VOGAN

Lowest Eigenvalue of the Laplacian on locally symmetric spaces (after J-S Li).

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Chicago

G connected, linear, simple; $\Gamma < G$ discrete, cocpt, torsion free; $K \underset{mc}{<} G$

G/K Riemannian, noncpt type.

$\rightsquigarrow \Delta$ Laplacian $[\leftrightarrow -\sum \partial_{x_i}^2$. (so eigenvalues are positive)$]$.

Given by action on $C^\infty(G/K)$ (on right) of $-\Omega$, the Casimir for G.

Γ acts (on left) of G/K properly discontinuously. Δ descends to $\Gamma \backslash G/K$. cpt Riemannian locally sym

Look at eigenvalues of Δ of $C^\infty(\Gamma \backslash G/K)$:

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

Problem: Find lower bound for λ_1 independent of Γ.

Repn Theory: $C^\infty(\Gamma\backslash G/K) = C^\infty(\Gamma\backslash G)^{K\text{-inv (right)}}$

integers ≥ 0.

$C^\infty(\Gamma\backslash G) =$ repn of G on right $= \bigoplus_{\pi \in \hat{G}_{\text{unitary}}} \boxed{m_\pi(\Gamma)} V_\pi$

Interested in repns of G with a K-fixed vector.

$G = KAN$ $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ $\rightsquigarrow \mathrm{Ind}_{MAN}^G(1 \otimes e^\nu \otimes 1) =: I(\nu)$ repn of G. (Normalized)

(1) $I(\nu)$ has unique K-fixed line $\Rightarrow$ $I(\nu)$ has unique irred subquotient (2) $J(\nu)$ s.t. $J(\nu)^K \neq 0$.

Thm. (3) Every irred admissible spherical repn with a K-fixed vector is a $J(\nu)$

little Weyl group.

(4) $J(\nu) \cong J(\nu') \Leftrightarrow \nu' \in W_A \cdot \nu$

(5) $\pi(\nu)[\Omega] = \langle \nu,\nu \rangle - \langle \rho_A, \rho_A \rangle$. ← little ρ.

($\leftarrow$ 0 if ν unitary.)

Write $\nu = \nu_{RE} + \nu_{IM}$.

$$= \underbrace{\langle \nu_{re}, \nu_{re} \rangle}_{\text{positive}} - \overbrace{\langle \nu_{im}, \nu_{im} \rangle}^{\text{negative}} - \langle \rho_A, \rho_A \rangle.$$

(6) [Helgason-Johnson]

[$\Leftrightarrow$ spherical functi bounded]

If $J(\nu)$ is unitary, then $\boxed{\nu_{re} \in \text{cnvx hull of } W_A \cdot \rho_A}$

(and hence $\langle \nu_{re}, \nu_{re} \rangle \leq \langle \rho_A, \rho_A \rangle$

(7) $J(\nu)$ unitary $\Rightarrow J(\nu_{re})$ unitary.

Conclusion: Eigenvalues of Δ on $\Gamma \backslash G/K$:

$$[\langle \rho_A, \rho_A \rangle - \langle \nu_{re}, \nu_{re} \rangle] + \langle \nu_{im}, \nu_{im} \rangle$$

for (unitary) $J(\nu)$ appearing in $L^2(\Gamma \backslash G)$.

Remark: Eigenvalue 0: $\mathbb{C} = J(\rho_A)$. $\nu = \nu_{re} = \rho_A$.

$\boxed{\text{Repn theoretic question: What's the biggest } \nu_{re} \text{ ~~[illegible]~~ (other than } \rho_A) \text{ with } J(\nu_{re}) \text{ unitary?}}$

Li solved this for G classical and shwed $J(\nu_{max})$ actually appears in some $L^2(\Gamma \backslash G/K)$. [Kazhdan: proved there is some $J(\nu_{max})$ but didn't say anything abt it.]

Today: Just the representation theory problem.

Sometimes won't get best answer, but works for G exceptional.

$G = SL(2,\mathbb{R})$.

$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}$ $\quad \mathfrak{a} \cong \mathbb{R}, \; \mathfrak{a}_c^* \cong \mathbb{C}. \quad \rho_A = 1.$

For ν real, $J(\nu)$ unitary $\iff \nu \in [-1,1]$.

Question above has no answer: $\pi(\nu)(-\Omega)$ arbitrary small for some unitary $\pi(\nu)$.

Happens for $SO(n,1)$, $SU(n,1)$ NOT otherwise (mostly Kazhdan).
Kostant: $Sp(n,1)$ $F_{4,1}$.

General Question: What can K-types of a unitary representation be (Control K-type in terms of infl char, etc.).

Our case: Know $J(\nu_{re})$ has
- trivial K-type
- some other K-types.

what can you deduce about Casimir eigenvalue?

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition; orthogonal for $\langle , \rangle$ ← positive on $\mathfrak{p}$ ← negative on $\mathfrak{k}$.

$$\Omega = -\sum_{i=1}^{\dim k} z_i^2 + \sum_{j=1}^{\dim p} y_j^2 .$$

v = K-fixed vector

so $z_i v = 0$.

$$\text{so} \quad \langle \Omega v, v \rangle = \sum_{j=1}^{\dim p} \langle Y_j^2 v, v \rangle = \overbrace{-\sum \langle Y_j v, Y_j v \rangle}^{\leq 0}$$

$$\pi(\Omega)\langle v, v \rangle.$$

Conclusion: $\pi(\Omega) \leq 0$.   Already knew that, but O.K.

Use other K-types:

(stupid) other K-types exist $\Rightarrow$ some $Y_j v \neq 0$ $\Rightarrow$ Casimir e-value ≤ 0.
Need some better ideas:

(1) Make precise the statement "other K-types occur"

Pick $T \subset K$ max'l torus, $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$:

$\hat{K} \longleftrightarrow$ dominant weights of T.

β = highest weight of K on $\mathfrak{p}_{\mathbb{C}}$.

Propn. If $J(\nu)$ is unitary & nontrivial. Then highest weights $m\beta$ occur in $J(\nu)$.

Parthasarathy: Dirac operator.

Thm. Suppose π unitary repn of G, contains a repn of K of highest weight $\mu \in \mathfrak{t}_{\mathbb{C}}^*$. Pick $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \rightsquigarrow \rho$.

Pick $w \in W(K,T)$ s.t. $w(\mu - \rho_n)$ dominant. Then

$$\pi(\Omega) \leq \langle w(\mu-\rho_n)+\rho_c, w(\mu-\rho_n)+\rho_c \rangle - \langle \rho, \rho \rangle.$$

Beautiful ~~Nice~~ Aside: (Pandzic-Huang) If equality holds, then the inf'l character of π must be given by $w(\mu-\rho_n)+\rho_c$.

Way to bound $\pi(\nu)(-2)$ above: Take $\mu = m\beta$, cleverly chosen $m \geq 1$,

Choose clever $\Delta^+(\mathfrak{t}_0, \mathfrak{k}_0)$.

Agrees with Li's bounds lots of the time; e.g.

$Sp(p,q)$ – always if $p \neq q$.

$SL(n,\mathbb{R})$ – n odd

$\vdots$

even when it fails, it's very close.