Vogan "Dirac Operators & Unitary Repns I" Lie groups 10/22/97

Goal: Analog for Dirac-operators of the Casselman-Osborne theorem about Lie algebra cohomology.
[Eventually will formulate a precise conjecture & give some evidence]

X Riemannian, homogeneous for G, G acting by isometries.
$= G/H$

Riemann structure gives H-inv't inner product B on $\mathfrak{g}/\mathfrak{h} \cong T_{eH}(G/H)$; i.e.

$$\boxed{Ad: H \to \underbrace{O(\mathfrak{g}/\mathfrak{h}, B)}_{\text{orthogonal group}}}$$

[Conversely: any H-inv't quadratic form B on $\mathfrak{g}/\mathfrak{h}$ gives a G-inv't Riemannian structure]

Spinors:

V real vector space, B symmetric bilinear form on V [Think $V = \mathfrak{g}/\mathfrak{h}$]

$C(V)$ = Clifford algebra of V

$C(V)$ = real associative algebra generated by V subject to

$$vw + wv = -2B(v,w) \quad (*)$$

= quotient of tensor algebra of V

∴ Inherits filtration

$$C_k(V) = \text{span of products of} \le k \text{ elements of } V \subset C_{k+j}(V)$$

Then

$$gr(C) = \sum_j C_j/C_{j-1} \cong \wedge V. \quad [(*) \text{ translates into structure constants of } \wedge V].$$

so $\dim C = 2^{\dim V}$. $\cong \mathbb{R}\cdot 1 \oplus$ (nilpotent ideal)

so its a pretty boring algebra

While $gr(C)$ is boring, C isn't. It's semisimple & central simple if V is even dimensional [center $= \mathbb{R}$].

Defn A space of spinors for V is a complex simple module S for $C(V)$. A generalized space of spinors is a complex module for $C(V)$.
(not necessarily simple)

Lemma. S always exists. It's unique if $\dim(V)$ is even; if $\dim V$ is odd, there are two (which will be interchangeable for later purposes).

$$\dim_{\mathbb{C}} S = 2^{[\dim V/2]}$$

$O(V)$ acts on $Cl(V)$ by automorphism (it respects (*)).

Lemma. There's a ~~natural map~~ subgroup $\mathrm{Spin}(V) \subseteq C^\times$ (units in $C(V)$) so that
(a) conjugation by $g \in \mathrm{Spin}\,V$ preserves $V \subset Cl(V)$, so defines a homomorphism $\pi: \mathrm{Spin}(V) \to O(V)$
(b) π is double cover of $SO(V)$.

Defn. A spin repn of $\mathrm{Spin}(V)$ is the action of $\mathrm{Spin}(V) \subset C^\times$ on a space S of spinors.

One way to define Spin:

- $v \in V$, $B(v,v)=1 \rightsquigarrow v$ is invertible in $C(V)$, inverse is $-v$.
- Compute $v\cdot w\cdot(-v) = -$(reflection of w through hyperplane $\perp v$)
- Define $\mathrm{Pin}(V)$ = group generated by all v's of norm 1.
  (So $\mathrm{Pin}(V)$ satisfies condition (a) of the Lemma)
- Take $\mathrm{Spin}(V)$ to be the identity component of $\mathrm{Pin}(V)$.

X Riemannian manifold.

TX : a bundle of inner product spaces over X.

$C(TX)$: Clifford bundle, vector bundle of dimension $2^{\dim X}$.

Defn (X Riemannian manifold) A bundle of spinors is a complex bundle of modules for $C(TX)$, simple at every point of X.
[A bundle of generalized spinors ... drop simple requirement.]

N.B. Bundles of spinors need not exist on arbitrary X. But bundles of generalized spinors always exist (e.g. $C(TX)$).

[BACK TO $X = G/H$].

Recall: G-eqvt bundle on G/H = representation of H.

$$G \times_H V = \mathcal{V} \longleftrightarrow \text{fiber } V \text{ at } eH.$$

$$\text{Section of } G \times_H V = \{\varphi : G \xrightarrow{\text{smooth}} V \mid \varphi(gh) = h^{-1}\cdot\varphi(g)\}.$$

In particular: Clifford bundle $\longleftrightarrow$ $C(\mathfrak{g}/\mathfrak{h}, B)$

~~$C(TX)$~~

$C(T(G/H))$

ASSUME: G/H has a G-invariant orientation; that is

$$\begin{array}{ccc}
\mathrm{Ad}: H & \longrightarrow & SO(\mathfrak{g}/\mathfrak{h}, B). \\
\uparrow & & \uparrow \\
\widetilde{H} & \longrightarrow & \mathrm{Spin}(\mathfrak{g}/\mathfrak{h}, B) \\
\uparrow & & \uparrow \\
\{1, \varepsilon\}. & & \{1, \varepsilon\}.
\end{array}$$

[i.e. define $\widetilde{H}$ to be pullback of spin double cover.]

The spin <u>representation</u> of $\widehat{H}$ is

$$\begin{tikzcd} \sigma: \widehat{H} \arrow[r] \arrow[dr] & \mathrm{End}(S) \\ & \mathrm{Spin}(\mathfrak{g}/\mathfrak{h}, B) \arrow[u] \end{tikzcd}$$

This representation satisfies that $\sigma(\varepsilon) = -1$.

<u>Lemma</u>. G/H has a G-inv ~~space~~ bundle of spinors iff $\widehat{H}$ has a one dimensional character χ such that $\chi(\varepsilon) = -1$.

<u>Propn</u> The G-inv bundles of generalized spinors on G/H correspond to representations τ of $\widehat{H}$ s.t. $\tau(\varepsilon) = -1$.

<u>Pf</u>. Fix (σ, S): space of spinors for $\widehat{H}$. Suppose $\mathcal{V}$ is a G-inv bundle of generalized spinors on G/H, so it gives rise to a repn V of H. (and a module for $C\ell(\mathfrak{g}/\mathfrak{h})$)
Form $W = \mathrm{Hom}_{C\ell(\mathfrak{g}/\mathfrak{h})}(S, V)$

Make $\widehat{H}$ act on W by its action σ on S & action of H on V.
Call that τ; then $\tau(\varepsilon) = -1$.

To go the other way: $V \cong W \otimes S$. //

SO: to each repn (τ, W) of $\tilde{H}$ with $\tau(\varepsilon) = -1$, get a bundle of generalized spinors $G \times_H (W \otimes S)$. [$W \otimes S$ descends to H]

FURTHER assume: $\mathfrak{g}$ has an $Ad(H)$-inv't decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. (Automatic if $\mathfrak{h}$ is cpt.)

Suppose $H \to SO(\mathfrak{g}/\mathfrak{h})$; $\overset{H\text{-inv't}}{\cancel{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{p}}$. Then fix (τ, W) a repn of $\tilde{H}$ s.t. $\tau(\varepsilon) = -1$.

Then the Dirac Operator is a first order G-inv't diff'l operator on sections of $G \times_H (W \otimes S) = \mathcal{V}$

Definition: take basis $X_1, \cdots, X_n$ for $\mathfrak{p}$
dual basis $X^1, \cdots, X^n$ w.r.t B: $B(X_i, X^j) = \delta_{ij}$.

Given a section φ of $\mathcal{V}$, ie $\varphi : G \to W \otimes S$, $\varphi(gh) = (\tau \otimes \sigma)(h^{-1}) \varphi(g)$.

$$D\varphi = \sum_{i=1}^{n} \left[\rho(X_i) \otimes C(X^i) \right] \cdot \varphi .$$

$\rho(X_i)$: differentiate φ on the right by X_i

$C(X^i)$: apply the Clifford algebra action of X^i on the S coordinate

Must check that $(D\varphi)$ has the correct transformation properties & that it's independent of the choice of basis X_i. [EASY]

Note: D commutes with the left action of G on sections of $\mathcal{V}$.
So get repns of G on kernel of D, image of D, ...

Next time: Study these repns.

Vogan: Dirac Operators & Unitary Repns II 11/5/97
(Lie Groups Seminar)

G Lie group.
K cpt subgroup $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Ad(K)-inv't decomposition.
Choose K-inv't positive definite symmetric form B on $\mathfrak{p}$ (i.e. G-inv't Riemannian structure on G/K).

$C(\mathfrak{p})$ = Clifford Alg of $\mathfrak{p}$.
S = space of spinors for $C(\mathfrak{p})$ [complex simple module for $C(\mathfrak{p})$].
S is unique (up to $\cong$) if dim $\mathfrak{p}$ even.
If dim($\mathfrak{p}$) odd, can form
S' = same vector space, different action of $C(\mathfrak{g})$.
$C'(Z) \cdot s^{\in S'} = -C(Z)s$ $Z \in \mathfrak{p}$ $C(Z)$ = module action of $Z \in \mathfrak{p}$.

Last time: defined $\mathrm{Spin}(\mathfrak{p}) \subset C(\mathfrak{p})^\times$ S is a repn σ of $\mathrm{Spin}(\mathfrak{p})$.
$$\mathrm{Spin}(\mathfrak{p}) \xrightarrow{2-1} SO(\mathfrak{p})$$

FACTS: (1) σ is independent of choice of S.
(2) σ is irred if dim $\mathfrak{p}$ odd; splits into two inequivalent irred pieces if dim ($\mathfrak{p}$) is even (write $S = S_+ \oplus S_-$ & $\sigma = \sigma_+ \oplus \sigma_-$ (+/- aren't defined canonically)
[Everything true for $\langle,\rangle$ not positive: get $SO(p,q)$'s, etc... & Maybe $\mathrm{Spin}(\mathfrak{p}) \to SO(\mathfrak{p})$ isn't surjective.]

ASSUME: action of K on $\mathfrak{p}$ gives $K \to SO(\mathfrak{p})$. Then pull back to
$$\tilde{K} \longrightarrow \mathrm{Spin}(\mathfrak{p})$$
double cover of K.
FIX repn τ of $\tilde{K}$ s.t. $\tau(\tilde{z}) = -\mathrm{id}$.

Then $\sigma\otimes\tau$ is a repn of K on $E := V_\tau \otimes S$.

Get $\begin{matrix}\mathcal{E}\\ \downarrow \\ G/K\end{matrix}$, bundle induced by E. The Dirac operator D acts on section of this bundle. Section were smooth $f : G \longrightarrow V_\tau \otimes S$ + trans. law. $f(gk) = (\sigma\otimes\tau)(k^{-1}) f(g)$

and D was gotten by choosing an o/n basis $Z_1 \cdots Z_n$ of $\mathfrak{p}$ and defining

$$D = \sum_{i=1}^{n} \underbrace{c(Z_i)}_{\text{act'g on } S} \otimes \underbrace{\rho(Z_i)}_{\text{right translation action}}$$

[terms don't, but sum does preserve transformation law.]

D is a 1st order elliptic operator which commuted with left action of G. So $\mathrm{Ker}(D)$, $\mathrm{Im}(D)$, ... are repns of G.

<u>GOAL</u>: Understand unitary repns related to kernel (D).

<u>PARTHASARATHY'S "PHILOSOPHY"</u>: Think of $C^\infty(G)$ built from matrix coeffs of smooth irred repns X_π. Given $v \in X_\pi$ & $\lambda \in X_\pi^*$, get matrix coeff $f_{\lambda\otimes v}(g) = \lambda(\pi(g)v)$. Gives a map

$$X_\pi^* \otimes X_\pi \longrightarrow C^\infty(G)$$

$\pi^*\otimes\pi$ action intertwines (left reg) $\otimes$ (right regular)

Contribution to sections of $\mathcal{E} \longrightarrow G/K$: look at $\varphi \in \mathrm{Hom}_K(E^*, X_\pi) \cong (X_\pi \otimes E)^K$; induces

$\Phi : X_\pi^* \longrightarrow$ sections of $\mathcal{E}$ [here $\varphi = \sum v_i \otimes e_i$]

$\lambda^\varphi \longmapsto \sum (f_{\lambda\otimes v_i}) e_i$.

[FROBENIUS]

<u>Idea</u>: $\mathrm{Hom}_K(E^*, X_\pi)$ controls occurrences of X_π^* in sections of E.
Dirac operator morally should correspond to an operator on $\mathrm{Hom}_K(V_\tau^* \otimes S^*, X_\pi)$. This idea works.

$$\cong \mathrm{Hom}_{\tilde{K}}(V_\tau^*, X_\pi \otimes S).$$

<u>Defn</u> X repn of $\mathfrak{g}$ with a compatible K action. Form $\tilde{X} = X \otimes S$; this carries a $\tilde{K}$ action (action on $X \otimes \sigma$). The Dirac operator on $\tilde{X}$ is

$$D = \sum Z_i \otimes c(Z_i)$$

acts in X ; c acts on S.

It commutes with $\tilde{K}$ action, so descends to

$$\tilde{X}_\tau \overset{\text{defn}}{=} \mathrm{Hom}_{\tilde{K}}(V_\tau, X \otimes S).$$

Morally: $\ker(D_\tau), \ldots$ should control appearance of X^* in a related space on G/K.

Assume G reductive, K maximally cpt.
X irred $(\mathfrak{g}, K)$-module. Then each $\tilde{X}_\tau$ is finite diml (since irred $(\mathfrak{g}, K)$ repns are admissible).

If X is unitary, then $\tilde{X}$ inherits a p.d. inner product & D_τ is self-adjoint.
<u>GOAL</u>: Understand the unitary X for which $\ker(D_\tau) \neq 0$.
[Has L^2 analog... but we'll stick to repn theory]

Hotta - Parthasarathy: most discrete series repns have this property for an appropriate τ.

... Schmid: All discrete series have this property.

construction of τ: $G \supset K \supset T$ maxl torus in K.

X discrete series $\rightsquigarrow$ global character $\Theta_X \big|_{T_{reg}} = \dfrac{\pm \sum_{w \in W_K} \varepsilon(w) e^{w\lambda}}{\text{(Weyl denominator) for } G}$

Schmid: choose τ s.t. $\boxed{\Theta_\tau = \dfrac{\sum_{w \in W_K} \varepsilon(w) e^{w\lambda}}{(\text{denom for } K)}}$ doesnt make sense (only up to sign) so really must pass to covering $\tilde{K}$ & that's what τ is supposed to be.

$[\lambda = \tau + \rho_c$ WCF$]$

Schmid's main idea: formula of Parthasarathy:

X $(\mathfrak{g}, K)$ module

~~XXXX~~ D acts on $X \otimes S$ (repn of $\tilde{K}$).

Fact: $D^2 = \Omega_G \otimes 1 - \Omega_{\tilde{K}} + C_{G/K}$.

Ω_G: casimir for G; $\Omega_{\tilde{K}}$: casimir for $\tilde{K}$ (acting diagonally) (use same form on $\mathfrak{p}$ extended to $\mathfrak{g}$, restricted to $\mathfrak{k}$)

FACT: D has a kernel $\iff$ D^2 does a kernel.

$\iff$ $X \otimes S$ has a $\tilde{K}$ τ-type whose Casimir eigenvalue "matches" Casimir eigenvalue for G.

("matches" up to constant $C_{G/K}$.)

In that case: $D_\tau^2 = 0$.

$K \supset T$ maxl torus

Write $\tau \in \mathfrak{t}^*$ for highest weight of repn of $\tilde{K}$.

$\rho_c \in \mathfrak{t}^*$ half sum of pos roots for $\tilde{K}$.

$(\mathrm{rk}\, G = \mathrm{rk}\, K)$ H-C isom $\mathfrak{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{t})^W$

infl char of $X \longleftrightarrow$ wt $\lambda \in \mathfrak{t}^*$

Thm
$$\boxed{\begin{array}{c} D_\tau^2 = 0 \\ \Updownarrow \\ \langle \tau+\rho_c, \tau+\rho_c \rangle = \langle \lambda, \lambda \rangle. \end{array}}$$

CONJECTURE: If X is unitary irred $(\mathfrak{g}, K)$-module & $D_\tau^2 = 0$, then inf'l char of X is equal to $\tau + \rho_c$

11/26: evidence, possible proof.

Certainly false for non-unitary repns.

How to prove something algebraically about unitary repns?

Vogan, "Dirac Operators III" (LieGroupsSem) 11/26/97

$T \subset K$ max'l ...

$\mathfrak{g}^t = \mathfrak{h}$ Cartan s...

$= \mathfrak{t} \oplus \mathfrak{p}^t$

$\mathfrak{t}^* \hookrightarrow \mathfrak{h}^*$

$\mathfrak{Z}(\mathfrak{g}) \cong S(\mathfrak{h})^W$

"infl char" = "homs $Z(\mathfrak{g}) \to \mathbb{C}$ are param'd by $\mathfrak{h}^*/W$

G connected real reductive, $\mathfrak{g}_0 = \mathrm{Lie}(G) = \mathfrak{k}_0 \overset{\theta}{\oplus} \mathfrak{p}_0$

K max'l cpt

$\langle , \rangle$ $\mathrm{Ad}(G)$ invariant symmetric nondegenerate form, $\mathfrak{k}_0 \perp \mathfrak{p}_0$.

$\langle , \rangle|_{\mathfrak{k}_0}$ negative, $\langle , \rangle|_{\mathfrak{p}_0}$ positive

$\rightsquigarrow C(\mathfrak{p}_0)$ Clifford algebra, $\overset{\mathbb{C}\text{-simple } C(\mathfrak{p})}{\text{module}}$ S

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & \mathrm{Spin}(\mathfrak{p}_0) \\ \downarrow & & \downarrow \\ K & \longrightarrow & SO(\mathfrak{p}_0) \end{array} \qquad 0 \to \{1, \varepsilon\} \to \tilde{K} \to K \to 0$$

σ repn of $\tilde{K}$ or $\mathrm{Spin}(\mathfrak{p}_0)$ on S.

X $(\mathfrak{g}, K)$-module

$\tilde{X} = X \otimes S$

Dirac operator $D = \sum Z_i \otimes Z_i$. $\{Z_i\}$ orthonormal basis of $\mathfrak{p}_0$

Can think of $D \in \mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}_0)$; acts on $\tilde{X}$.

(τ, V_τ) repn of $\tilde{K}$ with $\tau(\varepsilon) = -1$. Look at

$\tilde{X}_\tau = \mathrm{Hom}_{\tilde{K}}(\tau, \tilde{X})$.

D preserves $\tilde{X}_\tau$ [commutes with diagonal action of $\tilde{K}$, even in $\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p}_0)$.]

(i.e. $X \otimes S$)

Call the restriction D_τ.

<u>Conjecture</u> A Suppose X irred unitary, $\ker D_\tau \neq 0$. Then X has infl char $\tau + \rho_c \in \mathfrak{t}^* \hookrightarrow \mathfrak{h}^* \to \mathfrak{h}^*/W$. (Fixed pos roots for $\mathfrak{t}$ in $\mathfrak{k}$, defines ρ_c. Then τ is h/w of V_τ wrt this choice of roots.)

Remark. Once X fixed, conjecture says only finitely many τ with $D_\tau \neq 0$. Some evidence: There's only one τ.

Ex. X = discrete series repn, HC parameter $\lambda \in \mathfrak{t}^*$.

$\lambda \rightsquigarrow \Delta^+$ pos roots for $\mathfrak{t}$ in $\mathfrak{g}$.

$\rho = \rho_c + \rho_n$

Hecht-Schmid: X has a K-type μ of highest weight $\lambda + \rho - 2\rho_c = \lambda + \rho_n - \rho_c$.

Kostant (others) unpublished: S has a $\tilde{K}$ of lowest weight $-\rho_n$ [Borel-Wallach book]

Conclude: $X \otimes S$ has $\tilde{K}$ type of extremal weight $(\lambda + \rho_n - \rho_c) + (-\rho_n)$ $= \lambda - \rho_c$. This is dominant for K, so automatically a highest weight $\tau = \lambda - \rho_c$. So $\tau + \rho_c = \lambda$ = i/c of discrete series.

Conjecture suggests (in fact; it's true).

$D_\tau = 0$ on nonzero space X_τ. so this is some evidence

Zuckermann-Vogan: Classification of unitary repns with $\ker D_\tau \neq 0$ assuming $|\langle \tau + \rho_c, \alpha^\vee \rangle| \geq 1$ for all roots α.

(i.e. G linear, equivalently, $\tau + \rho_c$ regular)

All have infl. character $\tau + \rho_c$.

This is to be interpreted as more serious evidence.

Recall: D_τ^2 is a scalar operator (if X irreducible)

scalar = (eigenvalue of casimir operator for G in X)
— (what the eigenvalue would be for $\tau + \rho_c$).

(*) Conclusion: ~~D_τ has a kernel iff conjecture i~~

Fix X with $\tilde{X}_\tau \neq 0$. Then D_τ has a nonzero kernel iff Casimir eigenvalue for X is as predicted by conjecture

This is to be interpreted as bad & problematic. Here's why:

Ex. $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$K = SO(2) \times SO(2)$, $\tilde{K} = K \times \{1, \varepsilon\}$.

$\hat{K} = \mathbb{Z} \times \mathbb{Z}$, S = 4 diml repn of $\tilde{K}$; ε acts by -1,

K acts by 4 weights ~~$(\pm 1, 0), (0, \pm 1)$~~ $(\pm, \pm)$,

Define $J(\nu)$ = irred spherical repn of G $(\nu \in \mathbb{C}^2)$.

Know: $J(\nu)$ contains trivial K-type, has inf'l character (ν_1, ν_2) / sign changes in both coor

Take $X = J(\nu)$

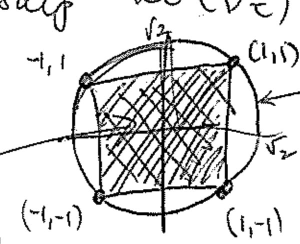
$\tilde{X} = J(\nu) \otimes S$ has $\tilde{K}$-type $\overbrace{(1,1)}^{=\tau}$, e.g..

Conjecture says: • if $J(\nu)$ is unitary, and $\ker D_\tau \neq 0$. Then $J(\nu)$ has i/c $(1,1)$.

• So says: $J(\nu)$ unitary $\Rightarrow$ $\nu = (1,1)$.

• On the other hand conclusion (*) says $\ker(D_\tau) \neq 0$ iff

$\nu_1^2 + \nu_2^2 = 1^2 + 1^2 = 2$.



circle of principle series repns with $\ker D_\tau \neq 0$.

unitary ones are in this square.

$J(\nu) \overset{\nu \text{ real}}{\Longleftrightarrow} \nu \in [-1,1] \times [-1,1]$.

So conjecture survives, but definitely false for non-unitary reps. (E.g. $\nu = (0, \sqrt{2})$, say.)

got nontrivial $\ker D_\tau$, but can't control i/c.

Turns out: ~~operator D_τ has~~

- take $\nu = (\nu_1, \nu_2)$ with $\nu_1^2 + \nu_2^2 = 2$. If $\nu \neq (\pm 1, \pm 1)$. Then D_τ is nilpotent ($D_\tau^2 = 0$) has Jordan blocks of size 2; i.e. $\ker D_\tau = \mathrm{Im}(D_\tau)$.

Suggest the following:

Conjecture B Suppose ~~$0 \neq \ker D_\tau$ AND~~ $\ker D_\tau \not\subset \mathrm{im}\, D_\tau$ (stronger than assuming its nonzero; equivalent when D_τ is self-adjoint, which it is if X is self-adjoint). Then X has i/c $\tau + \rho_c$.

This statement has nothing to do with unitarity. Here's a sketch of a proof.

Work in algebra $U(\mathfrak{g}) \otimes_{\mathbb{C}} C(\mathfrak{p}) \ni D$. $\overset{\text{diagonal } K}{K_\Delta}$ acts by $\mathrm{Ad} \otimes (\text{action on } \mathfrak{p})$; $D \in [U \otimes C(\mathfrak{p})]^{K_\Delta}$.

Now $[U \otimes C(\mathfrak{p})]^{K_\Delta} \supset Z(\mathfrak{g}) \otimes 1$ (or even $U(\mathfrak{g})^K \otimes 1$).

Have Lie algebra map $\mathfrak{k} \longrightarrow C(\mathfrak{p})$

$\mathfrak{k} \longrightarrow so(\mathfrak{p}) \longrightarrow C(\mathfrak{p})$.

Have trivial $\overset{\text{Lie alg}}{\text{map}}$: $\mathfrak{k} \xrightarrow{\text{Lie}} \mathcal{U}(\mathfrak{g})$

So get $\mathfrak{k}_\Delta \xrightarrow{\text{Lie}} \mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p})$

$$Z \longmapsto Z \otimes 1 + 1 \otimes \varphi(Z).$$

Universality : $\mathcal{U}(\mathfrak{k}_\Delta) \xrightarrow{\text{alg}} \mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p})$.

Centralizer of $\mathcal{U}(\mathfrak{k}_\Delta) = [\mathcal{U} \otimes C(\mathfrak{p})]^{K_\Delta}$. So

$$\mathfrak{Z}(\mathfrak{k}_\Delta) \longleftrightarrow [\mathcal{U} \otimes C(\mathfrak{p})]^{K_\Delta}$$

In algebra $[\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p})]^{K_\Delta}$, have ~~have~~

- element D
- subalgebra $\mathfrak{Z}(\mathfrak{g}) \otimes 1$ $\quad (\cong S(\mathfrak{h})^W)$
- subalgebra $\mathfrak{Z}(\mathfrak{k}_\Delta)$ (sits diagonally).

<u>CONJECTURE</u> C Suppose $z \in \mathfrak{Z}(\mathfrak{g})$. Then $\exists$ $\xi(z) \in \mathfrak{Z}(\mathfrak{k}_\Delta)$ so that

$$z \otimes 1 = \xi(z) + a_1 D + D a_2$$

for some $a_1, a_2 \in [\mathcal{U}(\mathfrak{g}) \otimes C(\mathfrak{p})]^{K_\Delta}$. [Like the H-C homomorphism...]

<u>Propn</u> Conjecture C $\Rightarrow$ Conjecture B.

<u>Pf.</u> Assuming $\ker D_\tau \neq 0$, $\not\subseteq \operatorname{im} D_\tau$. Means $\exists$ $\tilde{x} \in X \otimes S$ which: transforms by τ under $\tilde{K}$; is killed by D; is not in $\operatorname{im}(D)$.

~~Apply $z \otimes 1$ to $\tilde{x}$~~

Then $\xi(z) \in Z(\mathfrak{k}_\Delta)$ so acts by scalar $\tau(\xi(z))$ on τ:

Now apply

$$z \otimes 1 - \tau(\xi(z))$$

to $\tilde{x}$.

$$\left[z\otimes 1 - \tau(\xi(2))\right]\hat{x}$$

Get: $\left[\begin{matrix}\text{infl char}\\ \text{of } X \text{ in } z\end{matrix} - (\tau(\xi(2)))\right]\hat{x}$ on one hand.

On the other hand, get:

$$[\xi(z) - \underbrace{\cancel{\tau(\xi(2))}]}_{\substack{0\\ \text{by assumption}}}\cdot\hat{x} \; \underbrace{\cancel{+ a_1 D\hat{x}}}_{0}^{\hat{x}\in\ker D} + \boxed{Da_2\hat{x}.}$$ (from Conjecture C)

So conclude:

$$\text{Scalar}\cdot\hat{x} = D(\overbrace{a_2\hat{x}}^{\text{something}})$$

So conclude (since $\hat{x}\notin \operatorname{Im} D$), scalar $=0$. So

z acts by $\tau(\xi(2))$

To see that gives the infl char. Zariski denseness of the ~~...~~ positive examples ("generic τ") gives the conclusion. ∎

<u>Remark</u>. Conjecture C is true (for stupid reasons) for Z = Casimir operator. True for $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$.

<u>Remark</u>. Very similar to Casselman–Osborne theorem. There should be a universal approach to both.

<u>Remark</u> $\xi(2)$ in conjecture C <u>is</u> unique. (Not obvious but true. Can see from "generic case" & Zariski denseness.)

<u>Lusztig's Q</u>: Do D, $Z(\mathfrak{g})\otimes 1)$ & $\mathfrak{z}(\mathfrak{k}_0)$ generate whole algebra??

<u>Vogan</u>: replace $Z(\mathfrak{g})$ by $U(\mathfrak{g})^{k}$???