UNDERSTANDING THE UNITARY DUAL

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1. Introduction

Let G be a semisimple Lie group. One of the basic unsolved problems in non-commutative harmonic analysis is to describe the set $\hat{G}_u$ of equivalence classes of irreducible unitary representations of G. In those notes I want to advertise some conjectures about the general form of this solution: one old (Conjecture 4.4), one new (Conjecture 6.4), one borrowed (Conjecture 6.2), and one at least that is likely to be true (Conjecture 4.5). Before embarking on a technical discussion of them, I will try to explain why such conjectures are worth considering, since their resolution may not contribute much to the determination of $\hat{G}_u$.

An analogy with the theory of highest weight modules may be helpful. One of the fundamental problems in that theory is the character problem: determination of the formal characters of the irreducible highest weight modules. It is not very hard to show that, for a fixed semisimple Lie algebra $\mathfrak{g}$, this problem can be solved by a finite algebraic computation: one has to determine the ranks of a finite number of matrices, which depend on the structure constants of $\mathfrak{g}$ and on the highest weights in question. As a practical project, this is unpleasant even for $\mathfrak{sl}(3)$; but it is a kind of answer. Using more sophisticated ideas, various people have tabulated the formal characters for $\mathfrak{g}$ of dimension at most 51. (The ideas are in [11].) Such tables are another kind of solution of the character problem, for those $\mathfrak{g}$ for which they exist. For most mathematicians, the best solution is the Kazhdan-Lusztig conjecture, proved by Kazhdan-Lusztig, Beilinson-Bernstein, and Brylinski-Kashiwara ([12], [13], [5], [7]). This is an algorithm for computing formal characters from the "structure constants" of Weyl groups. As a tool for hand computation in small Lie algebras, it is not as good as Jantzen's techniques in [11]. It is easier to use than the method of computing ranks; but that is not a very important point. What sets it apart is that it makes sense; it connects the formal characters with other mathematical objects, and reveals structure in them that was not obvious before.

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All of that is important, but hardly new. The point is that the problem of finding $\hat{G}_u$ presents similar difficulties. It can be reduced (for a fixed G) to the solution of a countable set of polynomial inequalities on vector spaces. Experience indicates that when these inequalities are written explicitly, they can be solved by a finite process. If this were proved (and it should not be difficult), one would have a computable description of $\hat{G}_u$. Partly because of interest from physics, such calculations have become a fairly popular indoor sport. They lead to lists of unitary representations, and complete algebraic information about them. These lists -- notably Bargmann's [4], but also the later ones -- have been critical to the development of the subject, but they have obvious drawbacks. One doesn't usually want complete information, and having it can be a hindrance.

This problem was alleviated by Langlands' classification of $\hat{G}$ (the possibly non-unitary admissible representations) in [16]. One can now try to describe $\hat{G}_u$ as a subset of $\hat{G}$, and the answer is often more comprehensible than a description of $\hat{G}_u$ by itself. (The unitary spherical series for $SL(2, \mathbb{R})$ is represented by the "unitary cross" in this way, rather than as a certain allowed range of eigenvalues for the Casimir operator.) This technology, which is described in [14], is analogous to Jantzen's work on highest weight modules. As it stands, however, it does not determine $\hat{G}_u$ in general; and it would not be completely satisfactory if it did. (The reason is that there are nice families of unitary representations, such as those found by Flensted-Jensen in [9], which do not look like families in the Langlands parameters.)

What is needed is some major insight analogous to the Kazhdan-Lusztig conjecture for the character problem. Perhaps the best existing candidate is the Kirillov-Kostant "orbit method", which suggests that $\hat{G}_u$ is related to the set of integral orbits in the real dual $\mathfrak{g}_0^*$ of the Lie algebra $\mathfrak{g}_0$ of G. One problem with this method is that complementary series representations are often not attached to any orbit; but since complementary series are a thorn in the side of most known approaches, this objection is not too important. More serious is the fact that some integral orbits seem to have no attached representation (see [21]); and the orbit-to-representation correspondence is difficult to make precise in any case.

The conclusion I draw from this is that no one knows what $\hat{G}_u$ should look like. If this is granted, one should try to guess as much as possible about $\hat{G}_u$, to narrow and direct the search for its complete structure. The idea behind all of the conjectures here is Langlands' functoriality principle. This can be interpreted (optimistically) as suggesting that under certain conditions one can establish a correspondence between representations of another reductive group L and those of G. Still more optimistically, one can hope that, under more restrictive conditions, unitary representations will correspond to unitary representations.

The greatest success of this idea is in Langlands' classification of $\hat{G}$. He shows that every irreducible representation of G corresponds to a Cartan subgroup H of G, and a character $\Gamma \in \hat{H}$. (This is made precise in Section 2.) If H is fixed, this defines a correspondence between $\hat{H}$ and a subset of $\hat{G}$. In this case unitary characters of H do correspond to unitary (in fact to tempered) representations of G. The difficulty is that non-tempered unitary representations of G correspond to non-unitary characters Γ of H; the most obvious example is the trivial representation of G. If there were no other examples, we would be happy. What happens, however, is that there are other unitary representations of G, neither tempered nor trivial. What we want to do is make these intermediate cases look like they are attached to unitary representations of groups intermediate between H and G. Stated a little less ambitiously, the problem is this: which unitary representations of G are built from unitary representations of smaller groups? (The interpretation of the word "built" is to be regarded as part of the problem.) My claim is that a good description of $\hat{G}_u$ must be compatible with this problem; and, therefore, that an understanding of this problem may shed some light on the structure of $\hat{G}_u$.

The kind of correspondence we want to consider is this. Fix a Cartan subgroup H of G. Let L be another reductive group containing a Cartan subgroup isomorphic to (and henceforth identified with) H. The group L should be smaller than (though not necessarily a subgroup of) G. To each character $\Gamma \in \hat{H}$, there correspond irreducible representations X^L and X^G of L and of G. (Here we are being vague, since the Langlands classification is a little more subtle than we have indicated so far.) There are at least two things we can ask of (G,H,L,Γ):

(1.1) the algebraic structure of X^L determines that of X^G

(1.2) X^L unitary $\Longrightarrow$ X^G unitary

(For "algebraic structure", one may read "distribution character".) The conjectures describe various hypotheses on (G,H,L,Γ), which should make one or both of these assertions true. The idea that such hypotheses exist is what I mean by "Langlands functoriality" for unitary representations.

The first example of such a phenomenon is when L is a Levi factor of a real parabolic subgroup P of G. In that case

$$X^G \subseteq \mathrm{Ind}_P^G X^L,$$

so (1.2) is automatic. (1.1) means that the induced representation should be irreducible; many conditions are known which will guarantee that. This situation is discussed in Section 3.

The second example is when L is the centralizer of a compact torus in G. In that case, [17] provides good conditions for (1.1). Essentially the same conditions should give (1.2); this conjecture, and a little of the evidence for it, are described in Section 4.

Section 5 examines what these ideas imply about $\hat{G}_u$. The case of Sp(n,1) is considered (following [1]); and we find several unitary representations still unaccounted for. Section 6 is a collection of incomplete ideas which deal with these mysterious representations and related problems. There is one precise conjecture about Langlands functoriality for groups L not contained in G (Conjecture 6.4), but it is intended only as a hint about this intriguing phenomenon.

Since the main point is to formulate conjectures, almost all proofs are omitted. The ideas owe a great deal to conversations with, and the work of, almost everyone in the bibliography. (I hope they will be satisfied with my decision to attribute very little to anyone, rather than risking errors and omissions.) It is a pleasure particularly to thank Tony Knapp and Jeff Adams, with whom I have had extensive discussions as this paper was being written.

2. Notation and the Langlands classification

Let G be a reductive group in Harish-Chandra's class (see [10]; in particular, G is allowed to be non-linear or disconnected).. Fix a Cartan involution θ of G, with fixed point set K (a maximal compact subgroup). Put

$$
\begin{aligned}
G_0 &= \text{identity component of } G \\
\mathfrak{g}_0 &= \mathrm{Lie}(G) \\
\mathfrak{g} &= \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \\
U(\mathfrak{g}) &= \text{universal enveloping algebra of } \mathfrak{g} \\
Z(\mathfrak{g}) &= \text{center of } U(\mathfrak{g}).
\end{aligned}
\tag{2.1}
$$

We consider only $(\mathfrak{g},K)$ modules (instead of the corresponding group representations). Put

(2.2) $F(\mathfrak{g},K)$ = category of $(\mathfrak{g},K)$ modules of finite length

(see [22]). Then $\hat{G}$ is the set of equivalence classes of irreducibles in $\mathrm{Ob}(\mathcal{F}(\mathfrak{g},K))$. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, and define

$$\begin{aligned} \Delta(\mathfrak{g},\mathfrak{h}) &= \text{root system of } \mathfrak{h} \text{ in } \mathfrak{g} \\ W(\mathfrak{g},\mathfrak{h}) &= \text{Weyl group of } \mathfrak{h} \text{ in } \mathfrak{g} \\ \xi: \ Z(\mathfrak{g}) &\to S(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})} \quad \text{(Harish-Chandra map).} \end{aligned} \tag{2.3}$$

Suppose $\lambda \in \mathfrak{h}^*$. We say that a $\mathfrak{g}$ module has infinitesimal character λ if $Z(\mathfrak{g})$ acts on it by ξ composed with evaluation at λ.

Definition 2.4 Suppose $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ is a θ-stable Cartan subalgebra, and H is its centralizer in G. Write

$$\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$$

for the $+1$ and -1 eigenvalues of θ, and

$$T = H \cap K$$
$$A = \exp(\mathfrak{a}_0).$$

Then A is a vector group, and

$$H = T \times A,$$

a direct product. A regular pseudocharacter of H (briefly, a regular character) is a pair

$$\gamma = (\Gamma, \bar{\gamma})$$

subject to the following conditions.

R-1) $\Gamma \in \hat{H}$ (that is, Γ is an irreducible representation of H; recall that it may be non-abelian) and $\bar{\gamma} \in \mathfrak{h}^*$.

R-2) Suppose α is an imaginary root of $\mathfrak{h}$ in $\mathfrak{g}$. Then $\langle\alpha,\bar{\gamma}\rangle$ is real and non-zero. Write Ψ for the unique system of positive imaginary roots making $\bar{\gamma}$ dominant.

R-3) Write $\rho(\Psi)$ for half the sum of all the roots in Ψ, and $\rho_c(\Psi)$ for half the sum of the compact ones. Then

$$d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi).$$

Write $\mathcal{P}(H)$ (or $\mathcal{P}^G(H)$) for the set of regular characters of H.

This definition is discussed in [17] or [22]. ($\mathcal{P}(H)$ was called $\hat{H}'$ there, a notation which has proved disastrous.) Write L for the

centralizer of A in G. If $\gamma \in \mathcal{P}(H)$, then $\bar{\gamma}$ is the Harish-Chandra parameter of a discrete series representation X^L of L. We can choose X^L so that Γ is the highest weight of the lowest $L\cap K$-type of X^L; this determines X^L uniquely. Choose a real parabolic subgroup $P = LN$ of G, with Levi factor L, in such a way that $\mathrm{Re}(\bar{\gamma}|_{\mathfrak{a}_0})$ is negative on the roots of $\mathfrak{a}$ in $\mathfrak{n}$. Then we define

$$X(\gamma) = X^G(\gamma) = \mathrm{Ind}_{LN}^G(X^L \otimes 1), \tag{2.5}$$

the standard representation with parameter γ ([17]). $X(\gamma)$ has finite length, and infinitesimal character $\bar{\gamma}$. It is convenient to have a few more standard representations.

Definition 2.6 Suppose $H = TA$ is a θ-stable Cartan subgroup of G. A limit pseudocharacter (or limit character) of H is a triple

$$\gamma = (\Psi, \Gamma, \bar{\gamma})$$

with the following properties.

L-1) Ψ is a positive system for the imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\Gamma \in \hat{H}$, and $\bar{\gamma} \in \mathfrak{h}^*$.

L-2) If $\alpha \in \Psi$, then $\langle\alpha,\bar{\gamma}\rangle \geq 0$.

L-3) $d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi)$.

Write $\mathcal{P}_{\mathrm{lim}}(H)$ (or $\mathcal{P}^G_{\mathrm{lim}}(H)$) for the set of limit characters on H. A limit character γ is called final if it also satisfies

F-1) Suppose α is a simple root of Ψ, and $\langle\alpha,\bar{\gamma}\rangle = 0$. Then α is noncompact.

F-2) Suppose α is a real root of $\mathfrak{h}$ in $\mathfrak{g}$, and $\langle\alpha,\bar{\gamma}\rangle = 0$. Then α does not satisfy the parity condition ([22], Definition 8.3.11).

Write $\mathcal{P}_f(H)$ (or $\mathcal{P}^G_f(H)$) for the set of final limit characters on H.

Attached to every limit character γ is a standard (limit) representation

$$X(\gamma) = X^G(\gamma) \tag{2.7}$$

defined in analogy with (2.5). It is induced from a limit of discrete series on G^A; a discussion may be found in [17] or [15].

Proposition 2.8 Suppose $\gamma \in P_{lim}(H)$ (Definition 2.6).

a) γ satisfies F-1) $\Longleftrightarrow X(\gamma) \neq 0$.

b) If γ does not satisfy F-2), then $X(\gamma)$ is a direct sum of two standard limit representations attached to a more compact Cartan subgroup.

c) If γ is final, then $X(\gamma)$ has a unique irreducible submodule.

For connected linear G, this is proved in [15]. We omit the general proof.

Whenever γ satisfies F-1), write

$$\overline{X}^G(\gamma) = \overline{X}(\gamma) = \mathrm{soc}(X(\gamma)) \tag{2.9(a)}$$
$$= \text{largest completely reducible submodule of } X(\gamma).$$

Proposition 2.8 says that $\overline{X}(\gamma)$ is irreducible if γ is final. If γ does not satisfy F-1), write

$$\overline{X}(\gamma) = 0. \tag{2.9(b)}$$

Theorem 2.10 (Langlands, Knapp-Zuckerman [16], [15]).

a) Suppose $\gamma_i \in P_f(H_i)$ $(i = 1,2)$. Then $\overline{X}(\gamma_1) \cong \overline{X}(\gamma_2)$ if and only if (H_1,γ_1) is conjugate to (H_2,γ_2) under K.

b) Suppose X is an irreducible $(\mathfrak{g},K)$ module. Then there are a θ-stable Cartan subgroup H, and a $\gamma \in P_f(H)$, such that $X \cong \overline{X}(\gamma)$.

The Knapp-Zuckerman result was formulated differently, and proved only for connected linear groups. This version is a consequence of the results on translation functors in [17] and [20]. Langlands' part of the theorem may also be found in [6].

3. Functoriality and Mackey induction.

The results in this section are all more or less obvious; they are included as motivation for the conjectures which follow.

Definition 3.1 A reductive subgroup $M \subseteq G$ is called real polarizable if one of the following equivalent conditions is satisfied.

a) M is a Levi factor of a parabolic subgroup of G.

b) There is a reductive abelian subalgebra $\mathfrak{a}_0 \subseteq \mathfrak{g}_0$ such that $\mathrm{ad}(X)$ has real eigenvalues for $X \in \mathfrak{a}_0$; and M is the centralizer of $\mathfrak{a}_0$ in G.

c) M is conjugate to the centralizer of an abelian subalgebra of $\mathfrak{p}_0$ (the -1 eigenspace of θ).

A real polarization of M is a parabolic subgroup P of G with Levi factor M.

Definition 3.2 Suppose M is a θ-stable, real polarizable reductive subgroup of G, and P is a real polarization of it. Let H be a θ-stable Cartan subgroup of M, and suppose

$$\gamma = (\Psi,\Gamma,\overline{\gamma}) \in P^G_{lim}(H).$$

Define

$$\gamma_P = (\Psi,\Gamma,\overline{\gamma}) \in P^M_{lim}(H),$$

that is, γ_P is γ regarded as a limit parameter for M.

Lemma 3.3 In the setting of Definition 3.2,

a) $\gamma \in P^G_f(H) \Longrightarrow \gamma_P \in P^M_f(H)$

b) Every summand of $\overline{X}^G(\gamma)$ occurs in

$$\mathrm{Ind}^G_P(\overline{X}^M(\gamma_P)) \quad \text{(normalized induction)}.$$

Part (a) is trivial. Part (b) is a well-known consequence of either "asymptotics" or "lowest K-types"; see [16] or [19].

Corollary 3.4 Suppose M is a θ-stable, real polarizable reductive subgroup of G, with real polarization P. Suppose $H \subseteq M$ is a θ-stable Cartan subgroup, and $\gamma \in P^G_{lim}(H)$. Assume that $\overline{X}^M(\gamma_P)$ is unitarizable (Definition 3.2). Then $\overline{X}^G(\gamma)$ is as well.

This corollary can be sharpened slightly.

Lemma 3.5 In the setting of Definition 3.2, write H = TA, P = MN. Assume that for every root α of $\mathfrak{a}$ in $\mathfrak{n}$,

a) $\mathrm{Im}\langle\alpha,\overline{\gamma}\rangle \neq 0$.

Then if

$$\overline{X}^M(\gamma_P) = \bigoplus_{i=1}^{r} Y_i,$$

we have

$$\overline{X}^G(\gamma) = \bigoplus_i \mathrm{Ind}^G_P(Y_i).$$

This well-known result is due to Harish-Chandra; it gives one version of the desideratum (1.1).

Corollary 3.6 *Suppose M is a θ-stable, real polarizable reductive subgroup of G, with real polarization P. Assume that $H \subseteq M$ is a θ-stable Cartan subgroup, and that $\gamma \in P^G_{\lim}(H)$ satisfies (3.5)(a). Then $X^G(\gamma)$ is unitarizable if and only if $\overline{X}^M(\gamma_P)$ is.*

This follows from Lemma 3.5 by standard techniques.

4. Functoriality and holomorphic induction.

An important feature of Corollaries 3.4 and 3.6 is that their statements are purely formal. We can therefore hope to formulate conjectural analogues of them without a deep understanding of the analogues of induction which might be necessary for proofs.

Definition 4.1 A reductive subgroup $L \subseteq G$ is called complex polarizable if one of the following equivalent conditions is satisfied.

a) There is a parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$ (the complexified Lie algebra of G) such that L is the normalizer of $\mathfrak{q}$ in G.
b) There is a reductive abelian subalgebra $t_0 \subseteq \mathfrak{g}_0$ such that ad(X) has purely imaginary eigenvalues for $X \in t_0$; and L is the centralizer of t_0 in G.
c) L is the centralizer of a compact (connected) torus.
d) L is conjugate to the centralizer of an abelian subalgebra of k_0.

A complex polarization of L is a choice of $\mathfrak{q}$ as in (a). By (d) there is no loss of generality in assuming that L is θ-stable; and in that case, any complex polarization will be θ-stable as well.

Definition 4.2 Suppose L is a θ-stable, complex polarizable reductive subgroup of G, with complex polarization $\mathfrak{q} = \ell + u$. Let $H \subseteq L$ be a θ-stable Cartan subgroup, and suppose

$$\gamma = (\Psi, \Gamma, \overline{\gamma}) \in P^G_{\lim}(H).$$

Define the one dimensional representation V of H (trivial on the split component) as in [22], Lemma 8.1.1, using Ψ for $\Delta^+(m,t)$. Put

$$\gamma_{\mathfrak{q}} = (\Psi_{\mathfrak{q}}, \Gamma_{\mathfrak{q}}, \overline{\gamma}_{\mathfrak{q}}) \in P^L_{\lim}(H),$$

with

$$\Psi_{\mathfrak{q}} = \Psi \cap \Delta(\ell, h)$$
$$\overline{\gamma}_{\mathfrak{q}} = \overline{\gamma} - \rho(u)$$
$$\Gamma_{\mathfrak{q}} = \Gamma \otimes V^*.$$

Here $\rho(u)$ is half the sum of the roots of h in u. That $\gamma_{\mathfrak{q}}$ belongs to $P^L_{\lim}(H)$ is proved in exactly the same way as Lemma 8.1.2 of [22]. The differential of the character of H on V is determined by this fact, which is why we have not bothered to reproduce the (rather complicated) definition of V. In Definition 3.2, γ_P does not depend on the choice of P. Obviously $\gamma_{\mathfrak{q}}$ does depend on $\mathfrak{q}$, but the dependence is not too serious.

Lemma 4.3 *In the setting of Definition 4.2, let $\mathfrak{q}' = \ell + u'$ be another complex polarization of L. Then*

$$\Gamma_{\mathfrak{q}'} = \Gamma_{\mathfrak{q}} \otimes \Lambda^{\text{top}}(u'/u \cap u')^*$$
$$\overline{X}^L(\gamma_{\mathfrak{q}'}) \cong X^L(\gamma_{\mathfrak{q}}) \otimes \Lambda^{\text{top}}(u'/u \cap u')^*.$$

In particular, $X^L(\gamma_{\mathfrak{q}'})$ is unitarizable if and only if $X^L(\gamma_{\mathfrak{q}})$ is.

Analogy with Corollary 3.4 suggests

Conjecture 4.4 In the setting of Definition 4.2, assume that $\overline{X}^L(\gamma_{\mathfrak{q}})$ is unitarizable and non-zero. Then $\overline{X}^G(\gamma)$ is unitarizable or zero.

The hypothesis that $\overline{X}^L(\gamma_{\mathfrak{q}})$ be non-zero is necessary; $\gamma_{\mathfrak{q}}$ may fail to satisfy F-1) (Definition 2.6) even if γ does. In that case $\overline{X}^L(\gamma_{\mathfrak{q}})$ is zero, and $\overline{X}^G(\gamma)$ may be unitarizable or not. Even as stated, however, this conjecture may be too optimistic; there is little evidence for it except in the following special case.

Conjecture 4.5 In the setting of Definition 4.2, assume that $\overline{X}^G(\gamma)$ is non-zero and irreducible, and that $\overline{\gamma}$ satisfies the positivity hypothesis

$$(P_0) \qquad \mathrm{Re}\langle\alpha, \overline{\gamma}\rangle \geq 0, \text{ all } \alpha \in \Delta(u,h).$$

Then $\overline{X}^G(\gamma)$ is unitarizable if and only if $\overline{X}^L(\gamma_{\mathfrak{q}})$ is. (A version of (1.1) is established under these hypotheses in [17] -- see Proposition 6.1.) The usefulness of such a conjecture in organizing the unitary dual may be seen in the following result. (Semisimplicity is assumed only to make S finite.)

Proposition 4.6 *Assume that G is semisimple. Then there is a finite subset $S \subseteq \hat{K}$ with the following property. Suppose X is an irreducible unitarizable $(\mathfrak{g},K)$ module whose lowest K-types do not belong to the set S. Then there is a complex polarizable, reductive, proper subgroup L of G, with complex polarization $\mathfrak{q}$; a θ-stable Cartan subgroup $H \subseteq L$; and $\gamma \in P_{lim}(H)$, such that $X = \overline{X}^G(\gamma)$, and (P_0) of Conjecture 4.5 holds.*

This says that, except for those whose lowest K-type is in S, all unitary representations of G should come from unitary representations of proper subgroups. The proposition is a consequence of the following facts. Suppose $\gamma \in P_{lim}(H)$, and H = TA. Given any C > 0, we can find a finite set $S_C \subseteq \hat{K}$ such that if no lowest K-type of $\overline{X}^G(\gamma)$ belongs to S_C, then

$$(4.7)(a) \qquad \left|\overline{\gamma}\big|_{\mathfrak{t}}\right| \geq C.$$

This is clear from [9]. Next, there is a constant N, depending only on G, such that if $\overline{X}^G(\gamma)$ is unitary (or even if its matrix coefficients are bounded), then

$$(4.7)(b) \qquad \left|\mathrm{Re}\ \overline{\gamma}\big|_{\mathfrak{a}}\right| \leq N.$$

This is proved in [6]. If N is fixed, and C is large enough (depending on N), then (4.7) implies the existence of $\mathfrak{q}$ (not equal to $\mathfrak{g}$) satisfying (P_0). We leave the (elementary but not trivial) argument to the reader. This method gives a terrible estimate for S, one which is more or less useless for describing $\hat{G}_u$.

Problem 4.8 Find a description of the best possible set S in Proposition 4.6, even conjecturally.

Using the Dirac inequality, one can (with substantial effort) find the following possibilities for S in some examples.

TABLE 4.9

G	K	S
SO(n,1)	SO(n)	{trivial representation}
SU(n,1)	S(U(n)×U(1))	{trivial representation}
Sp(n,1)	Sp(n)×Sp(1)	$\{\text{trivial} \otimes S^k(\mathbb{C}^2), 0 \leq k \leq n-2\}$
$Sp(2,\mathbb{C})$	Sp(2)	$\{\text{trivial}, \mathbb{C}^4\}$
$SL(3,\mathbb{C})$	SU(3)	{trivial}
$SL(3,\mathbb{R})$	SO(3)	$\{\text{trivial}, \mathbb{C}^3\}$
$GL(4,\mathbb{R})$	O(4)	$\{\Lambda^i\mathbb{C}^4, 0 \leq i \leq 4\}$

It would not be very difficult to extend this table, finding the smallest set S allowed by the Dirac inequality for some other groups. However, the Dirac inequality does not by itself give the best possible estimate for S in higher rank groups; so such a calculation is not very helpful for Problem 4.8.

We conclude this section with some evidence for Conjecture 4.5. If $\overline{X}^G(\gamma)$ is non-zero, it is tempered if and only if $\Gamma \in \hat{H}$ is unitary ([16],[18]). Consequently, we have

Proposition 4.9 *In the setting of Definition 4.2, assume that $\overline{X}^L(\gamma_{\mathfrak{g}})$ and $\overline{X}^G(\gamma)$ are both non-zero. Then $\overline{X}^G(\gamma)$ is tempered if and only if $\overline{X}^L(\gamma_{\mathfrak{g}})$ is tempered.*

This is evidence even for Conjecture 4.4, since we do not assume (4.5) (P_0). The next result relates the functoriality of Definition 4.2 to unitary induction.

Proposition 4.10 *In the setting of Definition 4.2 assume that L is contained in a real polarizable reductive subgroup M, with real polarization P = MN (Definition 3.1). Then*

a) *$\overline{X}^G(\gamma) \subseteq \mathrm{Ind}_P^G(\overline{X}^M(\gamma))$. If $\overline{\gamma}$ satisfies (4.5)(P_0), then equality holds.*
b) *$\overline{X}^L(\gamma_{\mathfrak{g}})$ differs from $\overline{X}^L(\gamma_{\mathfrak{g}\cap m})$ by tensoring with a unitary character.*
c) *If $\overline{X}^M(\gamma)$ is unitary, so is $\overline{X}^G(\gamma)$. If $\overline{\gamma}$ satisfies (4.5) (P_0), then the converse is also true.*
d) *Conjectures 4.4 and 4.5 for (G,L,γ) follow from their analogues for (M,L,γ).*

The only non-formal assertion is the second part of (a); and that is a consequence of the results in [17]. (This is not obvious, but it is routine.) If G is a complex Lie group, then any complex polarizable subgroup L is automatically real polarizable: take t_0 as in Definition 4.1(b), and let $\mathfrak{a}_0$ be $(\sqrt{-1})(t_0)$. This $\mathfrak{a}_0$ shows that L satisfies Definition 3.1(b). Since Conjectures 4.4 and 4.5 are trivial when L = G, we deduce

Corollary 4.11 *Conjectures 4.4 and 4.5 are true when G is complex.*

For most real groups (for example, whenever rkG = rkK), the only real polarizable subgroup M containing a complex polarizable one is M = G; so Proposition 4.10 is of no use.

Any unitarizable $(\mathfrak{g},K)$ module carries an invariant Hermitian structure; so we can investigate the behaviour of this property under functoriality.

Definition 4.12 Suppose $\gamma = (\Psi,\Gamma,\bar{\gamma}) \in P_{lim}(H)$, and $H = TA$. Define γ^h, the Hermitian dual of γ, by

$$\gamma^h = (\Psi,\Gamma^h,\bar{\gamma}^h),$$

with

$$\bar{\gamma}^h(x) = \text{complex conjugate of } \bar{\gamma}(x) \qquad (x \in h_0),$$

and analogous notation for Γ. For $t \in \mathbb{R}$, define

$$\gamma^t = (\Psi,\bar{\gamma}^t,\Gamma^t),$$

with

$$\bar{\gamma}^t\big|_t = \bar{\gamma}\big|_t$$
$$\operatorname{Im}(\bar{\gamma}^t\big|_{\mathfrak{a}}) = \operatorname{Im}(\bar{\gamma}\big|_{\mathfrak{a}})$$
$$\operatorname{Re}(\bar{\gamma}^t\big|_{\mathfrak{a}}) = t\cdot\operatorname{Re}(\bar{\gamma}\big|_{\mathfrak{a}})$$
$$\Gamma^t\big|_T = \Gamma\big|_T$$
$$\Gamma^t\big|_A = \exp(\bar{\gamma}\big|_{\mathfrak{a}_0}).$$

Thus $\gamma^1 = \gamma$, and γ^0 is a parameter for a tempered representation.

Lemma 4.13 Suppose $\gamma \in P^G_{lim}(H)$, and $\bar{X}^G(\gamma) \neq 0$. Then $\bar{X}^G(\gamma)$ admits a non-degenerate invariant Hermitian form if and only if there is an element $w \in W(G,H)$ of order two, taking γ to γ^h (Definition 4.12).

For $\gamma \in P(H)$, this is well known. The general case is not entirely trivial, but we will not digress to prove it here.

Proposition 4.15 In the setting of Definition 4.2,

a) Assume that $\bar{X}^L(\gamma_{\mathfrak{g}})$ is non-zero and admits a non-degenerate invariant Hermitian form. Then $\bar{X}^G(\gamma)$ admits a non-degenerate invariant Hermitian form.

b) Assume that $\bar{X}^G(\gamma)$ is non-zero, and that $\bar{\gamma}$ satisfies (4.5)(P_0). Then $\bar{X}^G(\gamma)$ admits a non-degenerate invariant Hermitian form if and only if $\bar{X}^L(\gamma_{\mathfrak{g}})$ does.

Part(a) and the "if" part of (b) are more or less obvious from Lemma 4.13. The "only if" part of (b) is a little more subtle; it becomes easy if the inequality in (4.5)(P_0) is required to be strict.

Using complementary series tricks, this can be promoted to a result about unitarity.

Lemma 4.16 Suppose $\bar{X}^G(\gamma)$ admits a non-degenerate invariant Hermitian form, and that $X^G(\gamma^t)$ is irreducible for $0 \le t < 1$ (Definition 4.12). Then $\bar{X}^G(\gamma)$ is unitarizable.

This is well-known.

Lemma 4.17 ([17], Section 4). Under the hypothesis (4.5) (P_0), suppose that $X^L(\gamma_{\mathfrak{g}})$ is irreducible. Then $X^G(\gamma)$ is as well.

Proposition 4.18 In the setting of Definition 4.2, suppose that $\bar{X}^L(\gamma_{\mathfrak{g}})$ is unitarizable because of Lemma 4.16 (with L in place of G), and that $\bar{\gamma}$ satisfies (4.5)(P_0). Then $\bar{X}(\gamma)$ is unitarizable.

Corollary 4.19 The "if" part of Conjecture 4.5 is true whenever every noncompact simple factor of ℓ_0 is $\mathfrak{su}(n,1)$ or $\mathfrak{so}(n,1)$.

5. Life after the conjecture.

In this section, we will try to describe the picture of $\hat{G}_u$ which emerges from Conjecture 4.5 and Problem 4.8. Of course it is inductive in nature, relying on previous knowledge of the unitary duals of certain proper subgroups of G. We may as well assume G is semisimple. Fix a θ-stable Cartan subgroup $H = TA$; a positive system Ψ for t in the imaginary roots of h in $\mathfrak{g}$; and a pair

$$(5.1)(a) \qquad (\bar{\lambda},\Lambda) \in t^* \times \hat{T}.$$

Consider the limit characters $\gamma(\nu)$, for $\nu \in \mathfrak{a}^*$, defined by

$$(5.1)(b) \qquad \gamma(\nu) = (\Psi,(\bar{\lambda},\nu),(\Lambda\times e^{\nu})).$$

Clearly the condition for $\gamma(\nu)$ to be a limit character is a condition on $(\Psi,\bar{\lambda},\Lambda)$ (see Definition 2.6), and we assume that it is satisfied. To rule out trivialities, we may as well assume that (F-1) is satisfied (see Proposition 2.8). Write

$$(5.1)(c) \qquad \begin{aligned}\bar{X}(\nu) &= \bar{X}^G(\gamma(\nu)) \\ X(\nu) &= X^G(\gamma(\nu))\end{aligned} \qquad \text{(see 2.9)).}$$

We want to describe qualitatively the set of ν for which $\bar{X}(\nu)$ is unitary. If $\operatorname{Im}(\nu) \neq 0$, Corollary 3.6 reduces this question to the proper subgroup

$M(\nu) \supseteq H$, defined by

$$\Delta(m(\nu),h) = \{\alpha \in \Delta(\mathfrak{g},h) \mid \mathrm{Im}\langle\alpha,\nu\rangle = 0\}.$$

So we may as well restrict attention to the case $\mathrm{Im}(\nu) = 0$; that is,

(5.1)(d) $$\nu \in \mathfrak{a}_0^* \qquad \text{(the real dual of } \mathfrak{a}_0\text{)}.$$

The set of lowest K-types of $\overline{X}(\nu)$ depends only on $(\Psi,\overline{\lambda},\Lambda)$, and not on ν. If this set does not meet the set S of Proposition 4.6 (and Problem 4.8), then Conjecture 4.5 would reduce the unitarity question for $\overline{X}(\nu)$ to a proper subgroup. So we may assume that the lowest K-types meet S. This can happen for only finitely many choices of $(H,\Psi,\overline{\lambda},\Lambda)$. For each choice, we must describe

(5.2)(a)
$$Q = Q(H,\Psi,\overline{\lambda},\Lambda) \subseteq \mathfrak{a}_0^*,$$
$$Q = \{\nu \in \mathfrak{a}_0^* \mid \overline{X}(\nu) \text{ is unitarizable}\}.$$

Fix $w \in W(G,H)$ of order 2, and suppose that

(5.2)(b) $$w(\Psi,\overline{\lambda},\Lambda) = (\Psi,\overline{\lambda},\Lambda).$$

Set

(5.2)(c)
$$(\mathfrak{a}_0^*)_w = \{\nu \in \mathfrak{a}_0^* \mid w\nu = -\nu\}$$
$$Q_w = Q \cap (\mathfrak{a}_0^*)_w.$$

By Lemma 4.13

$$Q = \bigcup_{\substack{w \text{ satisfies} \\ (5.2)(b)}} Q_w.$$

We therefore fix w subject to (5.2)(b), and confine our attention to Q_w. We may as well assume that $(\mathfrak{a}_0^*)_w$ is not properly contained in any $(\mathfrak{a}_0^*)_{w'}$. (If (as often happens) we can find w satisfying (5.2)(b), such that

$$w\big|_{\mathfrak{a}_0^*} = -I,$$

then $(\mathfrak{a}_0^*)_w = \mathfrak{a}_0^*$. There is little harm in thinking only of this case.) Write

(5.3)
$$\begin{aligned} H &= H(H,\Psi,\overline{\lambda},\Lambda) \\ &= \text{set of maximal affine subspaces of } \mathfrak{a}_0^* \\ &\qquad \text{along which } X(\nu) \text{ is reducible.} \end{aligned}$$

Every element of H is contained in an affine hyperplane of the form

$$\{\nu \in \mathfrak{a}_0^* \mid \langle\check{\alpha},\nu\rangle = m/N\}.$$

Here α is a root of h in $\mathfrak{g}$; $m \in \mathbb{Z}$; and N is an integer depending only on G. (If G is linear, then N is 1.) Say that $\nu_1 \sim \nu_2$ if ν_1 and ν_2 lie in exactly the same elements of H. A <u>facet</u> of $(\mathfrak{a}_0^*)_w$ is by definition a connected component of an equivalence class, under this equivalence relation. Each facet is a bounded open subset of an affine subspace of $(\mathfrak{a}_0^*)_w$, defined by strict linear inequalities.

<u>Lemma 5.4</u> <u>Suppose F is a facet in $(\mathfrak{a}_0^*)_w$, $\nu_1 \in F$, and $\nu_2 \in \overline{F}$. If $\overline{X}(\nu_1)$ is unitarizable, so is $\overline{X}(\nu_2)$.</u>

This is well-known. As a consequence, we get

<u>Proposition 5.5</u> <u>In the setting just described, the region $Q \subseteq \mathfrak{a}_0^*$ of unitarity is a closed, bounded union of (finitely many) facets. It may be defined by finitely many polynomial inequalities.</u>

To describe $\hat{G}_u$ using Conjecture 4.5, we must therefore supply the following data (for subgroups of G as well): the finite set $S \subseteq \hat{K}$ of Problem 4.8; and for each of the (finitely many) $(H,\Psi,\overline{\lambda},\Lambda)$ as in (5.1) such that the lowest K-types of $\overline{X}(\nu)$ meet S, the unitarity region Q. By Proposition 5.5, this region admits a finite description. To give a <u>nice</u> description of $\hat{G}_u$, we need nice descriptions of S and of the regions Q. (For the latter, one possibility would be to find polynomials as in Proposition 5.5.) If we want to know characters for all elements of $\hat{G}_u$, we need to give them for each facet in every Q; again this is a finite problem.

Here are two examples. Suppose $G = SU(n,1)$, so that S consists only of the trivial representation of K (Table 4.9). We may consider only the maximally split Cartan $H = TA$, together with any choice of Ψ, and

$$\overline{\lambda} = \rho(\Psi)$$
$$\Lambda = \text{trivial character of } T.$$

Then $\{X(\nu)\}$ is the spherical principal series of $SU(n,1)$. Identify $\mathfrak{a}_0^*$ with $\mathbb{R}$ by sending the unique real root of h to 1. Then the reducibility set of (5.3) is (as computed by Kostant)

$$H = \{\pm(\tfrac{n}{2}+k),\ k = 0,1,2,\ldots\}.$$

The facets are

$$\left(-\tfrac{n}{2},\tfrac{n}{2}\right),\ \pm\left(\tfrac{n}{2}+k,\tfrac{n}{2}+k+1\right) \qquad (k = 0,1,2,\dots)$$

and the points of H. The set Q is

$$Q = [\tfrac{n}{2},\tfrac{n}{2}].$$

The precise structure of $\overline{X}(\nu)$ for $\nu \in Q$ is also easy to find: we have

$$\overline{X}(\nu) = X(\nu),\ \nu \in \left(-\tfrac{n}{2},\tfrac{n}{2}\right)$$
$$\overline{X}(\pm\tfrac{n}{2}) \cong \text{trivial representation.}$$

Next, suppose $G = Sp(n,1)$. Again we may consider only the maximally split Cartan $H = TA$. Fix $(\Psi,\overline{\lambda},\Lambda)$ as in (5.1), and suppose that the lowest K-types of $X(\nu)$ meet the set S of Table 4.9. Calculation shows that $X(\nu)$ has a unique lowest K-type, say

$$\text{trivial} \otimes S^r(\mathbb{C}^2),$$

with some r between 0 and n-2. (This r determines $(\Psi,\overline{\lambda},\Lambda)$ up to conjugation.) Again we identify $\mathfrak{a}_0^*$ with $\mathbb{R}$ by sending the real root to 1; for clarity, write $X_r(\nu)$ instead of $X(\nu)$. By [3],

$$H_r = \{\pm[n-(\tfrac{r+1}{2})+k],\ k = 0,1,2,\dots\}.$$

By [1], the unitarity set is

$$Q_r = \left[-\left(n-\left(\tfrac{r+1}{2}\right)\right), n+\left(\tfrac{r+1}{2}\right)\right]$$

if $r \neq 0$; and if $r = 0$,

$$Q_r = [-(n-\tfrac{1}{2}), n-\tfrac{1}{2}] \cup \{\pm(n+\tfrac{1}{2})\}.$$

Most of the unitarizable $\overline{X}_r(\nu)$ have simple structure:

$$\overline{X}_r(\nu) = X_r(\nu),\ \nu \in \text{interior of } Q_r$$
$$\overline{X}_0\left(\pm\left(n+\tfrac{1}{2}\right)\right) \cong \text{trivial representation.}$$

It remains only to describe the representations $\overline{X}_r\left(\pm\left(n-\left(\tfrac{r+1}{2}\right)\right)\right)$. We will do this in Section 6.

If Conjecture 4.5 is accepted as a natural construction of unitary representations, we see that all the unitary representations of SU(n,1) and Sp(n,1) are obtained as follows. Begin with unitary characters (one dimensional unitary representations), and the mysterious representations $\overline{X}_r\left(\pm\left(n-\left(\tfrac{r+1}{2}\right)\right)\right)$ of Sp(n,1). Apply as necessary

1) Unitary induction from real parabolic subgroups
2) Holomorphic induction (Conjecture 4.5)
3) Deformation (Lemma 5.4 and obvious generalizations).

6. Mysterious hints.

With a suitable (finite) set of mysterious representations, the picture of $\hat{G}_u$ given for SU(n,1) and Sp(N,1) at the end of the last section seems to persist for many of the (few) known examples. A starry-eyed dreamer might now formulate a conjecture to that effect. (Knapp has informed me of counterexamples to several very simple-minded formulations of the conjecture, but the picture remains.) Without being so brave, we can still consider the obstacles which would remain to finding $\hat{G}_u$ if this hypothetical conjecture were true. The principal practical one is that the application of Lemma 5.4 (deformation) requires a complete knowledge of the reducibility sets of all induced-from-unitary representations. This is provided by the Kazhdan-Lusztig conjecture [22] in the case of linear groups; but the calculations involved even for Sp(n,1) are formidable.

The most interesting obstruction is of course the description of the mysterious representations. These are of several kinds. The first kind consists of dull representations in disguise. To describe them, suppose we are in the setting of Definition 4.2. Recall from [22], Chapter 6, Zuckerman's cohomological parabolic induction functors

$$R^i_{\mathfrak{g}}:\ F(\ell, L\cap K) \to F(\mathfrak{g},K),\ 0 \le i \le S = \dim \mathfrak{u}\cap\mathfrak{k} \text{ (notation 2.4).}$$

<u>Proposition 6.1</u> <u>In the setting of Definition 4.2, assume that $\overline{\gamma}$ satisfies (4.5)(P_0). Then</u>

$$R^i_{\mathfrak{g}}(\overline{X}^L(\gamma_{\mathfrak{g}})) = \begin{cases} 0 & i \neq S \\ \overline{X}^G(\gamma) & i = S \end{cases}$$

This follows from [17] and technical results in [22]. What we want to do is generalize Conjecture 4.5 in some way that fits well with Proposition 6.1. It is not clear how to do this in maximal generality; but here is a special case, representing joint work with Jeffrey Adams.

<u>Conjecture 6.2</u> In the setting of Definition 4.2, assume that $\phi \in P^L_{\lim}(H)$; put $\overline{\gamma} = \overline{\phi} + \rho(\mathfrak{u})$. (This ϕ will play the role of $\gamma_{\mathfrak{g}}$, but γ need not actually exist as an element of $P^G_{\lim}(H)$. Assume that

a) $\overline{X}^L(\phi)$ is a one dimensional unitary representation;

b) $\langle\alpha,\overline{\gamma}-\rho(\ell)\rangle \geq 0$, all $\alpha \in \Delta(u,h)$;
here $\rho(\ell)$ is half the sum of the positive roots of h in ℓ (chosen to make $\overline{\gamma}$ dominant). Then

1) $R^i(\overline{X}^L(\phi)) = 0$, $i \neq S$;
2) $R^S(\overline{X}^L(\phi)) = Y(\mathfrak{g},\phi)$ is completely reducible (or zero). If strict inequality holds in (b), it is irreducible or zero;
3) $Y(\mathfrak{g},\phi)$ is unitarizable.

It should be emphasized that when (4.5)(P_0) fails, there may not be a γ with $\phi = \gamma_{\mathfrak{g}}$; so this conjecture is not a special case even of Conjecture 4.4. Adams has amassed a good deal of evidence for Conjecture 6.2, using the theory of dual reductive pairs; this will appear in a future paper. His work shows that, as ϕ varies subject to (6.2)(a), (b), the representations $Y(\mathfrak{g},\gamma)$ should be regarded as a single family, despite the fact that their Langlands parameters may vary irregularly. (It is in this sense that the Langlands classification is not a perfect context for discussing $\hat{G}_u$.)

Those mysterious representations which are of the form $Y(\mathfrak{g},\phi)$ are the ones we called dull representations in disguise. (We should also include in this class representations arising from some generalization of Conjecture 6.2 in which $\overline{X}^L(\phi)$ need no longer be one dimensional. This requires changing (6.2)(b) in some way which is still unclear.) As an example, let $G = Sp(n,1)$, $L = Sp(n-1,1) \times \mathbb{T}^1$. The choice of is unique up to conjugation. Fix $H = TA \subseteq L$. Up to $W(L,H)$ conjugacy, there are exactly n parameters $\phi \in P^L_{\lim}(H)$ which satisfy (6.2)(a) and (b), but not (4.5)(P_0). Call them $\phi_0,\phi_1,\ldots,\phi_{n-1}$, labelled by the integer $\langle\check{\alpha},\overline{\gamma}-\rho(\ell)\rangle$ for any $\alpha \in \Delta(u,h)$. It can be shown that Conjecture 6.2 is true in this case, and that the mysterious representations of $Sp(n,1)$ discussed in Section 5 are

$$\overline{X}_r\left(\pm\left(n-\left(\frac{r+1}{2}\right)\right)\right) \cong Y(\mathfrak{g},\phi_{n-r-1}).$$

The second kind of mysterious representation appears in Duflo's work [8] on complex groups of rank 2: he has the two components of the metaplectic representation for $Sp(2,\mathbb{C})$, and three unidentified representations for the complex G_2 as "mysterious representations" in the picture at the end of Section 5. To explain these, we need a more exotic analogue of Conjecture 4.5. It is not clear exactly what that should be, but here is the idea.

<u>Definition 6.3</u> Suppose $H \subseteq G$ is a θ-stable Cartan subgroup.

A reductive group M is called an <u>L-subgroup</u> of G if the following conditions hold.

a) M has a θ-stable Cartan subgroup isomorphic to, and henceforth identified with, H.
b) There is an inclusion $\Delta(m,h) \subseteq \Delta(\mathfrak{g},h)$, preserving the notions of compact and noncompact for imaginary roots.
c) There is a weight $\xi \in h^*$ such that

$$\Delta(m,h) = \{\alpha \in \Delta(\mathfrak{g},h) \mid \langle\check{\alpha},\xi\rangle \in \mathbb{Z}\}.$$

In the setting of algebraic groups, we are essentially requiring (among other things) that ${}^LM^0$ be the centralizer of a semisimple element in ${}^LG^0$. Both real polarizable and complex polarizable subgroups of G are L-subgroups. To give analogues of Conjecture 4.5 and Corollary 3.4, we need a correspondence between $P^G_{\lim}(H)$ and $P^L_{\lim}(H)$. This promises to be hard to define; the "ρ-shift" should be some special representation of H whose differential ξ satisfies (6.3)(c). Here is an example of what is wanted: we will take G complex, M as large as possible, and try to push the trivial representation of M up to G.

So assume G is a complex, connected, simple group, with $H = TA$ a θ-stable Cartan subgroup. The root system $\Delta(\mathfrak{g},h)$ is a union of two disjoint simple subsystems, interchanged by θ:

$$\Delta(\mathfrak{g},h) = \Delta^L(\mathfrak{g},h) \cup \Delta^R(\mathfrak{g},h).$$

Fix a positive system

$$(\Delta^L)^+(\mathfrak{g},h) \subseteq \Delta^L(\mathfrak{g},h),$$

and put

$$\begin{aligned}\Delta^+(\mathfrak{g},h) &= (\Delta^L)^+(\mathfrak{g},h) \cup (-\theta)(\Delta^L)^+(\mathfrak{g},h)\\ &= (\Delta^L)^+ \cup (\Delta^R)^+.\end{aligned}$$

Let β^L be the highest short root in $(\Delta^L)^+$. Fix once and for all a simple root α^L for $(\Delta^L)^+$, and set

$$m = \text{multiplicity of } (\alpha^L)^\vee \text{ in } (\beta^L)^\vee.$$

We may as well assume $m > 1$ (or all subsequent constructions are trivial). Define $\lambda^L \in h^*$ by

$$\langle\check{\alpha}^L,\lambda^L\rangle = 1$$
$$\langle\check{\varepsilon},\lambda^L\rangle = 0,\ \varepsilon \text{ a simple root} \neq \alpha^L.$$

Define

$$\Delta^L(m,h) = \{\varepsilon \in \Delta(\mathfrak{g},h) \,|\, \tfrac{1}{m}\langle \check{\varepsilon}, \lambda^L\rangle \in \mathbb{Z}\}$$
$$\Delta(m,h) = \Delta^L(m,h) \cup \theta(\Delta^L(m,h))$$
$$\Delta^+(m,h) = \Delta(m,h) \cap \Delta^+(\mathfrak{g},h).$$

There is a complex semisimple group M, with Cartan subgroup H and root system $\Delta(m,h)$. Its trivial representation has a Langlands parameter $\gamma_M \in P^M_{\lim}(H)$ characterized by

$$\bar{\gamma}_M = \rho(\Delta^+(m,h)) \in \mathfrak{a}^*.$$

Fix an element $w \in W(\Delta^L(\mathfrak{g},h))$ such that

$$w((\Delta^L)^+(m,h)) = (\Delta^L)^+(m,h).$$

Put

$$\lambda^w = \lambda^L - \theta(w\lambda^L) \in h^*$$
$$\bar{\gamma}^w = \bar{\gamma}_M + \frac{1}{m}\lambda^L .$$

It turns out that $\bar{\gamma}^w$ comes from a Langlands parameter

$$\gamma^w \in P^G_{\lim}(H),$$

at least if G is simply connected. The character of $\overline{X}^G(\gamma^w)$ is very nicely related to that of $\overline{X}^M(\gamma_M)$ (the trivial representation of M).

> Conjecture 6.4 Suppose G is a complex simple, simply connected Lie group. Fix a simple root α^L and a Weyl group element w as above, and define γ^w. Then $\overline{X}^G(\gamma^w)$ is unitarizable.

When $G = Sp(n,\mathbb{C})$ and α^L is long, then $M = SO(2n,\mathbb{C})$. There are two choices for w; the representations $\overline{X}(\gamma^w)$ are the two components of the metaplectic representation. For type G_2, one gets a total of three representations; they are Duflo's mysterious representations, and are unitarizable.

Even functoriality from L-subgroups is not enough to produce all mysterious representations, however. Consider for example $G = SO(2n,\mathbb{C})$, with $n \geq 4$. Induce from the one dimensional representation of a real parabolic with Levi factor $SO(2,\mathbb{C}) \times SO(2n-2,\mathbb{C})$. At the end of the interval of irredubility around zero, the subquotient X containing a K-fixed vector is a very singular unitary representation. (It is one of those studied in [21].) Its character is not nicely related to any characters on L-subgroups; apparently it must be regarded as a third kind of mysterious representation. One may perhaps expect to find such representations attached to nilpotent coadjoint orbits which are special

in the sense of Lusztig, and which have nothing to do with semisimple orbits in some sense. (For complex groups, the sense is that of Lusztig-Spaltenstein; for real groups the sense is not clear.)

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