{g}}\tilde{Y}$ *has a non-zero intersection with* $\tilde{Y}$.

b) $\tilde{v}$ *generates* $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}\tilde{Y}$.

c) *The map* ϕ *of Lemma* 5.16(b) *is an isomorphism.*

(This will be generalized in Section 8.)

Proof. For (a), we can find an element $x \in i(\mathfrak{l}_0 \cap \mathfrak{k}_0)$ such that x commutes with $\mathfrak{l}$ and $L \cap K$, and $\mathrm{ad}(x)$ has positive eigenvalues on $\mathfrak{u}$ ([V2], (6.3.13)). The eigenvalues of x in $\mathrm{ind}\,\tilde{Y}$ are

$$(\lambda + \rho(\mathfrak{u}))(x) \qquad\qquad (\text{on } \tilde{Y}),$$

$$(\lambda + \rho(\mathfrak{u}))(x) + r, \quad \text{for some } r > 0 \qquad (\text{on } \mathfrak{u}\,\mathrm{ind}\,\tilde{Y}).$$

Let $(\lambda + \rho(\mathfrak{u}))(x) + r_0$ be an eigenvalue of x on S, such that r_0 is minimal; write Y' for the corresponding eigenspace. It is enough to show that $r_0 = 0$. Necessarily Y' is an $(\mathfrak{l}, L \cap K)$ submodule of $\mathrm{ind}\,\tilde{Y}$. By the minimality of r_0, it is annihilated by $\bar{\mathfrak{u}}$. There is therefore a non-zero map

$$\mathrm{ind}\,Y' \to \mathrm{ind}\,\tilde{Y}. \tag{5.18}$$

As an $(\mathfrak{l}, L \cap K)$ module, Y' is a subquotient of

$$S(\mathfrak{u}) \otimes \tilde{Y}.$$

By Kostant's theorem ([V2], Corollary 7.1.13), Y' has a summand of infinitesimal character

$$\lambda - \rho(\bar{\mathfrak{u}}) + \psi,$$

with ψ a sum of roots of $\mathfrak{h}$ in $\mathfrak{u}$. Therefore $\mathrm{ind}\,Y'$ has a summand of infinitesimal character $\lambda + \psi$. By (5.18), this is equal to the infinitesimal character λ of $\mathrm{ind}\,\tilde{Y}$; so

$$\lambda + \psi = w\lambda, \quad \text{for some } w \in W(\mathfrak{g}, \mathfrak{h}).$$

By [V2], Lemma 6.3.28, (1.2)(a) implies that

$$w\lambda = \lambda + \sum_{\alpha \in \Delta(\bar{\mathfrak{q}}, \mathfrak{h})} z_\alpha \alpha, \operatorname{Re} z_\alpha \geq 0.$$

Since ψ is a sum of roots in $\mathfrak{u}$, this forces $\psi = 0$. The element x now has eigenvalue

$$(\lambda - \rho(\bar{\mathfrak{u}}) + \psi)(x) = (\lambda + \rho(\mathfrak{u}))(x)$$

on Y', which proves $r_0 = 0$. Part (b) is similar, and (c) follows. Q.E.D.

Proposition 5.4 follows from Lemmas 5.15 and 5.16. To fill the gap at the end of the proof of Corollary 5.5, we record also the following result.

LEMMA 5.19 (Shapovalov). *Suppose $\tilde{Y}$ is an $(\mathfrak{l}, L \cap K)$ module of finite length, with an invariant Hermitian form $\langle\, ,\rangle^L$. Then* $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}} \tilde{Y}$ *acquires a natural invariant Hermitian form $\langle\, ,\rangle^{\mathfrak{g}}$, as follows. Write*

$$\xi_{\ell}\colon \tilde{Y} \to \tilde{Y}^h$$

for the map defined by the form on $\tilde{Y}$,

$$\psi\colon \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} \tilde{Y} \to \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} \tilde{Y}^h$$

for the induced map, and

$$\psi\colon \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} \tilde{Y} \to \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}} \tilde{Y}^h \cong \left(\operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}} \tilde{Y}\right)^h$$

for the composition of ψ with the map ϕ of Lemma 5.15(b). *Then the form on* $\operatorname{ind} \tilde{Y}$ *is*

$$\langle x, x'\rangle^{\mathfrak{g}} = \langle x, \psi x'\rangle \qquad (x, x' \in \operatorname{ind} \tilde{Y}).$$

If the original form on $\tilde{Y}$ is non-degenerate, then the radical of the form on $\operatorname{ind} \tilde{Y}$ *is the kernel of ψ.*

Proof. This pairing obviously satisfies (a)–(c) of Definition 2.10; we only have to verify (d) of Definition 2.11. Define σ as in the proof of Lemma 5.13, and

$$(5.20) \qquad \rho\colon U(\mathfrak{g}) \to U(\mathfrak{l}), \qquad \rho(\mathfrak{u}U(\mathfrak{g}) + U(\mathfrak{g})\bar{\mathfrak{u}}) = 0.$$

Suppose $u, u' \in U(\mathfrak{g})$, and $y, y' \in \tilde{Y}$, so that $u \otimes y$ and $u' \otimes y'$ belong to $\operatorname{ind} \tilde{Y}$. The pairing constructed in the lemma may easily be computed in terms of ρ and σ: it is

$$\langle u \otimes y, u' \otimes y'\rangle^{\mathfrak{g}} = \langle \rho(\sigma(u')u)y, y'\rangle^L.$$

Condition (d) of Definition 2.11 therefore amounts to

$$(5.21) \qquad \sigma[\rho(\sigma(u')u)] = \rho(\sigma(u)u').$$

Because σ is an anti-automorphism, and $\sigma^2 = 1$, (5.21) follows from

$$\sigma \circ \rho = \rho \circ \sigma;$$

and this in turn is obvious from the definitions. Q.E.D.

6. Proof of Theorem 1.3

We continue to fix a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$.

LEMMA 6.1. *Suppose $\gamma_{\mathfrak{q}} \in \mathscr{P}^L_{\lim}(H)$; put*

$$\lambda = \bar{\gamma}_{\mathfrak{q}} + \rho(\mathfrak{u}).$$

$$\lambda_t = (\bar{\gamma}_{\mathfrak{q}})_t + \rho(\mathfrak{u}) \qquad (t \in \mathbf{R};\ \textit{see Definition } 2.16).$$

Assume that λ satisfies (1.2)(a) (*respectively*, (1.2)(b)). *Then λ_t does as well, for $|t| \leq 1$.*

Proof. The set of μ satisfying either condition is obviously convex and θ-stable. But

$$\lambda_t = s\lambda + (1-s)(\theta\lambda), \qquad s = \frac{1+t}{2}. \qquad \text{Q.E.D.}$$

LEMMA 6.2. *Suppose $\gamma_{\mathfrak{q}} \in \mathscr{P}_f^L(H)$ satisfies* (4.9)(a) (*that is, that $\bar{\gamma}_{\mathfrak{q}} + \rho(\mathfrak{u})$ satisfies* (1.2)(a)), *and that $\bar{X}^L(\gamma_{\mathfrak{q}})$ admits a non-zero Hermitian form $\langle\, ,\rangle^L$. Write the signature of $\langle\, ,\rangle^L$ as*

$$\left(\sum r_i^+ \Theta_{L\cap K}(Z_i^L), \sum r_j^- \Theta_{L\cap K}(Z_j^L)\right)$$

as in Theorem 1.5, *with each Z_i^L an irreducible tempered $(\mathfrak{l}, L\cap K)$ module with real infinitesimal character. Write*

$$Z_i^L = \bar{X}^L(\gamma_{\mathfrak{q}}^i), \qquad \gamma_{\mathfrak{q}}^i \in \mathscr{P}_f^L(H_i).$$

Then each $\gamma_{\mathfrak{q}}^i$ satisfies (4.9)(a).

If $\bar{\gamma}_{\mathfrak{q}} + \rho(\mathfrak{u})$ satisfies (1.2)(b), *then all the $\bar{\gamma}_{\mathfrak{q}}^i + \rho(\mathfrak{u})$ do as well.*

Proof. This is a consequence of Lemma 6.1 and the proof of Theorem 1.5. Q.E.D.

We will actually prove the following result, which includes Theorem 1.3.

THEOREM 6.3. *Let Y be an $(\mathfrak{l}, L\cap K)$ module of finite length, and generalized infinitesimal character $\lambda - \rho(\mathfrak{u})$. Assume that λ satisfies* (1.2)(a), *and that Y admits a non-degenerate invariant Hermitian form $\langle\, ,\rangle^L$. Write the signature $\langle\, ,\rangle^L$ as*

$$\left(\sum r_i^+ \Theta_{L\cap K}(Z_i^L), \sum r_j^- \Theta_{L\cap K}(Z_j^L)\right),$$

with notation and conventions as in Lemma 5.2; *in particular,*

$$Z_i^L = \bar{X}^L(\gamma_{\mathfrak{q}}^i).$$

Let $\gamma^i \in \mathscr{P}_{\lim}^G(H_i)$ be the limit character corresponding to $\gamma_{\mathfrak{q}}^i$ by Definition 4.9 (*which exists by Lemma* 5.2); *and set*

$$Z_i = \bar{X}^G(\gamma^i) = \mathscr{R}^S(Z_i^L).$$

Then

a) *Z_i is tempered, with real infinitesimal character; and it is irreducible or zero.*

b) *If λ satisfies* (1.2)(b), *Z_i is irreducible.*

c) *$\mathscr{R}^S Y$ admits a non-degenerate invariant Hermitian form $\langle\, ,\rangle^G$ of signature*

$$\left(\sum r_i^+ \Theta_K(Z_i), \sum r_j^- \Theta_K(Z_j)\right).$$

Proof. The proof of Theorem 1.5 produces some finite set of possible Z_i^L, depending only on composition series information; of course some of these occur with coefficient zero in the final formula for the signature of Y. Nevertheless, we use that largest possible set of Z_i^L throughout the proof. Parts (a) and (b) of Theorem 6.3 follow from Lemma 6.2 and Proposition 4.18. By Corollary 4.21, the various Z_i's are exactly the representations produced by the proof of Theorem 1.5 for $\mathscr{R}^S Y$. Consequently, the signature of any non-degenerate Hermitian form on $\mathscr{R}^S Y$ is of the form

$$\left(\sum s_i^+ \Theta_K(Z_i), \sum s_j^- \Theta_K(Z_j)\right). \tag{6.4}$$

The idea of the proof of (c) is that (6.4) and Corollary 3.45 allow us to restrict attention to the lowest K-type of some Z_i, where the form $\langle\, ,\rangle^G$ is easy to understand.

The existence of $\langle\, ,\rangle^G$ is an immediate consequence of Proposition 5.4 and Corollary 5.5; that is, it is due to Enright and Wallach. All of the ideas in Section 5 can be applied to K instead of G. We begin by recalling a result of [E] on the signature of the resulting form in that case.

LEMMA 6.5. *Suppose $(\delta_{\mathfrak{q}}, V_{\delta_{\mathfrak{q}}})$ is an irreducible representation of $L \cap K$. Fix a Cartan subgroup T^c of $L \cap K$, and let $\mu_{\mathfrak{q}} \in (\mathfrak{t}^c)^*$ be the differential of an extremal weight of $\delta_{\mathfrak{q}}$. Set*

$$\mu = \mu_{\mathfrak{q}} + 2\rho(\mathfrak{u} \cap \mathfrak{p}) \in (\mathfrak{t}^c)^*.$$

There are two possibilities.

I) *For every $\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t}^c)$,*

$$\langle \mu, \alpha \rangle \geq 0.$$

In that case, there is an irreducible representation (δ, V_δ) of K such that

a) $$\operatorname{Hom}_{L \cap K}\left(V_\delta^{\mathfrak{u} \cap \mathfrak{k}}, V_{\delta_{\mathfrak{q}}} \otimes \Lambda^{\text{top}}(\mathfrak{u} \cap \mathfrak{p})\right) \neq 0.$$

II) *For some $\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k}, \mathfrak{t}^c)$,*

$$\langle \mu, \alpha \rangle < 0.$$

In that case, no representation satisfying (a) *exists.*

This is just a version of highest weight theory. Because of the disconnectedness of K, δ need not be unique.

LEMMA 6.6. *In the setting of Lemma 6.5, fix a positive root system $\Delta^+(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t}^c)$. Then every infinitesimal character for $\mathfrak{l} \cap \mathfrak{k}$ occurring in $V_{\delta_{\mathfrak{q}}} \otimes (\Lambda^{\text{top}} \mathfrak{u})$ is of the form $(\mu + \rho_c) + \rho(\mathfrak{u} \cap \mathfrak{k})$ with $\mu_{\mathfrak{q}}$ a highest weight of $V_{\delta_{\mathfrak{q}}}$, and $\mu = \mu_{\mathfrak{q}} + 2\rho(\mathfrak{u} \cap \mathfrak{p})$.*

This is a trivial calculation.

LEMMA 6.7. *In the setting of Lemma* 6.5, *suppose* Z *is any* $(\mathfrak{k}, K)$ *module. Set*

$$W = \operatorname{ind}_{\bar{\mathfrak{q}} \cap \mathfrak{k}}^{\mathfrak{k}}\left(V_{\delta_{\mathfrak{q}}} \otimes \Lambda^{\mathrm{top}}\mathfrak{u}\right).$$

Then

$$\operatorname{Hom}_K(Z, \Gamma^S W) \cong \operatorname{Hom}_{L \cap K}\left(Z^{\mathfrak{u} \cap \mathfrak{k}}, V_{\delta_{\mathfrak{q}}} \otimes \Lambda^{\mathrm{top}}\mathfrak{u} \cap \mathfrak{p}\right).$$

This is a well-known version of the Borel-Weil theorem; it follows from [V2], Corollary 6.3.4.

PROPOSITION 6.8 ([E]). *Suppose* $(\delta_{\mathfrak{q}}, V_{\delta_{\mathfrak{q}}})$ *is an irreducible representation of* $L \cap K$ *with a non-degenerate invariant Hermitian form* $\langle\,,\rangle^{L \cap K}$. *Then the Hermitian form* $\langle\,,\rangle^{K}$ *on*

$$V = \Gamma^S\left(\operatorname{ind}_{\bar{\mathfrak{q}} \cap \mathfrak{k}}^{\mathfrak{k}} V_{\delta_{\mathfrak{q}}} \otimes \Lambda^{\mathrm{top}}\mathfrak{u}\right)$$

induced by $\langle\,,\rangle^{L \cap K}$ *(Corollary* 5.5 *for* K*) is always nondegenerate. We can choose the constant of* (5.12)(b) *in such a way that if* $\langle\,,\rangle^{L \cap K}$ *is positive definite, then* $\langle\,,\rangle^{K}$ *is as well.*

Proof. By Lemma 6.7, we may as well assume that we are in case I of Lemma 6.5; for otherwise V is zero, and there is nothing to prove. In that case, Lemma 6.6 guarantees that every infinitesimal character of $\mathfrak{l} \cap \mathfrak{k}$ in $V_{\delta_{\mathfrak{q}}} \otimes \Lambda^{\mathrm{top}}\mathfrak{u}$ is of the form $\lambda + \rho(\mathfrak{u} \cap \mathfrak{k})$, with $\lambda = \mu + \rho_c$. The hypothesis of case I of Lemma 6.5 gives

$$\langle \lambda, \alpha \rangle > 0, \quad \text{for all } \alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k}),$$

the analogue of (1.2) for $\mathfrak{k}$. Now Proposition 5.4 and Corollary 5.5 (for $\mathfrak{k}$) imply that $\langle\,,\rangle^{K}$ is non-degenerate. The positivity statement is Lemma 8.7 of [E]. Q.E.D.

Now we use this result to get a little information on $\mathfrak{g}$.

Definition 6.9. Suppose Y is an $(\mathfrak{l}, L \cap K)$ module. Put

$$W = \operatorname{ind}_{\bar{\mathfrak{q}} \cap \mathfrak{k}}^{\mathfrak{k}}(Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u}) = U(\mathfrak{k}) \underset{\bar{\mathfrak{q}} \cap \mathfrak{k}}{\otimes} (Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u}),$$

$$X = \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u}) = U(\mathfrak{g}) \underset{\bar{\mathfrak{q}}}{\otimes} (Y \otimes \Lambda^{\mathrm{top}}\mathfrak{u}),$$

and let

$$\psi: W \hookrightarrow X$$

be the natural inclusion. *The* $\mathfrak{q}$*-bottom layer of K-types of* $\Gamma^S X$ is the image of the induced map

$$\psi^S\colon \Gamma^S W \to \Gamma^S X.$$

This map is always injective, by the analogue of [V], Corollary 6.3.21 for ind instead of pro (cf. Lemma 5.2(b)).

PROPOSITION 6.10. *In the setting of Definition* 6.9, *assume that* Y *carries an invariant form* $\langle\, ,\rangle^L$. *Let* $\langle\, ,\rangle^K$ *be the form on* $\Gamma^S W$ *induced by* $\langle\, ,\rangle^L$ *and* $\langle\, ,\rangle^G$ *the form on* $\Gamma^S X$ (*Corollary* 5.5). *Then the map* ψ^S *of Definition* 6.9 *is unitary: for* $u, v \in \Gamma^S W$,

$$\langle u, v\rangle^K = \langle \psi^S u, \psi^S v\rangle^G.$$

Proof. By (5.11), it suffices to show that

$$\langle x, y\rangle^{\mathfrak{k}} = \langle \psi x, \psi y\rangle^{\mathfrak{g}}$$

for $x, y \in W$. (Here $\langle\, ,\rangle^{\mathfrak{k}}$ and $\langle\, ,\rangle^{\mathfrak{g}}$ are the forms on W and X constructed in Lemma 5.19.) But this formula in turn is obvious from the definitions; we leave the verification to the reader. Q.E.D.

The theory of the bottom layer of K-types becomes much clearer if we assume

(6.11) $\qquad L \cap K$ meets every connected component of K.

It suffices to prove Theorem 6.3(c) under this additional hypothesis, for every $(\mathfrak{g}, K)$ module occurring in the theorem is induced from a $(\mathfrak{g}, (L \cap K)\cdot K_0)$ module ([V2], Definition 6.2.9), and this induction obviously respects signatures.

LEMMA 6.12. *Under assumption* (6.11), *the correspondence* $\delta \leftrightarrow \delta_{\mathfrak{q}}$ *of Lemma* 6.5 *is a bijection from* $\hat{K}$ *onto the subset of* $(L \cap K)^{\wedge}$ *falling in case* I *of that lemma. We have*

(a) $\delta_{\mathfrak{q}}$ = *representation of* $L \cap K$ *on* $\mathfrak{u} \cap \mathfrak{k}$ *invariants in*

$$V_\delta \otimes (\Lambda^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{p}))^*,$$

(b) $$V_\delta = \Gamma^S\left(\mathrm{ind}_{\bar{\mathfrak{q}}\cap\mathfrak{k}}^{\mathfrak{k}}\left(V_{\delta_{\mathfrak{q}}} \otimes \Lambda^{\mathrm{top}}\mathfrak{u}\right)\right).$$

This is immediate from Lemmas 6.5 and 6.7.

LEMMA 6.13. *Suppose we are in the setting of Definition* 6.9, *and that* (6.11) *holds. Use the notation of Lemma* 6.12.

a) *The multiplicity of any* $\delta \in \hat{K}$ *in the* $\mathfrak{q}$*-bottom layer of* $\Gamma^S X$ *is equal to the multiplicity of* $\delta_{\mathfrak{q}}$ *in* Y.

b) *Suppose* $\phi \in \hat{K}$ *occurs in* $\Gamma^S X/\Gamma^S W$. *Then* $\phi_{\mathfrak{q}}$ *occurs in*

$$Y \otimes S^j(\mathfrak{u} \cap \mathfrak{p}) \quad \textit{for some } j > 0.$$

Proof. Part (a) is Lemma 6.7. Part (b) follows from the proof of Theorem 6.3.12 of [V2]. Q.E.D.

PROPOSITION 6.14. *Choose an element $x \in i(\mathfrak{l}_0 \cap \mathfrak{k}_0)$ as in the proof of Lemma* 5.17: *x is central in $\mathfrak{l}$, commutes with $L \cap K$, and has positive eigenvalues on $\mathfrak{u}$. In the setting of Definition* 5.21, *assume that x acts by some scalar c on Y (as it must if Y has an infinitesimal character). Assume* (6.11), *and use the notation of Lemma* 6.12. *Then a representation $\delta \in \hat{K}$ occurs in $\Gamma^S X$ only if*

$$\delta_{\mathfrak{q}}(x) \geq c.$$

If equality holds, δ can occur only in the bottom layer $\Gamma^S W$. If the inequality is strict, δ can occur only in $\Gamma^S X/\Gamma^S W$.

This is immediate from Lemma 6.13. The proposition implies that Definition 6.9 is compatible with Definition 4.12 of [S-V].

COROLLARY 6.15. *In the setting of Proposition* 6.14, *assume that Y has a non-degenerate Hermitian form $\langle\, ,\rangle^L$. Write $\langle\, ,\rangle^G$ for the induced form on $\Gamma^S X$ (Corollary* 5.5). *Fix $\delta \in \hat{K}$ such that $\delta_{\mathfrak{q}}(x) = c$. Then $\langle\, ,\rangle^G$ is non-degenerate on $(\Gamma^S X)^\delta$ (cf. Definition* 1.4); *its signature $(p(\delta), q(\delta))$ is equal to $(p(\delta_{\mathfrak{q}}), q(\delta_{\mathfrak{q}}))$.*

This is immediate from Propositions 6.14, 6.10 and 6.8.

PROPOSITION 6.16. *Suppose Y is an irreducible $(\mathfrak{l}, L \cap K)$ module of infinitesimal character $\lambda - \rho(\mathfrak{u})$, and that λ satisfies* (1.2)(a). *Assume* (6.11), *and use the notation of Lemma* 6.12 *and Definition* 6.9.

a) *If δ is a lambda-lowest K-type of $\Gamma^S X$, then $\delta_{\mathfrak{q}}$ is a lambda-lowest $L \cap K$-type of Y.*

b) *If $\delta_{\mathfrak{q}}$ is a lambda-lowest $L \cap K$-type of Y, and the corresponding δ exists, then δ is a lambda-lowest K-type of $\Gamma^S X$.*

This is deduced from Lemma 6.13 by essentially the same argument as for Lemma 8.8 in [V1]; it appears also as the proof of Theorem 6.5.9(b) in [V2]. We leave the transcription to the reader.

LEMMA 6.17. *Fix x as in Lemma* 6.12. *Then x acts by the same scalar c in all the representations Z_i^L of Theorem* 6.3, *and in Y itself.*

The scalar is of course

$$(\lambda - \rho(\mathfrak{u}))(x).$$

We leave the trivial proof to the reader.

We can now complete the proof of Theorem 6.3; recall that we are assuming (6.11). Write the signature of $\langle\, ,\rangle^G$ as in (6.4). By Lemma 6.17, Corollary 6.15, and Lemma 6.13(a), we find

(6.18) Suppose $\delta \in \hat{K}$ satisfies $\delta_{\mathfrak{q}}(x) = c$. Then the multiplicity of δ in $\sum r_i^{\pm}\Theta_K(Z_i)$ is equal to its multiplicity in $\sum s_i^{\pm}\Theta_K(Z_i)$.

By Proposition 6.16, these two K-characters agree on the lowest K-type of each Z_i. By Corollary 3.45, they coincide. Q.E.D.

7. Analytic continuation in the discrete parameters

Examples from the theory of dual reductive pairs, and discrete series for semisimple symmetric spaces, as well as the philosophy of coadjoint orbits, indicate that the hypothesis (1.2) on λ is too strong. At least when Y is sufficiently degenerate, $\mathscr{R}^S Y$ seems to be unitary under natural weaker hypotheses. Our goal in this section is to generalize Theorem 1.3 to cover some of these cases. The proof in Section 6 cannot easily be extended, because it involves also various representations $\mathscr{R}^S Z$, with Z non-degenerate. To generalize Theorem 1.3, we must therefore adopt a different approach. This new approach quickly leads to an apparently impossible calculation (cf. (7.29)). But we will see that the calculation is the same whether we are in the generalization of Theorem 1.3 or the original version. Since the original version is already proved, we will be done. Here is the statement we want.

THEOREM 7.1. *Fix notation as in Theorem* 1.2. *Define*

(i) $$\Phi = \textit{character of } L \textit{ on } \Lambda^{\mathrm{top}}\mathfrak{u},$$

(ii) $$\phi = \textit{differential of } \Phi,$$

(iii) $$\mathbf{C}_{t\phi} = \textit{one dimensional } \mathfrak{l} \textit{ module of weight } t\phi \ (t \in \mathbf{R}),$$

(iv) $$Y_t = Y \otimes \mathbf{C}_{t\phi} \qquad (\textit{an } \mathfrak{l} \textit{ module}),$$

(v) $$I_t = \mathrm{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(Y \otimes \mathbf{C}_{t\phi} \otimes \Lambda^{\mathrm{top}}\mathfrak{u}) \qquad (\textit{a } \mathfrak{g} \textit{ module}),$$

(vi) $$J_t = \mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(Y \otimes \mathbf{C}_{t\phi} \otimes \Lambda^{\mathrm{top}}\mathfrak{u}) \qquad (\textit{a } \mathfrak{g} \textit{ module}),$$

(vii) $$\psi_t \colon I_t \to J_t \textit{ the map of Lemma } 5.15(\mathrm{b}).$$

Assume that

a) *Y is unitarizable; and*

b) *for all $t \geq 0$, ψ_t is an injection.*

Then

c) *$\mathscr{R}^S Y$ is isomorphic to $\mathscr{L}^S Y$ under the map of Proposition* 5.4.

d) *$\mathscr{R}^i Y = 0$, $i \neq S$.*

e) *The Hermitian form $\langle\, ,\rangle^G$ of Corollary* 5.5 *is positive definite on $\mathscr{L}^S Y$.*

(Some conditions sufficient for (b) are given in Section 8. The deformation parameter t should not be confused with the parameter used in Theorem 3.8.)

The main tool used in the proof is a generalization of Sublemma 3.18 (Proposition 7.7). We begin by recalling

PROPOSITION 7.2 (Poincaré duality). *Let*

a)
$$0 \to V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{2S-1}} V_{2S} \to 0$$

be a complex of finite dimensional complex vector spaces. Assume that we are given a non-degenerate Hermitian form $\langle\, , \rangle_V$ *on* V_S, *and non-degenerate Hermitian pairings*

b)
$$\langle\, , \rangle_V \colon V_i \times V_{2S-i} \to \mathbf{C}, \qquad 0 \le i < S,$$

so that

c)
$$V_i \cong V_{2S-i}^h$$

(*cf. Definition* 2.10). *Assume that*

d)
$$d_i^* = d_{2S-i-1};$$

here $*$ *denotes the adjoint of a linear map. Write* H^i *for the* i^{th} *cohomology space of* (a). *Then there are non-degenerate Hermitian pairings*

e)
$$\langle\, , \rangle_H \colon H^i \times H^{2S-i} \to \mathbf{C}, \qquad 0 \le i \le S.$$

The pairing on H^S *is a Hermitian form.*

This is standard, and quite straightforward; it is a key ingredient in the Enright-Wallach proof of Theorem 5.3.

In the setting of Proposition 7.2, define

$$N_i = \dim V_i,$$
$$h_i = \dim H^i,$$

(7.3)
$$(P_S, Q_S) = \text{signature of Hermitian form on } V_S,$$
$$(p_S, q_S) = \text{signature of Hermitian form on } H^S$$

By Proposition 7.2(c) and (e),

(7.4)
$$N_i = N_{2S-i},$$
$$h_i = h_{2S-i},$$
$$P_S + Q_S = N_S,$$
$$p_S + q_S = h_S.$$

The Euler-Poincaré principle asserts that

(7.5)
$$\sum_{i=0}^{2S} (-1)^i N_i = \sum_{i=0}^{2S} (-1)^i h_i.$$

Using (7.4), we may rewrite this as

$$P_S + Q_S + 2\sum_{i=0}^{S-1}(-1)^{S-i}N_i = p_S + q_S + 2\sum_{i=0}^{S-1}(-1)^{S-i}h_i. \tag{7.6}$$

What we need is a refinement of this identity.

PROPOSITION 7.7. *In the setting of Proposition* 7.2, *use the notation* (7.3). *Then*

a) $$P_S + \sum_{i=0}^{S-1}(-1)^{S-i}N_i = p_S + \sum_{i=0}^{S-1}(-1)^{S-i}h_i,$$

b) $$Q_S + \sum_{i=0}^{S-1}(-1)^{S-i}N_i = q_S + \sum_{i=0}^{S-1}(-1)^{S-i}h_i.$$

Proof. We proceed by induction on S, then for fixed S, on the dimension N_0 of V_0. If $N_0 = 0$, the complex is equivalent to one of length $2S - 2$, and the proposition follows by inductive hypothesis. So we assume $N_0 > 0$. If $S = 0$, the result is trivial; so suppose $S > 0$. Next, suppose that d_0 is not injective. Consider the complex

$$0 \to V'_0 \xrightarrow{d'_0} V'_1 \to \cdots \to V'_{2S} \to 0$$

defined by

$$\begin{aligned} V'_0 &= V_0/\ker d_0, & \\ V'_{2S} &= (\ker d_0)^{\perp} \subseteq V_{2S} & &(\text{cf. Proposition 7.2(c)}), \\ V'_i &= V_i, & &0 < i < 2S, \\ d'_0 &= \text{map induced by } d_0 & \\ d'_{2S-1} &= (d'_0)^*, & \\ d'_i &= d_i, & &0 < i < 2S-1. \end{aligned} \tag{7.8}$$

Then the terms defined in (7.3) for V' are the same as those for V, except that

$$\begin{aligned} N'_0 &= N_0 - \dim\ker d_0, \\ h'_0 &= h_0 - \dim\ker d_0. \end{aligned} \tag{7.9}$$

The identities in Proposition 7.7 with and without primes are equivalent. Those with primes are known by induction; so we are done in this case. We may therefore assume d_0 is injective. We therefore regard V_0 as a subspace of V_1. Now there are two cases.

First, suppose $S = 1$. If v_0 and v'_0 are in V_0, then

$$\langle d_0v_0, d_0v'_0\rangle_V = \langle d_1d_0v_0, v'_0\rangle_V$$

(by Proposition 7.2(d))

$$= \langle 0, v_0' \rangle_V = 0$$

(since V is a complex). The subspace V_0 of V_1 is therefore totally isotropic. The proposition in this case amounts to Sublemma 3.18.

Next, suppose $S > 1$. Define a new complex by

$$\begin{aligned} V_0' &= \{0\}, \\ V_1' &= V_1/V_0, \\ V_{2S-1}' &= V_0^{\perp} \subseteq V_{2S-1}, \\ V_{2S}' &= \{0\}, \end{aligned} \tag{7.10}$$

with the obvious maps. The primed terms in (7.3) are then all equal to the unprimed terms, except that

$$\begin{aligned} N_0' &= 0, \\ N_1' &= N_1 - N_0, \\ N_{2S}' &= 0, \\ N_{2S-1}' &= N_{2S-1} - N_{2S}. \end{aligned} \tag{7.11}$$

Once again, therefore, the identities in Proposition 7.7 follow from their analogues for V', which are available by inductive hypothesis. Q.E.D.

Proof of Theorem 7.1. We assume G is connected; then $L \cap K$ is as well. For $n \in \mathbf{N}$, $\mathbf{C}_{n\phi}$ (cf. (iii) of Theorem 7.1) is in a natural way an $(\mathfrak{l}, L \cap K)$ module (the n^{th} power of Φ). Consequently,

$$\begin{aligned} &Y_n \text{ is an } (\mathfrak{l}, L \cap K) \text{ module}, \\ &I_n \text{ and } J_n \text{ are } (\mathfrak{g}, L \cap K) \text{ modules}. \end{aligned} \tag{7.12}$$

By hypothesis,

$$\psi_n \colon I_n \to J_n \text{ is an isomorphism.} \tag{7.13}$$

Now by (4.6) and (5.1)

$$\begin{aligned} \mathscr{R}^i Y_n &= \Gamma^i J_n, \\ \mathscr{L}^j Y_n &= \Gamma^j I_n; \end{aligned} \tag{7.14}$$

so (7.13) gives

$$\mathscr{R}^i Y_n \cong \mathscr{L}^i Y_n, \quad \text{for all } i \qquad (n \in \mathbf{N}). \tag{7.15}$$

By Lemma 4.7(b) and Lemma 5.2(b), this amounts to

$$\mathscr{R}^i Y_n = \mathscr{L}^i Y_n = \{0\}, \quad i \neq S \qquad (n \in \mathbf{N}). \tag{7.16}$$

We have now proved (c) and (d) of Theorem 7.1.

By Lemma 5.19 and hypothesis (b) in Theorem 7.1,

(7.17) $\quad I_t$ carries a non-degenerate Hermitian form $\langle\, ,\rangle_t \qquad (t \geq 0)$.

Now I_t is a semisimple $\mathfrak{l} \cap \mathfrak{k}$ module with finite multiplicities. We may therefore define a formal signature

$$\left(\sum_{\gamma \in (\mathfrak{l} \cap \mathfrak{k})^\wedge} \tilde{p}_t(\gamma)\gamma, \sum \tilde{q}_t(\gamma)\gamma \right) \tag{7.18}$$

for I_t, as in Definition 1.4. The $\mathfrak{l} \cap \mathfrak{k}$-module isomorphism

$$I_t \cong U(\mathfrak{u}) \otimes Y \otimes \mathbf{C}_{t\phi} \otimes \Lambda^{\mathrm{top}}\mathfrak{u} \tag{7.19}$$

shows that

(7.20) $\quad$ multiplicity of γ in I_t = multiplicity of $\gamma \otimes \mathbf{C}_{-t\phi}$ in I_0.

In a basis constructed from (7.19), the form $\langle\, ,\rangle_t$ depends in a polynomial way on t (see the proof of Lemma 5.19). Now (7.17) and (7.20) show that, in the notation of (7.18),

$$\begin{aligned} \tilde{p}_t(\gamma) &= \tilde{p}_0(\gamma \otimes \mathbf{C}_{-t\phi}), \\ \tilde{q}_t(\gamma) &= \tilde{q}_0(\gamma \otimes \mathbf{C}_{-t\phi}). \end{aligned} \tag{7.21}$$

Define

$$\begin{aligned} (P_t(\gamma), Q_t(\gamma)) &= \text{signature of Hermitian form on } \gamma \\ &\qquad \mathfrak{l} \cap \mathfrak{k}\text{-type of } \Lambda^S(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})^* \otimes I_t, \\ N_t^i(\gamma) &= \text{multiplicity of } \gamma \text{ in } \Lambda^i(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})^* \otimes I_t. \end{aligned} \tag{7.22}$$

Here we endow $\Lambda^S(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k})^*$ with the Hermitian form

$$\langle \omega, \tau \rangle = \phi(\omega \wedge \bar{\tau})$$

discussed in (5.10)–(5.12). All these numbers are computable from $\tilde{p}_t$ and $\tilde{q}_t$ by formulas independent of t; so by (7.21),

$$\begin{aligned} P_t(\gamma) &= P_0(\gamma \otimes \mathbf{C}_{-t\phi}), \\ Q_t(\gamma) &= Q_0(\gamma \otimes \mathbf{C}_{-t\phi}), \\ N_t^i(\gamma) &= N_0^i(\gamma \otimes \mathbf{C}_{-t\phi}). \end{aligned} \tag{7.23}$$

Finally, for $\delta \in \hat{K}$, and $n \in \mathbf{N}$, define

$$P_n(\delta) = \sum_{\gamma \in (\mathfrak{l} \cap \mathfrak{k})^\wedge} [\gamma \colon \delta|_{\mathfrak{l} \cap \mathfrak{k}}] P_n(\gamma), \tag{7.24}$$

and similarly for $Q_n(\delta)$, $N_n^i(\delta)$. Thus

$$N_n^i(\delta) = \dim \operatorname{Hom}_{L\cap K}\left(\Lambda^i(\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k}), V_\delta^* \otimes I_n\right), \tag{7.25}$$

$$\left(P_n(\delta), Q_n(\delta)\right) = \text{signature of } \operatorname{Hom}_{L\cap K}\left(\Lambda^S(\mathfrak{k}/\mathfrak{l}\cap\mathfrak{k}), V_\delta^* \otimes I_n\right). \tag{7.26}$$

Define

$$\begin{aligned}\left(p_n(\delta), q_n(\delta)\right) &= \text{signature of } \operatorname{Hom}_K\left(V_\delta, \mathscr{L}^S Y_n\right)\\ &= \text{signature of } H^S(\mathfrak{k}, L\cap K; V_\delta^* \otimes I_n)\end{aligned} \tag{7.27}$$

(cf. (5.8)). Theorem 7.1(e), which is what we still have to prove, asserts that

$$q_0(\delta) = 0. \tag{7.28}$$

By Proposition 7.7, (7.25), (7.26), and (7.16) we have

$$q_0(\delta) = Q_0(\delta) + \sum_{i=0}^{S-1} (-1)^{S-i} N_0^i(\delta). \tag{7.29}$$

Proving that the right-hand side is zero is the "apparently impossible calculation" mentioned at the beginning of this section. Now if n is large enough (say $n \geq n_0$), then Y_n satisfies the hypothesis (1.2)(a). By Theorem 1.3(a), $\mathscr{R}^S Y_n \cong \mathscr{L}^S Y_n$ is unitary for $n \geq n_0$. That is,

$$q_n(\delta) = 0, \quad n \geq n_0. \tag{7.30}$$

But the analogue of (7.29) is also true; so

$$Q_n(\delta) + \sum_{i=0}^{S-1} (-1)^{S-i} N_n^i(\delta) = 0, \quad n \geq n_0. \tag{7.31}$$

To prove (7.29), it certainly suffices to prove that

$$Q_0(\gamma) + \sum_{i=0}^{S-1} (-1)^{S-i} N_0^i(\gamma) = 0, \quad \text{for all } \gamma \in (\mathfrak{l}\cap\mathfrak{k})^\wedge \tag{7.32}$$

(see (7.24)). By (7.22) and (7.19), every term in (7.32) is zero unless γ actually exponentiates to $L\cap K$; so we confine our attention to such γ henceforth. So fix $\gamma \in (L\cap K)^\wedge$, and choose a weight μ of γ. Define the *height of* γ to be

$$h(\gamma) = \langle \mu, \phi \rangle; \tag{7.33}$$

here ϕ is as in (ii) of Theorem 7.1. Since ϕ is orthogonal to the roots of $\mathfrak{l}$, $h(\gamma)$ is independent of the choice of μ. Clearly the height function takes values in a lattice in $\mathbf{R}$. If $h(\gamma)$ is sufficiently negative, (7.19) shows that all terms in (7.32) are zero. Because of these two facts, we can prove (7.32) by induction on $h(\gamma)$.

Now fix $\gamma \in (L\cap K)^\wedge$, and assume that

(7.34) (7.32) holds for all $\gamma' \in (L\cap K)^\wedge$ with $h(\gamma') < h(\gamma)$.

Let μ be the highest weight of γ. Choose $n \geq n_0$ so large that $\mu + n\phi$ is dominant for K. This is certainly possible. Let (δ, V_δ) be the representation of K of highest weight $\mu + n\phi$. Then

(7.35)(a) $$V_\delta^{\mathfrak{u} \cap \mathfrak{k}} = \gamma \otimes \mathbf{C}_{n\phi}$$

(cf. Lemma 6.5). If $\gamma' \otimes \mathbf{C}_{n\phi}$ is any $L \cap K$-type occurring in $V_\delta / V_\delta^{\mathfrak{u} \cap \mathfrak{k}}$, then clearly

(7.35)(b) $$h(\gamma') < h(\gamma).$$

By (7.31),

$$Q_n(\delta) + \sum_{i=0}^{S-1} (-1)^{S-i} N_n^i(\delta) = 0.$$

By (7.24) and (7.35)(a),

$$Q_n(\gamma \otimes \mathbf{C}_{n\phi}) + \sum (-1)^{S-i} N_n^i(\gamma \otimes \mathbf{C}_{n\phi})$$
$$= -\sum_{\gamma'} \left[\gamma' \otimes \mathbf{C}_{n\phi} : V_\delta / V_\delta^{\mathfrak{u} \cap \mathfrak{k}}\right] \left(Q_n(\gamma' \otimes \mathbf{C}_{n\phi}) + \sum (-1)^{S-i} N_n^i(\gamma' \otimes \mathbf{C}_{n\phi})\right).$$

By (7.23), this may be rewritten as

$$Q_0(\gamma) + \sum (-1)^{S-i} N_0^i(\gamma)$$
$$= -\sum_{\gamma'} \left[\gamma' \otimes \mathbf{C}_{n\phi} : V_\delta / V_\delta^{\mathfrak{u} \cap \mathfrak{k}}\right] \left(Q_n(\gamma') + \sum (-1)^{S-i} N_n^i(\gamma')\right).$$

By (7.35)(b) and (7.34), the right side is zero. This is (7.32). Q.E.D.

8. Irreducibility theorems for generalized Verma modules

In this section we will give a useful sufficient condition for hypothesis (b) of Theorem 7.1. Because the most general form of this condition (Proposition 8.18) is probably quite hard to understand, we will state two weaker versions as well (Propositions 8.5 and 8.17). The reader should bear in mind Lemma 5.17, which is the result we seek to generalize.

Throughout this section (except in some examples) we drop the assumption (1.1), and fix the following notation:

(8.1)(a) $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ a parabolic subalgebra of $\mathfrak{g}$,

(8.1)(b) $\bar{\mathfrak{q}} = \mathfrak{l} + \bar{\mathfrak{u}}$ the opposite parabolic subalgebra,

(8.1)(c) $\mathfrak{h} \subseteq \mathfrak{l}$ a Cartan subalgebra,

(8.1)(d) $\Delta^+(\mathfrak{l}, \mathfrak{h})$ a positive root system, (8.1)(e) $\Delta^+ = \Delta^+(\mathfrak{l}, \mathfrak{h}) \cup \Delta(\mathfrak{u}, \mathfrak{h})$,

(8.2)(a) $\lambda \in \mathfrak{h}^*$,

(8.2)(b) Y a representation of $\mathfrak{l}$ of infinitesimal character $\lambda - \rho(\mathfrak{u})$,

(8.2)(c) $\tilde{Y} = Y \otimes \Lambda^{\text{top}}\mathfrak{u}$,

(8.3)(a) $I_t = \operatorname{ind}_{\bar{\mathfrak{q}}}^{\mathfrak{g}}(\tilde{Y} \otimes \mathbf{C}_{t\phi})$ (cf. Theorem 7.1(iii)),

(8.3)(b) $J_t = \operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}(\tilde{Y} \otimes \mathbf{C}_{t\phi})$,

(8.3)(c) $\psi_t\colon I_t \to J_t$ the map of Lemma 5.15(b).

Here (8.3)(b) requires a little clarification, since pro was defined in (4.4) using a finiteness condition which makes sense only when Y is an $(\mathfrak{l}, L \cap K)$ module. One possibility is to drop the finiteness condition; since we seek only conditions which make ψ_t (cf. (8.3)(c)) injective, it is harmless to enlarge the target space. Another is to define (with $\mathfrak{c}(\mathfrak{l})$ the center of $\mathfrak{l}$)

$$\operatorname{pro}_{\mathfrak{q}}^{\mathfrak{g}}(\tilde{Y} \otimes \mathbf{C}_{t\phi}) = \operatorname{Hom}_{\mathfrak{q}}\big(U(\mathfrak{g}), \tilde{Y} \otimes \mathbf{C}_{t\phi}\big)_{\mathfrak{c}(\mathfrak{l})\text{-finite}}. \tag{8.4}$$

We leave the choice to the reader.

PROPOSITION 8.5. *With notation* (8.1)–(8.3), *assume that*
a) Y *is one-dimensional; and*
b) $\operatorname{Re}\langle \alpha, \lambda - \rho(\mathfrak{l})\rangle \geq 0$ *for all* $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$.
Then the map ψ_t *of* (8.3)(c) *is injective for* $t \geq 0$.

An elementary argument about roots and weights shows that the hypothesis of this proposition, for one dimensional Y, is weaker than (1.2)(a).

The argument is identical to that for the general result; so we will phrase the lemmas in appropriate generality.

LEMMA 8.6. *With notation* (8.1)–(8.3), *the* $\mathfrak{g}$ *module* I_0 *has infinitesimal character* λ. *It is locally finite for* $\bar{\mathfrak{u}}$; *that is, for any* $x \in I_0$, *we can find* $k \geq 0$ *so that if* $\{U_1, \ldots, U_k\} \subseteq \bar{\mathfrak{u}}$, *then*

$$(U_1 \ldots U_k) \cdot x = 0.$$

It is also locally finite for the center $Z(\mathfrak{l})$ *of* $U(\mathfrak{l})$. *Any infinitesimal character for* $Z(\mathfrak{l})$ *which occurs is of the form*

$$\lambda - \rho(\bar{\mathfrak{u}}) + \sum_{\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})} n_\alpha \alpha,$$

with each n_α *a non-negative integer.*

Proof. The first assertion is standard (compare [V2], proof of Proposition 6.3.11). The second is obvious. The assertions about $Z(\mathfrak{l})$ have essentially appeared in the proof of Lemma 5.17. Q.E.D.

LEMMA 8.7. *With notation* (8.1)–(8.3), *suppose S is a non-zero $\mathfrak{g}$ submodule of I_0. Let $\tilde{Y}'$ denote a non-zero eigenspace of $Z(\mathfrak{l})$ on the $\mathfrak{u}$ invariants $S^{\mathfrak{u}}$ (which exists by Lemma* 8.6), *say of infinitesimal character $\lambda' - \rho(\bar{\mathfrak{u}})$. Then*

a) $\tilde{Y}' \subseteq \tilde{Y} \otimes_{\mathbf{C}}[\sum_{k=0}^{r} S^k(\mathfrak{u})]$ *as an $\mathfrak{l}$ module, for some r.*

b) $\lambda' = \lambda + \sum_{\alpha \in \Delta(\mathfrak{u},\mathfrak{h})} n_\alpha \alpha$, $n_\alpha \geq 0$.

c) *If the sum in* (b) *is empty, then* $\tilde{Y}' \subseteq \tilde{Y}$.

This is proved by the argument for Lemma 5.17. These two lemmas lead at once (as in the proof of Lemma 5.17) to the following result.

LEMMA 8.8. *With notation* (8.1)–(8.3), *suppose ψ_0 (cf.* (8.3)(c) *is* not *an injection. Then there is an $\mathfrak{l}$ module Y', of infinitesimal character $\lambda' - \rho(\mathfrak{u})$, with the following properties.*

a) *λ' is conjugate under $W(\mathfrak{g},\mathfrak{h})$ to λ.*

b) $\lambda' = \lambda + \sum_{\alpha \in \Delta(\mathfrak{u},\mathfrak{h})} n_\alpha \alpha$, *with n_α a non-negative integer and some n_α non-zero.*

c) *$Y' \subseteq Y \otimes F$, for some finite dimensional $\mathfrak{l}$ module F.*

Proof of Proposition 8.5. Because $\lambda + t\phi$ satisfies (b) of the proposition for all $t \geq 0$, it is enough to prove that ψ_0 is injective. Suppose not. Write

(8.9) $\mathfrak{s} = [\mathfrak{l},\mathfrak{l}] \cap \mathfrak{h}$ (the semisimple part of $\mathfrak{h}$ for $\mathfrak{l}$),

$\mathfrak{c} = $ center of $\mathfrak{l} \subseteq \mathfrak{h}$.

Then

(8.10) $$\mathfrak{h} = \mathfrak{s} + \mathfrak{c},$$

an orthogonal direct sum. Write

(8.11) $$\lambda = \lambda_{\mathfrak{s}} + \lambda_{\mathfrak{c}}$$

accordingly; since Y is one dimensional,

(8.12) $$\lambda_{\mathfrak{s}} = \rho(\mathfrak{l}), \quad \lambda_{\mathfrak{c}} = \lambda - \rho(\mathfrak{l}).$$

Choose Y' and λ' as in Lemma 8.8, and write

(8.13) $$\lambda' = \lambda'_{\mathfrak{s}} + \lambda'_{\mathfrak{c}}$$

in accordance with (8.10). By Lemma 8.8(c), Y' is finite dimensional, so that

$$\lambda'_{\mathfrak{s}} = \rho(\mathfrak{l}) + \mu,$$

with μ a dominant weight for $\mathfrak{l}$. In particular (by (8.12)),

(8.14) $$\langle \lambda'_{\mathfrak{s}}, \lambda'_{\mathfrak{s}} \rangle \geq \langle \rho(\mathfrak{l}), \rho(\mathfrak{l}) \rangle = \langle \lambda_{\mathfrak{s}}, \lambda_{\mathfrak{s}} \rangle.$$

By Lemma 8.8(b),

$$\begin{aligned}\lambda_c' &= \lambda_c + \Big(\sum_{\alpha\in\Delta(\mathfrak{u},\mathfrak{h})} n_\alpha\alpha\Big)\Big|_{\mathfrak{c}}\\ &= (\lambda - \rho(\mathfrak{l})) + \Big(\sum n_\alpha\alpha\Big)\Big|_{\mathfrak{c}},\\ \langle\lambda_c',\lambda_c'\rangle &= \langle\lambda_c,\lambda_c\rangle + \Big\langle \Big(\sum n_\alpha\alpha\Big)\Big|_{\mathfrak{c}}, \Big(\sum n_\alpha\alpha\Big)\Big|_{\mathfrak{c}}\Big\rangle + 2\Big\langle \lambda - \rho(\mathfrak{l}), \sum n_\alpha\alpha\Big\rangle.\end{aligned}$$

By Proposition 8.5(b), the last term has non-negative real part, and Lemma 8.8(b) says that the second term is positive. Consequently,

$$\operatorname{Re}\langle\lambda_c',\lambda_c'\rangle > \operatorname{Re}\langle\lambda_c,\lambda_c\rangle. \tag{8.15}$$

Adding this to (8.14) gives

$$\operatorname{Re}\langle\lambda',\lambda'\rangle > \operatorname{Re}\langle\lambda,\lambda\rangle,$$

which contradicts Lemma 8.8(a). Q.E.D.

The first generalization of Proposition 8.5 simply abstracts the property of one dimensional representations used in the preceding proof.

Definition 8.16. A representation Y of $\mathfrak{g}$ is called *weakly unipotent* if the following conditions are satisfied:

1) Y has an infinitesimal character $\lambda \in \mathfrak{h}^*$ which lies in the real span of the roots.

2) If F is any finite dimensional representation of $\mathfrak{g}$ and the infinitesimal character λ' occurs in $Y \otimes F$, then

$$\langle\lambda',\lambda'\rangle \geq \langle\lambda,\lambda\rangle.$$

Here are some examples of weakly unipotent representations: the trivial representation; the metaplectic representation; the spherical principal series representation with $\mathfrak{a}$ parameter zero. A great many more are constructed in [B-V].

PROPOSITION 8.17. *With notation* (8.1)–(8.3), *assume that*

a) $Y|_{[\mathfrak{l},\mathfrak{l}]}$ *is weakly unipotent* (*Definition* 8.16);

b) $\operatorname{Re}\langle\alpha, \lambda|_{center\ of\ \mathfrak{l}}\rangle \geq 0$, *for all* $\alpha \in \Delta(\mathfrak{u},\mathfrak{h})$.

Then the map ψ_t *of* (8.3)(c) *is injective for* $t \geq 0$.

The proof is almost word-for-word the same as that of Proposition 8.5, so we omit it.

It is Proposition 8.17 which seems to me to be the most natural condition to use in Theorem 7.1. (More precisely, "weakly unipotent" ought to be replaced by "unipotent", a term which unfortunately is so far defined only for finite Chevalley groups. Whatever the correct definition of unipotent turns out to be over **R**, it should imply weakly unipotent.) Nevertheless, there is a useful generalization.

PROPOSITION 8.18. *With notation* (8.1)–(8.3), *assume that the Cartan subalgebra* $\mathfrak{h}$ *has an orthogonal decomposition*

$$\mathfrak{h} = \mathfrak{s} + \mathfrak{c}$$

with the following properties.

a) $\mathfrak{s} \subseteq [\mathfrak{l}, \mathfrak{l}]$,

b) $\mathfrak{c} \supseteq$ *center of* $\mathfrak{l}$,

c) $\mathrm{Re}\langle \alpha, \lambda|_{\mathfrak{c}} \rangle \geq 0$, *for all* $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$,

d) *Suppose* F *is any finite dimensional representation of* $\mathfrak{l}$, *and* Y' *is a submodule of* $Y \otimes F$ *having infinitesimal character* λ'. *Then, after replacing* λ' *by a* $W(\mathfrak{l})$ *conjugate, we may arrange*

$$\mathrm{Re}\langle \lambda|_{\mathfrak{s}}, \lambda|_{\mathfrak{s}} \rangle \leq \mathrm{Re}\langle \lambda'|_{\mathfrak{s}}, \lambda'|_{\mathfrak{s}} \rangle.$$

Then the map ψ_t *of* (8.3)(c) *is injective for* $t \geq 0$.

Again the proof is identical to that of Proposition 8.5, so we omit it.

Here is an example of how (d) of the proposition can be satisfied. Let $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ be an Iwasawa-Cartan subalgebra of $\mathfrak{l}_0$, and let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{l}$. Set

$$\mathfrak{s} = [\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{h} \subseteq \mathfrak{t}.$$

Then (d) will be satisfied whenever Y has an $(\mathfrak{l} \cap \mathfrak{k})$-fixed vector. (This follows from Harish-Chandra's subquotient theorem.)

All of the results in this section can be extended slightly using the fact that the injectivity of ψ_t is controlled by the set of roots integral on $\lambda + t\phi$. The crudest result of this nature is

PROPOSITION 8.19. *With notation* (8.1)–(8.3), *assume that* ψ_t *is injective for* $t \geq 0$. *Let* t_0 *be the smallest strictly positive number such that*

$$\langle \check{\alpha}, \lambda - t_0\phi \rangle \in \mathbf{Z}, \quad \textit{for some } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}).$$

Then ψ_t *is injective for* $t > - t_0$.

Because I know of no nice applications, its proof and generalizations will be left to the reader's imagination (or to a future paper).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASS.

REFERENCES

[A] J. ADAMS, Unitary highest weight modules, preprint.

[B-V] D. BARBASCH and D. VOGAN, Unipotent representations of complex semisimple groups, preprint.

[E] T. J. ENRIGHT, Unitary representations for two real forms of a semisimple Lie algebra: A theory of comparison, to appear in Proc. Special Year in Harmonic Analysis, University of Maryland.

[E-P-W-W] T. J. ENRIGHT, R. PARTHASARATHY, N. R. WALLACH, and J. A. WOLF, Classes of unitarizable derived functor modules, preprint.

[E-W] T. J. ENRIGHT AND N. R. WALLACH, Notes on homological algebra and representations of Lie algebras, Duke Math. J. **47** (1980), 1–15.

[J] J. C. JANTZEN, *Modulen mit einem höchsten Gewicht*, Lect. Notes in Math. **750**, Springer: Berlin-Heidelberg-New York (1979).

[K-S] A. W. KNAPP and E. STEIN, Intertwining operators for semisimple groups, II, Invent. Math. **60** (1980), 9–84.

[K-Z1] A. W. KNAPP and G. ZUCKERMAN, Classification theorems for representations of semisimple Lie groups, in *Non-Commutative Harmonic Analysis*, Lect. Notes in Math. **587**, 138–159, Springer: Berlin-Heidelberg-New York (1977).

[K-Z2] ______, Classification of irreducible tempered representations of semisimple groups, Ann. of Math. **116** (1982), 389–501.

[L] R. P. LANGLANDS, On the classification of irreducible representations of real algebraic groups, mimeographed notes, Institute for Advanced Study, Princeton (1973).

[P1] R. PARTHASARATHY, A generalization of the Enright-Varadarajan modules, Compositio Math. **36** (1978), 53–73.

[P2] ______, Criteria for unitarizability of some highest weight modules, Proc. Indian Acad. Sci. **89** (1980), 1–24.

[P3] ______, Unitary modules with non-vanishing relative Lie algebra cohomology, to appear in Proc. Int. Cong. Math., Warsaw, 1983.

[R-S-W] J. RAWNSLEY, W. SCHMID, and J. A. WOLF, Singular unitary representations and indefinite harmonic theory, J. Funct. Anal. **51** (1983), 1–114.

[S-V] B. SPEH and D. VOGAN, Reducibility of generalized principal series representations, Acta Math. **145** (1980), 227–299.

[V1] D. VOGAN, The algebraic structure of the representations of semisimple Lie groups I, Ann. of Math. **109** (1979), 1–60.

[V2] ______, *Representations of Real Reductive Lie Groups*, Birkhäuser: Boston-Basel-Stuttgart (1981).

[V3] D. VOGAN, Understanding the unitary dual, to appear in Proc. Special Year in Harmonic Analysis, University of Maryland.

[V-Z] D. VOGAN and G. ZUCKERMAN, Unitary representations with non-zero cohomology, to appear in Compositio Math.

(Received January 2, 1984)