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David A. Vogan, Jr.
**Representations of
Real Reductive
Lie Groups**

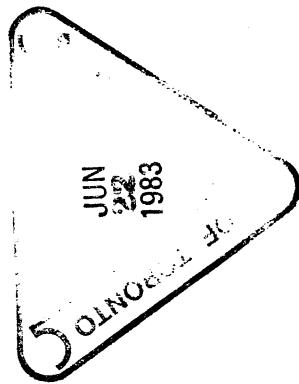
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To Jonathan, for eating the first draft.

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Preface

Since this manuscript was completed in August, 1980, a great deal of progress has been made on some of the topics treated. Working independently, Brylinski-Kashiwara and Beilinson-Bernstein have established Conjecture 2.1.7 on the composition series of Verma modules. Lusztig and I have used the work of Beilinson and Bernstein to prove Conjectures 2.2.12 and 2.3.11 on Harish-Chandra modules, computing all composition series in the case of integral infinitesimal character. [This settles Problems 9 and 10 of Section 9.7.] The non-integral case remains open. The proofs rely on the Weil conjectures and the Deligne-Goresky-MacPherson intersection homology theory. It is not yet clear how these developments will affect the material presented in this book -- they do not provide any obvious substantial simplifications, although in some cases they suggest significant improvements in the formulation of the theory.

I would like to thank Janet Ellis for typing and proof-reading a long and messy manuscript with great care and skill in the midst of many other responsibilities. I have the dubious satisfaction of knowing that no errors remain which are not of my own devising.

Cambridge, Massachusetts

April, 1981

Introduction

This book is a survey of some recent work on the (non-unitary) infinite dimensional representations of a real reductive Lie group G . There are three major topics. The first is Langlands' theorem on the classification and realization of the irreducible representations of G (Theorem 6.5.12). They are described in terms of certain "standard representations" (Definition 6.5.2), which generalize the principal series and are sometimes reducible. The second topic is the reducibility of these standard representations. The main results (Theorem 8.6.6 and Proposition 8.7.6) are due to B. Speh and the author; they are not quite decisive. The third topic is a conjecture which describes explicitly the decomposition of standard representations into irreducible representations -- or, equivalently, the Harish-Chandra characters of the irreducible representations. This generalizes a conjecture of Kazhdan and Lusztig for Verma modules, and is described in Section 9.6.

Since the theory of non-unitary representations of G was created by Harish-Chandra essentially as a means to study unitary representations, some apology might seem to be required for a book in which unitary representations play almost no part. The first explanation for this omission is simply a lack of space. The first two topics at least are very important for recent work on unitary representations. For example, the study of unitary representations with non-zero continuous cohomology (see [31]) has been advanced by the algebraic study of certain representations which are still

not known to be unitary. (They are included in the conjecturally unitary representations of (6.5.17) below.) The theory of "complementary series" of unitary representations depends on (among other things) an understanding of the reducibility of standard representations. Thus Theorem 8.6.6 provides a large number of unitary representations, and a proof of the conjecture of Section 9.6 would provide even more.

The real explanation, however, is that non-unitary representation theory is interesting enough not to require any such justification. Such a claim has to be supported in the text and not in the introduction; but Chapter 2 attempts to describe the nature of the main results without too much technical clutter.

In a little more detail, the book is organized as follows. The reader is assumed to be quite familiar with the structure and finite dimensional representation theory of complex reductive Lie algebras; this is really a prerequisite even for understanding the statements of most of the results. Logically, the book also depends on Harish-Chandra's basic theory relating group representations and Lie algebra representations ([12]) and on his subquotient theorem ([13]). These topics are treated in [49] or [50], and the results are summarized in Sections 0.3 and 4.1 of this book. Since the ideas needed to prove them will not be used here, the reader who is willing to take them on faith should have little difficulty. The first three sections of Chapter 1 summarize, with some proofs, the representation

theory of $SL(2, \mathbb{R})$. This is intended to provide examples as guides to the rather abstract and technical treatment of the general case. Some results are proved in general by reduction to $SL(2, \mathbb{R})$, and the necessary special cases of these are discussed in the rest of Chapter 1.

Chapter 2 contains a detailed statement of the Langlands classification of irreducible representations of G in a special case; and a geometric formulation of the conjecture of Section 9.6 on composition series of standard representations. (The two formulations are not known to be equivalent.) The entire chapter is meant as an extended introduction to the rest of the book.

The main technical tool used here to study representations is Lie algebra cohomology (of the nil radical of a parabolic subalgebra, with coefficients in a representation). Chapter 3 contains two fundamental theorems in that subject: the Casselman-Osborne theorem relating cohomology and the center of the enveloping algebra, and Kostant's formulation of the Bott-Borel-Weil theorem.

Chapter 4 discusses that part of the classification of irreducibles which can be obtained from ordinary principal series representations. In addition to more standard intertwining operator techniques, it uses the Bernstein-Gelfand-Gelfand theory of fine representations (see [2]); a detailed account of this theory is given in Section 4.3.

Chapters 5 and 6 complete the classification of irreducible representations. The method is discussed in some detail in the introductions to those chapters. Essentially

it is a generalization of the highest weight theory of finite dimensional representations, with highest weight vectors replaced by a more general kind of cohomology classes for the representation. The main problem is to find a construction of representations, generalizing induction, which is nicely related to cohomology. This was done (for reductive groups) by G. Zuckerman (Definition 6.3.1). This book uses only one special case of this definition, in addition to ordinary induction. It seems likely that one can do much more with the idea.

Chapter 7 is devoted to the Jantzen-Zuckerman "translation principle" and related matters. This says that all irreducible representations come in nice families (like the principal series, or finite dimensional representations). Many results can therefore be proved by reducing to the case when the representation is in "general position" in some sense. This is helpful technically, but the translation principle plays an even more fundamental role. Roughly speaking, it provides a connection between the structure of irreducible representations, and the combinatorial structure of the Weyl group. This is discussed more carefully in Section 7.3; the key result is Theorem 7.3.16.

In Chapter 8, the basic theorem on reducibility of standard representations is proved. This is in principle a trivial consequence of the results of Chapter 7, but requires some messy calculations. (For example, we need the Hecht-Schmid character identities for discrete series from [15].) Exactly the same ideas, with the judicious addition

of a technical conjecture (Conjecture 7.3.25), lead to the algorithm for computing composition series; this is the content of Chapter 9.

The book differs from the existing literature in several ways. Most importantly, the standard representations are not constructed by Langlands' method (that is, ordinary induction from discrete series). We use instead Zuckerman's "cohomological" induction from principal series. This gives isomorphic standard representations (Theorem 6.6.15), but that fact is not proved in the text (or used). As far as the classification itself goes, this choice is simply a matter of taste. However, the conjecture on composition series seems to be comprehensible only in the realization we have given. (There is an obvious way to try to use ordinary induction, but then any simple analogue of the critical Theorem 9.5.1 is false.) This is not to say that Zuckerman's realization is better; for analytic problems, it is Zuckerman's method which encounters obstacles.

There are many less substantive changes. What I had perceived as the main theorem of [42], which relates cohomology and $U(\mathfrak{g})^{\bar{K}}$, does not appear here; the argument has been rearranged slightly to eliminate the need for it. The definition of lowest K -type in [42] (Definition 5.4.18 below) has been replaced by a much more technical one (Definition 5.4.1); they are equivalent for irreducibles but not for general $(\mathfrak{g}, K)$ modules. The advantage of the new definition is that a number of subsequent technical arguments become much simpler when it is used.

The proof of the Knappp-Stein reducibility theorem (Corollary 4.4.11) is new, avoiding both the rather delicate analysis used in [26] and the long case-by-case computation in the unpublished second part of [42].

The main result about translation "across a wall" is Theorem 7.3.16; it is (or was) the only non-formal part of the proof of the theorem on reducibility of standard representations. About one-third of [39] is devoted to a proof of it, which might charitably be described as sketchy. A second proof was given in [45], which was fairly short, but used Duflo's main theorem on primitive ideals (see [8]). The proof given here is entirely trivial; but the result is labelled as a theorem in memory of [39].

Each chapter begins with an introduction; these provide a more detailed guide to the main results. Section 9.7 summarizes some open problems.

This book is based on lectures given at MIT during the 1979-80 academic year. I would like to thank those who attended for helpful comments and dogged perseverance. Much of Chapter 6 is unpublished work of G. Zuckerman. I thank him for explaining it to me, and allowing it to appear here. J. Vargas provided a list of errors in the first draft, which was very helpful. Many people have pointed out particular errors and shortcomings; I apologize for those which undoubtedly remain.

The author was supported in part by a grant from the National Science Foundation during the preparation of this book.

Chapter 0. Preliminaries

General references for this chapter are [40] and [50].

§1. Assumptions on G

Notation 0.1.1.1 Suppose H is a real Lie group. Write

H_0 = identity component of H ;

$\mathfrak{h}_0$ = Lie algebra of H ;

$\text{Ad}: H \rightarrow \text{End}(\mathfrak{h}_0)$, the adjoint representation of H ;

$\mathfrak{h} = (\mathfrak{h}_0)_{\mathbb{C}}$, the complexification of $\mathfrak{h}$;

$U(\mathfrak{h})$ = universal enveloping algebra of $\mathfrak{h}$.

$\mathfrak{h}$ is given the complex structure bar, defined by

$$\overline{X + iY} = X - iY \quad (X, Y \in \mathfrak{h}_0).$$

This notation will be applied to groups denoted by other Roman letters in the same way without comment.

By a real reductive linear group, we will mean a real Lie group G (not necessarily connected), a maximal compact subgroup K of G , and an involution θ of $\mathfrak{g}_0$, satisfying the following conditions.

- | | | |
|---------|---|---|
| (0.1.2) | { | <ul style="list-style-type: none"> (a) $\mathfrak{g}_0$ is a real reductive Lie algebra; (b) If $g \in G$, the automorphism $\text{Ad}(g)$ of $\mathfrak{g}$ is inner
(for the corresponding complex connected group); (c) The fixed point set of θ is $\mathfrak{h}_0$; (d) Write $\mathfrak{p}_0$ for the -1 eigenspace of θ; then the
map $K \times \mathfrak{p}_0 \rightarrow G, (k, X) \rightarrow k \cdot \exp(X)$ is a diffeomorphism; (e) G has a faithful finite dimensional representation; |
|---------|---|---|

{f) Let $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ be a Cartan subalgebra, and let H be the centralizer of $\mathfrak{h}_0$ in G. Then H is abelian.

(Notice that (d) forces G to have only finitely many components.)

Throughout this book, G will denote a real reductive

linear group. We will from time to time assume that G satisfies additional conditions, but these at least must always be met. Conditions (a) - (d) define Harish-Chandra's category of real reductive groups (cf. [40], §5) and require no further justification here; we will use inductive arguments which lead from connected groups to disconnected ones. Assumption (e) is in some sense only a convenience -- most of the results we obtain can be gotten without it, although sometimes this requires more work and a less satisfactory formulation of the theorems. However, one of our main goals is the formulation of the Kazhdan-Lusztig conjectures discussed in the introduction; and this has not been carried out for non-linear groups. (The problems do not seem to be very deep, but they are quite messy.) Assumption (f) is included chiefly to make the Knapp-Stein "commutativity of intertwining operators" theorem (Corollary 6.5.14) hold. (The simplest case where it fails has $|G/G_0| = 4$, $G_0 = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$; (a) - (e) are satisfied, but (f) is not.)

Definition 0.1.3 Make θ an involution of G by setting $\theta(k \cdot \exp(X)) = k \cdot \exp(-X)$ ($X \in \mathfrak{g}_0$, $k \in K$). We call θ the Cartan involution of G or $\mathfrak{g}_0$.

Example 0.1.4

- a) $G =$ connected real linear semisimple group;
- b) $G = \mathrm{GL}(n, \mathbb{R})$, $K = \mathrm{O}(n)$, $\theta(g) = {}^t g^{-1}$;
- c) $G =$ real points of a reductive algebraic group defined over $\mathbb{R}$;

d) If $G_{\mathbb{C}}$ is a complex semisimple Lie algebra, and $\mathfrak{g}_0$ is a real form of its Lie algebra $\mathfrak{g}$, then

$$G = \text{normalizer in } G_{\mathbb{C}} \text{ of } \mathfrak{g}_0;$$

$$\text{e) } G = \mathrm{SL}(2, \mathbb{R}) \cup \mathrm{SL}(2, \mathbb{R}) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

f) Suppose G is a real reductive linear group, and

H is a θ -stable abelian subgroup. Then the centralizer G^H of H in G is a real reductive linear group.

Of course, these examples overlap enormously. We will make constant use of example (f); its (easy) verification is left to the reader.

Example (b) illustrates an annoying technical problem: K need not be a real reductive group. The problem is that the orthogonal group $\mathrm{O}(n)$ does not satisfy (0.1.2)(b). Thus the Cartan-Weyl highest weight theory does not apply directly to K ; we do not have a priori a perfect grasp on $\hat{K}$. This is circumvented by using the relation between K and G ; a description of $\hat{K}$ is given in Section 5.1.

Definition 0.1.5 A Cartan subgroup of G is the centralizer in G of a Cartan subalgebra of $\mathfrak{g}_0$. A parabolic subgroup of G is the normalizer in G of a parabolic subalgebra of $\mathfrak{g}_0$.

Lemma 0.1.1.6

- a) Suppose $P \subseteq G$ is a parabolic subgroup. Then $G = K \cdot P$.
- b) Suppose $H \subseteq G$ is a Cartan subgroup. Then H is conjugate under G to a θ -stable Cartan subgroup H' (that is, $\theta H' = H'$).
- c) Two θ -stable Cartan subgroups of G are conjugate under G if and only if they are conjugate under K .

These results are trivial consequences of the corresponding ones for connected groups (cf. [50], 1.2 and 1.3). They are included mostly as an illustration; we will use other structure theorems (Iwasawa decomposition, Langlands decomposition for parabolics, and so forth) as needed, although the references may treat only connected semisimple groups.

Finally, we fix once and for all a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, which we denote by $\langle \cdot, \cdot \rangle$. We can and do choose this form so that

$$(0.1.7) \quad \begin{aligned} &\text{the Cartan decomposition } \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \\ &\text{is orthogonal for } \langle \cdot, \cdot \rangle; \\ &\langle \cdot, \cdot \rangle \text{ is positive definite on } \mathfrak{p}_0 \text{ and negative definite on } \mathfrak{k}_0. \end{aligned}$$

This form will be restricted, complexified, and transferred to dual vector spaces without change of notation.

§2. Roots and $Z(\mathfrak{g})$.

Through Lemma 0.2.2, H will be a fixed θ -stable Cartan subgroup of G . We write V^* for the dual of a vector space V .

Definition 0.2.1 The vector part of H is

$$A = H \cap \exp(-\mathfrak{p}_0) = \exp(\mathfrak{k}_0 \cap \mathfrak{p}_0).$$

The compact part of H is

$$T = H \cap K.$$

The set of roots of $\mathfrak{h}$ in $\mathfrak{g}$ is written $\Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$.

A root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is called real if $\alpha|_{\mathfrak{t}} = 0$; imaginary if $\alpha|_{\mathfrak{a}} = 0$; and complex if it is neither real nor imaginary. Suppose α is an imaginary root; let $X_\alpha \in \mathfrak{g}$ be a root vector for it. Then α is called compact imaginary if $X_\alpha \in \mathfrak{k}$, and noncompact imaginary otherwise. (Harish-Chandra calls these latter roots singular imaginary.)

Lemma 0.2.2 ([50], 1.3 and 1.4). In the notation just defined:

- a) $H = T A$, a direct product of Lie groups. A is isomorphic to its Lie algebra via $\exp$, and T is a compact abelian Lie group (possibly disconnected);
- b) As linear functionals on $\mathfrak{h}_0$, the roots take real values on $\mathfrak{a}_0$ and purely imaginary values on $\mathfrak{t}_0$; so a root is real, imaginary, or complex as its values on $\mathfrak{h}_0$ are;
- c) If we make the Cartan involution θ act on $\mathfrak{h}^*$ by duality, then $\Delta(\mathfrak{g}, \mathfrak{h})$ is θ -stable. Furthermore, if α is a root, then

$$\alpha \text{ is real} \iff \theta\alpha = -\alpha$$

$$\alpha \text{ is imaginary} \iff \theta\alpha = \alpha$$

α is complex $\Leftrightarrow \theta\alpha \neq \pm\alpha$;

d) If we make the complex structure bar act on $\mathfrak{h}^*$ by duality (notation (0.1.1)), then

$$\bar{\alpha} = -\theta\alpha.$$

Sketch of Proof. Part (a) follows from hypothesis (0.1.2) (d) on G. One knows that $\mathfrak{a}_0$ has real eigenvalues on $\mathfrak{g}_0$; and a character of a compact group (such as a root regarded as a character of T) has a purely imaginary differential. This proves (b). Since θ is an automorphism of $\mathfrak{g}$, (c) is obvious. Part (d) follows from (b). Q.E.D.

Definition 0.2.3 Let H be an abelian Lie group, and V a finite dimensional semisimple representation of H. (This means that the action of H on V is diagonalizable.) If χ is a character of H (that is, $\chi \in \hat{H}$), define the χ weight space of V

$$V(\chi) = \{v \in V \mid h \cdot v = \chi(h)v, \text{ all } h \in H\},$$

and the set of weights of H in V

$$\Delta(V) = \Delta(V, H) = \{\chi \in \hat{H} \mid V(\chi) \neq 0\}.$$

We may often regard $\Delta(V)$ as a "multiset"; that is, we assume that χ occurs in $\Delta(V)$ with multiplicity $\dim V(\chi)$. Similarly, if V is just a representation of the Lie algebra $\mathfrak{h}$ of H, and $\lambda \in \mathfrak{h}^*$, define the λ -weight space and the set of weights of V by

$$V(\lambda) = \{v \in V \mid h \cdot v = \lambda(h)v, \text{ all } h \in \mathfrak{h}\}$$

$$\Delta(V) = \Delta(V, \mathfrak{h}) = \{\lambda \in \mathfrak{h}^* \mid V(\lambda) \neq 0\}.$$

Finally define,

$$\rho(\Delta(V)) = \rho(V) = \frac{1}{2} \sum_{\lambda \in \Delta(V)} \lambda \in \mathfrak{h}^* \quad (\text{with multiplicities}).$$

So if $h \in \mathfrak{h}$, $\rho(V)(h)$ is half the trace of h on V. We include multiplicities in $\Delta(V)$ so that formulas like $\det_V(h) = \prod_{\chi \in \Delta(V)} \chi(h)$ ($h \in H$) hold. A very important special case is when $\mathfrak{h} \subseteq \mathfrak{g}$, and $\mathfrak{s} \subseteq \mathfrak{g}$ is stable under $\text{ad}(\mathfrak{h})$; then expressions like $\rho(\mathfrak{s})$, half the sum of the weights of $\mathfrak{h}$ in $\mathfrak{s}$, will appear often. The most important special case is when $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and

$$\mathfrak{h} = \mathfrak{h} + \mathfrak{n}$$

is a Borel subalgebra of $\mathfrak{g}$, with nil radical $\mathfrak{n}$. When the choice of such a $\mathfrak{h}$ is clear, we may write simply

$$(0.2.4) \quad \rho = \rho(\mathfrak{n}) = \rho(\Delta^+(\mathfrak{g}, \mathfrak{h})) = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha;$$

here $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is the positive root system corresponding to $\mathfrak{n}$.

Definition 0.2.5 Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra. If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$, the simple reflection about α is the automorphism of $\mathfrak{h}^*$ given by

$$\lambda \mapsto s_\alpha(\lambda) = \lambda - (2\langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle) \alpha.$$

The coroot corresponding to α is

$$\check{\alpha} = 2\alpha / \langle \alpha, \alpha \rangle.$$

(Thus we can write

$$s_{\alpha}(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha.$$

The (complex) Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$, $W = W(\mathfrak{g}, \mathfrak{h})$ is the group generated by the various s_{α} ; it is regarded as a group of automorphisms of $\mathfrak{h}^*$ or (by duality) of $\mathfrak{h}$.

Definition 0.2.6 Suppose $H \subseteq G$ is a θ -stable Cartan subgroup. The Weyl group of H in G , $W(G, H)$ is by definition the quotient of the normalizer of H in G by H :

$$\begin{aligned} W(G, H) &= N_G(H)/H \\ &= N_K(H)/(H \cap K); \end{aligned}$$

the second equality is [50], Proposition 1.4.2.1. We regard $W(G, H)$ as a subgroup of $W(\mathfrak{g}, \mathfrak{h})$ when this is convenient.

Definition 0.2.7 The center of the universal enveloping algebra of $\mathfrak{g}$ is written $Z(\mathfrak{g})$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ (with nil radical $\mathfrak{n}$). Let $\mathfrak{n}^-$ denote the opposite nil radical, so that

$$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}$$

a) [a triangular decomposition of G]. By Poincaré-Birkhoff-Witt, write

$$b) \quad U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$$

$$c) \quad = U(\mathfrak{h}) \otimes (\pi^{-1}U(\mathfrak{g}) + U(\mathfrak{g})\pi),$$

and let

$$\xi : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})$$

be the projection on the first factor in (c). Let

$$\rho = \rho(\pi)$$

be as in (0.2.4). Regard $U(\mathfrak{h})$ as the algebra of polynomial functions on $\mathfrak{h}^*$, and define

$$(T_{\rho}f)(\lambda) = f(\lambda - \rho) \quad (f \in U(\mathfrak{h}), \lambda \in \mathfrak{h}^*).$$

The Harish-Chandra map from $Z(\mathfrak{g})$ into $U(\mathfrak{h})$ is by definition

$$\xi = T_{\rho} \circ \check{\xi}.$$

Theorem 0.2.8 (Harish-Chandra) see [22], p. 130).

In the setting of Definition 0.2.7, the map ξ is an algebra isomorphism from $Z(\mathfrak{g})$ onto $U(\mathfrak{h})^W$, the algebra of Weyl group-invariant elements of $U(\mathfrak{h})$. It depends only on $\mathfrak{h}$ (and not on the choice of $\mathfrak{b}$).

Definition 0.2.9 Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra and

$$\xi : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})$$

is the Harish-Chandra map. If $\lambda \in \mathfrak{h}^*$, let

$$\xi_{\lambda} : Z(\mathfrak{g}) \rightarrow \mathbb{C}$$

be the composition of ξ with evaluation at λ ; here we continue to regard elements of $U(\mathfrak{h})$ as functions on $\mathfrak{h}^*$.

Corollary 0.2.10 Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Every homomorphism from $Z(\mathfrak{g})$ to $\mathbb{C}$ is of the form ξ_λ , for some $\lambda \in \mathfrak{h}^*$. Furthermore

$$\xi_\lambda = \xi_{\lambda'}, \iff \text{there is } w \in W \text{ such that } \lambda' = w\lambda.$$

Definition 0.2.11 Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. A weight $\lambda \in \mathfrak{h}^*$ is called nonsingular (or regular) if

$$\langle \alpha, \lambda \rangle \neq 0 \quad (\text{all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h})).$$

The infinitesimal character ξ_λ is called nonsingular if λ is; by Corollary 0.2.10, this concept depends only on ξ_λ , and not on the choice of $\mathfrak{h}$ and λ . If λ is nonsingular, we can associate to it a positive root system

$$\begin{aligned} \text{a) } \Delta_\lambda^+(\mathfrak{g}, \mathfrak{h}) &= \{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \operatorname{Re} \langle \alpha, \lambda \rangle > 0, \text{ or} \\ &\quad \operatorname{Re} \langle \alpha, \lambda \rangle = 0, \text{ and } \operatorname{Im} \langle \alpha, \lambda \rangle > 0 \}; \end{aligned}$$

we call this the positive system defined by λ . Conversely, if $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{h})$ is a fixed positive root system, set

$$\begin{aligned} \text{b) } C_{\Delta^+} &= \{ \lambda \in \mathfrak{h}^* \mid \lambda \text{ is nonsingular, and} \\ &\quad \Delta^+ = \Delta_\lambda^+ \} \quad \text{--} \end{aligned}$$

this is the positive Weyl chamber defined by Δ^+ . An element λ of the closure $\overline{C}_{\Delta^+}$ of this positive Weyl chamber is called dominant for Δ^+ ; and Δ^+ is called positive for λ .

Lemma 0.2.12 Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, and $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ a positive root system. Then the closed

Weyl chamber $\overline{C}_{\Delta^+}$ (Definition 0.2.11) is a fundamental domain for the action of W on $\mathfrak{h}^*$.

This result is most familiar for the real span of the roots in $\mathfrak{h}$; but the general case is easily reduced to that.

§3. Group representations and Harish-Chandra modules.

Definition 0.3.1 A (continuous) representation of a Lie group H is a pair $(\pi, \mathcal{H})$, with $\mathcal{H}$ a complex Hilbert space, and

$$\pi : H \rightarrow B(\mathcal{H})$$

a continuous homomorphism of G into the semigroup $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$ (endowed with the weak topology); we assume that $\pi(1)$ is the identity operator.

An invariant subspace of $(\pi, \mathcal{H})$ is a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ which is left invariant by all the operators in $\pi(H)$. Such a subspace is called proper if it is not equal to $\{0\}$ or $\mathcal{H}$. The representation $(\pi, \mathcal{H})$ is called irreducible if $\mathcal{H} \neq 0$, and there are no proper closed invariant subspaces. It is called unitary if the operators in $\pi(H)$ are all unitary.

At some critical points in Chapter 4, we will need to deal with some representations on spaces of smooth functions, which are, of course, only Fréchet spaces. However, it seems reasonable to deal then with some minor technical problems, rather than outline the theory in that generality.

Definition 0.3.2 Suppose $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are continuous representations of a Lie group H . Define

$$\begin{aligned}\text{Hom}_H(\pi, \pi') &= \text{Hom}_H(\mathcal{H}, \mathcal{H}') \\ &= \{L : \mathcal{H} \rightarrow \mathcal{H}' \mid L \text{ is} \\ &\quad \text{continuous, linear, and} \\ &\quad \pi'(g)L = L\pi(g) \text{ for all } g \in H\},\end{aligned}$$

the space of intertwining operators between $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$. We say that the representations are boundedly equivalent if there is an invertible intertwining operator between them.

Definition 0.3.3 Suppose H is a direct product of a compact group and an abelian group, so that every irreducible representation of H is finite dimensional. Define $\hat{H}$, the dual object of H , to be the set of bounded equivalence classes of irreducible representations of H . If H is abelian, so that every irreducible representation is one-dimensional, we make $\hat{H}$ a group as usual, and call it the group of characters of H .

For infinite dimensional (non-unitary) representations, bounded equivalence fails to identify some representations which morally ought to be equivalent. There is no completely satisfactory way around this in general; but for reductive groups, the right notion of equivalence is provided by the work of Harish-Chandra in [12]. Its formulation requires some preparation.

Definition 0.3.4 Suppose $(\pi, \mathcal{H})$ is a continuous representation of G . A vector $v \in \mathcal{H}$ is called K-finite if

$$\dim \langle \pi(K)v \rangle < \infty;$$

here we write $\langle \pi(K)v \rangle$ for the linear span of all vectors of the form $\pi(k)v$, for $k \in K$. Put

$$\mathcal{H}_K = \{v \in \mathcal{H} \mid v \text{ is K-finite}\}.$$

Fix $(\delta, V_\delta) \in \hat{K}$, and define

$$\mathcal{H}_K(\delta) = \bigcup_{L \in \text{Hom}_K(\delta, \pi)} L(V_\delta)$$

the δ K-type or δ -primary subspace of $\mathcal{H}$; it is a (possibly infinite) direct sum of copies of δ . We say that $(\pi, \mathcal{H})$ is admissible if

$$\dim \mathcal{H}_K(\delta) < \infty, \text{ all } \delta \in \hat{K}.$$

In that case, the multiplicity of δ in π is by definition $m(\delta, \pi) = \dim \text{Hom}_K(V_\delta, \mathcal{H}_K)$.

Theorem 0.3.5 (Harish-Chandra [12]). Let $(\pi, \mathcal{H})$ be an admissible representation of G . If $v \in \mathcal{H}_K$ (Definition 0.3.4) and $x \in \mathcal{G}_0$, the limit

$$\lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp(tx))v - v)$$

exists; call it $\pi(x)v$. Then

$$\pi(x)v \in \mathcal{H}_K,$$

and this defines a representation of the Lie algebra $\mathcal{G}_0$ on $\mathcal{H}_K$. There is a lattice isomorphism between

the closed invariant subspaces of $\mathcal{H}$, and (arbitrary) K -invariant, $\mathcal{O}_0$ -invariant subspaces of $\mathcal{H}_K$; if $\mathcal{S} \subseteq \mathcal{H}$ is closed and invariant, then

$$\mathcal{S} \leftrightarrow \mathcal{S}_K = \mathcal{S} \cap \mathcal{H}_K; \mathcal{S} = \overline{\mathcal{S}_K}.$$

A nice exposition of this may be found in [49]. One does not know whether an arbitrary irreducible representation is admissible; but there are various technical results which show that admissibility is a very weak condition on an irreducible representation (cf. [12]). As an illustration, we mention

Theorem 0.3.6 (Harish-Chandra [12]). Any irreducible unitary representation of G is admissible.

Theorem 0.3.5 gives to $\mathcal{H}_K$ the structure of a representation of $\mathcal{O}_0$; and since $\mathcal{H}$ is complex, this structure can be complexified to a (complex linear) representation of $\mathcal{O}$. At the same time $\mathcal{H}_K$ is a representation of the group K . These two representations satisfy the following (obvious) conditions:

- (0.3.7) $\left\{ \begin{array}{l} \text{(a) Every vector } v \in \mathcal{H}'_K \text{ is } K\text{-finite; that is,} \\ \dim \langle \pi(K)v \rangle < \infty. \\ \text{(b) The differential of the representation of } K \text{ is} \\ \text{the restriction of the } \mathcal{O} \text{ representation to } \mathcal{K}_0; \\ \text{that is, if } v \in \mathcal{H}_K \text{ and } x \in \mathcal{K}_0, \\ \pi(x)v = \lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp(tx))v - v). \\ \text{(c) If } x \in \mathcal{O} \text{ and } k \in K, \text{ then} \end{array} \right.$

$$\pi(\text{Ad}(k)x) = \pi(k)\pi(x)\pi(k)^{-1}.$$

Definition 0.3.8 (Lepowsky) A $(\mathcal{O}, K)$ module is a pair (π, X) with X a complex vector space and π a map

$$\pi: \mathcal{O} \cup K \rightarrow \text{End}(X),$$

satisfying

- a) $\pi|_{\mathcal{O}}$ is a complex linear Lie algebra representation, and $\pi|_K$ is a group representation; and
b) the compatibility conditions 0.3.7 (a) - (c) hold, with X replacing $\mathcal{H}_K$.

The K -types of X and their multiplicities, and the notion of admissible, are defined exactly as for group representations (Definition 0.3.4). We write $\mathcal{M}(\mathcal{O}, K)$ for the category of $(\mathcal{O}, K)$ modules, and $\mathcal{A}(\mathcal{O}, K)$ for the subcategory of admissible modules. The $(\mathcal{O}, K)$ module $(\pi, \mathcal{H}_K)$ attached to an admissible group representation $(\pi, \mathcal{H})$ by Theorem 0.3.5 is called the Harish-Chandra module of π .

We will often use module notation for $(\mathcal{O}, K)$ modules, writing $x \cdot v$ instead of $\pi(x)v$. The category structure referred to above is the obvious one. Part of it is given by

Definition 0.3.9 Suppose (π, X) and (π', X') are two $(\mathcal{O}, K)$ modules. Define

$$\begin{aligned}\text{Hom } \mathfrak{g}_{K,K}(\pi, \pi') &= \text{Hom } \mathfrak{g}_{K,K}(X, X') \\ &= \{L : X \rightarrow X' \mid L \text{ is complex} \\ &\quad \text{linear, and } \pi' L = L \pi'\},\end{aligned}$$

the $(\mathfrak{g}, K)$ -module maps (or intertwining operators) from π to π' . We call X and X' equivalent if there is an invertible $(\mathfrak{g}, K)$ -module map between them. We write $\hat{G}$ for the set of equivalence classes of irreducible $(\mathfrak{g}, K)$ modules. Two representations of G are called infinitely-mally equivalent if their Harish-Chandra modules are equivalent.

The study of $(\mathfrak{g}, K)$ modules instead of group representations is justified by Theorems 0.3.5 and 0.3.6, and the next result.

Theorem 0.3.10 (Harish-Chandra, Lepowsky, Rader -- see [49]). Every irreducible $(\mathfrak{g}, K)$ module is the Harish-Chandra module of an irreducible admissible representation of G . Two irreducible unitary representations of G are boundedly equivalent if and only if they are infinitesimally equivalent.

In particular, an irreducible $(\mathfrak{g}, K)$ module is automatically admissible.

When G is connected, the group K plays a less serious role in the structure of a $(\mathfrak{g}, K)$ module: for example, condition (0.3.7)(c) is automatic, and $(\mathfrak{g}, K)$ -module maps are just $\mathfrak{g}$ -module maps. Best of all is

Theorem 0.3.11 (Harish-Chandra [12]). Suppose G is connected. Let (π, X) and (π', X') be irreducible $(\mathfrak{g}, K)$ modules. Fix $(\delta, V_\delta) \in \hat{K}$, and suppose

$$X(\delta) \neq 0.$$

Let $U(\mathfrak{g})^k$ denote the centralizer of k in $U(\mathfrak{g})$. Then X and X' are equivalent if and only if $X(\delta)$ and $X'(\delta)$ are isomorphic $U(\mathfrak{g})^k$ modules. Furthermore, the $U(\mathfrak{g})^k$ module

$$\text{Hom}_K(V_\delta, X)$$

(where $U(\mathfrak{g})^k$ acts on the range only) is irreducible (and finite dimensional).

Probably one can find a good substitute for this theorem in the case of disconnected groups, but I have not been able to do so. To some extent this may be an advantage, because it prevents us from spending a lot of effort trying to understand the (hideously complicated) non-commutative ring $U(\mathfrak{g})^k$. For the reader whose tastes run in that direction, however, we point out that [42] contains a proof of the classification of $(\mathfrak{g}, K)$ modules (for connected G), in which $U(\mathfrak{g})^k$ plays a central role.

We need some generalities about composition series. Although the admissible representations we need all have finite composition series, this will not be proved for some time, and we need to work with them to prove it; so our definitions allow for infinite composition series.

Definition 0.3.12 Suppose (π, X) is a $(\mathfrak{g}, K)$ module, and (π^0, X^0) is an irreducible $(\mathfrak{g}, K)$ module. We say that π^0 is a subquotient or composition factor of π , or that π^0 occurs in π if X has two submodules M_1 and M_2 such that

$$M_1 \supseteq M_2, \quad M_1/M_2 \cong X^0 \quad (\text{as } (\mathfrak{g}, K) \text{ modules}).$$

Suppose (π, X) is admissible. The multiplicity of π^0 in π is the largest integer n such that there is a chain of submodules

$$M_0 \subseteq M_1 \subseteq \dots \subseteq M_N \subseteq X,$$

with

$$M_i/M_{i-1} \cong X^0$$

for n distinct values of i . We write

$$n = m(\pi^0, \pi) = m(X^0, X),$$

and say that π^0 occurs n times in π . If (π', X') is another admissible $(\mathfrak{g}, K)$ module, we say that π and π' have equivalent composition series if for every irreducible $(\mathfrak{g}, K)$ module (π^0, X^0) ,

$$m(\pi^0, \pi) = m(\pi^0, \pi').$$

To see that this all makes sense, one looks at one K -type at a time. For example, to prove that the largest integer n exists, pick $\delta \in K$ so that $X(\delta) \neq 0$; then clearly

$$n \leq \text{multiplicity of } \delta \text{ in } \pi.$$

Definition 0.3.13 The $(\mathfrak{g}, K)$ module (π, X) is said to have finite length if it admits a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_N = X,$$

such that M_i/M_{i-1} is irreducible for $i = 1, \dots, N$. We write $\mathcal{F}(\mathfrak{g}, K)$ for the category of $(\mathfrak{g}, K)$ modules of finite length.

By the remark after Theorem 0.3.10, any $(\mathfrak{g}, K)$ module of finite length is admissible; so

$$(0.3.14) \quad \mathcal{F}(\mathfrak{g}, K) \subseteq \mathcal{Q}(\mathfrak{g}, K) \subseteq \mathcal{M}(\mathfrak{g}, K).$$

Lemma 0.3.15 Suppose (π, X^0) is a fixed irreducible $(\mathfrak{g}, K)$ module. Then

$$X \rightarrow m(X^0, X) \quad (X \in \mathcal{Q}(\mathfrak{g}, K))$$

is exact in X ; that is, if

$$0 \rightarrow X^1 \rightarrow X^2 \rightarrow X^3 \rightarrow 0$$

is an exact sequence in $\mathcal{Q}(\mathfrak{g}, K)$,

$$m(X^0, X^2) = m(X^0, X^1) + m(X^0, X^3).$$

This is trivial.

Definition 0.3.16 The Grothendieck group of $\mathcal{F}(\mathfrak{g}, K)$ is the abelian group generated by $(\mathfrak{g}, K)$ modules of finite length, modulo the equivalence relations

$$X \sim Y + Z$$

whenever there is a short exact sequence

$$0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0 \quad .$$

We write it as $\mathcal{V}(\mathcal{G}, K)$, and call it the group of virtual $(\mathcal{G}, K)$ modules or virtual characters of G . Although we will not, strictly speaking, need to do so, it may be helpful to think of virtual characters in more concrete terms. The first level of concreteness is

Lemma 0.3.17 Suppose X is a $(\mathcal{G}, K)$ module of finite length. Then, in the Grothendieck group $\mathcal{V}(\mathcal{G}, K)$, we have the relation

$$X = \sum_{Y \in \hat{G}} m(Y, X) Y \quad .$$

This gives rise to an isomorphism of $\mathcal{V}(\mathcal{G}, K)$ with the free abelian group having $\hat{G}$ as its set of generators; so for any $X \in \mathcal{V}(\mathcal{G}, K)$ and $Y \in \hat{G}$, $m(Y, X)$ is a well-defined integer.

Again, we can leave the easy proof to the reader. We can be still more concrete.

Definition 0.3.18 A $\mathcal{G}$ module X is called quasisimple if $Z(\mathcal{G})$ acts by scalars on X (Definition 0.2.7). The corresponding homomorphism

$$\chi: Z(\mathcal{G}) \rightarrow \mathbb{C} \quad ,$$

defined by

$$Z \cdot X = \chi(Z) X \quad (X \in X, Z \in Z(\mathcal{G}))$$

is called the infinitesimal character of X . More

generally, X is called $Z(\mathcal{G})$ -finite if there is an ideal $I \subseteq Z(\mathcal{G})$ of finite codimension which annihilates X . If I can be chosen to be primary (that is, $I \subseteq (\ker \chi)^n$ for some positive integer n and homomorphism

$$\chi: Z(\mathcal{G}) \rightarrow \mathbb{C} \quad ,$$

then we say that X has generalized infinitesimal character χ . This means that

$$(z - \chi(z))^n X = 0, \quad \text{all } X \in X, z \in Z(\mathcal{G}).$$

If X is a $Z(\mathcal{G})$ -finite $\mathcal{G}$ module, and χ is a character of $Z(\mathcal{G})$ as above, we define

$$P_\chi(X) = \{X \in X \mid \text{for some } n > 0, \text{ and all } z \in Z(\mathcal{G}), (z - \chi(z))^n X = 0\},$$

the χ -primary component of X . If χ corresponds to $\lambda \in \mathfrak{h}^*$ (cf. 0.2.9), we may write $P_\lambda(X)$ instead of $P_\chi(X)$.

Proposition 0.3.19

- Any irreducible $(\mathcal{G}, K)$ module is quasisimple.
- Any $(\mathcal{G}, K)$ module of finite length is $Z(\mathcal{G})$ -finite.
- If X is a $Z(\mathcal{G})$ -finite $\mathcal{G}$ module, then

$$X = \bigoplus_{\chi \in \text{Spec } Z(\mathcal{G})} P_\chi(X).$$

- If X is a $Z(\mathcal{G})$ -finite $(\mathcal{G}, K)$ module, then each $P_\chi(X)$ is a $(\mathcal{G}, K)$ -submodule of X .

Proposition 0.3.23 The map

$$V \rightarrow \Theta(V)$$

from $\mathcal{U}(\mathcal{G}, K)$ to distributions on G is an isomorphism onto the free abelian group with generators

$$\{\Theta(\pi) \mid \pi \in \hat{G}\}.$$

Proof. We only need to check that $\Theta(V)$ is non-zero when V is; but this follows from the linear independence of the irreducible characters (Theorem 0.3.21). Q.E.D.

We may therefore refer to $\mathcal{U}(\mathcal{G}, K)$ or the corresponding distributions as the lattice of virtual characters of G .

Proposition 0.3.24 Suppose $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are admissible representations of G , satisfying the conditions of Theorem 0.3.21. Then $\mathcal{H}_K$ and $\mathcal{H}'_K$ have equivalent composition series (Definition 0.3.12) if and only if $\Theta(\pi) = \Theta(\pi')$.

The "only if" part follows from the formula for $\Theta(\pi)$ in Theorem 0.3.21. The "if" part is a slight strengthening of the linear independence of irreducible characters, and can be proved by exactly the same method (cf. [14]). We leave this to the reader.

We conclude with a simple technique for constructing representations of disconnected groups; it is used in Definition 1.4.7 to construct the discrete series for $SL^\pm(2, \mathbb{R})$ from those for $SL(2, \mathbb{R})$, and for analogous purposes later in the book.

Definition 0.3.25 Suppose G_1 is a subgroup of G of finite index (which therefore contains the identity component G_0). Let $K_1 = K \cap G_1$. Suppose (π_1, X_1) is a $(\mathcal{G}, K_1)$ module. Define a $(\mathcal{G}, K)$ module

$$(\pi, X) = \text{Ind}_{\mathcal{G}, K_1}^{\mathcal{G}, K} (\pi_1, X_1)$$

as follows.

$$\begin{aligned} X &= \text{Ind}_{K_1}^K (X_1) \\ &= \{f : K \rightarrow X \mid f(kk_1) = \\ &\quad \pi_1(k_1^{-1})f(k) \text{ for } k \in K, \\ &\quad k_1 \in K_1\}. \end{aligned}$$

If f is in X , k and k_0 are in K , and x is in $\mathcal{G}$,

$$\begin{aligned} [\pi(k)f](k_0) &= f(k_1^{-1}k_0) \\ [\pi(x)f](k_0) &= \pi_1(\text{Ad}(k_0^{-1})x) \cdot (f(k_0)). \end{aligned}$$

It is easy to check that (π, X) is really a $(\mathcal{G}, K)$ module. Its basic properties are given by

Lemma 0.3.26 (Frobenius reciprocity). Fix notation as in Definition 0.3.25.

a) Suppose $\delta \in \hat{K}$. In the notation (0.3.4),

$$m(\delta, \text{Ind}_{\mathcal{G}, K_1}^{\mathcal{G}, K} (\pi_1)) = \sum_{\delta_1 \in K_1} m(\delta_1, \delta) m(\delta_1, \pi_1).$$

More precisely, there is a natural isomorphism

$$\text{Hom}_K(V_\delta, \text{Ind}_{\mathcal{G}, K_1}^{\mathcal{G}, K} (X_1)) \cong \text{Hom}_{K_1}(V_\delta|_{K_1}, X_1).$$

b) Let (ρ, Y) be any $(\mathfrak{g}, K)$ module. Then there is a natural isomorphism

$$\text{Hom}_{\mathfrak{g}, K}(Y, \text{Ind}_{\mathfrak{g}, K_1}^{\mathfrak{g}, K}(X_1)) \cong \text{Hom}_{\mathfrak{g}, K_1}(Y|_{(\mathfrak{g}, K_1)}, X_1).$$

The proof, which is exactly like that of Frobenius reciprocity for compact groups, is left to the reader.

§4. Finite dimensional representations.

Because G is not necessarily connected, the extension of the Cartan-Weyl theory of finite dimensional representations to G requires a few comments.

Definition 0.4.1 A θ -stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ is called maximally split if

$$\sigma_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$$

is a maximal abelian subalgebra of $\mathfrak{p}_0$. A Cartan subgroup is called maximally split if its Lie algebra is.

We often write H^S for such a subgroup.

Lemma 0.4.2 Any two θ -stable maximally split Cartan subgroups of G are conjugate under K_0 . Fix such a subgroup H^S , and write

$$T^S = H^S \cap K, \quad A^S = \exp(\mathfrak{h}_0^S \cap \mathfrak{p}_0).$$

Then

a) T^S meets every component of G .

$$\begin{aligned} \text{b) } W(G, H) &= W(G_0, H^S \cap G_0) \\ &= \{w \in W(\mathfrak{g}, \mathfrak{h}^S) \mid \theta w = w\theta\} \end{aligned}$$

$$= \{w \in W(\mathfrak{g}, \mathfrak{h}^S) \mid w(\sigma_0^S) = \sigma_0^S\}.$$

Proof. The conjugacy statement is in [50], Corollary 1.3.1.5. The last two equalities in (b) are [50], Proposition 1.1.3.3. For (a), fix a component C of G , and pick an element $k \in K \cap C$. (This is possible by (0.1.2) (d).) Then $\text{Ad}(k)(H^S)$ is another maximally split Cartan subgroup of G , so we can find $k_0 \in K_0$ so that

$$\text{Ad}(k_0)[\text{Ad}(k)H^S] = H^S.$$

So $k_0 k$ normalizes H^S . But $k_0 k$ also lies in C ; so we may as well replace k by $k_0 k$, and assume that k normalizes H^S . Thus k defines an element

$$w = kH^S \in W(G, H^S).$$

Since $\theta k = k$, w commutes with the action of θ on $\mathfrak{h}^S$. By the part of (b) we already know, it follows that

$$w \in W(G_0, H^S \cap G_0).$$

This step in the argument completes the proof of (b)). So we can find $k_0 \in K_0$ normalizing $H^S \cap G_0$, so that $k_0 k$ actually centralizes $\mathfrak{h}_0$ (that is, defines the trivial element of $W(G, H^S)$). Thus $k_0 k$ is an element of T^S . Since $k_0 k$ also lies in C , this proves (a). Q.E.D.

Next, we recall the theory for Lie algebras.

Proposition 0.4.3 (Cartan-Weyl) Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra, and F an irreducible finite dimensional representation of $\mathfrak{g}$. Write $W = W(\mathfrak{g}, \mathfrak{h})$, and let $\Delta(F)$ denote the set of

weights of $\mathfrak{h}$ in F (Definition 0.2.3).

a) $\Delta(F)$ is stable under W . If $\mu \in \Delta(F)$, and α is a root,

$$\langle \alpha, \mu \rangle \in \mathbb{Z} \quad \text{Definition (0.2.5).}$$

b) There is a unique W orbit

$$W \cdot \lambda \subseteq \Delta(F)$$

characterized by

$$\langle \lambda, \lambda \rangle - \langle \mu, \mu \rangle \geq 0, \quad \text{all } \mu \in \Delta(F);$$

equality holds if and only if $\mu \in W \cdot \lambda$. The weights in $W \cdot \lambda$ are called extremal weights of F . Every weight lies in the convex hull of the extremal weights.

c) Every extremal weight has multiplicity one.

d) If F' is any other irreducible finite dimensional representation of $\mathfrak{g}$, then F' is isomorphic to F if and only if F' and F have the same extremal weights.

e) Fix a positive root system $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$. Then there is a unique extremal weight λ which is dominant for Δ^+ :

$$\langle \lambda, \alpha \rangle \geq 0, \quad \text{all } \alpha \in \Delta^+.$$

We call λ the highest weight of F .

f) The highest weight space of F is characterized by

$$F(\lambda) = \{v \in F \mid X_\alpha v = 0, \text{ all } \alpha \in \Delta^+\};$$

here X_α denotes a root vector for the root α .

g) If μ is any weight of F , then

$$\mu = \lambda - \sum_{\alpha \in \Delta^+} n_\alpha \alpha \quad (n_\alpha \text{ a non-negative integer}).$$

There is also an existence result.

Proposition 0.4.4 (Cartan) Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra, and $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ a positive root system. Fix a weight $\lambda \in \mathfrak{h}^*$, and suppose that

$$\langle \alpha, \lambda \rangle \in \mathbb{Z}, \quad \text{all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}).$$

Then there is a unique irreducible finite dimensional representation F of extremal weight λ . If λ is dominant, i.e.,

$$\langle \alpha, \lambda \rangle \geq 0, \quad \text{all } \alpha \in \Delta^+,$$

then λ is the highest weight of F .

Lemma 0.4.5 Let $H \subseteq G$ be a Cartan subgroup, and let

$\Lambda^r \subseteq \mathfrak{h}^*$ be the lattice generated by the roots. Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$; let X_α be a root vector. Define $\tilde{\alpha} \in \hat{H}$ to be the character by which H acts on $\mathbb{C}X_\alpha$. Suppose

$$\sum_{\alpha \in \Delta} n_\alpha \alpha = \sum_{\alpha \in \Delta} m_\alpha \alpha$$

for some integers n_α, m_α . Then (writing $\hat{H}$ additively)

$$\sum n_\alpha \tilde{\alpha} = \sum m_\alpha \tilde{\alpha}.$$

Proof. Because the simple roots form a $\mathbb{Z}$ basis of Λ^r , and the root vectors for simple roots and their negatives generate $\mathfrak{g}$, the relations of the form in the lemma are all integer

combinations of relations

$$\begin{aligned}\alpha + \beta &= \gamma & (\alpha, \beta, \gamma \in \Delta(\mathfrak{g}, \mathfrak{h})) \\ \alpha + (-\alpha) &= 0 & (\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}))\end{aligned}$$

So it is enough to prove the corresponding relations for the

$\tilde{\alpha}$. In the first case,

$$[X_\alpha, X_\beta] = cX_\gamma \quad (c \neq 0);$$

so if $h \in H$,

$$\begin{aligned}c\tilde{\gamma}(h)X_\gamma &= c \operatorname{Ad}(h)X_\gamma \\ &= \operatorname{Ad}(h)[X_\alpha, X_\beta] \\ &= [\operatorname{Ad}(h)X_\alpha, \operatorname{Ad}(h)X_\beta] \\ &= [\tilde{\alpha}(h)X_\alpha, \tilde{\beta}(h)X_\beta] \\ &= \tilde{\alpha}(h)\tilde{\beta}(h)[X_\alpha, X_\beta] \\ &= (\tilde{\alpha} + \tilde{\beta})(h)(cX_\gamma).\end{aligned}$$

It follows that $\tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}$. In the second case, $[X_\alpha, X_{-\alpha}]$ is a non-zero element of $\mathfrak{h}$, and the proof is similar. Q.E.D.

Definition 0.4.6 Let $H \subseteq G$ be a Cartan subgroup.

The root lattice in $\mathfrak{h}^*$ is the lattice generated by the roots; and the root lattice in $\hat{H}$ is the subgroup of $\hat{H}$ generated by the elements $\tilde{\alpha}$ of Lemma 0.4.5. We identify these lattices by

$$\sum n_\alpha \alpha \leftrightarrow \sum n_\alpha \tilde{\alpha}$$

(which makes sense by the lemma), and may therefore

write

$$\sum n_\alpha \alpha \in \hat{H}.$$

Proposition 0.4.7 Let F be a finite dimensional irreducible representation of G . Then the restriction of F to G_0 , or equivalently, the differentiated representation of $\mathfrak{g}$, is irreducible.

Proof. Choose a maximally split Cartan subgroup H^S of G , and a positive root system Δ^+ for $\mathfrak{h}^S$ in $\mathfrak{g}$. Fix an irreducible $\mathfrak{g}$ subrepresentation F_0 of F , and let $\gamma \in (\mathfrak{h}^S)^*$ be the highest weight of F_0 . By Lemma 0.4.5 (f),

$$F^\gamma = \{v \in F(\gamma) \mid X_\alpha v = 0 \text{ for all } \alpha \in \Delta^+\}$$

is non-zero. Clearly F^γ is stable under H^S , so we can choose a weight vector $v \in F^\gamma$ for H^S . Let

$$F_1 = U(\mathfrak{g}) \cdot v,$$

the smallest $\mathfrak{g}$ -invariant subspace containing v . Because it is generated by a highest weight vector, F_1 is an irreducible representation of $\mathfrak{g}$. We will show that $F_1 = F$. If $h \in H^S$, and $u \in U(\mathfrak{g})$,

$$\begin{aligned}h \cdot (uv) &= (\operatorname{Ad}(h)u)(hv) \\ &= c(\operatorname{Ad}(h)u)v \in F_1\end{aligned}$$

since v is a weight vector for H^S . So F_1 is H^S -invariant. By exponentiation, F_1 is G_0 invariant; so by Lemma 0.4.2, F_1 is invariant under $H^S \cdot G_0 = G$. Since F is irreducible, $F = F_1$. Q.E.D.

Definition 0.4.8 Suppose F is an irreducible representation of G , $H \subseteq G$ is a Cartan subgroup, and $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ is a positive root system. The extremal weights of H

in F are the weights of H on the extremal $\mathfrak{h}$ -weight spaces (Proposition 0.4.3(b)). The highest weight of H in F is the weight of H on the highest $\mathfrak{h}$ -weight space (Proposition 0.4.3(e)).

These definitions make sense because of Proposition 0.4.7.

Proposition 0.4.9 Suppose $H \subseteq G$ is a Cartan subgroup, and $\Delta^+ \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$ is a positive system. Let F be an irreducible finite dimensional representation of G , of highest weight $\lambda \in \hat{H}$.

- a) Any two weights of H in F are congruent modulo the root lattice in $\hat{H}$.
- b) Every weight of H in F is of the form

$$\lambda - \sum_{\alpha \in \Delta^+} n_{\alpha} \alpha.$$

- c) H acts by scalars on each $\mathfrak{h}$ -weight space of F .
- d) Suppose H^S is maximally split, and F' is another irreducible finite dimensional representation of G .
Then F and F' are isomorphic if and only if they have the same highest H^S -weight.

Proof. For (a), let $\Lambda^r \subseteq \hat{H}$ denote the root lattice. Define

$$F_0 = \bigoplus_{\gamma \in (\lambda + \Lambda^r)} F(\gamma),$$

the span of the H -weights of F congruent to $\lambda \bmod \Lambda^r$. We want to show $F = F_0$; by Proposition 0.4.7, it is enough to show that F_0 is $\mathfrak{g}$ -invariant. Clearly, it is $\mathfrak{h}$ -invariant; and if X_{α} is a root vector for the root α , $\gamma \in (\lambda + \Lambda^r)$, and

$v \in F(\gamma)$, then clearly

$$X_{\alpha} \cdot v \in F(\gamma + \alpha).$$

Since $\alpha \in \Lambda^r$, $(\gamma + \alpha) \in (\lambda + \Lambda^r)$; so

$$X_{\alpha} \cdot v \in F_0.$$

Since $\mathfrak{g}$ is spanned by $\mathfrak{h}$ and the root vectors, F_0 is $\mathfrak{g}$ -invariant. Part (b) follows from (a) and Proposition 0.4.3(g). For (c), suppose $\mu_0 \in \mathfrak{h}^*$ is a weight of F . Let λ_0 be the differential of λ . Then we can write

$$\mu_0 = \lambda_0 - \sum n_{\alpha} \alpha.$$

By (a) and Lemma 0.4.5, H acts on $F(\mu_0)$ by the weight

$$\mu = \lambda + \sum n_{\alpha} \alpha \in \hat{H}.$$

For (d), the "only if" is clear; so suppose F' also has highest weight λ . Let

$$I : F \rightarrow F'$$

be an isomorphism of $\mathfrak{g}$ modules. Then I preserves the $\mathfrak{h}^S$ weight spaces. By the proof of (c), H^S acts by the same scalars in a given $\mathfrak{h}^S$ weight space for F and F' ; so I intertwines the action of H^S . By exponentiation, I intertwines the action of G_0 ; so it intertwines $H^S G_0 = G$. Q.E.D.

The existence result corresponding to Proposition 0.4.4 is a little tricky. The following result about Lie algebra representations will be needed.

Lemma 0.4.10 Suppose $\mathfrak{g}$ is a complex reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, and F is a finite dimensional representation of $\mathfrak{g}$. Fix a weight

$\lambda \in \Delta(F, \mathfrak{h})$. Then there is an integer n so that the

finite dimensional irreducible representation of extremal weight λ occurs in

$$F \otimes T^n(\mathfrak{g}).$$

Here $T^n(\mathfrak{g})$ is the n^{th} tensor power of $\mathfrak{g}$, and $\mathfrak{g}$ acts on it by the adjoint representation.

The reader can supply a proof without serious difficulty.

Definition 0.4.11 Suppose $H \subseteq G$ is a Cartan subgroup.

The weight lattice in $\hat{H}$ is the subgroup Λ of $\hat{H}$ consisting of weights of finite dimensional representations of G . The weight lattice in $\mathfrak{h}^*$ is the additive subgroup of $\mathfrak{h}^*$ consisting of differentials of elements of the weight lattice in $\hat{H}$. If $\lambda \in \Lambda$, write $\bar{\lambda} \in \mathfrak{h}^*$ for the differential of λ .

If G has compact center, the weight lattice in $\mathfrak{h}^*$ really is a lattice; and if in addition G is connected, the weight lattice in $\hat{H}$ is really a lattice.

Proposition 0.4.12 Suppose $H \subseteq G$ is a Cartan subgroup, and $\lambda \in \hat{H}$ lies in the weight lattice. Then there is an irreducible finite dimensional representation F of G of extremal weight λ . If H is maximally split, F is unique.

Proof. Fix a finite dimensional representation F_1 of G containing the weight λ ; we may as well assume F_1 is irreducible. By Proposition 0.4.9 (a), all weights of F_1 are congruent to λ modulo the root lattice. Choose n as in Lemma 0.4.10, and regard $T^n(\mathfrak{g})$ as a representation of G by the adjoint action. Then every weight of $T^n(\mathfrak{g})$ lies in the root lattice for H ; so all weights of

$$F_2 = F_1 \otimes T^n(\mathfrak{g})$$

are congruent to λ modulo the root lattice. By Lemma 0.4.10, F_2 contains an irreducible G module F with an extremal weight μ having the same differential as λ . By Lemma 0.4.5, $\mu = \lambda$; so F has the desired property. The uniqueness statement is Proposition 0.4.9(d). Q.E.D.

In general, it can be shown that the number of distinct irreducible representations of extremal weight λ is

$$|W(G, H) / W(G_0, H \cap G_0)|.$$

For example, $SL^+(2, \mathbb{R})$ (see Section 1.4) has two representations of each positive dimension, and these have the same weights under the compact Cartan subgroup $SO(2)$. We will not use this, so the proof is left to the reader.

Chapter 1. $SL(2, \mathbb{R})$.

One of the most pleasant techniques in the theory of reductive groups is reduction to three dimensional simple subgroups. We will need a very detailed understanding of the structure and representation theory of the group $SL(2, \mathbb{R})$, and various closely related groups. This material will be quite familiar to most readers, except perhaps in the disconnected case; and they can refer to this chapter as needed or not at all. For other readers, the first three sections of this chapter can be taken as an introduction; some material from Chapter 0 has been repeated here for clarity.

§1. Group representations and Lie algebra representations.

Throughout the first three sections, G will denote the group

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

Definition 1.1.1 A (continuous) representation of the group G on a Hilbert space is a pair $(\pi, \mathcal{H})$, with $\mathcal{H}$ a complex Hilbert space and

$$\pi : G \rightarrow B(\mathcal{H})$$

a homomorphism from G to the semigroup of bounded operators on $\mathcal{H}$. We assume that π is continuous with respect to the weak topology on $B(\mathcal{H})$; that is, we assume that the map

$$G \times \mathcal{H} \rightarrow \mathcal{H}, \quad (g, v) \mapsto \pi(g)v$$

is continuous. The representation is called unitary

if, for each $g \in G$, $\pi(g)$ is a unitary operator on $\mathcal{H}$; it is called (topologically) irreducible if there is no proper closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ which is invariant under all of the operators $\pi(g)$.

We may often refer to π or $\mathcal{H}$ alone as a representation of G , if the other is clear from the context.

Example 1.1.2 Let

$$\mathcal{H} = L^2 \left[\mathbb{R}, \frac{dx}{x^2+1} \right]$$

be the Hilbert space of measurable functions f on the real line such that

$$\|f\|^2 = \int_{-\infty}^{\infty} \frac{|f(x)|^2}{x^2+1} dx < \infty.$$

If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, f \in \mathcal{H}$$

define

$$[\pi(g)f](x) = f \left(\frac{dx-b}{-cx+a} \right).$$

Then $(\pi, \mathcal{H})$ is a continuous representation of G , as can be checked by a messy calculus computation (the only messy part is continuity). It is not irreducible: the constant functions clearly form a one-dimensional invariant subspace $\mathcal{H}_0$. It is less obvious that the natural quotient representation $(\bar{\pi}, \bar{\mathcal{H}}/\mathcal{H}_0)$ (whose definition we leave to the reader) is also reducible: it is a direct sum of two infinite dimensional irreducible representations. We will prove this in a moment.

For the rest of this section, K will denote the circle group

$$SO(2) = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\} \subseteq SL(2, \mathbb{R}).$$

Lemma 1.1.3 Suppose $\mathcal{H}$ is a Hilbert space, and

$$\rho: K \rightarrow U(\mathcal{H})$$

is a (continuous) unitary representation of K (that is,

a weakly continuous homomorphism into the group of

unitary operators on $\mathcal{H}$). Then there is a unique

collection of closed, mutually orthogonal subspaces

$$\{\mathcal{H}_n \subseteq \mathcal{H} \mid n \in \mathbb{Z}\}$$

such that

a) $\mathcal{H} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n$ (Hilbert space direct sum);

b) Each $\mathcal{H}_n$ is an invariant subspace for $\rho(K)$;

c) If $v \in \mathcal{H}_n$, and

$$k = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in K,$$

then

$$\rho(k)v = e^{in\theta} v.$$

This can be regarded as a very special case of the spectral theorem. The subspace $\mathcal{H}_n$ is the image of the projection operator P_n defined as follows. Set

$$k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Then

$$P_n(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \rho(k(\theta))v \, d\theta.$$

We leave the details to the reader.

Definition 1.1.4 In the setting of Lemma 1.1.3, define

the space of K -finite vectors in $\mathcal{H}$ by

$$\mathcal{H}_K = \{v \in \mathcal{H} \mid \text{the span of } \rho(K) \cdot v \text{ is finite dimensional}\}$$

$$= \text{algebraic direct sum of the } \mathcal{H}_n,$$

a dense subspace of $\mathcal{H}$.

Definition 1.1.5 Let $(\pi, \mathcal{A})$ be a representation of G , and suppose that the restriction of π to K is unitary.

Define the subspaces $\mathcal{H}_n$ as in Lemma 1.1.3. Then π is called admissible if

$$\dim \mathcal{H}_n < \infty, \quad \text{all } n \in \mathbb{Z}.$$

Write $\mathcal{G}_0$ for the Lie algebra of G .

Proposition 1.1.6 (Harish-Chandra [12]). Suppose $(\pi, \mathcal{H})$

is an admissible representation of G . Fix $x \in \mathcal{G}_0$ and

$v \in \mathcal{H}_K$ (Definition 1.1.4). Then

$$\lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp tx)v - v) = \pi(x)v$$

exists, and defines a Lie algebra representation π of

$\mathcal{G}_0$ on $\mathcal{H}_K$. There is a bijection between the set of closed, G -invariant subspaces of $\mathcal{H}$, and arbitrary $\mathcal{G}_0$ -invariant subspaces of $\mathcal{H}_K$; a closed subspace $\mathcal{S}$ corresponds to

$$\mathcal{S}_K = \mathcal{S} \cap \mathcal{H}_K,$$

and

$$\mathcal{S} = \overline{\mathcal{S}_K}.$$

In particular, $\mathcal{H}$ is topologically irreducible if and only if $\mathcal{H}_K$ is algebraically irreducible.

Example 1.1.2 continued. In the representation defined earlier,

$$[k(\theta) \cdot f](\gamma) = f(-\tan \theta) .$$

So the condition $f \in \mathcal{H}_n$ means at least that

$$\begin{aligned} f(-\tan \theta) &= [k(\theta) \cdot f](0) \\ &= e^{in\theta} \cdot f(0) \end{aligned}$$

Taking $\theta = \pi$, we find

$$f(0) = (-1)^n f(0) ,$$

which is possible only if $n = 2m$ is even. In that case, f must be $f(0)$ times the function f_{2m} defined by

$$f_{2m}(x) = e^{-2im(\tan^{-1} x)} .$$

An elementary calculation shows that

$$\left\{ \frac{1}{\sqrt{\pi}} f_{2m} \mid m \in \mathbb{Z} \right\}$$

is an orthonormal basis of $\mathcal{H}$. We can now compute the action of $\mathcal{A}_0$ in this basis. Consider first the element

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{A}_0; \quad \exp(ta) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} .$$

We have

$$\begin{aligned} [\pi(a)f](x) &= \lim_{t \rightarrow 0} \frac{1}{t} [\pi(\exp ta)f(x) - f(x)] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left\{ f(e^{-2t}x) - f(x) \right\} \\ &= -2x \frac{df}{dx} \end{aligned}$$

by the chain rule. So

$$[\pi(a)f_{2m}](x) = -\frac{4imx}{x^2+1} e^{2im(\tan^{-1} x)} .$$

Write $\theta = \tan^{-1} x$; then

$$\begin{aligned} -\frac{x}{x^2+1} &= (\tan \theta)/(\tan^2 \theta + 1) = \sin \theta \cos \theta , \\ &= \frac{1}{4i} \left[e^{2i\theta} - e^{-2i\theta} \right] . \end{aligned}$$

This leads to the formula

$$\pi(a) f_{2m} = -m f_{2m+2} + m f_{2m-2} .$$

Similarly one calculates

$$\begin{aligned} \left[\pi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} f \right](x) &= -\frac{df}{dx} \\ \pi \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} f_{2m} &= \frac{im}{2} f_{2m+2} + im f_{2m} + \\ &\quad + \frac{im}{2} f_{2m-2} \\ \left[\pi \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} f \right](x) &= x^2 \frac{df}{dx} \\ \pi \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} f_{2m} &= +\frac{im}{2} f_{2m+2} - im f_{2m} + \\ &\quad + \frac{im}{2} f_{2m-2} . \end{aligned}$$

From these formulas, one checks easily that $\mathcal{H}_K$ has exactly three proper invariant subspaces: $\mathcal{H}_0$ (as defined earlier, the span of f_0), and

$$\mathcal{H}_K^+ = \text{span} \{f_{2m} \mid m \geq 0\}$$

$$\mathcal{H}_K^- = \text{span} \{f_{2m} \mid m \leq 0\} .$$

Write $\mathcal{A}^\pm$ for the closure of $\mathcal{A}_K^\pm$ in $\mathcal{H}$. Then Proposition 1.1.6 implies that the representations of G on $\mathcal{H}^\pm/\mathcal{H}_0$ are both irreducible, as claimed earlier.

Proposition 1.1.6 provides an algebraic way to study representations of G . To complete this method, a converse of some sort is required. This can be formulated as follows.

Definition 1.1.7 A $(\mathfrak{g}_0, K)$ module (π, V) is a Lie algebra representation of $\mathfrak{g}_0$, and a representation of K , both denoted by π and both on the same complex vector space V . We require also that there be subspaces

$$\{V_n \subseteq V \mid n \in \mathbb{Z}\}$$

such that

- a) $V = \bigoplus_{n \in \mathbb{Z}} V_n$ (algebraic direct sum);
- b) each V_n is invariant under $\pi(K)$;
- c) if $v \in V_n$ and $\theta \in \mathbb{R}$, then

$$\pi(K(\theta))v = e^{in\theta} v;$$

- d) the action of the Lie algebra $\mathfrak{k}_0$ of K in the Lie algebra representation is the differential of the action of K . More explicitly, if $v \in V_n$, and

$$h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}_0,$$

$$\pi(h)v = in \cdot v.$$

We say that (π, V) is admissible if

$$\dim V_n < \infty \quad (n \in \mathbb{Z}).$$

If $(\pi, \mathcal{H})$ is an admissible representation of G , the $(\mathfrak{g}_0, K)$ module $(\pi, \mathcal{H}_K)$ defined by Proposition 1.1.6 is called the Harish-Chandra module of π . If $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are two admissible representations of G , we say that they are infinitesimally equivalent if their Harish-Chandra modules $(\pi_1, (\mathcal{H}_1)_K)$ are isomorphic.

The converse we want for Proposition 1.1.6 is

Proposition 1.1.8 Every irreducible admissible $(\mathfrak{g}_0, K)$ module arises as the Harish-Chandra module of an irreducible representation of G .

The representations of G in question will be constructed in Section 1.3.

There is an obvious notion of equivalence of representations of G (the existence of a bounded invertible intertwining operator). This implies infinitesimal equivalence. The converse is not obvious, and may in fact be false; but it is true in an important special case.

Proposition 1.1.9 (Harish-Chandra [12]). Any unitary irreducible representation of G is admissible. Two such representations are infinitesimally equivalent if and only if they are unitarily equivalent.

The importance of this proposition is that it allows us to study unitary representations by algebraic methods: the definitions of "admissible" and "infinitesimal equivalence", which were arranged for algebraic convenience, neither omit

any unitary representations, nor overlook some subtlety of their structure. Since it is unitary representations which arise most often in applications, this shows that we have not defined all of the interesting problems out of existence.

§2. Structure of Harish-Chandra modules.

Because of Proposition 1.1.8, we want to describe all the irreducible admissible $(\mathfrak{g}_0, K)$ modules. (Actually, an irreducible $(\mathfrak{g}_0, K)$ module is automatically admissible, but we will not need this fact.) I have made no attempt to attribute the ideas of this section; most go back to [1]. Let

$G = SL(2, \mathbb{R})$, $K = SO(2)$ as before, and put

$$\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}},$$

the complexified Lie algebra of G . Representations of a real Lie algebra on a complex vector space can be complexified; so we may as well consider $(\mathfrak{g}, K)$ modules (defined in precisely the same way, with $\mathfrak{g}$ replacing $\mathfrak{g}_0$ in Definition 1.1.7). Our first step is to choose a convenient basis of

$\mathfrak{g}$. We take

$$(1.2.1) \quad \begin{aligned} H &= -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & [H, X] &= 2X \\ X &= \frac{1}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] & [H, Y] &= -2Y \\ Y &= \frac{1}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] & [X, Y] &= H, \end{aligned}$$

which are easily seen to be a basis satisfying the indicated commutation relations.

Lemma 1.2.2 (cf. [22], Lemma 7.2) Let (π, V) be a $(\mathfrak{g}, K)$ module (Definition 1.1.7), with

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

Then if $v \in V_n$,

$$\begin{aligned} \pi(H)v &= nv \\ \pi(X)v &\in V_{n+2} \\ \pi(Y)v &\in V_{n-2}. \end{aligned}$$

Proof. The first formula is (d) of Definition 1.1.7. For the second,

$$\begin{aligned} \pi(H)\pi(X)v &= \pi(X)\pi(H)v + 2\pi(X)v && \text{(by 1.2.1)} \\ &= n\pi(X)v + 2\pi(X)v \\ &= (n+2)\pi(X)v. \end{aligned}$$

So, again, by 1.1.7(d),

$$\pi(X)v \in V_{n+2}.$$

The last formula is proved in the same way. Q.E.D.

Example 1.2.3 Let (π, V) be the Harish-Chandra module of the representation of Example 1.1.2. If we combine the calculations before (1.1.7) with the definitions in (1.2.1), we find the formulas

$$\begin{aligned} \pi(H)f_{2m} &= 2m \cdot f_{2m} \\ \pi(X)f_{2m} &= -m \cdot f_{2m+2} \\ \pi(Y)f_{2m} &= m \cdot f_{2m-2}. \end{aligned}$$

Lemma 1.2.4 The element

$$\begin{aligned}
 \Omega &= H^2 - 2H+1 + 4XY \\
 &= H^2 + 1 + 2XY + 2YX \\
 &= H^2 + 2H+1 + 4YX \\
 &= (H-1)^2 + 4XY \\
 &= (H+1)^2 + 4YX
 \end{aligned}$$

is in the center of the enveloping algebra $U(\mathfrak{g})$. It acts by a scalar on any irreducible admissible $(\mathfrak{g}, K)$ module (π, V) . Fix a square root λ of this scalar. Then if $v \in V_n$, we have

$$\begin{aligned}
 \pi(XY)v &= \frac{1}{4}(\lambda^2 - (n-1)^2)v \\
 \pi(YX)v &= \frac{1}{4}(\lambda^2 - (n+1)^2)v.
 \end{aligned}$$

Proof. The fact that the various formulas for Ω are equivalent follows from the identity in $U(\mathfrak{g})$

$$XY = YX + H,$$

which is part of (1.2.1). To see that Ω is in the center of $U(\mathfrak{g})$, we must show that it commutes with X, Y , and H . Using the commutation relations in (1.2.1), we find

$$\begin{aligned}
 [X, \Omega] &= [X, H^2] - 2[X, H] + 4[X, XY] \\
 &= [X, H]H + H[X, H] - 2[X, H] + 4X[X, Y] \\
 &= -2XH - 2HX + 4X + 4XH \\
 &= -2XH - (2XH + 4X) + 4X + 4XH \\
 &= 0.
 \end{aligned}$$

The calculations for Y and H are similar. (The element Ω is the Casimir element plus one -- cf. [22], 22.1.)

Since Ω is central in $U(\mathfrak{g})$, any eigenspace of $\pi(\Omega)$ is a $\mathfrak{g}$ -invariant subspace of V , and so (by irreducibility) is 0 or V . So to show that $\pi(\Omega)$ is a scalar, we only have to show that it has an eigenvector. Fix n so that V_n is non-zero. Since Ω commutes with H , it preserves the finite dimensional space V_n ; so it must have an eigenvector.

The last two formulas are simply the last two formulas for Ω , together with the information

$$\pi(\Omega)v = \lambda^2 v, \quad \pi(H)v = n v. \quad \text{Q.E.D.}$$

Before treating arbitrary $(\mathfrak{g}, K)$ modules, we need to take care of a small technical problem.

Lemma 1.2.5 Suppose (π, V) is an irreducible admissible $(\mathfrak{g}, K)$ module, and $v \in V_n$ is non-zero.

a) If $\pi(X)v = 0$, then the elements

$$w_{n-2m} = \pi(Y)^m v$$

span V ; and in particular, V_{n+2} is zero.

b) If $\pi(Y)v = 0$, then the elements

$$w_{n+2m} = \pi(X)^m v$$

span V ; and in particular, V_{n-2} is zero.

c) If $\pi(YX)v = 0$, then $\pi(X)v = 0$.

d) If $\pi(XY)v = 0$, then $\pi(Y)v = 0$.

Proof. Consider (a). Let W denote the span of $\{w_{n-2m}\}$.

Since v is in W , W is non-zero. We claim that W is invariant under $\mathfrak{g}$. Since V is irreducible, it will follow that

$W = V$. By Lemma 1.2.2,

$$w_{n-2m} \in V_{n-2m};$$

so

$$\pi(H)w_{n-2m} = (n-2m)w_{n-2m}.$$

So $\pi(H)$ leaves W invariant; and (trivially) $\pi(Y)$ does also.

By hypothesis

$$\pi(X)w_n = \pi(X)v = 0 \in W;$$

and if $m > 0$,

$$\begin{aligned} \pi(X)w_{n-2m} &= \pi(XY)w_{n-2(m-1)} \\ &= c \cdot w_{n-2(m-1)}, \end{aligned}$$

where c is a scalar (computed in Lemma 1.2.4). So $\pi(X)$ also leaves W invariant, proving that $W = V$. Since w_{n-2m} lies in V_{n-2m} , V_{n+2} is zero. Part (b) is proved in exactly the same way. For (c), suppose (to get a contradiction) that

$$\pi(X)v = v_1 \neq 0.$$

By Lemma 1.2.2, v_1 lies in V_{n+2} ; and by hypothesis

$$\pi(Y)v_1 = \pi(YX)v = 0.$$

If we apply (b) with v replaced by v_1 , we conclude that

$$V_{(n+2)-2} = V_n = 0,$$

contradicting $v \neq 0$. This proves (c), and (d) is identical.

Q.E.D.

Obviously, parts (a) and (b) of this lemma depend on the irreducibility of V . What is not so obvious is that (c) and d) do as well. This can be seen (for (c), say) by taking $(\cdot, V)$ as in Example 1.2.3, with $v = f_{-2}$.

Lemma 1.2.6 (cf. [22], Lemma 7.2). Suppose (π, V) is an irreducible admissible $(\mathfrak{g}, K)$ module, λ is as in Lemma 1.2.4, and $v \in V_n$ is non-zero. Define (inductively, for $m \geq 0$)

$$\begin{aligned} w_n &= v \\ w_{n+2(m+1)} &= \begin{cases} \left(\frac{2}{\lambda + (n+2m+1)} \right) \pi(X)w_{n+2m} & \text{if } \lambda \neq -n-2m-1 \\ 0 & \text{if } \lambda = -n-2m-1. \end{cases} \\ w_{n-2(m+1)} &= \begin{cases} \left(\frac{2}{\lambda - (n-2m-1)} \right) \pi(Y)w_{n-2m} & \text{if } \lambda \neq n-2m-1. \\ 0 & \text{if } \lambda = n-2m-1. \end{cases} \end{aligned}$$

Then, whenever $w_j \neq 0$,

$$\begin{aligned} \text{a)} \quad \pi(H)w_j &= j \cdot w_j \\ \text{b)} \quad \pi(X)w_j &= \frac{1}{2}(\lambda + (j+1)) w_{j+2} \\ \text{c)} \quad \pi(Y)w_j &= \frac{1}{2}(\lambda - (j-1)) w_{j-2}. \end{aligned}$$

In particular, the non-zero w_j form a basis of V .

Proof. Since w_j is a multiple of either

$$\pi(X)^{\frac{1}{2}(j-n)} v$$

or of

$$\pi(Y)^{\frac{1}{2}(n-j)} v,$$

formula (a) follows from Lemma 1.2.2. Consider next (b). There are several possibilities. Suppose first that $j \geq n$; write $j = n + 2m$. If $-\lambda \neq n+2m+1$, then (b) is just the definition of w_{n+2m+2} . If $-\lambda = n+2m+1$, then by Lemma 1.2.4,

$$\begin{aligned}\pi(YX)w_{n+2m} &= \frac{1}{4}(\lambda^2 - (n+2m+1)^2)w_{n+2m} \\ &= 0.\end{aligned}$$

By Lemma 1.2.5(c),

$$\pi(X)w_{n+2m} = 0.$$

Since w_{n+2m+2} is zero in this case, (b) holds. Next, suppose j is negative; write

$$j = n-2m-2,$$

with $m \geq 0$. If $\lambda \neq n-2m-1$, then by definition of w_{n-2m-2} ,

$$\pi(X)w_{n-2m-2} = \pi(X)\pi(Y)\left\{\frac{2}{\lambda-(n-2m-1)}\right\}w_{n-2m}.$$

By Lemma 1.2.4, the right side is

$$\begin{aligned}& \frac{1}{4}(\lambda^2 - (n-2m-1)^2) \frac{2}{\lambda-(n-2m-1)} w_{n-2m} \\ &= \frac{1}{2}(\lambda + (n-2m-1)) w_{n-2m} \\ &= \frac{1}{2}(\lambda + (j+1)) w_{j+2},\end{aligned}$$

as claimed. So suppose finally that $\lambda = n-2m-1$. In this case $w_j = w_{n-2m-2}$ is zero by definition, so there is nothing to prove. So formula (b) is completely established. The proof of (c) is very similar. These formulas show that the span of $\{w_j\}$ is invariant under $\mathcal{J}$. Since it contains the non-zero element $w_n = v$, it must be all of V . Q.E.D.

Lemma 1.2.7 In the setting of Lemma 1.2.6, suppose $m \geq 0$. Then

- a) $\frac{w_{n+2(m+1)}}{w_{n-2(m+1)}} = 0$ if and only if $\frac{w_{n+2m}}{w_{n-2m}} = 0$, or $\lambda = \pm(n+2m+1)$.
 b) $\frac{w_{n-2(m+1)}}{w_{n-2m-1}} = 0$ if and only if $\frac{w_{n-2m}}{w_{n-2m-1}} = 0$, or $\lambda = \pm(n-2m-1)$.

Proof. Consider (a). We may as well suppose $w_{n+2m} \neq 0$ (for, if not, w_{n+2m+2} is also zero, and (a) holds). If $\lambda = -(n+2m+1)$, then by definition $w_{n+2(m+1)} = 0$, and (a) holds; so we may suppose that $\lambda \neq -(n+2m+1)$. In this case, $w_{n+2(m+1)}$ is a non-zero multiple of $\pi(X)w_{n+2m}$; so we want to show

$$\pi(X)w_{n+2m} = 0 \iff \lambda = n+2m+1.$$

By Lemma 1.2.5(c), and the last formula in Lemma 1.2.4,

$$\begin{aligned}\pi(X)w_{n+2m} = 0 &\iff \pi(YX)w_{n+2m} = 0 \\ &\iff (\lambda^2 - (n+2m+1)^2)w_{n+2m} = 0 \\ &\iff \lambda = \pm(n+2m+1), \text{ or } w_{n+2m} = 0.\end{aligned}$$

The proof of (b) is similar.

Q.E.D.

Thus the entire structure of V can be deduced from two things: the scalar $\pi(\Omega)$, and a single integer n such that $V_n \neq 0$. We state that as

Corollary 1.2.8 Suppose (π, V) and (π', V') are irreducible admissible $(\mathcal{O}, K)$ modules. Then they are isomorphic if and only if

- a) $\pi(\Omega) = \pi'(\Omega)$ (cf. Lemma 1.2.4);

b) there is an $n \in \mathbb{Z}$ such that V_n and V'_n are both non-zero.

Lemma 1.2.9 For each complex number c , and each integer n , there is an irreducible admissible $(\mathfrak{g}, K)$ module (π, V) such that

- a) $\pi(\Omega) = c$,
- b) $V_n \neq 0$.

Proof. Lemmas 1.2.6 and 1.2.7 describe the structure of V very explicitly. One only has to check that the formulas there really define a $(\mathfrak{g}, K)$ module. This amounts to checking that 1.2.6(a) - (c) define a Lie algebra representation. The only relation which offers any difficulty is

$$(\pi(X)\pi(Y) - \pi(Y)\pi(X))w_j = \pi(H)w_j.$$

Here one must consider several cases, depending on which of $w_{j \pm 2}$ are zero. In all cases the calculation is easy. Q.E.D.

These results completely describe all the irreducible admissible $(\mathfrak{g}, K)$ modules. It is convenient, however, to organize the results a little differently.

Definition 1.2.10 Let (π, V) be a $(\mathfrak{g}, K)$ module. The set of K -types of (π, V) is

$$\{n \in \mathbb{Z} \mid V_n \neq 0\}.$$

An integer $n \in \mathbb{Z}$ is called a lowest K -type of (π, V) if it is a K -type of (π, V) and $|n|$ is minimal with respect to that property.

Lemma 1.2.11 Suppose (π, V) is an irreducible admissible $(\mathfrak{g}, K)$ module of lowest K -type n , and $|n| > 1$. Then

$$\pi(\Omega) = (|n| - 1)^2,$$

so that, in particular, (π, V) is uniquely determined by n .

Proof. Suppose for definiteness that $n > 0$; the other case is similar. Fix $v \in V_n$ not equal to zero. Since $|n| > 1$,

$$|n - 2| < |n|;$$

so by Definition 1.2.10, V_{n-2} is zero. By Lemma 1.2.2,

$$\pi(Y)v = 0.$$

By the next to last formula for Ω in Lemma 1.2.4,

$$\begin{aligned} \pi(\Omega)v &= \pi((H-1)^2)v + 4\pi(XY)v \\ &= \pi((H-1)^2)v \\ &= (n-1)^2v; \end{aligned}$$

so

$$\pi(\Omega) = (n-1)^2,$$

as claimed. The uniqueness statement is Corollary 1.2.8.

Q.E.D.

Lemma 1.2.12 Suppose $n \in \mathbb{Z}$, and $|n| \geq 1$. Let (π, V) be the irreducible admissible $(\mathfrak{g}, K)$ module such that $V_n \neq 0$, and

$$\pi(\Omega) = (|n| - 1)^2.$$

Then the set of K -types of (π, V) is

$$\{n + 2(\operatorname{sgn} n)m \mid m = 0, 1, 2, \dots\}.$$

In particular, (π, V) has lowest K -type n .

Proof. Again we suppose for definiteness that n is positive. Using (say) Lemma 1.2.6, we see that it is enough to show that

$$w_{n-2} = 0, \quad w_{n+2m} \neq 0 \quad (m = 1, 2, 3, \dots).$$

But this is immediate from Lemma 1.2.7. Q.E.D.

Definition 1.2.13 Suppose $n \neq 0$. Define

$$(\pi, X_d(n)),$$

the discrete series representation with parameter n , to be the unique irreducible admissible $(\mathfrak{g}, K)$ module with lowest K -type $n + \operatorname{sgn} n$. (This exists by the two preceding lemmas.) If $\mu = 0, 1$, or -1 , and λ is a complex number, define the continuous series representation

$$(\pi, \bar{X}_C(\lambda)(\mu))$$

to be the unique irreducible admissible $(\mathfrak{g}, K)$ module containing the K -type μ , such that $\pi(\Omega) = \lambda^2$. (This exists by Lemma 1.2.9).

Proposition 1.2.14 Let (π, V) be an irreducible admissible $(\mathfrak{g}, K)$ module, and let $\mu \in \mathbb{Z}$ be a lowest K -type of it.

a) If $|\mu| > 1$, then

$$V \cong X_d(\mu - \operatorname{sgn} \mu)$$

In this case, μ is the unique lowest K -type of V .

b) Suppose $|\mu| \leq 1$; let λ be a square root of $\pi(\Omega)$ (cf. Lemma 1.2.4). Then

$$V \cong \bar{X}_C(\lambda)(\mu).$$

Furthermore,

$$\bar{X}_C(\lambda)(\mu) \cong \bar{X}_C(\lambda')(\mu) \iff \lambda = \pm \lambda'.$$

If $\mu = 0$, then μ is the unique lowest K -type of

V . If $\mu = \pm 1$, and $\lambda \neq 0$, then the set of lowest K -types of V is

$$\{\mu, -\mu\}.$$

In this case

$$\bar{X}_C(\lambda)(\mu) \cong \bar{X}_C(\lambda)(-\mu) \quad (\lambda \neq 0).$$

If $\mu = \pm 1$ and $\lambda = 0$, then μ is the unique lowest K -type of V .

This proposition gives a precise classification of the irreducible admissible $(\mathfrak{g}, K)$ modules, although the more complete structural information of Lemmas 1.2.6 and 1.2.7 has been lost. A result very similar to this proposition will be proved for general reductive groups, however (Theorem 6.5.12); and there seems to be little hope at present of finding reasonable generalizations of Lemmas 1.2.6 and 1.2.7.

Proof. Part (a) is Lemma 1.2.11. For (b), the fact that V is isomorphic to the indicated standard representation is

Corollary 1.2.8; and since

$$\pi(\Omega) = \lambda^2,$$

the second assertion is also included in that corollary. If $\mu = 0$, the remaining assertion is trivial. Suppose then for definiteness that $\mu = 1$. Obviously, the only other candidate for a lowest K-type of V is -1 . By Lemma 1.2.7(b) (with $n = 1$ and $m = 0$), this K-type occurs if and only if

$$\lambda \neq \pm(n+2m-1) = 0.$$

It remains only to show that if $\lambda \neq 0$,

$$\bar{X}_C(\lambda)(\mu) \cong \bar{X}_C(\lambda)(-\mu).$$

By what we have just shown, these representations have the K-types $\pm\mu$ in common; and Ω acts by λ^2 in both. By Corollary 1.2.8, they are isomorphic. Q.E.D.

Lemma 1.2.12 gives a satisfactory description of the discrete series of $(\mathfrak{g}, K)$ modules; we want to record the corresponding result for the continuous series.

Lemma 1.2.15 Fix $\mu = 0, 1$, or -1 , and $\lambda \in \mathfrak{C}$.

a) If λ is not an integer of the same parity as $\mu+1$, then the K-types of $\bar{X}_C(\lambda)(\mu)$ are

$$\{\mu + 2m | m \in \mathbb{Z}\}.$$

b) If λ is a non-zero integer of the same parity as $\mu+1$, then $\bar{X}_C(\lambda)(\mu)$ is the finite dimensional representation of highest weight $|\lambda| - 1$ (cf.

[22], Theorem 7.2). Its K-types are

$$\{|\lambda| - 1, |\lambda| - 3, \dots, -|\lambda| + 1\}.$$

c) If $\lambda = 0$ and $\mu = \pm 1$, then the K-types of

$\bar{X}_C(\lambda)(\mu)$ are

$$\{\mu + 2(\operatorname{sgn} \mu)m | m = 0, 1, 2, \dots\}.$$

Proof. Take $n = \mu$ in Lemma 1.2.6, and construct a basis of $\bar{X}_C(\lambda)(\mu)$. By Lemma 1.2.7, all the $w_{\mu+2m}$ are non-zero unless

$$\lambda = \pm\mu \pm 2m \pm 1$$

for some m . This proves (a). In case (b), suppose for definiteness that $\mu = 0$ and λ is positive; the other cases are similar. By hypothesis, λ is odd; so

$$\lambda = 2r + 1,$$

with r a non-negative integer. By Lemma 1.2.7,

$$w_{\mu+2m} = w_{2m} \neq 0 \iff m = 0, \pm 1, \dots, \pm r.$$

Thus the K-types of $\bar{X}_C(\lambda)(\mu)$ are

$$\{2r, 2r-2, \dots, -2r+2, -2r\},$$

which agrees with the statement in the lemma. Finally, (c) is contained in Lemma 1.2.12. Q.E.D.

§3. The principal series.

The formulas of Lemma 1.2.6 are so nice that the somewhat complicated condition for w_j to be non-zero seems unnatural. In fact, one can make a perfectly good representation without this condition. We retain the notation of Sec.2.

Definition 1.3.1 Fix $\lambda \in \mathbb{C}$ and a "parity" $\varepsilon = \pm 1$.

The principal series representation (or $(\mathfrak{g}, \kappa)$ module)

with parameters ε and λ , $X_C(\varepsilon \otimes \lambda)$, is defined as follows

a) $X_C(\varepsilon \otimes \lambda)$ has a basis

$$\{w_n \mid n \equiv \varepsilon \pmod{2} \mathbb{Z}\}.$$

b) If $k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, then

$$\pi(k(\theta)) \cdot w_n = e^{in\theta} w_n.$$

c) The basis (1.2.1) of $\mathfrak{g}$ acts by the formulas of Lemma 1.2.6; that is,

$$\begin{aligned} \pi(H)w_n &= n w_n \\ \pi(X)w_n &= \frac{1}{2}(\lambda + (n+1)) w_{n+2} \\ \pi(Y)w_n &= \frac{1}{2}(\lambda - (n-1)) w_{n-2}. \end{aligned}$$

To see that this definition makes sense, we need only check that the formulas of (c) define a Lie algebra representation. This is very easy to do, but the following argument also works. The relations to be proved are obviously algebraic in λ . If λ is not an integer, then Lemma 1.2.15(a) shows that the module we want is a certain $\bar{X}_C(\lambda)(\mu)$ (whose existence was proved by the calculation we are now avoiding). So the relations in question hold when λ is not an integer; so they always hold.

Example 1.3.2 Take $\lambda = -1$, $\varepsilon = +1$. Then the defining relations for $X_C(1 \otimes (-1)) = V$ are

$$\pi(H)w_{2m} = 2m \cdot w_{2m}$$

$$\pi(X)w_{2m} = m \cdot w_{2m+2}$$

$$\pi(Y)w_{2m} = -m \cdot w_{2m-2}.$$

Comparing this with the calculation in Example 1.2.3, we see that (π, V) is isomorphic to the Harish-Chandra module of the representation defined in Example 1.1.2; the isomorphism takes w_{2m} to $(-1)^m f_{2m}$.

All of the principal series have a similar analytic realization, as we will see in a moment. First we will describe the subspace structure of the principal series representations.

Proposition 1.3.3 Fix $\lambda \in \mathbb{C}$ and a parity $\varepsilon = \pm 1$; and let (π, V) denote the principal series representation $\bar{X}_C(\varepsilon \otimes \lambda)$ (Definition 1.3.1). Let μ be 0 if $\varepsilon = 1$, and either ± 1 if $\varepsilon = -1$.

a) $\pi(\Omega) = \lambda^2$ (cf. Lemma 1.2.4).
 b) If λ is not an integer of parity $-\varepsilon$, then V is irreducible, and isomorphic to $\bar{X}_C(\lambda)(\mu)$ (cf. Definition 1.2.13).

c) If λ is a positive integer of parity $-\varepsilon$, then V contains as a submodule $X_d(\lambda) \oplus X_d(-\lambda)$ (cf. Definition 1.2.13). This submodule, and its two summands, are the only proper invariant subspaces of V . The quotient is

$$V/(X_d(\lambda) \oplus X_d(-\lambda)) \cong \bar{X}_C(\lambda)(\mu),$$

the finite dimensional module of highest weight $\lambda - 1$.

d) If λ is a negative integer of parity $-\varepsilon$, then V contains as its unique irreducible submodule the finite dimensional representation $\bar{X}_C(\lambda)(\mu)$ of highest weight $-\lambda - 1$. The quotient is

$$V/\bar{X}_C(\lambda)(\mu) \cong X_d(\lambda) \oplus X_d(-\lambda).$$

e) If $\lambda = 0$ and $\varepsilon = -1$, then

$$V \cong \bar{X}_C(0)(\mu) \oplus \bar{X}_C(0)(-\mu).$$

Proof. For (a), we use the last formula for Ω in Lemma 1.2.4 and Definition 1.3.1. This gives

$$\begin{aligned} \pi(\Omega)w_n &= \pi((H+1)^2)w_n + 4\pi(YX)w_n \\ &= (n+1)^2w_n + 2\pi(Y)(\lambda+(n+1))w_{n+2} \\ &= [(n+1)^2 + (\lambda-n-1)(\lambda+n+1)]w_n \\ &= [(n+1)^2 + \lambda^2 - (n+1)^2]w_n \\ &= \lambda^2w_n, \end{aligned}$$

as claimed. For the rest of the proposition, it is an elementary exercise to read off the invariant subspace structure of $X_C(\varepsilon \otimes \lambda)$ from the structure constants given in Definition 1.3.1. To identify the various irreducible composition factors, we only need to know their K -types and $\pi(\Omega)$ (by Corollary 1.2.8). We have that information, so the results can easily be checked. We leave the details to the reader. Q.E.D.

Corollary 1.3.4 Fix $\lambda \in \mathbb{C}$, and $\varepsilon = \pm 1$; and suppose

Re $\lambda > 0$. Let $\mu = 0$ if $\varepsilon = +1$, and $\mu = \pm 1$ if $\varepsilon = -1$.

Then there is a unique (up to a scalar) non-zero

homomorphism of $(\mathfrak{g}, K)$ modules

$$A : X_C(\varepsilon \otimes \lambda) \rightarrow X_C(\varepsilon \otimes -\lambda).$$

The image of A is isomorphic to $\bar{X}_C(\lambda)(\mu)$; in particular,

A is non-zero on the K -type μ .

Proof. Suppose first that λ is not an integer of parity $-\varepsilon$. By Proposition 1.3.3(b) and Proposition 1.2.14(b),

$$X_C(\varepsilon \otimes \lambda) \cong \bar{X}_C(\lambda)(\mu) \cong \bar{X}_C(-\lambda)(\mu) \cong X_C(\varepsilon \otimes -\lambda).$$

So the result is true in this case; A must even be an isomorphism. Next suppose λ is an integer of parity $-\varepsilon$; by hypothesis it is positive. By Proposition 1.3.3(c) and (d), $\bar{X}_C(\lambda)(\mu)$ is a quotient of $X_C(\varepsilon \otimes \lambda)$ and a submodule of $X_C(\varepsilon \otimes -\lambda)$; so there is a map A with the desired properties. Now let B be any other homomorphism between these principal series. By Proposition 1.3.3(c), $X_C(\varepsilon \otimes \lambda)$ contains $X_d(\pm\lambda)$ as a submodule; so $X_C(\varepsilon \otimes -\lambda)$ contains

$$B(X_d(\pm\lambda)) = Y_{\pm}$$

as a submodule. But the only irreducible submodule of $X_C(\varepsilon \otimes -\lambda)$ is $\bar{X}_C(-\lambda)(\mu)$ (by Proposition 1.3.3(d)); so $Y_{\pm} = 0$. So B factors as

$$X_C(\varepsilon \otimes \lambda) \xrightarrow{B} X_C(\varepsilon \otimes -\lambda) \xrightarrow{\bar{B}} \bar{X}_C(\lambda)(\mu)$$

Since $\bar{X}_C(\lambda)(\mu)$ is irreducible and occurs just once as a submodule of $X_C(\varepsilon \otimes -\lambda)$, the map

$$\bar{X}_C(\lambda)(\mu) \rightarrow X_C(\varepsilon \otimes -\lambda)$$

is unique up to a scalar. So $\bar{B}$ must be a multiple of $\bar{A}$, so B is a multiple of A . Q.E.D.

We turn now to the analytic description of the principal series representations.

Definition 1.3.5 Define the following subgroups of

$G = SL(2, \mathbb{R})$:

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}, a > 0 \right\}$$

$$M = \text{centralizer of } A \text{ in } K = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$P = MAN = \left\{ \begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix} \mid 0 \neq a \in \mathbb{R}, t \in \mathbb{R} \right\}.$$

If $\varepsilon = \pm 1$, and $\lambda \in \mathbb{C}$, define a character

$$\varepsilon \otimes \lambda : P \rightarrow \mathbb{C}$$

by

$$(\varepsilon \otimes \lambda) \begin{pmatrix} a & t \\ 0 & a^{-1} \end{pmatrix} = (\varepsilon) (\text{sgn } a) |a|^\lambda.$$

Let $\mathcal{H}_{\varepsilon \otimes \lambda}$ denote the Hilbert space of complex valued functions f on G , satisfying

a) if $g \in G$ and $p \in P$,

$$f(gp) = [\varepsilon \otimes (\lambda + 1)](p^{-1}) f(g);$$

b) the restriction of f to K lies in $L^2(K)$.

(We make $\mathcal{H}_{\varepsilon \otimes \lambda}$ a Hilbert space using the restriction mapping into the Hilbert space $L^2(K)$.) The principal

series (group) representation with parameters ε and λ ,

$(\pi, \mathcal{H}_{\varepsilon \otimes \lambda})$, is defined by

$$[\pi(g)f](g_0) = f(g^{-1}g_0) \quad (g, g_0 \in G, f \in \mathcal{H}_{\varepsilon \otimes \lambda}).$$

If λ is purely imaginary, so that $\varepsilon \otimes \lambda$ is a unitary character of P , then

$$(\pi, \mathcal{H}_{\varepsilon \otimes \lambda}) = \text{Ind}_P^G (\varepsilon \otimes \lambda),$$

the usual induced representation in Mackey's sense [50]. If $\varepsilon \otimes \lambda$ is not purely imaginary, $\varepsilon \otimes \lambda$ is not unitary; and there is no good general definition of non-unitary induced representations. In the present setting, however, Definition 1.3.5 is a reasonable one, and we may speak of $\mathcal{H}_{\varepsilon \otimes \lambda}$ as the induced representation.

There are still several things to check: that the operators $\pi(g)$ are bounded, that the representation is continuous, and that its Harish-Chandra module is $X_c(\varepsilon \otimes \lambda)$. All of this is based on

Lemma 1.3.6 (Iwasawa decomposition). Any element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL(2, \mathbb{R})$ can be written uniquely as a product

$$g = k(\theta) \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} \quad (\theta \in \mathbb{R}, y > 0, x \in \mathbb{R}).$$

Here

$$k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$y = \sqrt{a^2 + c^2}, \quad \cos \theta = \frac{a}{\sqrt{a^2 + c^2}}, \quad \sin \theta = -\frac{c}{\sqrt{a^2 + c^2}}$$

Proof. By matrix multiplication,

$$k(\theta) \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} y \cos \theta & x \cos \theta + y^{-1} \sin \theta \\ -y \sin \theta & -x \sin \theta + y^{-1} \cos \theta \end{pmatrix}.$$

If this is equal to g , then clearly

$$a^2 + c^2 = y^2 \cos^2 \theta + y^2 \sin^2 \theta = y^2;$$

so since y is assumed to be positive,

$$y = \sqrt{a^2 + c^2}.$$

(Since g is invertible, a and c are not both zero.) This, in turn, forces

$$\cos \theta = \frac{a}{y}, \quad \sin \theta = -\frac{c}{y}.$$

So $k(\theta)$ and y are uniquely determined; and x is fixed by

$$(k(\theta))^{-1} g = \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix}.$$

So the decomposition is unique. To prove that it exists, define $k(\theta)$ by the formulas above. Then

$$\begin{aligned} (k(\theta))^{-1} g &= \begin{pmatrix} ay^{-1} & cy^{-1} \\ -cy^{-1} & ay^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} (a^2+c^2)y^{-1} & x \\ 0 & (ad-bc)y^{-1} \end{pmatrix} \\ &= \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix} \end{aligned}$$

since $a^2 + c^2 = y^2$, and $ad - bc = 1$; here x is some real number. Multiplying by $k(\theta)$ proves the lemma. Q.E.D.

Corollary 1.3.7 If $\varepsilon = \pm 1$, let $\mathcal{H}_\varepsilon$ denote the Hilbert space of functions in $L^2(K)$ of parity ε , i.e.,

$$\mathcal{H}_\varepsilon = \{f \in L^2(K) \mid f(-x) = \varepsilon f(x)\}.$$

Then restriction to K defines an isomorphism

$$\mathcal{H}_{\varepsilon \otimes \lambda} \cong \mathcal{H}_\varepsilon.$$

Proof. The element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of G lies in $P \cap K$, and acts by ε in the character $\varepsilon \otimes \lambda$ of P . So, by definition 1.3.5, if $\varepsilon \in \mathcal{H}_{\varepsilon \otimes \lambda}$ and $x \in K$,

$$f(-x) = f \left(x \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) = \varepsilon f(x).$$

Conversely, suppose $\bar{f}$ lies in $\mathcal{H}_\varepsilon$. Extend $\bar{f}$ to a function f on G by

$$f(g) = \bar{f}(k)y^{-\lambda-1} \quad (g = k \cdot \begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix}, \quad k \in K).$$

Then f restricted to K is $\bar{f}$, so f satisfies condition (b) of Definition 1.3.5; and f was constructed to satisfy (a). Q.E.D.

Proposition 1.3.8 The principal series representation

$(\pi, \mathcal{H}_{\varepsilon \otimes \lambda})$ of Definition 1.3.5 is an admissible group representation (Definition 1.1.5). Its Harish-Chandra module (Definition 1.1.7) is isomorphic to $X_C(\varepsilon \otimes \lambda)$ of Definition 1.3.1.

Proof. Lemma 1.3.6 shows that the homogeneous space G/AN (with A and N as in Definition 1.3.5) is naturally isomorphic to K , by the inclusion

$$K = K/(K \cap AN) \rightarrow G/AN.$$

Therefore, there is an action of G on K , by

$$(1.3.9) \quad g \cdot k = k' \quad (gk = k' \begin{bmatrix} Y & x \\ 0 & Y^{-1} \end{bmatrix}, Y > 0)$$

Define a function

$$a: G \rightarrow \{Y \in \mathbb{R} \mid Y > 0\}$$

by

$$(1.3.10) \quad a(g) = Y \text{ if } g = k \cdot \begin{bmatrix} Y & x \\ 0 & Y^{-1} \end{bmatrix} \quad (k \in K, Y > 0).$$

Lemma 1.3.6 gives a formula for this function:

$$a \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sqrt{a^2 + c^2}.$$

It also gives a formula for the action of G on K , which we will not write down; but we conclude by inspection of these formulas that the map

$$G \times K \rightarrow (K \times \mathbb{R}) \\ (g, k) \mapsto (g^{-1} \cdot k, a(g^{-1} k))$$

is continuous. Suppose now that f lies in $\mathcal{H}_\varepsilon \subseteq L^2(K)$. By Definition 1.3.5 and (1.3.9) and (1.3.10),

$$[\pi(g)f](k) = f(g^{-1} \cdot k) a(g^{-1}k)^{-\lambda-1}.$$

It is now clear that the operator $\pi(g)$ is bounded (by the maximum of the Jacobian of the diffeomorphism of K given by the action of g , times the maximum on K of the function $a(g^{-1}k)^{-\lambda-1}$); one simply performs a change of variable in the integration over K which defines $||\pi(g)f||$. For similar reasons π is continuous. Recall that $\mathcal{H}_\varepsilon$ is the space of

functions on K (the circle group) of parity ε . By Fourier series, $\mathcal{H}_\varepsilon$ has an orthonormal basis given by functions $\{f_n \mid n \in \mathbb{Z} \text{ has parity } \varepsilon, \text{ and } f_n(k(\theta)) = e^{in\theta}\}$.

Defining $\mathcal{H}_n$ as in Lemma 1.1.3 (for $\mathcal{H}_\varepsilon$), we find

$$\mathcal{H}_n = \begin{cases} 0 & \text{if } n \text{ has parity } -\varepsilon \\ \mathbb{C} \cdot f_n & \text{if } n \text{ has parity } \varepsilon. \end{cases}$$

In particular, $(\pi, \mathcal{H}_{\varepsilon \otimes \lambda})$ is admissible, and its Harish-Chandra module has basis $\{f_n\}$. By Lemma 1.2.2

$$(1.3.11) \quad \begin{aligned} \pi(H)f_n &= nf_n \\ \pi(X)f_n &= a_n f_{n+2} \\ \pi(Y)f_n &= b_n f_{n-2}; \end{aligned}$$

so we want to compute a_n and b_n . Since all the functions f_n are one at the identity element,

$$(1.3.12) \quad a_n = [\pi(X)f_n](1).$$

To compute this, we regard f_n as extended to a function on all of G , in accordance with condition (a) of Definition 1.3.5.

Then

$$f_n \begin{pmatrix} Y & x \\ 0 & Y^{-1} \end{pmatrix} = Y^{-\lambda-1} \quad (Y > 0).$$

Therefore

$$\begin{aligned} \left[\pi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f_n \right](1) &= \lim_{t \rightarrow 0} \frac{1}{t} \left[f_n \begin{pmatrix} 0 & -t \\ 0 & 0 \end{pmatrix} \right] - f_n(1) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left[f_n \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} - f_n(1) \right] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} [1 - 1] \\ &= 0; \end{aligned}$$

and similarly,

$$\begin{aligned} \left[\pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f_n \right] (1) &= \lim_{t \rightarrow 0} \frac{1}{t} \left[f_n \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} - f_n(1) \right] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left(e^{t(\lambda+1)} - 1 \right) \\ &= \lambda + 1. \end{aligned}$$

Finally, the first formula of 1.3.11 gives

$$\left[\pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f_n \right] (1) = i n.$$

By (1.2.1),

$$X = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

so the computations above and (1.3.12) show that

$$a_n = \frac{1}{2}(\lambda + 1) + \frac{n}{2} = \frac{1}{2}(\lambda + (n+1)).$$

Similarly, we find

$$b_n = \frac{1}{2}(\lambda - (n-1)).$$

These are exactly the structure constants for $X_C(\varepsilon \otimes \lambda)$

(Definition 1.3.1), completing the proof of the proposition.

Q.E.D.

§4. $SL^{\pm}(2, \mathbb{R})$

For the remainder of this chapter, we return to our previous hypotheses on the level of experience of the reader; and in particular we provide no more detailed proofs. In

this section, G will denote the group

$$SL^{\pm}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc = \pm 1 \right\}.$$

Obviously G has two connected components, and the identity component G_0 is $SL(2, \mathbb{R})$. Define

$$(1.4.1) \quad \left\{ \begin{aligned} K = O(2) &= \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ \mp \sin \theta & \pm \cos \theta \end{pmatrix} \right\} \\ T = SO(2) &= K_0 \\ A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} \\ M &= \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} = \text{centralizer of } A \text{ in } K. \\ P = MAN &= \left\{ \begin{pmatrix} a & x \\ 0 & \pm a^{-1} \end{pmatrix} \mid 0 \neq a \in \mathbb{R}, x \in \mathbb{R} \right\}. \end{aligned} \right.$$

Lemma 1.4.2 The set of irreducible unitary representations of $K, \hat{K}$, may be parametrized as follows: if $n \in \mathbb{Z}$, let

$$\chi_n : K_0 \rightarrow \mathbb{C}, \chi_n(k(\theta)) = e^{in\theta}$$

be the indicated character; here

$$k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

If n is a positive integer, define

$$\mu_n = \text{Ind}_{K_0}^K(\chi_n) \cong \text{Ind}_{K_0}^K(\chi_{-n});$$

then μ_n is a two-dimensional irreducible unitary representation, and

$$\mu_n|_{K_0} = \chi_n \oplus \chi_{-n}.$$

Let μ_0^+ denote the trivial representation, and μ_0^- the other one dimensional representation of K (which is trivial on K_0); then

$$\mu_0^\pm \Big|_{K_0} = \chi_0, \quad \text{Ind}_{K_0}^K(\chi_0) = \mu_0^+ \oplus \mu_0^-.$$

Then

$$\hat{K} = \{\mu_0^+, \mu_0^-, \mu_1, \mu_2, \mu_3, \dots\}.$$

The proof is left to the reader.

Definition 1.4.3 Identify the set $\hat{M}$ of the characters of M with pairs

$$\delta = (\delta_1, \delta_2) \quad (\delta_i = 0 \text{ or } 1)$$

by

$$\delta \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} = \prod_{\varepsilon_i} \delta_i.$$

(For example, if $\delta = (1, 0)$, then

$$\delta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1, \quad \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1.)$$

If $\delta \in \hat{M}$ and $\nu \in \mathbb{C}$, define the principal series representation with parameters δ and ν by

$$(\pi, \mathcal{H}_{\delta \otimes \nu}) = \text{Ind}_P^G(\delta \otimes \nu).$$

Here we regard ν as a character of A as in Definition 1.3.5, and we make $\delta \otimes \nu$ a character of $P = MAN$ by making it trivial on N . More specifically,

$$\mathcal{H}_{\delta \otimes \nu} = \{f: G \rightarrow \mathbb{C} \mid f(gp) = f(g) [(\delta \otimes (\nu+1))(p^{-1})] \text{ for all } g \in G \text{ and } p \in P; \text{ and } f|_K \text{ lies in } L^2(K)\}$$

$$[\pi(g)f](g_0) = f(g^{-1}g_0) \quad (g, g_0 \in G, f \in \mathcal{H}'_{\delta \otimes \nu}).$$

Lemma 1.4.4 Fix $\delta \in \hat{M}$ and $\nu \in \mathbb{C}$.

$$a) \quad \mathcal{H}_{\delta \otimes \nu} \Big|_K = \begin{cases} \mu_0^+ \oplus \mu_2 \oplus \mu_4 \oplus \dots & (\delta = (0, 0)) \\ \mu_1 \oplus \mu_3 \oplus \mu_5 \oplus \dots & (\delta = (1, 0) \text{ or } (0, 1)) \\ \mu_0^- \oplus \mu_2 \oplus \mu_4 \oplus \dots & (\delta = (1, 1)) \end{cases}$$

b) Let ε be the restriction of δ to $M \cap G_0$. Then

$$\mathcal{H}_{\delta \otimes \nu} \Big|_{\text{SL}(2, \mathbb{R})} \cong \mathcal{H}_{\varepsilon \otimes \nu}^{G_0};$$

here the representation on the right is the principal series for $\text{SL}(2, \mathbb{R})$ defined in Definition 1.3.5.

Definition 1.4.5 If $\delta \in \hat{M}$ and $\nu \in \mathbb{C}$, define the

principal series $(\mathfrak{g}, K)$ module with parameters

δ and ν , $X(\delta \otimes \nu)$, to be Harish-Chandra module of $\mathcal{H}_{\delta \otimes \nu}$.

We know the $\mathfrak{g}$ -invariant subspaces of $\mathcal{H}_{\delta \otimes \nu}$ from Proposition 1.3.3(c) and Lemma 1.4.4(b); and the K -invariant subspaces can be read off from Lemma 1.4.4(a). So we can immediately compute the whole composition series of $\mathcal{H}_{\delta \otimes \nu}$. To express the results, some definitions are helpful.

Definition 1.4.6 Suppose $\delta \in \hat{M}$. Define a (one element) subset $A(\delta) \subseteq \hat{K}$ by

$$A(\delta) = \{\mu \in \hat{K} \mid \mu = \mu_1, \mu_0^+, \text{ or } \mu_0^-, \text{ and } \delta \text{ occurs in } \mu|_M\}$$

(so

$$A(0, 0) = \{\mu_0^+\}, A(0, 1) = A(1, 0) = \{\mu_1\}, A(1, 1) = \{\mu_0^-\}.)$$

If $\mu \in A(\delta)$, and $\nu \in \mathbb{C}$, let

$$\bar{X}(\delta \otimes \nu)(\mu) = \bar{X}(\delta \otimes \nu)$$

be the unique irreducible subquotient of $X(\delta \otimes \nu)$ containing the K-type μ ; this exists by Lemma 1.4.4.

Definition 1.4.7 Fix a positive integer n ; and define

$$\begin{aligned} X_d(n) &= \text{Ind}_{(\mathfrak{g}, K_0)}^{(\mathfrak{g}, K)} (X_d^{G_0}(n)) \\ &\cong \text{Ind}_{(\mathfrak{g}, K_0)}^{(\mathfrak{g}, K)} (X_d^{G_0}(-n)), \end{aligned}$$

the discrete series representation with parameter n .

Here the representations on the right are the discrete series for $SL(2, \mathbb{R})$, defined in Definition 1.2.13; the induction is explained in Definition 0.3.25, and verification of the indicated isomorphism is left to the reader.

Proposition 1.4.8 Fix $\nu \in \mathbb{C}$, $\delta \in \hat{M}$, and $\mu \in A(\delta)$ (Definition 1.4.6). Let

$$\varepsilon = \delta \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \pm 1.$$

- a) If ν is not a non-zero integer of parity $-\varepsilon$, then $X(\delta \otimes \nu)$ is irreducible, and isomorphic to $\bar{X}(\delta \otimes \nu)(\mu)$.
 b) If ν is a positive integer of parity $-\varepsilon$, then there is a non-split short exact sequence

$$0 \rightarrow X_d(\nu) \rightarrow X(\delta \otimes \nu) \rightarrow \bar{X}(\delta \otimes \nu)(\mu) \rightarrow 0;$$

the quotient $\bar{X}(\delta \otimes \nu)(\mu)$ is a finite dimensional module of highest weight $\nu-1$.

- c) If ν is a negative integer of parity $-\varepsilon$, then there is a non-split short exact sequence

$$0 \rightarrow \bar{X}(\delta \otimes \nu)(\mu) \rightarrow X(\delta \otimes \nu) \rightarrow X_d(-\nu);$$

the submodule $\bar{X}(\delta \otimes \nu)(\mu)$ is finite dimensional, of highest weight $-\nu-1$.

The proof was sketched before Definition 1.4.6. The most important difference between this result and Proposition 1.3.3 concerns the case $\varepsilon = -1$, $\nu = 0$. In that case the principal series for $SL(2, \mathbb{R})$ is a direct sum of two pieces; but these two pieces are permuted by K , and the representation of $SL^+(2, \mathbb{R})$ is irreducible.

Lemma 1.4.9 If $\delta = (\delta_1, \delta_2) \in \hat{M}$ (Definition 1.4.3), set $s \cdot \delta = (\delta_2, \delta_1)$.

- a) $X(\delta \otimes \nu)$ and $X(\delta' \otimes \nu')$ have the same set of irreducible composition factors if and only if

$$(\delta, \nu) = (\delta', \nu') \text{ or } (s\delta', -\nu);$$

- b) $\bar{X}(\delta \otimes \nu) \cong \bar{X}(\delta' \otimes \nu')$ if and only if

$$(\delta, \nu) = (\delta', \nu') \text{ or } (s\delta', -\nu).$$

Proof. The action of the Weyl group of A on $\hat{M}$ is given by s ; so (a) is a special case of Theorem 4.1.4. The "if" part of (b) is immediate. For the "only if", suppose the isomorphism holds. If either module is all of the corresponding principal series, the result follows from (a); so by Proposition 1.4.8, we may assume that ν is a non-zero integer of parity $-\varepsilon$, and ν'

is a non-zero integer of parity $-\varepsilon'$. (Here $\varepsilon = \delta \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\varepsilon' = \delta' \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.) The hypotheses and conclusion are unaffected if we replace (δ, ν) by $(s \cdot \delta, -\nu)$, and similarly for ν' ; so we may assume ν and ν' are positive. Since the restrictions to G_0 are equivalent, Proposition 1.2.14(b) gives $\nu = \nu'$, $\varepsilon = \varepsilon'$. So the composition factors of the two principal series are (by Proposition 1.4.8(b))

$$\begin{aligned} X(\delta \otimes \nu): & \quad \bar{X}(\delta \otimes \nu), X_d(\nu) \\ X(\delta' \otimes \nu'): & \quad \bar{X}(\delta' \otimes \nu'), X_d(\nu). \end{aligned}$$

These are the same by hypothesis, so (a) gives the result we want. Q.E.D.

The real issue in this proposition is the inequivalence of $\bar{X}(\delta \otimes \nu)$ and $\bar{X}(\delta' \otimes \nu')$ when $\nu \neq 0$ and $\delta \neq \delta'$. This can also be seen more explicitly, as follows. If $A(\delta) \neq A(\delta')$, the representations do not have the same restriction to K ; so the hard case is

$$\delta = (1, 0), \quad \delta' = (0, 1); \quad A(\delta) = \mu_1.$$

Choose a non-zero vector

$$w_1 \in \bar{X}(\delta \otimes \nu)$$

of weight 1 for K_0 . Then

$$w_{-1} = \pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot w_1, \quad \pi(Y)w_1 = v_{-1}$$

both have weight -1 , so

$$v_{-1} = cw_{-1}.$$

The constant c is clearly an invariant of the representation; it is $\pm \frac{1}{2}\nu$, and the sign distinguishes between δ and δ' . This method is much too subtle to use in more general situations, however, and we will have to rely on Theorem 4.1.4 (which is based on a calculation of global characters).

§5. An introduction to the R group.

In this section, G will be a reductive group satisfying the hypotheses of 0.1.2, and in addition

$$(1.5.1) \quad \left\{ \begin{array}{l} \text{a) the center of } G \text{ is compact} \\ \text{b) every simple ideal of } \mathfrak{g}_0 \text{ is isomorphic to } \mathfrak{sl}(2, \mathbb{R}). \end{array} \right.$$

Accordingly, we write

$$\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \dots \oplus \mathfrak{g}_0^n \oplus \mathfrak{C}_0,$$

with $\mathfrak{C}_0$ the center of $\mathfrak{g}_0$, and $\mathfrak{g}_0^i$ simple. Choose an isomorphism

$$\phi^i: \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}_0^i;$$

if θ denotes the Cartan involution of $\mathfrak{g}_0$, we may arrange for

$$\phi^i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \theta \phi^i(X);$$

and if $\mathfrak{g}_0$ is given a θ -stable split Cartan subalgebra

$$\mathfrak{h}_0 = \alpha_0 + \mathfrak{C}_0, \quad \text{we can arrange}$$

$$\phi^i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \alpha_0.$$

If such an $\mathfrak{h}$ is not given, we simply define

$$(1.5.2) \quad \begin{cases} \alpha_0 = \left[\oplus_{\mathbf{i}} \phi_{\mathbf{i}} \left(\mathbb{R} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right] \\ h_0 = \alpha_0 + c_0. \end{cases}$$

Our goal is to understand reducibility of unitary principal series for G . As we have seen, such reducibility exists for $SL(2, \mathbb{R})$ (Proposition 1.3.3(e)), but not for $SL^\pm(2, \mathbb{R})$ (Proposition 1.4.8(a)). The general case of course combines these two phenomena; the R -group will provide a precise measure of this combination, and thus of the reducibility.

If our group G comes endowed with a positive root system $\Delta^+(g, h)$, we may assume that

$$(1.5.3) \quad \phi^i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ lies in a positive root space.}$$

If $\Delta^+(g, h)$ is not given, we define it so that (1.5.3) holds. We need a great deal of explicit notation. Notice first that since G is linear, ϕ^i exponentiates to

$$(1.5.4) \quad \phi^i: SL(2, \mathbb{R}) \rightarrow G;$$

ϕ^i either is an isomorphism, or has kernel $\{\pm 1\}$. Set

$$(a) \quad \{\alpha_i\} = \Delta^+(g, h), \quad \alpha_i \in \Delta(g^i, h)$$

$$(b) \quad z_i = \phi^i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(c) \quad \mathcal{X}_0 = \oplus_{\mathbf{i}} \langle z_i \rangle \oplus \mathbb{C}_0$$

$$(d) \quad T = \text{centralizer of } \mathcal{X}_0 \text{ in } G$$

$$(e) \quad K = \text{normalizer of } \mathcal{X}_0 \text{ or } T \text{ in } G$$

$$(f) \quad A = \exp(\alpha_0)$$

$$(g) \quad M = \text{centralizer of } A \text{ in } K$$

- (h) $H = MA = \text{centralizer of } \alpha_0 \text{ or } h_0 \text{ or } A \text{ in } G$
- (i) $M' = \text{normalizer of } A \text{ in } K$
- (j) $W = W(H) = M'/M = \text{Weyl group of } A \text{ in } G$
- (k) $W_K = W_K(T) = K/T = \text{Weyl group of } T \text{ in } G$
- (l) $\mathcal{H}_{\mathbf{i}} = \oplus_{\mathbf{i}} \langle \phi^i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle$
- (m) $N = \exp \mathcal{H}_0$
- (n) $P = MAN$
- (o) $\sigma_{\mathbf{i}} = \exp(\frac{\pi}{2} z_i) = \phi^i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M'$
- (p) $s_i = \sigma_{\mathbf{i}} M \in W$
- (q) $m_i = \sigma_{\mathbf{i}}^2 = \exp(\pi z_i) = \phi^i \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in M$

Of course K is a maximal compact subgroup, T is a compact Cartan subgroup, H is a split Cartan subgroup, and P is a Borel subgroup (and therefore a minimal parabolic subgroup).

Definition 1.5.6 Suppose

$$\gamma = \delta \otimes \nu \in \hat{H} \quad (\delta \in \hat{M}, \nu \in \hat{A}).$$

(Recall that H is assumed to be abelian, so that γ is one-dimensional.) Define

$$(\pi(\gamma), H_\gamma) = \text{Ind}_P^G(\gamma),$$

the principal series representation with parameter γ .

Thus, if we write $\rho = (\det(\text{Ad}|_{\mathcal{H}}))^{1/2} \in \hat{A}$, and $\tilde{\gamma} = \delta \otimes (\nu + \rho)$,

$$H_\gamma = \{f: G \rightarrow \mathbb{C} \mid f(g \text{ man}) = f(g) \tilde{\gamma}(\text{ma})^{-1}, \text{ and} \\ \text{the restriction of } f \text{ to } K \text{ lies in } L^2(K)\},$$

and $\pi(\gamma)$ is the representation by left translation.

For emphasis, we may write things like $\pi^G(\gamma)$ or

$\pi(H, \gamma)$, when the groups are varying or are not clear from context. The principal series (σ, K) module with parameter γ , $X(\gamma)$, is the Harish-Chandra module of $\pi(\gamma)$.

Lemma 1.5.7 Suppose $\gamma = \delta \otimes \nu \in \hat{H}$. Let $\pi(\gamma)$ be the principal series representation with parameter γ .

$$a) \quad \pi(\gamma)|_K = \text{Ind}_M^K(\delta)$$

$$b) \quad \pi(\gamma)|_{G_0} = \pi_{G_0}^{G_0}(\gamma|_{G_0}),$$

the principal series representation of G_0 with parameter $\gamma|_{G_0} = \delta|_{(M \cap G_0)} \otimes \nu$.

Proof. These follow from the identifications

$$G/P \cong K/M \cong K_0/M \cap K_0 \cong G_0/P \cap G_0. \quad \text{Q.E.D.}$$

Recall from Proposition 1.3.5 that when ν is unitary, the principal series can be reducible only when ν is zero and δ is non-trivial. So we will concentrate on this case; some aspects of the general case will be reduced to it in Chapter 4. One difference between $SL(2, \mathbb{R})$ and $SL^\pm(2, \mathbb{R})$ was that W acts trivially on $\hat{M}$ in the first case, but not in the second. This suggests the simplest definition of the R-group; (Another is given by Corollary 1.5.24.)

Definition 1.5.8 (Knapp-Stein [26]). Suppose $\delta \in \hat{M}$, and $\delta(m_i) = -1$ for all i (cf. (1.5.5)(q)). Define

$$M'_\delta = \{x \in M' \mid x \cdot \delta \cong \delta\} \supseteq M$$

$$W_\delta = R_\delta = M'_\delta/M \subseteq \bar{W};$$

these are the stabilizer of δ in the Weyl group and the R-group for δ (which coincide in this case, but not in general).

(Recall that M' acts on $\hat{M}$ by

$$(x \cdot \delta)(m) = \delta(x^{-1}mx) \quad (\delta \in \hat{M}, m \in M);$$

this makes sense because M' normalizes M . Clearly M acts trivially, so this lifts to an action of W on $\hat{M}$.)

Since W is a product of copies of $\mathbb{Z}/2\mathbb{Z}$, the R-group is as well; so the dual group $\hat{R}_\delta$ is defined.

Proposition 1.5.9 (Knapp-Stein [26]). Suppose $\delta \in \hat{M}$, and

$$\delta(m_i) = -1, \text{ all } i \quad (\text{cf. (1.5.5)(q)});$$

and suppose $\nu \in \hat{A}$ is trivial. Then the unitary principal series representation $\pi(\delta \otimes \nu)$ decomposes as a direct sum of $|R_\delta|$ distinct irreducible representations (Definition 1.5.8). There is a natural simply transitive action of the dual group $\hat{R}_\delta$ on the set of constituents of $\pi(\delta \otimes \nu)$.

Knapp and Stein's proof works by studying analytically defined intertwining operators on $\pi(\delta \otimes \nu)$. We will use a more algebraic approach, which is designed to study the K-types of the constituents at the same time. This requires a little preparation.

Definition 1.5.10 Suppose $\delta \in \hat{M}$, and $\delta(m_i) = -1$ for all i . The set of fine K-types containing δ is defined

to be

$A(\delta) = \{\mu \in \hat{K} \mid \mu(\sqrt{-1} z_i) \text{ has eigenvalues}$

$\pm 1 \text{ for all } i, \text{ and } \delta \text{ occurs in } \mu|_M\}$.

(If $G = \text{SL}(2, \mathbb{R})$, $A(\delta) = \{\pm 1\}$; if $G = \text{SL}^+(2, \mathbb{R})$, $A(\delta) = \{\mu_1\}$; (Definition 1.4.6).)

Lemma 1.5.11 In the setting of Proposition 1.5.9,

suppose

$$\vec{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n.$$

Define

$$X_{\vec{i}} = \{v \in X(\delta \otimes v) \mid \pi(z_j)v = [\sqrt{-1} i_j]v\}.$$

Then $X_{\vec{i}}$ is zero unless all i_j are odd, in which case it has dimension one. If $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n$, set

$$X^\varepsilon = \bigoplus_{\substack{\vec{i} \in \mathbb{Z}^n \\ \text{sgn } i_j = \varepsilon_j}} X_{\vec{i}}$$

If $S \subseteq (\mathbb{Z}/2\mathbb{Z})^n$, set

$$X^S = \bigoplus_{\varepsilon \in S} X^\varepsilon.$$

Then each X^S is a $\mathcal{O}$ -invariant subspace of $X(\delta \otimes 1)$, and every $\mathcal{O}$ -invariant subspace is of this form.

Proof. This follows from Lemma 1.5.7(b) and Proposition 1.3.3(b). Q.E.D.

Corollary 1.5.12 In the setting of Lemma 1.5.11, set

$$X_f = \bigoplus_{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n} X_\varepsilon;$$

here we identify $(\mathbb{Z}/2\mathbb{Z})$ with $\{\pm 1\}$, and f stands for

"fine". Then X_f is K -invariant, and

$$a) \quad X_f|_K \cong \bigoplus_{\mu \in A(\delta)} \mu.$$

Identify the representation space V_μ of μ with a subspace of X_f , and define $S(\mu) \subseteq (\mathbb{Z}/2\mathbb{Z})^n$ by

$$V_\mu = \bigoplus_{\varepsilon \in S(\mu)} X_\varepsilon.$$

Then

$$X(\delta \otimes 1)(\mu) = X^S(\mu) \subseteq X(\delta \otimes 1)$$

is K -stable and $\mathcal{O}$ -stable; it is the unique irreducible constituent of $X(\delta \otimes 1)$ containing the K -type μ . Thus

$$X(\delta \otimes 1) = \bigoplus_{\mu \in A(\delta)} X(\delta \otimes 1)(\mu)$$

is the decomposition of $X(\delta \otimes 1)$ into irreducible components.

Proof. The element z_i (cf. 1.5.5(b)) is specified up to sign by the fact that it lies in $\mathcal{O}_0^i \cap k_0$, and its length in the killing form. These properties are preserved by $\text{Ad}(K)$; so if $k \in K$, we can define $q_i(k) = \pm 1$ by

$$1.5.13) \quad \text{Ad}(k)(z_i) = q_i(k)z_i.$$

In particular, this makes it clear that X_f is K -invariant. More precisely, define

$$1.5.14) \quad q: K \rightarrow (\mathbb{Z}/2\mathbb{Z})^n, \quad q(k) = (q_1(k), \dots, q_n(k)).$$

Then

$$1.5.15) \quad \pi(k)X_\varepsilon = X_{q(k)\varepsilon}.$$

Formula (a) of the corollary is clear from Lemma 1.5.7(a) and Definition 1.5.10. The other statements follow easily. Q.E.D.

Corollary 1.5.16 In the setting of Corollary 1.5.12 and (1.5.14), the sets $S(\mu)$ are the cosets of $q(K)$ in $(\mathbb{Z}/2\mathbb{Z})^n$. Thus the quotient group $(\mathbb{Z}/2\mathbb{Z})^n / q(K)$ acts in a natural, simply transitive way on the set of constituents of $X(\delta \otimes \nu)$: if

$$\bar{\varepsilon} = \varepsilon + q(K) \in (\mathbb{Z}/2\mathbb{Z})^n / q(K),$$

then

$$\bar{\varepsilon} \cdot X^S(\mu) = X^{\varepsilon S}(\mu).$$

To complete the proof of Proposition 1.5.9, we need only make an identification

$$(1.5.17) \quad \hat{R}_\delta \cong (\mathbb{Z}/2\mathbb{Z})^n / q(K).$$

To do this, set

$$P(n) = \{\text{subsets of } \{1, 2, \dots, n\}\}.$$

Regard $P(n)$ as the dual group of $(\mathbb{Z}/2\mathbb{Z})^n$ by the pairing

$$(1.5.18) \quad \varepsilon(A) = \prod_{i \in A} \varepsilon_i \quad (A \in P(n), \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n).$$

Now identify

$$W \cong P(n)$$

by the map

$$A \mapsto \prod_{i \in A} s_i \quad (A \in P(n), s_i \text{ as in 1.5.5(p)}).$$

The dual group of $R_\delta \cong W_\delta$ is then naturally isomorphic to the quotient

$$(\mathbb{Z}/2\mathbb{Z})^n / \text{Ann}(W_\delta);$$

here

$$\text{Ann}(W_\delta) = \{\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^n \mid \text{for all } A \in W_\delta,$$

$$\prod_{i \in A} \varepsilon_i = 1\}.$$

So what must be shown is that

$$(1.5.19) \quad \text{Ann}(W_\delta) = q(K) \subseteq (\mathbb{Z}/2\mathbb{Z})^n.$$

To prove this, notice first that the kernel of q is (almost by definition) the centralizer of $\mathcal{X}_0$ in K ; so

$$(1.5.20) \quad \begin{array}{l} \text{the map } q \text{ of (1.5.14) defines} \\ \text{an injection } \bar{q}: W(K/T) \rightarrow (\mathbb{Z}/2\mathbb{Z})^n. \end{array}$$

For us the main point is that q is trivial on $K_0 = T_0$, the identity component of K . Since M meets every component of G , and hence of K , it follows that

$$(1.5.21) \quad q(K) = q(M).$$

Lemma 1.5.22 Suppose $A \in P(n)$; let

$$w = \prod_{i \in A} s_i$$

be the corresponding element of W . Then if $m \in M$,

and $\delta(m_i) = -1$ for all i ,

$$\begin{aligned} (w \cdot \delta)(m) &= (q(m)[A])\delta(m) \quad \frac{(\text{notation (1.5.14)}, (1.5.18))}{=} \\ &= [\prod_{i \in A} q_i(m)]\delta(m). \end{aligned}$$

Proof. We first compute $w^{-1}mw$. By (1.5.13) and (1.5.5)(o),

$$\begin{aligned}
m\sigma_i m^{-1} &= \text{Ad}(m) \cdot \exp\left(\frac{\pi}{2} Z_i\right) \\
&= \exp\left[\frac{\pi}{2} \left(\text{Ad}(m) \cdot Z_i\right)\right] \\
&= \begin{cases} \sigma_i & \text{if } q_i(m) = 1 \\ \sigma_i^{-1} & \text{if } q_i(m) = -1 \end{cases}
\end{aligned}$$

Now $\sigma_i^2 = m_i$ (cf. (1.5.5)(q)), and $m_i^2 = 1$; so

$$\sigma_i^{-1} = \sigma_i m_i.$$

The preceding computation therefore gives

$$\sigma_i^{-1} m \sigma_i = \begin{cases} m & \text{if } q_i(m) = 1 \\ m m_i & \text{if } q_i(m) = -1. \end{cases}$$

So

$$(1.5.23) \quad w^{-1} m w = (m) \left(\prod_{i \in A} m_i \right) \cdot q_i(m) = -1$$

Finally, we calculate

$$\begin{aligned}
(w \cdot \delta)(m) &= \delta(w^{-1} m w) \\
&= \delta(m \cdot \prod_{i \in A} m_i) \\
&= \delta(m) \prod_{i \in A} q_i(m) \\
&= -1
\end{aligned}$$

since $\delta(m_i) = -1$.

Q.E.D.

As an immediate corollary, we see that

$$W_\delta = \text{Ann}(q(M)) = \text{Ann}(q(K))$$

(the last equality being (1.5.21)); and (1.5.19) follows.

The proof of Proposition 1.5.9 is now complete.

Corollary 1.5.24 In the setting of Proposition 1.5.9,

$$|R_\delta| = 2^n / |W_K(T)|;$$

here n is the number of simple factors of G . More precisely, there is a natural isomorphism

$$\hat{R}_\delta \cong W(\mathfrak{g}, \mathfrak{t}) / W_K(T).$$

Proof. This follows from (1.5.20), (1.5.19), and (1.5.17).

Q.E.D.

For a later application we will want another description of the representations $X(\delta \otimes \nu)(\mu)$.

Lemma 1.5.25 Define

$$\begin{aligned}
K^\# &= T \\
G^\# &= T G_0 = K^\# \text{AN} \\
M^\# &= M \cap T \\
P^\# &= M^\# \text{AN}
\end{aligned}$$

In the setting of Proposition 1.5.9, define

$$\delta^\# = \delta|_{M^\#}.$$

Then

$$a) \quad \pi_G(\delta \otimes \nu)|_{G^\#} \cong \pi_{G^\#}(\delta^\# \otimes \nu);$$

b) Given $\mu \in A(\delta)$, define $S(\mu)$ as in Corollary 1.5.12.

If $\varepsilon \in S(\mu)$, let $\mu_\varepsilon \in \hat{T}$ be the weight of T on X_ε (notation (1.5.11)). Then

$$X(\delta \otimes 1)(\mu) \Big|_{(\mathfrak{g}, K^\#)} = \bigoplus_{\varepsilon \in S(\mu)} X(\delta^\# \otimes 1)(\mu_\varepsilon);$$

c) If $\mu \in A(\delta)$, and $\varepsilon \in S(\mu)$, then

$$X(\delta \otimes 1)(\mu) = \text{Ind}_{\mathcal{G}, K}^{\mathcal{G}, K} (X(\delta^\# \otimes 1)(\mu_\varepsilon)).$$

Proof. Part (a) follows from the identifications

$$G/P \cong G_0/P \cap G_0 \cong G^\# / P^\#$$

just as in Lemma 1.5.7. Part (b) is clear from the description of $X(\delta \otimes 1)(\mu)$ given in Corollary 1.5.12. Part (c) follows from (b) by Lemma 0.3.26 (Frobenius reciprocity).

Q.E.D.

Proposition 1.5.26 In the setting of Proposition

1.5.9, fix $\mu \in A(\delta)$ and $\varepsilon \in S(\mu)$ (cf. (1.5.12)).

For each i between 1 and n (that is, for each simple factor of $\mathcal{G}$), recall the map

$$\phi^i: \mathcal{S}(2, \mathbb{R}) \rightarrow \mathcal{G}_0.$$

Define U_i to be either $\phi^i(X)$ or $\phi^i(Y)$ (notation (1.2.1)) in $\mathcal{G}$, making the choice so that

$$[Z_i, U_i] = 2\sqrt{-1} \varepsilon_i U_i$$

(notation (1.5.5)(b); compare (1.2.1)). Put

$$u = \oplus \mathbb{C} \cdot U_i \quad \mathcal{G} = \mathcal{T} + u$$

Make $\mathbb{C}_{\mu_\varepsilon}$, the representation space of μ_ε (see (1.5.25)(b)) a $\mathcal{G}$ module by making u act trivially (and $\mathcal{T}$ by the differential of μ_ε). Make

$$\text{Hom}_{\mathcal{G}}(U(\mathcal{G}), \mathbb{C}_{\mu_\varepsilon})$$

a $U(\mathcal{G})$ module by acting on the right in the first factor, and a T module by the adjoint action on $U(\mathcal{G})$ and the action on $\mathbb{C}_{\mu_\varepsilon}$; and write

$$\text{pro}_{\mathcal{G}, T}^{\mathcal{G}, T}(\mathbb{C}_{\mu_\varepsilon})$$

for the T -finite part of this module. (This definition is explained more carefully at (6.1.21).) Then, in the notation of Lemma 1.5.25,

$$\begin{aligned} X(\delta^\# \otimes 1)(\mu_\varepsilon) &\cong \text{pro}_{\mathcal{G}, T}^{\mathcal{G}, T}(\mathbb{C}_{\mu_\varepsilon}) \\ X(\delta \otimes 1)(\mu) &\cong \text{Ind}_{\mathcal{G}, T}^{\mathcal{G}, K} (\text{pro}_{\mathcal{G}, T}^{\mathcal{G}, T}(\mathbb{C}_{\mu_\varepsilon})). \end{aligned}$$

Proof. Write $\bar{u}$ for the algebra defined by $-\varepsilon$ as in the statement of the proposition. Clearly

$$\mathcal{G} = u + \mathcal{T} + \bar{u},$$

so by Poincaré-Birkhoff-Witt,

$$\begin{aligned} U(\mathcal{G}) &\cong U(u) \otimes U(\mathcal{T}) \otimes U(\bar{u}) \\ &= U(\mathcal{G}) \otimes U(\bar{u}). \end{aligned}$$

So

$$\begin{aligned} \text{Hom}_{\mathcal{G}}(U(\mathcal{G}), \mathbb{C}_{\mu_\varepsilon}) &\cong \text{Hom}_{\mathcal{G}}(U(\mathcal{G}) \otimes U(\bar{u}), \mathbb{C}_{\mu_\varepsilon}) \\ &\cong \text{Hom}_{\mathbb{C}}(U(\bar{u}), \mathbb{C}_{\mu_\varepsilon}) \\ &\cong \mathbb{C}_{\mu_\varepsilon} \otimes [S(\bar{u})]^*. \end{aligned}$$

This is an isomorphism of T modules, using the adjoint action of T on $\bar{u}$. Comparing this formula with Lemma 1.5.11, one checks easily that

$$(*) \quad \text{proj}_{\mathfrak{g}, T}^{\mathfrak{g}, T}(\mathbb{C}_{\mu_{\varepsilon}}) \Big|_T \cong X(\delta^{\#} \otimes 1)(\mu_{\varepsilon}) \Big|_T.$$

So to prove that these modules are isomorphic, we only have to find a non-zero $(\mathfrak{g}, T)$ -module map

$$\phi: X(\delta^{\#} \otimes 1)(\mu_{\varepsilon}) \rightarrow \text{Hom}_{\mathfrak{g}}(U(\mathfrak{g}), \mathbb{C}_{\mu_{\varepsilon}});$$

ϕ will then be injective since $X(\delta^{\#} \otimes 1)(\mu_{\varepsilon})$ is irreducible, and surjective by (*). To find such a map ϕ , first choose a non-zero T -module map

$$f: X(\delta^{\#} \otimes 1)(\mu_{\varepsilon}) \rightarrow \mathbb{C}_{\mu_{\varepsilon}}.$$

By the choice of $\mathfrak{g}$, if α is a root of T in $\mathfrak{u}$, then the weight $\mu_{\varepsilon} - \alpha$ of T does not occur in $X(\delta^{\#} \otimes 1)(\mu_{\varepsilon})$. It follows that

$$f(z \cdot x) = 0 \quad (x \in X(\delta^{\#} \otimes 1)(\mu_{\varepsilon}), \quad z \in \mathfrak{u}).$$

So f is actually a $\mathfrak{g}$ -module map. Now to define ϕ , we have to specify for each x in the $(\mathfrak{g}, T)$ module, a map $\phi(x)$ from $U(\mathfrak{g})$ to $\mathbb{C}_{\mu_{\varepsilon}}$. We define

$$\phi(x)(u) = f(u \cdot x) \quad (x \in X(\delta^{\#} \otimes 1)(\mu_{\varepsilon}), \quad u \in U(\mathfrak{g})).$$

To see that $\phi(x)$ is a $\mathfrak{g}$ -module map, suppose $q \in \mathfrak{g}$. Then

$$\begin{aligned} \phi(x)(qu) &= f(qu \cdot x) \\ &= f(q \cdot (ux)) \\ &= q \cdot f(u \cdot x) \quad (\text{since } f \text{ is a } \mathfrak{g}\text{-module map}) \\ &= q \cdot [\phi(x)(u)], \end{aligned}$$

as required. To check that ϕ is a $\mathfrak{g}$ module map, suppose

$\mathfrak{g} \in \mathfrak{g}$. Then

$$\begin{aligned} [\phi(gx)](u) &= f(u \cdot (gx)) \\ &= f(ug \cdot x) \\ &= \phi(x)(ug) \\ &= [g \cdot \phi(x)](u). \end{aligned}$$

So

$$\phi(gx) = g \cdot \phi(x),$$

as required. Similarly one sees that ϕ is a T module map. This proves the first claim of the proposition. The second follows from Lemma 1.5.25(c). Q.E.D.

The reader may have noticed that the key point in the construction of the map ϕ was the non-vanishing of

$$X/\mathfrak{u} \cdot X$$

(with $X = X(\delta^{\#} \otimes 1)(\mu_{\varepsilon})$). This group is nearly the same as

$$H^n(\mathfrak{u}, X).$$

In Chapter 6, we will generalize this construction, characterizing $(\mathfrak{g}, K)$ modules in terms of certain cohomology groups

$$H^R(\mathfrak{u}, X)$$

(no longer with $R = \dim \mathfrak{u}$).

Example 1.5.27 Let $G = \text{SO}(2, 2)$; this group has two connected components, and satisfies the hypotheses (1.5.1). We can regard G as the group of 4×4 matrices of determinant one, preserving the indefinite

form

$$||x|| = x_1x_2 + x_3x_4.$$

We can take for A the group of matrices with diagonal entries (x, x^{-1}, y, y^{-1}) (with x and y positive). The group M has order 4, and is generated by two elements

$$m_+ = \text{diag}(-1, -1, 1, 1)$$

$$m_- = \text{diag}(1, 1, -1, -1);$$

here "diag" is the diagonal matrix with indicated entries. The two elements m_i coming from simple $SL(2, \mathbb{R})$ factors coincide, and are equal to m_{+m_-} . (They would be distinct in the corresponding spin group.) The Weyl group of A has two generators. These act on diagonal elements by

$$w_1(x, x^{-1}, y, y^{-1}) = (x^{-1}, x, y^{-1}, y) \\ w_2(x, x^{-1}, y, y^{-1}) = (y, y^{-1}, x, x^{-1}).$$

Thus w_1 acts trivially on M, and w_2 interchanges m_+ and m_- . (The simple reflections in W are w_2 and w_1w_2). M has exactly two characters satisfying $\delta(m_i) = -1$; these are δ_+ and δ_- , defined by

$$\delta_+(m_+) = -1, \quad \delta_+(m_-) = 1$$

$$\delta_-(m_+) = 1, \quad \delta_-(m_-) = -1.$$

Clearly

$$w_1 \cdot \delta_{\pm} = \delta_{\pm} \\ w_2 \cdot \delta_{\pm} = \delta_{\mp}.$$

So if δ is either δ_+ or δ_- , $R_{\delta} = W_{\delta}$ is the two element group $\{1, w_1\}$. It can be shown (compare Lemma 4.3.46) that the R-group can always be made trivial by adding certain outer automorphisms to the group. (This does not apply to the more general R-groups to be defined for non-principal series representations.) A typical example is the passage from $SL(2, \mathbb{R})$ to $SL^{\pm}(2, \mathbb{R})$. In this case, one might guess that the right group is $O(2, 2)$. It is not, however. For one thing, the automorphism of the Lie algebra defined by permuting the first two coordinates (which lies in $O(2, 2)$) is not inner for the complexification -- it permutes the two simple factors. So $O(2, 2)$ does not belong to the category of groups we are considering. (In any case, it can be shown that the unitary principal series for $O(2, 2)$ can be reducible.) The right element to add to the group is

$$\tilde{m} = \text{diag}(1, -1, 1, -1);$$

that is, we should consider the group of matrices preserving the square of the quadratic form, and having determinant one. It is easy to check that

$$w_1 \cdot \tilde{m} = \tilde{m}(m_+m_-),$$

and therefore that w_1 cannot fix either extension of δ_+ (or of δ_-) to the group generated by M and $\tilde{m}$.

Chapter 2. Geometry of the Kazhdan-Lusztig Conjecture

In this chapter we will give a precise statement of part of the classification of irreducible representations of G , and a geometric interpretation of it. Lusztig and Zuckerman suggested a way in which this might lead to a geometric interpretation of the formulas for the characters of irreducible representations. A slight variant of their idea is presented in Section 3.

§1. The Kazhdan-Lusztig conjecture for Verma modules.

In this section $\mathfrak{g}$ will be a complex reductive Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ a fixed Cartan subalgebra, and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a fixed Borel subalgebra containing $\mathfrak{h}$, with nil radical $\mathfrak{h}$. We will assume familiarity with the most basic part of the theory of Verma modules; the material in [22] is sufficient, and [7] contains far more than we need. We first describe the set of $\mathfrak{g}$ -modules we will consider.

Definition 2.1.1.1 Let $(\hat{\mathfrak{g}})_1$ denote the set of equivalence classes of irreducible (possibly infinite dimensional)

$\mathfrak{g}$ -modules L such that

- the center $Z(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$ acts on L as it does in the trivial representation; and
 - L has a vector $v \neq 0$ which is annihilated by $\mathfrak{n}$.
- (Such a v is called a maximal vector in [22]; we call it a highest weight vector, which is more common.)

We will write

$$\rho = \rho(\mathfrak{n}) = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{n}, \mathfrak{h})} \alpha$$

$$W = W(\mathfrak{g}, \mathfrak{h}) = \text{Weyl group of } \mathfrak{h} \text{ in } \mathfrak{g}.$$

Theorem 2.1.1.2 Suppose $L \in (\hat{\mathfrak{g}})_1$. Then the highest weight vector v of L is unique up to a constant multiple. There is an element $w \in W$ such that if $H \in \mathfrak{h}$,

$$H \cdot v = -(w\rho + \rho)(H)v,$$

(that is, v has weight $-(w\rho + \rho)$ for $\mathfrak{h}$). This correspondence defines a bijection

$$(\hat{\mathfrak{g}})_1 \leftrightarrow W.$$

(We call $-(w\rho + \rho)$ the highest weight of L .)

This is an easy consequence of Harish-Chandra's theorem on $Z(\mathfrak{g})$; a proof may be found in [22], Theorems 20.2 and 23.3. The last statement is a short way of saying that

- for each $w \in W$ there is an $L \in (\hat{\mathfrak{g}})_1$ of highest weight

$$-(w\rho + \rho);$$

and

- two modules in $(\hat{\mathfrak{g}})_1$ are isomorphic if and only if they have the same highest weight.

Definition 2.1.1.3 Suppose $w \in W$. Let $L(w)$ denote the unique element of $(\hat{\mathfrak{g}})_1$ of highest weight $-(w\rho + \rho)$. Define $Z(w)$, the Verma module of highest weight $-(w\rho + \rho)$, by

$$Z(w) = U(\mathfrak{g}) \otimes \mathbb{C}_{-(w\rho+\rho)}$$

(cf. [22], 20.2, where this module is called $Z(-(w\rho+\rho))$).

Write

$$\Theta(w) = \text{ch}(Z(w)), \quad \bar{\Theta}(w) = \text{ch}(L(w))$$

for the formal characters of $L(w)$ and $Z(w)$ ([22], 22.5).

The formal characters are not to be taken seriously; we just want to be able to consider formal linear combinations of elements of $(\hat{\mathfrak{g}})_1$.

Theorem 2.1.4 Suppose $w \in W$. The module $Z(w)$ has

a finite composition series. All its irreducible

composition factors lie in $(\hat{\mathfrak{g}})_1$, so we can write

$$\Theta(w) = \sum_{y \in W} m(y, w) \bar{\Theta}(y).$$

Here $m(y, w)$ is a non-negative integer, the multiplicity of $L(y)$ in $Z(w)$. These formulas can be inverted, giving

$$\bar{\Theta}(w) = \sum_{y \in W} M(y, w) \Theta(y)$$

for some integers $M(y, w)$.

This is proved in [22], 24.2.

The first half of this book is largely devoted to proving analogues of Theorems 2.1.2 and 2.1.4 for $(\mathfrak{g}, K)$ modules. These are essentially due to Langlands and to Harish-Chandra, respectively; special cases will be formulated in the next section. The next interesting questions concern the numbers $m(y, w)$.

Definition 2.1.5 Let ℓ denote the length function on W :

$$\ell(w) = |\{\alpha \in \Delta(\pi, \ell) \mid w^{-1}\alpha \notin \Delta(\pi, \ell)\}|.$$

Let $w_0 \in W$ denote the unique element of length $|\Delta(\pi, \ell)|$, the longest element of W .

Proposition 2.1.6 Suppose $y, w \in W$.

a) $m(y, w) \neq 0$ only if $\ell(y) \leq \ell(w)$; and equality can occur only if $y = w$.

b) $M(y, w) \neq 0$ only if $\ell(y) \leq \ell(w)$; and equality can occur only if $y = w$.

c) $m(y, y) = M(y, y) = 1$.

d) $m(1, w) = 1$; that is, $L(1)$ occurs with multiplicity one in $M(w)$.

e) $M(y, w_0) = (-1)^{\ell(y) - \ell(w_0)}$; that is,

$$\bar{\Theta}(w_0) = \sum_{y \in W} (-1)^{\ell(y) - \ell(w_0)} \Theta(y).$$

Parts (a) and (b) are weak versions of results of Verma (see [7]); Proposition 24.2(b) of [22], which is still weaker, illustrates the idea adequately. Part (c) is in [22], 24.2. Part (d) is due to Verma (see [7]). Part (e) is just the Weyl character formula ([22], Theorem 24.3); for $L(w_0)$ is the trivial representation of G .

Jantzen has made a deep study of the numbers $m(y, w)$ and $M(y, w)$; one of his results in [23] is an enormous generalization of 2.1.6(d), which says exactly when $m(y, x) = 1$. (It is known (see [7]) that $m(y, w) \neq 0$ exactly when $y \leq w$ in the Bruhat ordering.) But his techniques are not (or at least not obviously) sufficient to compute all of the multiplicities

$m(y,w)$. In its basic form (cf. [24]), the Kazhdan-Lusztig conjecture is an algorithm for computing certain numbers $\tilde{M}(y,w)$ (in their notation, $(-1)^{\ell(y)-\ell(w)} P_{y,w}(1)$) which are conjecturally equal to $M(y,w)$. We will discuss the generalization of this algorithm to $(\mathcal{Y},K)$ modules in Chapter 9. For the moment, we want to describe some consequences of the conjecture, and their geometric formulation of it.

Conjecture 2.1.7 (Kazhdan-Lusztig [24], [25]).

A) The coefficients $M(y,w)$ in the character formula for $L(w)$ are the numbers $\tilde{M}(y,w)$ given by the Kazhdan-Lusztig algorithm.

B) $m(y,w) = [M(ww_0, yw_0)](-1)^{\ell(y)-\ell(w)}$

C) If $M(y,w) \neq 0$, then

$$\operatorname{sgn}(M(y,w)) = (-1)^{\ell(y)-\ell(w)}.$$

Parts (B) and (C) are consequences of (A). Proposition 2.1.6 was written to emphasize the duality between $m(y,w)$ and $M(y,w)$; this duality, which was apparently first noticed by Jantzen, is stated in its most general form in (B). Part (C) suggests that $M(y,w)$ is given by some sort of Euler characteristic. This is easily seen to be the case; the analogous fact for $(\mathcal{Y},K)$ modules is Proposition 9.4.7.

We turn now to the geometric version of the conjecture.

Let $G_{\mathbb{C}}$ be a complex reductive Lie group with Lie algebra $\mathfrak{g}$, and let B be the Borel subgroup with Lie algebra $\mathfrak{b}$.

Definition 2.1.8 The complex flag manifold of G , $\mathcal{B}$, is the variety of all Borel subgroups of G . If $w \in W$, the Schubert cell $\mathcal{B}_w$ is

$$\mathcal{B}_w = \{B' \in \mathcal{B} \mid B' = gBg^{-1}, \text{ some } g \in BwB\}.$$

(Here $\tilde{w}$ denotes a representative of w in the normalizer of H in G . The Schubert variety $\overline{\mathcal{B}}_w$ is the closure of $\mathcal{B}_w$ in $\mathcal{B}$.)

One knows that $\mathcal{B}$ is a smooth projective variety, and that $\overline{\mathcal{B}}_w$ is a subvariety which can have complicated singularities. The flag manifold is the disjoint union of all the $\overline{\mathcal{B}}_w$.

Deligne has shown how to attach to $\overline{\mathcal{B}}_w$ (in fact to much more general varieties) a collection $\{\mathcal{H}^i(\overline{\mathcal{B}}_w)\}$ of sheaves of vector spaces on $\overline{\mathcal{B}}_w$. If $\overline{\mathcal{B}}_w$ is smooth at a point x , then the stalks at x satisfy

$$\mathcal{H}_x^i(\overline{\mathcal{B}}_w) = \begin{cases} 0 & i \neq 0 \\ \mathbb{C} & i = 0. \end{cases}$$

If $\overline{\mathcal{B}}_w$ is singular at x , the stalks $\mathcal{H}_x^i(\overline{\mathcal{B}}_w)$ measure the failure of local Poincaré duality at x (that is, the failure of a punctured neighborhood of x to look like a homology sphere). (This is described precisely in [25].) The geometric version of the Kazhdan-Lusztig conjecture is

Conjecture 2.1.9 ([25]). If y is another element of W , and $x \in \mathcal{B}_y$, then

$$M(y,w) = (-1)^{\ell(y)-\ell(w)} \sum_i \dim \mathcal{H}_x^{2i}(\overline{\mathcal{B}}_w).$$

Here we interpret the right side to be zero if $x \notin \overline{\mathcal{B}}_w$.

(or, equivalently, we regard $\mathcal{H}^i(\overline{\mathcal{B}_w})$ as a sheaf on all of $\mathcal{B}_w$.)

This conjecture has been deduced from 2.1.7(A) by Kazhdan and Lusztig (cf. [25]); they also show that the sheaves $\mathcal{H}^i(\overline{\mathcal{B}_w})$ are zero for i odd.

§2. A special case of the classification.

It will be convenient for the rest of this chapter to strengthen our assumptions on G slightly.

Assumption 2.2.1 For the rest of this chapter, we fix a connected complex reductive algebraic group $G_{\mathbb{C}}$, defined over $\mathbb{R}$. We let

$$\sigma: G_{\mathbb{C}} \rightarrow G_{\mathbb{C}},$$

be the Galois action; the differential of σ is a conjugate-linear involutive automorphism of the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$. Write

$$G_{\mathbb{R}} = G_{\mathbb{C}}^{\sigma},$$

the set of real points. Assume that $G \subseteq G_{\mathbb{C}}$, and that $G \cap G_{\mathbb{R}}$ has finite index in G and in $G_{\mathbb{R}}$; thus G and $G_{\mathbb{R}}$ have the same Lie algebra $\mathfrak{g}_0$. Choose a Cartan involution

$$\tau: G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$$

for $G_{\mathbb{C}}$ which commutes with σ (as is possible), and set

$$\theta = \sigma\tau.$$

Then our final assumptions are that $\theta G = G$, and that the component group of any Cartan subgroup of G has exponent 2.

These assumptions imply that θ is a Cartan involution for G , and that all of our previous assumptions (0.1.2) hold.

We want to describe a special case of the classification theorem for $\hat{G}$.

Definition 2.2.2 For the rest of this chapter, we write $(\hat{G})_1$ for the set of infinitesimal equivalence classes (Definition 0.3.9) of irreducible admissible representations of G , on which the center $Z(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$ acts as it does in the trivial representation.

Next we describe the parametrizing set for $\hat{G}_1$.

Definition 2.2.3 A set of $\mathbb{Z}/2\mathbb{Z}$ data for G is a triple (H, Δ^+, χ) , consisting of a Cartan subgroup H of G , a positive root system Δ^+ for $\mathfrak{h}$ in $\mathfrak{g}$, and a group homomorphism

$$\chi: H \rightarrow \{\pm 1\}.$$

The group G acts on such sets in an obvious way; and we write $\mathcal{S}$ for the set of orbits, the equivalence classes of sets of $\mathbb{Z}/2\mathbb{Z}$ data for G .

Theorem 2.2.4 (Langlands). The set $(\hat{G})_1$ (Definition 2.2.2) is in one-to-one correspondence with the set $\mathcal{S}$ (Definition 2.2.3). More precisely, one can attach to each $\gamma \in \mathcal{S}$ an admissible representation $\pi(\gamma)$ of G (which need not be irreducible), having the following properties:

- a) $\pi(\gamma)$ has a finite composition series, and all of its irreducible subquotients belong to $(\hat{G})_1$.
- b) $\pi(\gamma)$ has a unique irreducible subrepresentation $\bar{\pi}(\gamma)$.

The various $\{\bar{\pi}(\gamma)\}$ exhaust $(\hat{G})_1$.

This is a consequence of Theorem 6.5.12 and is proved in Section 6.7. Our construction of $\pi(\gamma)$ (which is due to Zuckerman) is very different from that of Langlands; it yields the same result, but that fact is not proved in this book (cf. Section 6.6). Neither construction is easy to describe: for experts, one can say that one starts with $(\Delta^+) \in \mathcal{H}^*$, and applies Harish-Chandra's prescription for getting a representation from a character of a Cartan subgroup. The character χ has to be twisted a bit to fit into this construction properly.

Definition 2.2.5 Suppose $\gamma \in \mathcal{B}$ (Definition 2.2.3). Put

$$\begin{aligned}\Theta(\gamma) &= \text{character of } \pi(\gamma) \\ \bar{\Theta}(\gamma) &= \text{character of } \bar{\pi}(\gamma),\end{aligned}$$

either as distributions (Definition 0.3.22) or in the appropriate Grothendieck group (Definition 0.3.16). If $\mu \in \mathcal{D}$, define $m(\mu, \gamma)$ to be the multiplicity of $\bar{\pi}(\mu)$ in $\pi(\gamma)$.

Theorem 2.2.6 (Harish-Chandra, Zuckerman) In the notation of the preceding definition, the formulas

$$(a) \quad \Theta(\gamma) = \sum_{\mu \in \mathcal{D}} m(\mu, \gamma) \bar{\Theta}(\mu)$$

can be inverted, giving

$$(b) \quad \bar{\Theta}(\gamma) = \sum_{\mu \in \mathcal{D}} m(\mu, \gamma) \Theta(\mu)$$

for unique integers $M(\mu, \gamma)$.

This is a consequence of Proposition 6.6.7. Just as in the case of Verma modules, the interesting problem is to understand the integers $m(\mu, \gamma)$ and $M(\mu, \gamma)$. (For $SL(2, \mathbb{R})$, they have been computed in Chapter 1, as will be explained in the next example.) The reason that these numbers are important is that the representations $\pi(\gamma)$, like the Verma modules in Section 2.1, may be regarded as well understood: their distribution characters and restrictions to $\bar{K}$ are explicitly computable, at least in principle.

Example 2.2.7 Suppose $G = SL(2, \mathbb{R})$. The condition that $Z(\mathcal{G})$ act as in the trivial representation means (in the notation of Lemma 1.2.4) that $\pi(\emptyset) = 1$. By Proposition 1.2.14, there are four such irreducible $(\mathcal{G}, K)$ modules:

$$\hat{G}_1 = \{X_d(1), X_d(-1), \bar{X}_C(1)(0), \text{ and } \bar{X}_C(1)(1)\}.$$

Here $X_d(\pm 1)$ are two discrete series, $\bar{X}_C(1)(0)$ is the trivial representation, and $\bar{X}_C(1)(1)$ is an irreducible principal series representation. The set $\mathcal{D}$ can be described as follows. Write

$$\begin{aligned}H^S &= \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{R}, \lambda \neq 0 \right\} \\ \chi &: H^S \rightarrow \{\pm 1\}, \chi \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right] = \text{sgn } \lambda \\ \Delta_S^+ &\leftrightarrow \text{positive root vector } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

$$H^C = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$$

$\Delta_C^+ \leftrightarrow$ positive root vector X (cf. 1.2.1)

Here s means split, and c means compact. We can choose as representatives of the equivalence classes in $\mathcal{D}$ the four elements

$$\begin{aligned} \gamma_d^+ &= (H^C, \Delta_C^+, 1) & \gamma_f &= (H^S, \Delta_S^+, 1) \\ \gamma_d^- &= (H^C, -\Delta_C^+, 1) & \gamma_p &= (H^S, \Delta_S^+, \chi). \end{aligned}$$

Here d stands for discrete series, f for finite dimensional, and p for (irreducible) principal series. These correspond in the obvious way to the four elements of $(\hat{G})_1$ listed above. The formulas of Theorem 2.2.6(a) are

$$\begin{aligned} \overline{\Theta}(\gamma_d^+) &= \overline{\Theta}(\gamma_d^+) \\ \Theta(\gamma_f) &= \overline{\Theta}(\gamma_f) + \overline{\Theta}(\gamma_d^+) + \overline{\Theta}(\gamma_d^-) \\ \Theta(\gamma_p) &= \overline{\Theta}(\gamma_p) \end{aligned}$$

(Proposition 1.3.3). Inverting these formulas gives

$$\begin{aligned} \overline{\Theta}(\gamma_d^+) &= \Theta(\gamma_d^+) \\ \overline{\Theta}(\gamma_f) &= \Theta(\gamma_f) - \Theta(\gamma_d^+) - \Theta(\gamma_d^-) \\ \overline{\Theta}(\gamma_p) &= \Theta(\gamma_p), \end{aligned}$$

which are the formulas of Theorem 2.2.6(b).

A good analogue of Proposition 2.1.6 is available for group representations; to state it, we need an analogue for $\mathcal{D}$ of the length of function on W .

Definition 2.2.8 Suppose $\gamma \in \mathcal{D}$ (Definition 2.2.3).

Choose a representative (H, Δ^+, χ) for γ so that H is θ -stable (as we may by Lemma 0.1.6); we will call

(H, Δ^+, χ) a θ -stable representative of γ , even though Δ^+ need not be θ -stable. Let $d(H)$ be half the cardinality of a maximal orthogonal set of real roots of $\mathcal{h}$ in $\mathfrak{g}$ (Definition 0.2.1). Set

$$\ell(\gamma) = \frac{1}{2}(|\{\alpha \in \Delta^+ \mid \theta\alpha \notin \Delta^+\}| + d(H)),$$

the length of γ . Let H^S be a maximally split Cartan subgroup of G (Definition 0.4.1), and Δ_S^+ a positive root system for $\mathcal{h}^S$ in $\mathfrak{g}$ compatible with an Iwasawa decomposition of G . Define

$$\gamma_0 = \text{equivalence class of } (H^S, \Delta_S^+, 1),$$

the standard longest element of $\mathcal{D}$. Define

$$e(G) = \ell(\gamma_0).$$

Lemma 2.2.9 With notation as in Definition 2.1.8, $\ell(\gamma)$ is an integer; and

- a) $0 \leq \ell(\gamma) \leq \ell(\gamma_0) = e(G)$.
- b) $\ell(\gamma) = \ell(\gamma_0)$ if and only if γ is equivalent to

$$(H_S, \Delta_S^+, \chi)$$

for some character $\chi: H_S \rightarrow \{\pm 1\}$.

c) $\overline{\pi}(\gamma_0)$ is the trivial representation, and $\pi(\gamma_0)$ is the unique spherical principal series representation among the $\pi(\gamma)$.

- d) $\ell(\gamma) = 0$ if and only if γ has a representative (H, Δ^+, χ) , with H a θ -stable fundamental Cartan subgroup of G , and Δ^+ a θ -stable positive system ([50], Definition 1.3.3). If G has a compact Cartan subgroup, this means that $\pi(\gamma)$ is a discrete series representation.
- e) Let c_0 be one half the dimension of the split part of a fundamental Cartan subgroup of G . Then

$$e(G) = \frac{1}{2} \dim G/K - c_0.$$

The statements not involving the representations $\pi(\gamma)$ are elementary, and the others are immediate consequences of the definition of $\pi(\gamma)$. Most of the details are worked out in Lemmas 9.4.9 and 9.4.14.

The appropriate analogue of Proposition 2.1.6 is

Proposition 2.2.10 Suppose $\gamma, \mu \in \mathcal{A}$ (Definition 2.2.3).

In the notation of Theorem 2.2.6 and Definition 2.2.8,

- a) $m(\gamma, \mu) \neq 0$ only if $\ell(\gamma) \leq \ell(\mu)$; and equality can occur only if $\gamma = \mu$.
- b) $M(\gamma, \mu) \neq 0$ only if $\ell(\gamma) \leq \ell(\mu)$; and equality can occur only if $\gamma = \mu$.
- c) $m(\gamma, \gamma) = M(\gamma, \gamma) = 1$.
- d) Suppose γ has a representative (H, Δ^+, χ) . Then

$$M(\gamma, \gamma_0) = \begin{cases} (-1)^{\ell(\gamma) - \ell(\gamma_0)} & \text{if } \chi \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

Parts (a) - (c) are contained in Proposition 6.6.7 and 8.6.19. (Part (c) follows from Langlands' classification paper [32]; slightly weakened versions of (a) were noticed by several people simultaneously, although the first completely correct proof was given by B. Speh. Part (b) is a formal consequence of (a).) Part (d) is an unpublished result of Zuckerman, which will be proved later (Proposition 9.4.16). We have not given an analogue of 2.1.6(d), or even defined an analogue in $\mathcal{A}$ of the element $1 \in W$. This can be done; it is the right interpretation of the main theorem of [43]. However, this problem is too complicated technically to discuss here.

Example 2.2.11 In the notation of Example 2.2.7,

$$\ell(\gamma_d^+) = 0, \quad \ell(\gamma_f) = \ell(\gamma_p) = 1 = e(G).$$

The reader can easily verify that the formulas of that example are consistent with Proposition 2.2.10; in fact, except for the irreducibility of $\pi(\gamma_p)$, they can even be deduced from the proposition.

What we mean by the Kazhdan-Lusztig conjecture for real groups is

Conjecture 2.2.12

- A) The numbers $m(\gamma, \mu)$ are computed by the algorithm of Section 9.6.
- B) If $M(\gamma, \mu) \neq 0$, then

$$\operatorname{sgn} M(\gamma, \mu) = (-1)^{\ell(\gamma) - \ell(\mu)}.$$

These are analogues of 2.1.7(A) and (C). Section 2.3 is devoted to the analogue of Conjecture 2.1.9. There is an analogue for real groups of Conjecture 2.1.7(B); but it involves changing the group as well as the representations, and will not be formulated here.

§3. Geometry on the flag manifold.

Recall from Definition 2.1.8 the complex flag manifold

$\mathcal{B}$. The idea of Conjecture 2.1.9 was to use the parametrizing set W for $(\hat{\mathcal{G}})_1$ to define subvarieties $\overline{\mathcal{B}}_w$ of $\mathcal{B}$, whose geometry was related to the numbers $M(y, w)$. There are two ways to try to do this for real groups. The first is to use the orbits of G on $\mathcal{B}$, an idea which goes back to Kostant and Wolf [52]. It is (in hindsight) obvious and natural, but has the disadvantage that the cells it produces are only components of real algebraic varieties. The second is not as natural, but seems for our purposes to be a closer analogue of the Schubert cell decomposition. To state it, we need a new group.

Definition 2.3.1 Put $K = G^\theta$, with θ as in 2.2.1; and

$$(K_{\mathbb{C}})'_0 = \text{complex connected subgroup of } G_{\mathbb{C}} \text{ with} \\ \text{Lie algebra } \mathcal{K}$$

$$K_{\mathbb{C}} = K \cdot (K_{\mathbb{C}})'_0,$$

the complexification of K .

Since K has finitely many components, $K_{\mathbb{C}}$ is an algebraic group. It need not be connected, however; in fact its component group is the same as that of K (or G). A typical example is

$G = GL(n, \mathbb{R})$, $G_{\mathbb{C}} = GL(n, \mathbb{C})$; then $K_{\mathbb{C}}$ may be chosen to be the complex orthogonal group $O(n, \mathbb{C})$, which has exactly two connected components.

We are going to decompose $\mathcal{B}$ according to the orbits of $K_{\mathbb{C}}$. The next definition sets up the parameterization of the orbits, and its relation to the classification of representations.

Definition 2.3.2

a) A set of real data for G consists of a pair (H, Δ^+) with $H \subseteq G$ a Cartan subgroup, and Δ^+ a positive root system for $\mathcal{h}$ in $\mathfrak{g}$. We write $\overline{\mathcal{B}}_{\mathbb{R}}$ for the set of equivalence classes of real data, under conjugation by G .

b) A set of $(\mathbb{R}, \theta)$ data for G consists of a pair (H, Δ^+) , with $H \subseteq G$ a θ -stable Cartan subgroup, and Δ^+ a positive system for $\mathcal{h}$ in $\mathfrak{g}$. We write $\overline{\mathcal{B}}_{\mathbb{R}, \theta}$ for the set of equivalence classes of $(\mathbb{R}, \theta)$ data, under conjugation by K .

c) A set of $\mathbb{A}$ data for G consists of a pair $(H_{\mathbb{C}}, \Delta^+)$, with $H_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ a θ -stable Cartan subgroup, and Δ^+ a positive system for $\mathcal{h}$ in $\mathfrak{g}$. We write $\overline{\mathcal{B}}_{\theta}$ for the set of equivalence classes of θ data, under conjugation by $K_{\mathbb{C}}$.

Definition 2.3.3 Suppose $p = (H, \Delta^+)$, with H a Cartan subgroup of G or $G_{\mathbb{C}}$, and Δ^+ a positive system for $\mathcal{h}$ in $\mathfrak{g}$. Define $B_p \in \mathcal{B}$ to be the unique Borel subgroup of $G_{\mathbb{C}}$ whose Lie algebra contains $\mathcal{h}$ and the root vectors for Δ^+ . If $p \in \gamma \in \overline{\mathcal{B}}_{\mathbb{R}}$ is a set of real data for G , define

$$\mathcal{O}_Y^{\mathbb{R}} = \mathcal{O}_p^{\mathbb{R}} = \{g_B g_p^{-1} \mid g \in G\} \subseteq \mathcal{B},$$

the real orbit (or G orbit) defined by p . If $p \in \gamma \in \overline{\mathcal{B}}^\theta$ is a set of θ data for G , define

$$\mathcal{O}_Y^\theta = \mathcal{O}_p^\theta = \{k_B k_p^{-1} \mid k \in K_{\mathbb{Q}}\} \subseteq \mathcal{B},$$

the θ -stable orbit (or $K_{\mathbb{Q}}$ orbit) defined by p .

Proposition 2.3.4 (cf. [52], [33]). The natural maps

$$\overline{\mathcal{B}}^{\mathbb{R}, \theta} \rightarrow \overline{\mathcal{B}}^{\mathbb{R}}, \quad \overline{\mathcal{B}}^{\mathbb{R}, \theta} \rightarrow \overline{\mathcal{B}}^\theta$$

are bijections. The maps

$$\overline{\mathcal{B}}^{\mathbb{R}} \rightarrow \{\text{orbits of } G \text{ on } \mathcal{B}\}$$

$$\overline{\mathcal{B}}^\theta \rightarrow \{\text{orbits of } K_{\mathbb{Q}} \text{ on } \mathcal{B}\}$$

of Definition 2.3.3 are also bijections. If $p = (H, \Delta^+)$ is a set of $(\mathbb{R}, \theta)$ data for G , then the natural maps

$$\begin{aligned} H/H_0 &\rightarrow G \cap B_p / (G \cap B_p)_0 \\ H/H_0 &\rightarrow K_{\mathbb{Q}} \cap B_p / (K_{\mathbb{Q}} \cap B_p)_0 \end{aligned}$$

are isomorphisms.

This is proved in [33]. (Actually, Matsuki discusses orbits of B on $K_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$, but this is of course equivalent. In both cases the point is to describe $K_{\mathbb{Q}} \backslash G_{\mathbb{Q}}/B$.) In particular, we see that an equivalence class of sets of $\mathbb{Z}/2\mathbb{Z}$ data for G (Definition 2.2.3) defines in a natural way a $K_{\mathbb{Q}}$ orbit on $\mathcal{B}$. But the set of $\mathbb{Z}/2\mathbb{Z}$ data contains more information. To insert this in the geometric picture, we need the language of sheaves.

Definition 2.3.5 Suppose X is a topological space on which the topological group H act continuously. Recall that a sheaf of vector spaces on X is a correspondence $\mathcal{S}$ which assigns to every open subset $U \subseteq X$ a vector space $\mathcal{S}(U)$, subject to certain axioms. $\mathcal{S}(U)$ could be the space of all sections over U of a homogeneous vector bundle on X , or the space of all locally constant functions on U ; this second example is closest to what we will be using.) We say that $\mathcal{S}$ is an H -sheaf if for each open $U \subseteq X$, and each $g \in H$, we are given an isomorphism $i(U, g) : \mathcal{S}(U) \rightarrow \mathcal{S}(g \cdot U)$ subject to the cocycle condition

$$i(U, g_1 g_2) = i(g_2^U, g_1) \cdot i(U, g_2) \quad (g_1, g_2 \in H)$$

and compatibility with passing to open subsets. (Each of the sheaves mentioned above has a canonical H -sheaf structure).

Example 2.3.6 Let $I \subseteq H$ be a closed subgroup, and suppose $X = H/I$ is a homogeneous space. Let

$$\chi : I \rightarrow \mathbb{C}, \quad \chi|_{I_0} \text{ trivial}$$

be a character of the component group of I . The H -sheaf induced by χ , $\mathcal{S}_{I, \chi}$, is defined by

$$\begin{aligned} \mathcal{S}_{I, \chi}(U) &= \text{space of locally constant} \\ &\quad \text{functions } f \text{ on the inverse} \\ &\quad \text{image } \tilde{U} \text{ of } U \text{ in } H, \text{ satisfying} \\ &\quad f(hi) = f(h)\chi(i^{-1}) \text{ for all} \\ &\quad h \in H, i \in I. \end{aligned}$$

The condition that χ be trivial on I_0 is necessary to make $\mathcal{S}_{I,\chi}$ non-trivial, as the reader can check. If χ is trivial, then $\mathcal{S}_{I,\chi}$ is just the constant sheaf (the second example in Definition 2.3.5).

Definition 2.3.7 Suppose $\gamma \in \mathcal{D}^*$ (Definition 2.2.3).

Choose a θ -stable representative (H, Δ^+, χ) for γ

(Definition 2.2.8). Let B be the corresponding Borel subgroup (Definition 2.3.3), and

$$I = B \cap K_{\mathbb{C}}$$

the isotropy group of $K_{\mathbb{C}}$ acting on $\mathcal{B}$ at B . Put

$$O_Y^{\theta} = K_{\mathbb{C}} \cdot B \cong K_{\mathbb{C}}/I.$$

Let

$$\tilde{\chi} : I \rightarrow \{\pm 1\} \subseteq \mathbb{C}$$

be the character corresponding to χ under the isomorphisms of component groups given at the end of Proposition 2.3.4. Let $\mathcal{S}_{\tilde{\chi}}$ be the sheaf on O_Y^{θ} induced by $\tilde{\chi}$ (Example 2.3.6). We call a sheaf of vector spaces a sheaf of lines if its stalk at every point is one dimensional; clearly $\mathcal{S}_{\tilde{\chi}}$ is a sheaf of lines. Finally, we may also regard $\mathcal{S}_{\tilde{\chi}}$ as a sheaf on all of $\mathcal{B}$, by extending it to be zero off of O_Y^{θ} .

Proposition 2.3.8 The correspondences (Theorem 2.2.4 and Definition 2.3.7)

$$\overline{\pi}(\gamma) \rightarrow \gamma \rightarrow (\mathcal{S}_{\gamma}, O_Y^{\theta})$$

define a bijection between $(\hat{G})_1$ and the set of $K_{\mathbb{C}}$ sheaves of lines on orbits of $K_{\mathbb{C}}$ on $\mathcal{B}$.

One could state a similar result about G -sheaves on G orbits, but I do not know how to use it to formulate an analogue of Conjecture 2.1.9; so it is left to the reader.

Example 2.3.9 Suppose $G = \mathrm{SL}(2, \mathbb{R})$; we use the notation of Example 2.2.7. We may identify $\mathcal{B} = \mathbb{CP}^1$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$; the action of G is by linear fractional transformations. The orbits of $K_{\mathbb{C}} = \mathrm{SO}(2, \mathbb{C})$ are

$$O_{Y_d}^{\theta} = \{i\}, \quad O_{Y_d}^{\theta} = \{-i\}$$

$$O_{Y_f}^{\theta} = O_{Y_p}^{\theta} = \text{everything else}.$$

Each of the sheaves $\mathcal{S}_{\gamma}$ is the constant sheaf over O_Y^{θ} , except for $\mathcal{S}_{Y_p}$. That sheaf may be described as follows.

After a change of coordinates, we may identify

$$O_{Y_p}^{\theta} \cong \mathbb{C}^{\times} = \{\text{non-zero complex numbers}\}$$

$$\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^{\times}.$$

The action is

$$z \cdot x = z^2 x \quad (z \in \mathrm{SO}(2, \mathbb{C}), x \in O_{Y_p}^{\theta}).$$

There is an obvious non-trivial locally flat line bundle $\mathcal{L}$ over $O_{Y_p}^{\theta}$; moving once around the origin takes a point in $\mathcal{L}$ to its negative. Then $\mathcal{S}_{Y_p}$ is the sheaf of germs

of flat sections of $\mathcal{L}$. If $U \subseteq O_{Y_p}^\theta$ is a connected open set, then

$$S_{Y_p}(U) = \begin{cases} \mathbb{C} & \text{if } U \text{ contains no closed curve winding} \\ & \text{around } 0 \text{ an odd number of times,} \\ 0 & \text{otherwise.} \end{cases}$$

It is now clear how to formulate an analogue of Conjecture

2.1.9. To state it, we need a generalization of Deligne's construction as described in [25]. Although I have been assured by various experts that this can be done, it has not been written down; so we state it as:

Conjecture 2.3.10 Fix $\gamma \in \mathcal{D}$, and let n be the

dimension of $\overline{O}_Y^\theta$ (over $\mathbb{C}$). Then there is a complex

$$\cdots \rightarrow \mathcal{S}_Y^i \rightarrow \mathcal{S}_Y^{i+1} \rightarrow \cdots$$

of $K_{\mathbb{C}}$ -sheaves on $\overline{O}_Y^\theta$, uniquely defined as an object in an appropriate derived category by

A) the cohomology sheaf $\mathcal{H}_Y^i$ of $\mathcal{S}_Y^*$ is zero for

$i < 0$, and its support has dimension at most $n - i - 1$ for $i > 0$;

B) $\mathcal{S}_Y^*$ is self-dual in the derived category;

$$C) \quad \mathcal{H}_Y^i|_{O_Y^\theta} = \begin{cases} \mathcal{S}_Y & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

We regard $\mathcal{H}_Y^i$ as a sheaf on all of $\mathcal{B}$ by making it zero off of $\overline{O}_Y^\theta$.

Conjecture 2.3.11 Suppose $\gamma, \mu \in \mathcal{D}$. Then $\mathcal{H}_Y^i$ is the zero sheaf for i odd, and

$$M(\mu, \gamma) = (-1)^{\ell(\gamma) - \ell(\mu)} \sum_i \dim \operatorname{Hom}(\mathcal{S}_\mu, \mathcal{H}_\gamma^{2i}|_{O_\mu^\theta}).$$

Of course this would imply Conjecture 2.2.12(B). I do not know its relation to 2.2.12(A); this is a difficult, but probably interesting, geometry problem.

Example 2.3.12 We continue with Example 2.3.9.

Suppose first that $\gamma \neq \gamma_p$. Then $\mathcal{S}_\gamma^0$ is the constant sheaf on $\overline{O}_Y^\theta$, and $\mathcal{S}_\gamma^i$ is zero for $i \neq 0$. In particular, the stalk of $\mathcal{H}_{Y_f}^0$ at the point i is one dimensional; so the conjecture says

$$M(\gamma_d^+, \gamma_f) = (-1) \cdot 1,$$

which agrees with what was computed in Example 2.2.7.

If $\gamma = \gamma_p$, O. Gabber has computed that $\mathcal{S}_\gamma^0$ is just $\mathcal{S}_\gamma$ (extended by zero to $\overline{O}_Y^\theta$), and $\mathcal{S}_\gamma^i = 0$ for $i \neq 0$. Thus

$$M(\mu, \gamma_p) = 0 \quad (\mu \neq \gamma),$$

which also agrees with 2.2.7.

Example 2.3.13 Suppose G is arbitrary, and $\gamma = \gamma_0$ (Definition 2.2.8). Then $O_{Y_0}^\theta$ is all of $\mathcal{B}$, $\mathcal{S}_{Y_0}^0$ is the constant sheaf, and $\mathcal{S}_{Y_0}^i$ is zero for $i \neq 0$. If $\mu \in \mathcal{D}$ has a representative (H, Δ^+, χ) , then the conjecture says

$$M(\mu, \gamma_0) = \begin{cases} (-1)^{\ell(\mu) - \ell(\gamma_0)} & \text{if } \chi \text{ is trivial,} \\ 0 & \text{otherwise;} \end{cases}$$

this is Zuckerman's theorem (Proposition 2.2.10(d)).

Finally, the conjecture says that $\pi(\gamma)$ must be irreducible whenever O_γ^θ is closed; and this can also be shown to be true.

Chapter 3. Kostant's Borel-Weil Theorem.

§1. The Casselman-Osborne theorem.

Kostant's version of the generalized Borel-Weil theorem is a precise description of certain Lie algebra cohomology groups of finite dimensional representations. The main difficulty is to show that these groups cannot be too large. This he did by a complicated calculation, which in the end related the action of the Casimir operators in the original representation and the cohomology. The Casselman-Osborne theorem does this for all of $z(\mathfrak{g})$ at once.

Let $\mathfrak{g}$ be a complex reductive Lie algebra, and $\mathfrak{g} \subseteq \mathfrak{g}$ a parabolic subalgebra. Fix a Levi decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{u},$$

with $\mathfrak{u}$ the nil radical of $\mathfrak{g}$ and $\mathfrak{l}$ reductive. Let V be a $U(\mathfrak{g})$ -module. We recall from ([5], XIII, §8) the definition of the Lie algebra cohomology groups

$$H^i(\mathfrak{u}, V).$$

Let $\Lambda^i \mathfrak{u}$ denote the i^{th} exterior power of $\mathfrak{u}$. We make $\text{Hom}_{\mathbb{C}}(\Lambda^i \mathfrak{u}, V)$ into a cochain complex using a differential d .

If

$$f \in \text{Hom}_{\mathbb{C}}(\Lambda^i \mathfrak{u}, V),$$

then

$$df \in \text{Hom}_{\mathbb{C}}(\Lambda^{i+1} \mathfrak{u}, V).$$

More precisely, if $x_j \in \mathfrak{u}$, for $j = 0, 1, \dots, i$, then

$$\begin{aligned}
 \text{df}(X_0 \wedge \dots \wedge X_i) &= \sum_{j=0}^i (-1)^j X_j \cdot f(X_0 \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_i) \\
 (3.1.1) \quad &+ \sum_{j < k} (-1)^{j+k} f([X_j, X_k] \wedge X_0 \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge \hat{X}_k \wedge \dots \wedge X_i).
 \end{aligned}$$

As usual one can check that $d^2 = 0$; and we put

$$H^i(\mathcal{U}, V) = (\text{Ker } (d_i)) / (\text{Im } (d_{i-1})).$$

Here for simplicity we write d_i for the restriction of d to $\text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{U}, V)$. (A functorial version of this is given in Chapter 6, in Definition 6.1.13.)

The algebra $\mathcal{L}$ normalizes $\mathcal{U}$, and therefore acts on $\Lambda^i \mathcal{U}$ in a natural way. Furthermore $\mathcal{L}$ acts on V ; so it acts on $\text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{U}, V)$. This action commutes with d , so $H^i(\mathcal{U}, V)$ is in a natural way an $\mathcal{L}$ module.

As was first pointed out by Allan Cooper, one can make $H^i(\mathcal{U}, V)$ into a $Z(\mathcal{G})$ module in the following way. $Z(\mathcal{G})$ acts on V , and hence, on $\text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{U}, V)$ by acting on the range of a map. From (3.1.1), it is obvious that this action commutes with d ; so it defines an action of $Z(\mathcal{G})$ on $H^i(\mathcal{U}, V)$.

Proposition 3.1.2 Suppose

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

is a short exact sequence of $\mathcal{G}$ modules. Then there

is a long exact sequence

$$\dots \rightarrow H^i(\mathcal{U}, U) \rightarrow H^i(\mathcal{U}, V) \rightarrow H^i(\mathcal{U}, W) \rightarrow H^{i+1}(\mathcal{U}, U) \rightarrow \dots$$

The maps are $Z(\mathcal{G})$ and $U(\mathcal{L})$ module maps.

proof. Except for the statement about $Z(\mathcal{G})$, this may be found in [5]. The maps which do not increase the degree of the cohomology group are obtained by passage to the quotient from the obvious maps between the various

$$\text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{U}, *).$$

That they commute with the $Z(\mathcal{G})$ action is obvious. So we need only consider the "connecting homomorphism"

$$H^i(\mathcal{U}, W) \xrightarrow{\delta} H^{i+1}(\mathcal{U}, U).$$

Fix a class ω in the first group, and choose a representative

$$f \in \text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{U}, W).$$

The natural map

$$\text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{U}, V) \rightarrow \text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{U}, W)$$

is surjective since that from V to W is; choose a preimage h of f . Then

$$dh \in \text{Hom}_{\mathbb{Q}}(\Lambda^{i+1} \mathcal{U}, V).$$

Necessarily dh takes values in U (since $df = 0$) so we may regard dh as an element of

$$\text{Hom}_{\mathbb{Q}}(\Lambda^{i+1} \mathcal{U}, U).$$

It is closed, and $\delta(\omega)$ is the corresponding element of $H^{i+1}(\mathcal{U}, U)$. Now suppose $z \in Z(\mathcal{G})$. As a representative of $z \cdot \omega$, we can of course choose $z \cdot f$; and as a preimage of $z f$ we can choose zh . Since d and z commute,

$$d(zh) = z(dh).$$

This says that $\delta(z\omega)$ has a representative which is z applied to a representative of $\delta(\omega)$; so

$$\delta(z\omega) = z\delta(\omega),$$

which is what we wanted to prove.

Q.E.D.

Returning to our parabolic subalgebra $\mathfrak{g}$, we write in analogy with (0.2.7),

$$\begin{aligned} U(\mathfrak{g}) &= U(\mathfrak{u}^-) \otimes U(\mathfrak{l}) \otimes U(\mathfrak{u}) \\ &= U(\mathfrak{l}) \oplus (\mathfrak{u}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{u}). \end{aligned} \quad (3.1.3)$$

Here $\mathfrak{g}^- = \mathfrak{l} + \mathfrak{u}^-$ is the parabolic subalgebra opposite to $\mathfrak{g}$.

Define

$$\gamma: Z(\mathfrak{g}) \rightarrow U(\mathfrak{l})$$

to be the projection on the first factor according to the decomposition (3.1.3). Since the adjoint action of $\mathfrak{l}$ preserves both these factors, γ is actually a map into $Z(\mathfrak{l})$. Also, if $z \in Z(\mathfrak{g})$

$$z - \gamma(z) \in U(\mathfrak{g})\mathfrak{u} \quad (3.1.4)$$

(compare [22], p. 130). It follows easily that γ is a homomorphism.

Theorem 3.1.5 (Casselman-Osborne [6]). Suppose V is a $U(\mathfrak{g})$ module, and $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$. If $z \in Z(\mathfrak{g})$, and $\omega \in H^i(\mathfrak{u}, V)$, then

$$z \cdot \omega = \gamma(z)\omega.$$

Here notation is as above; the action on the left is that of $Z(\mathfrak{g})$, and the action on the right is that of $U(\mathfrak{l})$.

Proof. We proceed by induction on i . Suppose first that $i = 0$. It is immediate from (3.1.1) that

$$H^0(\mathfrak{u}, V) \cong V^{\mathfrak{u}},$$

the subspace of V consisting of vectors annihilated by $\mathfrak{u}$. Both $U(\mathfrak{l})$ and $Z(\mathfrak{g})$ act by the restriction of their actions on V . Because of (3.1.4),

$$(z - \gamma(z))\omega = 0,$$

which is what we wanted to show.

Suppose next that $i > 0$, and that the result is known for $i - 1$. By ([5], I, 3.3) we can find an injective $U(\mathfrak{g})$ module I containing V as a submodule. (This fact and the others used below are also proved in Section 6.1.) This gives a short exact sequence

$$0 \rightarrow V \rightarrow I \rightarrow Q \rightarrow 0.$$

Since $U(\mathfrak{g})$ is free as a $U(\mathfrak{u})$ module (by the Poincaré-Birkhoff-Witt theorem), I is also injective as a $U(\mathfrak{u})$ module ([5], II, 6.2a). It follows that the cohomology of I vanishes in positive degrees ([5], V, 3.1), and in particular, that $H^i(\mathfrak{u}, I) = 0$. By the long exact sequence of Proposition 3.1.2 we can therefore choose an element

$$\tilde{\omega} \in H^{i-1}(\mathfrak{u}, Q)$$

such that if δ is the connecting homomorphism, then $\delta(\tilde{\omega}) = \omega$. By inductive hypothesis,

$$z \cdot \tilde{\omega} = \gamma(z) \cdot \tilde{\omega}.$$

Applying δ to both sides, and using the fact that δ is a map of $Z(\mathfrak{g})$ and $U(\mathfrak{l})$ modules, we get the conclusion of the theorem. Q.E.D.

Corollary 3.1.6 Under the hypotheses of Theorem 3.1.5, suppose in addition that V is $Z(\mathfrak{g})$ -finite. Then $H^1(u, V)$ is $Z(\mathfrak{l})$ -finite. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$, and a positive root system

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) \supseteq \Delta(u, \mathfrak{h}).$$

Let

$$\{\lambda_1, \dots, \lambda_r\} \subseteq \mathfrak{h}^*$$

be the set of generalized infinitesimal characters occurring in V (Proposition 0.3.19 and Definition 0.2.9).

Then any $Z(\mathfrak{l})$ -infinitesimal character occurring in

$$H^1(u, V) \text{ is of the form } w\lambda_i - \rho(u), \text{ for some } w \text{ in } W(\mathfrak{g}, \mathfrak{h}).$$

Proof. Define

$$\begin{aligned} \tilde{\xi}: Z(\mathfrak{g}) &\rightarrow U(\mathfrak{h}) \\ \tilde{\xi}^\ell: Z(\mathfrak{l}) &\rightarrow U(\mathfrak{h}) \end{aligned}$$

as in Definition 0.2.7. If we let $\mathfrak{n}$ be the span of the root vectors for $\Delta^+(\mathfrak{g}, \mathfrak{h})$, and $\mathfrak{n}^\ell$ the span for $\Delta^+(\mathfrak{l}, \mathfrak{h})$, then

$$\mathfrak{n} = \mathfrak{n}^\ell + \mathfrak{u}.$$

It follows that, with γ defined by (3.1.4),

$$\begin{aligned} \tilde{\xi} &= \tilde{\xi}^\ell \circ \gamma \\ \rho &= \rho(u) + \rho(\mathfrak{n}^\ell) \end{aligned}$$

$$\rho = \rho(u) + \rho^\ell.$$

Now let ξ and ξ^ℓ denote the respective Harish-Chandra maps (Definition 0.2.7). We find

$$\begin{aligned} (3.1.7) \quad \xi &= T_\rho \circ \tilde{\xi} \\ &= T_\rho(u) \circ T_{\rho^\ell} \circ \tilde{\xi}^\ell \circ \gamma \\ \xi &= T_\rho(u) \circ \xi^\ell \circ \gamma \end{aligned}$$

Since V is annihilated by an ideal $I \subseteq Z(\mathfrak{g})$ of finite codimension, Theorem 3.1.5 says that $H^1(u, V)$ is annihilated by $\gamma(I) \subseteq Z(\mathfrak{l})$. We study $\gamma(I)$ by using the Harish-Chandra isomorphism ξ^ℓ of $Z(\mathfrak{l})$ onto

$$U(\mathfrak{h})^{W(\mathfrak{l}, \mathfrak{h})}.$$

By (3.1.7),

$$\xi^\ell(\gamma(I)) = T_{-\rho(u)}(\xi(I)).$$

By the Harish-Chandra isomorphism for $\mathfrak{g}$, the right side has finite codimension in

$$T_{-\rho(u)}(U(\mathfrak{h})^{W(\mathfrak{g}, \mathfrak{h})}),$$

and hence in all of $U(\mathfrak{h})$; so the left side has finite codimension in $U(\mathfrak{h})^{W(\mathfrak{l}, \mathfrak{h})}$. So $H^1(u, V)$ is annihilated by the ideal $\gamma(I)$, which has finite codimension. This proves the first claim.

For the second, suppose that the infinitesimal character $\mu \in \mathfrak{h}^*$ occurs in $H^1(u, V)$. We claim that μ is of the form $w\lambda_i - \rho(u)$, for some $w \in W(\mathfrak{g}, \mathfrak{h})$ and some i . Suppose not. Then we can find a polynomial function on $\mathfrak{h}^*$,

$$p \in U(\mathfrak{h})^{W(\mathfrak{g}, \mathfrak{h})}$$

such that

- (3.1.8) (a) $p(\mu + \rho(\mathfrak{u})) \neq 0$
 (b) $p(\lambda_i) = 0$, all i .

Choose $z \in Z(\mathfrak{g})$ so that $\xi(z) = p$ (as we may by Theorem 0.2.8). By (3.1.8) (b), z is killed by all the infinitesimal characters occurring in V ; so since V is $Z(\mathfrak{g})$ -finite, z^n annihilates V for some $n > 0$. Replacing p by p^n changes nothing, so we may assume z annihilates V . By Theorem 3.1.5, $\gamma(z)$ annihilates $H^1(\mathfrak{u}, V)$. Since the infinitesimal character μ is assumed to occur there,

$$[\xi^\lambda \circ \gamma(z)](\mu) = 0.$$

Comparing this with (3.1.7) gives

$$\begin{aligned} 0 &= [T_{-\rho}(\mathfrak{u}) \circ \xi(z)](\mu) \\ &= \xi(z)(\mu + \rho(\mathfrak{u})) \\ &= p(\mu + \rho(\mathfrak{u})) \end{aligned}$$

by the definition of z . This contradicts (3.1.8) (a), and proves the result.

Q.E.D.

§2. Kostant's theorem.

In this section, $\mathfrak{g}$ will continue to denote a complex reductive Lie algebra, and

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$$

the Levi decomposition of a parabolic subalgebra. The results will be applied mostly to the complexified Lie algebra $\mathfrak{L}$ of

the maximal compact subgroup of G . After (3.2.13), we will again assume that $\mathfrak{g}$ is the complexified Lie algebra of G . Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$, and let

$$W = W(\mathfrak{g}, \mathfrak{h})$$

denote the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ (Definition 0.2.5). Fix a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ containing $\Delta(\mathfrak{u}, \mathfrak{h})$. We recall Definition 2.1.5; if $w \in W$, set

$$\begin{aligned} \Delta_w^+(\mathfrak{u}) &= \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) \mid w^{-1}\alpha \notin \Delta^+(\mathfrak{g}, \mathfrak{h})\} \\ \ell(w) &= |\Delta_w^+(\mathfrak{u})|. \end{aligned} \quad (3.2.1)$$

Using our fixed parabolic $\mathfrak{g}$, we define

$$\begin{aligned} W^1 &= \{w \in W \mid \Delta^+(\mathfrak{u}) \subseteq \Delta(\mathfrak{u}, \mathfrak{h})\} \\ &= \{w \in W \mid \text{whenever } \mu \in \mathfrak{h}^* \text{ is dominant} \\ &\quad \text{for } \Delta^+(\mathfrak{g}, \mathfrak{h}), \text{ then } w\mu \text{ is dominant} \\ &\quad \text{for } \Delta^+(\mathfrak{l}, \mathfrak{h})\}. \end{aligned} \quad (3.2.2)$$

(The equivalence of the two definitions is easily checked.)

Theorem 3.2.3 (Kostant - see [50], Theorem 2.5.2.1)

Suppose F is an irreducible finite dimensional representation of $\mathfrak{g}$, of highest weight $\lambda \in \mathfrak{h}^*$ (Proposition

0.4.3 (e)). Then the irreducible finite dimensional

representation of $\mathfrak{l}$ of highest weight μ occurs in

$H^*(\mathfrak{u}, F)$ if and only if there is a $w \in W^1$ such that

$$\mu = w(\lambda + \rho) - \rho;$$

and in that case it occurs exactly once, in degree $\ell(w)$.

The proof is based on several simple but important facts. (Some of the next results will not be needed until later, but are included now for simplicity of exposition.)

Lemma 3.2.4 ([22], Corollary 10.2B) (Notation (0.2.5)).

Whenever $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h})$ is a simple positive root, we have

$$\langle \check{\alpha}, \rho \rangle = 1.$$

Corollary 3.2.5

a) Suppose α is a root of $\mathfrak{h}$ in $\mathfrak{l}$. Then

$$\langle \alpha, \rho(\mathfrak{u}) \rangle = 0.$$

b) Suppose α is a root of $\mathfrak{h}$ in $\mathfrak{u}$. Then

$$\langle \alpha, \rho(\mathfrak{u}) \rangle > 0.$$

Proof. For (a), since the simple roots of $\mathfrak{h}$ in $\mathfrak{l}$ span all the roots, we may assume α is simple. A simple root of $\mathfrak{h}$ in $\mathfrak{l}$ is also simple in $\mathfrak{g}$, since $\mathfrak{g}$ was assumed to be compatible with Δ^+ . So by Lemma 3.2.4,

$$\begin{aligned} \langle \check{\alpha}, \rho(\mathfrak{u}) \rangle &= \langle \check{\alpha}, \rho \rangle - \langle \check{\alpha}, \rho(\Delta^+(\mathfrak{l})) \rangle \\ &= 1 - 1 = 0. \end{aligned}$$

For (b), notice that every root in $\Delta(\mathfrak{u})$ is of the form

$$\beta + \sum_{\gamma \text{ simple}} n_{\gamma} \gamma,$$

with β a simple root not in $\Delta^+(\mathfrak{l})$, and n_{γ} a non-negative integer. By (a), therefore, we may assume α is simple. Write

$$\rho(\Delta^+(\mathfrak{l})) = \sum_{\gamma \text{ simple in } \Delta^+(\mathfrak{l})} a_{\gamma} \gamma;$$

then a_{γ} is a positive rational number. If γ is simple in $\Delta^+(\mathfrak{l})$

then $\alpha \neq \gamma$ (since α is a root in $\mathfrak{u}$). The inner product of distinct simple roots is non-positive, so

$$\langle \alpha, \gamma \rangle \leq 0.$$

Combining these gives

$$\begin{aligned} \langle \alpha, \rho(\mathfrak{u}) \rangle &= \langle \alpha, \rho \rangle - \langle \alpha, \rho(\Delta^+(\mathfrak{l})) \rangle \\ &= \langle \alpha, \rho \rangle - \sum a_{\gamma} \langle \alpha, \gamma \rangle. \end{aligned}$$

The first term is positive by Lemma 3.2.4, and the second is non-negative by the argument just given. This proves (b).

Q.E.D.

From now on, many similar arguments will be left to the reader.

Lemma 3.2.6 Let F be the irreducible finite dimensional representation of $\mathfrak{g}$ of highest weight λ . Then for all $\mu \in \Delta(F)$, (notation 0.2.3), we have

$$\langle \mu + \rho, \mu + \rho \rangle \leq \langle \lambda + \rho, \lambda + \rho \rangle.$$

Equality holds if and only if $\mu = \lambda$.

Proof. According to Proposition 0.4.3(b) and (g),

$$\begin{aligned} \langle \mu, \mu \rangle &\leq \langle \lambda, \lambda \rangle \\ \mu &= \lambda - \sum_{\alpha \in \Delta^+} n_{\alpha} \alpha, \end{aligned}$$

with n_{α} a non-negative integer. This equation and Lemma 3.2.4 imply

$$\langle \mu, \rho \rangle \leq \langle \lambda, \rho \rangle,$$

with equality if and only if $\mu = \lambda$. Adding the two inequalities leads to

* $\langle \mu, \mu \rangle + 2\langle \mu, \rho \rangle + \langle \rho, \rho \rangle \leq \langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle + \langle \rho, \rho \rangle$,
with equality if and only if $\mu = \lambda$. This is exactly the assertion of the lemma. Q.E.D.

Lemma 3.2.7 The irreducible finite dimensional representation F of highest weight λ has infinitesimal character $\lambda + \rho$.

Proof. Obviously $Z(\mathfrak{g})$ acts on the highest weight space $F(\lambda)$ by composition of ξ (Definition 0.2.7) and evaluation at λ ; for the positive root vectors annihilate $F(\lambda)$ by Proposition 0.4.3 (f). Since ξ is $T_\rho \circ \tilde{\xi}$ (Definition 0.2.7) the lemma follows. Q.E.D.

Lemma 3.2.8 Suppose $\lambda \in \mathfrak{h}^*$, and $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Delta^+$. Let $\{\alpha_i\}_{i=1}^m$ be any subset of Δ^+ . Then for all $w \in W$,

$$w(\lambda + \rho) - \rho + \sum \alpha_i, w(\lambda + \rho) - \rho + \sum \alpha_i \rangle \geq \langle \lambda, \lambda \rangle.$$

Equality holds if and only if $\{\alpha_i\} = \Delta^+(w)$.

Proof. We use the formula

$$(3.2.9) \quad \rho - w\rho = \sum_{\beta \in \Delta^+(w)} \beta$$

(see, for example, [50], 2.5.2.4). Notice also that

$$(3.2.10) \quad \Delta^+(w) = \{\alpha \in \Delta^+ \mid \langle \alpha, w\rho \rangle < 0\};$$

this is immediate from (3.2.1) and Lemma 3.2.4. Then

$$(*) \quad w(\lambda + \rho) - \rho + \sum \alpha_i = w\lambda - \sum_{\beta \in \Delta^+(w)} \beta + \sum \alpha_i.$$

Set

$$\begin{aligned} A &= \Delta^+(w) \cap \{\alpha_i\} \\ B &= \{\beta \mid -\beta \in \Delta^+(w) - A\} \\ C &= \{\alpha_i\} - A. \end{aligned}$$

Thus $\{\alpha_i\} = \Delta^+(w)$ if and only if $B \cup C$ is empty. Because of (3.2.10), if $\beta \in B$, then $\langle \beta, w\rho \rangle > 0$.

Similarly, if $\gamma \in C$, then γ is a positive root not in $\Delta^+(w)$; so

$$\langle \gamma, w\rho \rangle > 0.$$

On the other hand, (*) and (3.2.9) imply that

$$w(\lambda + \rho) - \rho + \sum \alpha_i = w\lambda + \sum_{\beta \in B \cup C} \beta.$$

Since λ and ρ are dominant, $w\lambda$ and $w\rho$ lie in the same closed Weyl chamber; so by the inequalities above,

$$\langle \sum_{\beta \in B \cup C} \beta, w\lambda \rangle \geq 0.$$

So expanding the left side gives

$$|w(\lambda + \rho) - \rho + \sum \alpha_i|^2 \geq |w\lambda|^2 = |\lambda|^2,$$

with equality if and only if $B \cup C$ is empty. Q.E.D.

Corollary 3.2.11 In the setting of Theorem 3.2.3,

the representation of $\mathfrak{l}$ of highest weight $w(\lambda + \rho) - \rho$ occurs in $\text{Hom}_{\mathfrak{g}}^*(\mathfrak{u}, F)$ exactly once, in degree $\ell(w)$.

Proof. Clearly the multiplicity of the weight μ in $\text{Hom}_{\mathfrak{g}}^*(\mathfrak{u}, F)$ is the number of expressions

$$\mu = \gamma - \sum \alpha_i,$$

with $\gamma \in \Delta(F)$ and $\{\alpha_i\} \subseteq \Delta(\mathfrak{u})$; here we count this expression

with the multiplicity of γ . Of course, the degree in $\Lambda^* \mathcal{U}$ corresponds to the cardinality of $\{\alpha_i\}$. In particular, consider the multiplicity of the weight $w(\lambda+\rho)-\rho$; fix an expression for it as above. By Lemma 3.2.8

$$\langle \gamma, \gamma \rangle \geq \langle \lambda, \lambda \rangle,$$

with equality only if $\{\alpha_i\} = \Delta^+(w)$. By Proposition 0.4.3(b), the reverse inequality also holds; so we have equality, and $\{\alpha_i\} = \Delta^+(w)$. Using (1.3.11), we get

$$w(\lambda+\rho)-\rho = \gamma + w\rho-\rho;$$

so γ is necessarily $w\lambda$, which we know has multiplicity one (by Proposition 0.4.3 (c)). So this weight occurs exactly once, in degree $|\Delta^+(w)| = \ell(w)$. Suppose α_0 is any positive root of $\mathcal{L}$ in ℓ . We claim that the weight

$$w(\lambda+\rho)-\rho + \alpha_0$$

cannot occur in $\text{Hom}_{\mathbb{Q}}(\Lambda^* \mathcal{U}, F)$. For we would then get an expression

$$w(\lambda+\rho)-\rho = \gamma - \sum_{i=0}^m \alpha_i,$$

with $\gamma \in \Delta(F)$, $\{\alpha_i \mid 1 \leq i \leq m\} \subseteq \Delta(\mathcal{U})$. The argument just given would then show

$$\{\alpha_i \mid 0 \leq i \leq m\} = \Delta^+(w);$$

so $\Delta^+(w)$ would meet $\Delta^+(\ell)$, contradicting $w \in W^1$. It follows that the weight $w(\lambda+\rho)-\rho$ is annihilated by all the positive root vectors of ℓ ; so by Proposition 0.4.3, it must generate the irreducible representation of ℓ of that highest weight.

Q.E.D.

Proof of Theorem 3.2.3 Let μ be the highest weight of a representation of ℓ occurring in $H^*(\mathcal{U}, F)$. This representation has infinitesimal character $\mu+\rho(\Delta^+(\ell))$ by Lemma 3.2.7. By Corollary 3.1.6, there is a $w \in W$ such that

$$\mu+\rho(\Delta^+(\ell)) = w(\lambda+\rho) - \rho(\mathcal{U});$$

or equivalently

$$\mu = w(\lambda+\rho) - \rho.$$

Since $\mu+\rho$ is dominant for $\Delta^+(\ell)$ (by Proposition 0.4.3(e) and Lemma 3.2.4), so is $w(\lambda+\rho)$. So if $\alpha \in \Delta^+(\ell)$,

$$\langle w^{-1}\alpha, \lambda+\rho \rangle = \langle \alpha, w(\lambda+\rho) \rangle > 0.$$

So $\Delta^+(w) \cap \Delta^+(\ell)$ is empty; so $w \in W^1$. Now Corollary 3.2.11 implies that the representation of highest weight μ occurs in $\text{Hom}_{\mathbb{Q}}(\Lambda^* \mathcal{U}, F)$ exactly once, in degree $\ell(w)$. Since the action of ℓ commutes with the coboundary map d , this holds also for the cohomology. Q.E.D.

One more "folk theorem" about finite dimensional representations will be of some use later; it is the technique of proof which is most important, however.

Proposition 3.2.12 Let F be the irreducible finite dimensional representation of $\mathfrak{g}$ of highest weight λ , and let V be another finite dimensional representation of $\mathfrak{g}$. Then the highest weight of any constituent of $F \otimes V$ is of the form $\lambda+\mu$, with $\mu \in \Delta(V)$.

This result can also be extended to G in analogy with Corollary 3.2.16; we leave this to the reader, although it is the extended version we usually need.

Proof. Let $\mathcal{L} = \mathcal{L} + \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to Δ^+ . We know from Proposition 1.3.2 that

$$H^0(\mathfrak{n}, F) \cong \mathbb{C}_\lambda,$$

the one dimensional $\mathcal{L}$ module of weight λ ; and what we must show is that any weight of

$$H^0(\mathfrak{n}, F \otimes V)$$

is of the form $\lambda + \mu$. Since $\mathcal{L}$ is solvable, we can choose a $\mathcal{L}$ -stable filtration

$$V = V_n \supseteq V_{n-1} \supseteq \dots \supseteq V_0 = \{0\},$$

such that

$$\dim V_i/V_{i-1} = 1.$$

We will show by induction on i that any weight of

$$H^0(\mathfrak{n}, F \otimes V_i)$$

is of the form $\lambda + \mu$, with $\mu \in \Delta(V)$. This is trivial if $i = 0$. Suppose then that $i \geq 1$, and the result is known for $i-1$. The short exact sequence

$$0 \rightarrow F \otimes V_{i-1} \rightarrow F \otimes V_i \rightarrow F \otimes V_i/V_{i-1} \rightarrow 0$$

of $\mathcal{L}$ modules gives rise (by Proposition 3.1.2) to a long exact sequence

$$0 \rightarrow H^0(\mathfrak{n}, F \otimes V_{i-1}) \rightarrow H^0(\mathfrak{n}, F \otimes V_i) \rightarrow H^0(\mathfrak{n}, F \otimes V_i/V_{i-1}) \rightarrow \dots$$

of $\mathcal{L}$ modules. Any weight of the middle term is therefore a weight of one of the end terms. If it is a weight of $H^0(\mathfrak{n}, F \otimes V_{i-1})$, we know by induction that it has the desired form. So we may suppose it is a weight of $H^0(\mathfrak{n}, F \otimes V_i/V_{i-1})$. Now V_i/V_{i-1} has dimension 1, and is therefore trivial as a module for the commutator subalgebra $\mathfrak{n}$ of $\mathcal{L}$. It follows very easily that

$$H^*(\mathfrak{n}, F \otimes V_i/V_{i-1}) \cong H^*(\mathfrak{n}, F) \otimes V_i/V_{i-1}$$

as $\mathcal{L}$ modules. Let μ be the weight of V_i/V_{i-1} as an $\mathcal{L}$ module; then $\mu \in \Delta(V)$, and

$$H^0(\mathfrak{n}, F) \otimes V_i/V_{i-1}$$

has weight $\lambda + \mu$, as desired. Q.E.D.

It will be convenient to have available a formulation of Theorem 3.2.3 for disconnected groups. This requires only some definitions.

Definition 3.2.13 Suppose G is a reductive linear group

(cf. 0.1.2). A parabolic subalgebra $\mathfrak{g}$ of $\mathfrak{g}$ is said to have a θ -stable real Levi factor if there is a Levi

decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$, with $\mathfrak{l} = (\mathfrak{l}_0)_\mathbb{C}$, and

$$\mathfrak{l}_0 \subseteq \mathfrak{g}_0, \quad \theta \mathfrak{l}_0 = \mathfrak{l}_0.$$

In that case we define

$$L = \{x \in G \mid \text{Ad}(x) \cdot \mathfrak{l} \subseteq \mathfrak{l}, \text{Ad}(x) \cdot \mathfrak{u} \subseteq \mathfrak{u}\}.$$

Lemma 3.2.14 In the setting of the preceding definition

- a) $\theta|_L = L$.
 b) L is a real reductive linear group, with Cartan involution θ .
 c) Any Cartan subgroup of L is a Cartan subgroup of G .
 d) If F is a finite dimensional representation of G , then the natural action of L on

$$\text{Hom}(\wedge^i \mathcal{U}, F)$$

commutes with the differential (3.1.1), and so defines a representation of L on $H^i(\mathcal{U}, F)$.

Proof. For (a), fix a complex reductive group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$, and let $Q_{\mathbb{C}}$ be the connected subgroup with Lie algebra $\mathfrak{q}$. Set

$$L_{\mathbb{C}} = \{x \in G_{\mathbb{C}} \mid \text{Ad}(x) \cdot \ell \subseteq \ell, \text{Ad}(x) \cdot \mathcal{U} \subseteq \mathcal{U}\}.$$

By the theory of complex reductive groups, $L_{\mathbb{C}}$ is the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra ℓ ; and by definition

$$L = \{x \in G \mid \text{Ad}(x) \in \text{Ad}(L_{\mathbb{C}})\}.$$

Since $L_{\mathbb{C}}$ is connected and ℓ is θ -stable,

$$\theta \text{Ad}(L_{\mathbb{C}}) = \text{Ad}(L_{\mathbb{C}}) ;$$

so

$$\begin{aligned} \theta L &= \{x \in G \mid \theta \text{Ad}(x) \in \text{Ad}(L_{\mathbb{C}})\} \\ &= \{x \in G \mid \text{Ad}(x) \in \theta \text{Ad}(L_{\mathbb{C}})\} \\ &= \{x \in G \mid \text{Ad}(x) \in \text{Ad}(L_{\mathbb{C}})\} \\ &= L \end{aligned}$$

This proves (a). The rest of the lemma is trivial. Q.E.D.

Definition 3.2.15 Let H be a Cartan subgroup of G , and $\Lambda \subseteq \hat{H}$ be the lattice of weights of finite dimensional representations of G (Definition 0.4.11). If $\lambda \in \Lambda$, write

$$\overline{\lambda} \in \mathfrak{h}^*$$

for the differential of λ . Make $W(\mathfrak{g}, \mathfrak{h})$ act on Λ as follows: if $w \in W(\mathfrak{g}, \mathfrak{h})$ and $\lambda \in \Lambda$, then we require

- a) $\overline{w\lambda} = w \cdot \overline{\lambda}$
 b) $w\lambda - \lambda$ is a sum of roots.

Then $w\lambda$ exists by Proposition 0.4.12, 0.4.3(a), and 0.4.9(b); it is unique by Lemma 0.4.5.

If $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is a positive system, and

$$\rho = \rho(\Delta^+),$$

then we define

$$w\rho - \rho \in \hat{\mathfrak{h}}$$

to be the unique sum of roots with differential $w\rho - \rho$; this exists by (3.2.9) and Lemma 0.4.5.

This definition will be extended in Definition 7.2.21.

Corollary 3.2.16 Suppose $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra, with $\mathfrak{l}$ a θ -stable real Levi factor (Definition 3.2.13). Fix a maximally split Cartan subgroup of H of L , and a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ containing $\Delta(\mathfrak{u}, \mathfrak{h})$. Let F be an irreducible finite dimensional representation of G , of highest weight $\lambda \in \hat{H}$ (Definition 0.4.8). Make L act on $H^*(\mathcal{U}, F)$

as in Lemma 3.2.14(d). Then the irreducible finite dimensional representation of L of highest weight $\mu \in \hat{H}$ occurs in $H^*(\mathfrak{u}, F)$ if and only if there is a $w \in W^1$ such that

$$\mu = w\lambda + (w\rho - \rho)$$

(notation (3.2.2), (3.2.15)); and in that case it occurs exactly once, in degree $\ell(w)$ (notation 3.2.1).

Proof. Every weight of H in $\Lambda^*\mathfrak{u}$ is a sum of roots; so by Proposition 0.4.9(b), every weight of

$$\text{Hom}(\Lambda^*\mathfrak{u}, F)$$

is of the form

$$\lambda + Q,$$

with Q a sum of roots. So suppose the representation of L of highest weight μ occurs in $H^*(\mathfrak{u}, F)$. By Theorem 3.2.3 (writing bar for differentials):

$$\bar{\mu} = w\bar{\lambda} + (w\rho - \rho);$$

and by the observation above, $\mu - \lambda$ is a sum of roots. By Definition 3.2.15,

$$\mu = w\lambda + (w\rho - \rho).$$

The rest of the corollary is now an immediate consequence of Theorem 3.2.3.

Q.E.D.

The Cartan subgroup was chosen to be maximally split because of Proposition 0.4.9(d); otherwise, we could not refer

to the representation of highest weight μ . Since H need not be maximally split in G , we find that F is not in general determined by $H^*(\mathfrak{u}, F)$ (in contrast to the Lie algebra or connected case). This happens already for $SL^\pm(2, \mathbb{R})$ when H is compact and $\mathfrak{g}_F$ is a Borel subalgebra (compare the remarks at the end of Section 0.4). This problem is inherited by larger groups (even connected ones) and infinite dimensional representations. It is the fundamental technical complication separating the theory of $(\mathfrak{g}, K)$ modules from that of highest weight modules; in the composition series conjectures of Section 9.6, it is dealt with after (9.6.5).

Chapter 4. Principal Series Representations and Quasisplit Groups.

As explained in the introduction, we are going to study representations by studying certain of their Lie algebra cohomology groups. In the form used here, this technique does not provide any reduction when G is a split group and the representation belongs to the principal series; so such representations (and, for convenience, a slightly larger class) must be treated separately. That is the goal of the present chapter. Its results could also be expressed in cohomological terms, and in fact we will need to do that in Chapter 6; but it seems more natural to phrase them analytically. This has the advantage of bringing out some of the main ideas in Langlands' approach to the classification of representations, a subject which we have otherwise neglected.

§1. The principal series.

For the purposes of this chapter, it is convenient to assign specific meanings for notation which is elsewhere used more flexibly. So we make

Definition 4.1.1 For Chapter 4 only, choose a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}_0$, and let A be the corresponding connected subgroup of G . Let

$$M = K^A = \text{centralizer of } A \text{ in } K$$

$\Delta = \Delta(\mathfrak{g}/m + \mathfrak{a}, \mathfrak{a})$ = non-zero roots of $\mathfrak{a}$ in $\mathfrak{g}$
(the restricted roots);

$$\bar{\Delta} = \{\alpha \in \Delta \mid \frac{1}{2}\alpha \notin \Delta\}$$

(the reduced roots)

$$W = W(\bar{\Delta})$$

(the Weyl group of $\mathfrak{a}$ in $\mathfrak{g}$).

$M' =$ normalizer of A in K

$$W = M'/M$$

(the Weyl group of A in G ; as the notation indicates, it is naturally isomorphic to $W(\bar{\Delta})$, as a group acting on $\mathfrak{a}$.) Fix a positive system $\Delta^+ \subseteq \Delta$, and let $\mathfrak{n}_0$ be the span of the root spaces for roots in Δ^+ . Let N be the connected group with Lie algebra $\mathfrak{n}_0$; we may call it the Iwasawa N . Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in \mathfrak{a}^*$$

$$P = MAN,$$

the minimal parabolic subgroup of G ; the module of P is trivial on MN , and has differential 2ρ on A .

We have an Iwasawa decomposition

$$(4.1.2) \quad G \cong K \times N \times A;$$

this means that multiplication defines an analytic diffeomorphism of the right side onto the left ([17], p. 234).

Notation 4.1.3 We identify $\hat{A}$ with $\mathfrak{a}^*$ by setting

$$\vee(\exp X) = \exp(\vee(X)) \quad (X \in \mathfrak{a}_0, \vee \in \mathfrak{a}^*);$$

if $a \in A$, we sometimes write $a^\vee$ instead of $\vee(a)$. We make $W = M'/M$ act on $\hat{M}$ and $\hat{A}$ as follows; fix (for definiteness)

a representation (δ, V_δ) of M , and an element $w \in W$. Choose a representative x of w in M' . Then $w \cdot \delta$ is a representation of M on V_δ , defined by

$$(w \cdot \delta)(m) = \delta(x^{-1}mx) \quad (m \in M)$$

(This makes sense since x normalizes M ; and one checks easily that the equivalence class of $w \cdot \delta$ is independent of x .)

Definition 4.1.3 Fix a representation (δ, V_δ) of M , and a character $\nu \in \hat{A}$. Define a Hilbert space

$$\mathcal{H}_{\delta \otimes \nu}^P = \mathcal{H}_{\delta \otimes \nu} = \{f: G \rightarrow V_\delta \mid f \text{ is measurable, } f|_K \text{ is square-integrable, and for all } m \in M, a \in A, n \in N, g \in G$$

$$f(gman) = a^{-(\nu+\rho)} \delta(m^{-1}) f(g)\};$$

the norm is given by

$$\|f\|^2 = \int_K \|f(k)\|^2 dk.$$

This space is invariant by left translation in G ; so we define a representation $\pi(\delta \otimes \nu) = \pi_G(P, \delta \otimes \nu)$ of G on $\mathcal{H}_{\delta \otimes \nu}$ by

$$[\pi(\delta \otimes \nu)(g) \cdot f](x) = f(g^{-1}x) \quad (g, x \in G);$$

this is the principal series representation with parameters δ and ν . We also use a subspace

$$\mathcal{H}_{\delta \otimes \nu}^{P, \infty} = \{f \in \mathcal{H}_{\delta \otimes \nu}^P \mid f \text{ is smooth}\},$$

and write $\pi^\infty(\delta \otimes \nu)$ for the restriction of $\pi(\delta \otimes \nu)$ to

this subspace. This is called the smooth principal series with parameters δ and ν . Finally, we write $X(\delta \otimes \nu)$ or $X_G(P, \delta \otimes \nu)$ for the Harish-Chandra module of $\pi(\delta \otimes \nu)$.

Theorem 4.1.4 (Harish-Chandra, Bruhat - see [50]).

With notation as above, $\pi(\delta \otimes \nu)$ is an admissible quasisimple representation of G . Two principal series representations $\pi(\delta \otimes \nu)$ and $\pi(\delta' \otimes \nu')$ have equivalent composition series (Definition 0.3.12) if and only if there is a $w \in W$ such that

$$(\delta', \nu') = (w \cdot \delta, w \cdot \nu).$$

A detailed proof may be found in [50]; but since some of the ideas will be useful later, we sketch part of the argument.

Lemma 4.1.5 In the setting of Definition 4.1.3, put

$$\mathcal{H}_\delta = \{f: K \rightarrow V_\delta \mid f \text{ is measurable and square-integrable, and for all } m \in M, k \in K,$$

$$f(km) = \delta(m^{-1})f(k)\};$$

this is the space of $\text{Ind}_M^K(\delta)$. Then restriction to

K defines a Hilbert space isomorphism of $\mathcal{H}_{\delta \otimes \nu}$ onto

$\mathcal{H}_\delta$, which intertwines the action of K . In particular, if μ is any representation of K , then the multiplicity of μ in $\pi(\delta \otimes \nu)$ is the multiplicity of δ in $\mu|_M$. This multiplicity in turn is at most $\dim(\mu)$.

Proof. The first assertion is a consequence of the Iwasawa decomposition (4.1.2). The second is Frobenius reciprocity.

For the last, simply observe that the δ -primary subspace of μ has dimension

$$m(\delta, \mu|_M) \cdot \dim \delta,$$

so this can be at most $\dim \mu$.

Q.E.D.

This shows that $\pi(\delta \otimes \nu)$ is admissible. Next, choose a Cartan subalgebra $\mathcal{L}_0^S$ of $\mathfrak{m}_0$; then

$$(4.1.6) \quad \mathcal{h} = \mathcal{L}_0^S + \mathfrak{a}$$

is a Cartan subalgebra of $\mathfrak{m} + \mathfrak{a}$ and of $\mathfrak{g}$. Choose a positive root system $\Delta^+(\mathfrak{m}, \mathcal{L}_0^S)$, and let

$$(4.1.7) \quad \Delta^+(\mathfrak{g}, \mathcal{h}) = \{\alpha \in \Delta(\mathfrak{g}, \mathcal{h}) \mid \alpha \in \Delta^+(\mathfrak{m}, \mathcal{L}_0^S) \text{ or } \alpha|_{\mathfrak{a}} \in \Delta^+\}.$$

Then

$$\rho(\Delta^+(\mathfrak{g}, \mathcal{h})) = \rho(\Delta^+(\mathfrak{m}, \mathcal{L}_0^S)) + \rho,$$

the first and second terms being the restrictions to $\mathcal{L}^S$ and $\mathfrak{a}$, respectively.

Lemma 4.1.8 With notation as above, suppose δ has highest weight $\lambda_0 \in (\mathcal{L}^S)^*$ (Definition 0.4.8). Put

$$\lambda = \lambda_0 + \rho(\Delta^+(\mathfrak{m}, \mathcal{L}_0^S)),$$

so that $\delta \otimes (\nu + \rho)$ has infinitesimal character $(\lambda, \nu + \rho)$ (Lemma 3.2.7). Then $\pi(\delta \otimes \nu)$ has infinitesimal character (λ, ν) .

Proof. Let

$$\gamma: Z(\mathfrak{g}) \rightarrow Z(\mathfrak{m} + \mathfrak{a})$$

be defined as in Section 3.1, using the parabolic $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$.

If $z \in Z(\mathfrak{g})$, then by (3.1.4),

$$(*) \quad z - \gamma(z) \in U(\mathfrak{g})\mathfrak{n}.$$

Now fix a K -finite element f of $\mathcal{H}_{\delta \otimes \nu}$; we want to compute $z \cdot f$. The action of $U(\mathfrak{g})$ on f is by right invariant differential operators. Since z is fixed by $\text{Ad}(G)$, we can compute $z \cdot f$ using left invariant differential operators. Since f is right invariant under N , $(*)$ implies

$$z \cdot f = \gamma(z) \cdot f,$$

with $\gamma(z)$ acting on the right. Let $\xi^{\mathfrak{m} + \mathfrak{a}}$ denote the Harish-Chandra map for $\mathfrak{m} + \mathfrak{a}$. Since f transforms by $\delta \otimes (\nu + \rho)$ on the right,

$$\gamma(z) \cdot f = [\xi^{\mathfrak{m} + \mathfrak{a}}(\gamma(z))](\lambda, \nu + \rho) \cdot f.$$

Comparing these two equations with (3.1.7) gives

$$z \cdot f = \xi(z)(\lambda, \nu),$$

where ξ is the Harish-Chandra map for G . Q.E.D.

In particular, $\pi(\delta \otimes \nu)$ is quasisimple. By Lemma 4.1.5 and Theorem 0.3.21, $\pi(\delta \otimes \nu)$ has a global distribution character. This character can be computed explicitly (see [50], 5.5.3.1). The last assertion of Theorem 4.1.4 follows from Proposition 0.3.24 by inspecting this explicit formula. This completes our sketch of the proof.

We will make the principal series fit into the classification of all representations using

Theorem 4.1.9 (Harish-Chandra's subquotient theorem -- see [50], 5.5.1.5, or [7]). Every irreducible admissible representation of G is infinitesimally equivalent to a

composition factor (Definition 0.3.12) of a principal series representation of G.

This is one of the deepest results which we will use without proof; but none of the various proofs which are known uses any ideas we will need later, so the reader needs no understanding of them to continue.

Occasionally, we will make use of induction in a slightly more general context. (The reader can omit this until later, however.) To describe it, fix a parabolic subgroup P^1 of G . (By definition, a parabolic subgroup is one which contains a minimal parabolic subgroup.) Let N^1 denote the unipotent radical of P^1 , and let

$$(4.1.10) \quad P^1 = M_{A^1}^1 N^1$$

be the Langlands decomposition of P^1 ([50], p. 81); here M^1 is a (θ -stable) reductive subgroup of G satisfying our basic assumptions 0.1.2, and

$$A^1 \subseteq \exp(-\mathcal{P}_0)$$

is isomorphic to a vector space as a Lie group.

Definition 4.1.11 (a) Suppose P^1 is as above. Suppose (δ^1, V_{δ^1}) is an admissible representation of M^1 on a

Hilbert space V_{δ^1} , satisfying the conditions of Theorem

0.3.21. (For example, δ^1 could be any admissible irreducible representation.) Define

$$\rho^1 = \rho(\pi^1, \sigma^1) \in (\sigma^1)^* \cong \hat{A}^1.$$

If $\nu^1 \in \hat{A}^1$, defines a Hilbert space

$$\begin{aligned} \mathcal{H}_{\delta^1 \otimes \nu^1}^{P^1} &= \{f: G \rightarrow V_{\delta^1} \mid f \text{ is measurable, } f|_K \text{ is} \\ &\quad \text{square integrable, and for all } m \in M^1, \\ &\quad a \in A^1, n \in N^1, g \in G, \\ &\quad f(gman) = a^{-(\nu^1 + \rho^1)} \delta^1(m^{-1}) f(g)\}; \end{aligned}$$

the norm is

$$||f||^2 = \int_K ||f(k)||^2 dk.$$

This space is invariant under left translation in G ; so we define a representation $\pi(\delta^1 \otimes \nu^1) = \pi_G(P^1, \delta^1 \otimes \nu^1)$ on $\mathcal{H}_{\delta^1 \otimes \nu^1}^{P^1}$ by

$$[\pi(\delta^1 \otimes \nu^1)(g)f](x) = f(g^{-1}x);$$

we call this construction parabolic induction, and write

$$(\pi(\delta^1 \otimes \nu^1), \mathcal{H}_{\delta^1 \otimes \nu^1}^{P^1}) = \text{Ind}_{P^1}^G (\delta^1 \otimes \nu^1 \otimes 1).$$

Define

$$\mathcal{H}_{\delta^1 \otimes \nu^1}^{P^1, \infty} = \{f \in \mathcal{H}_{\delta^1 \otimes \nu^1}^{P^1} \mid f \text{ is a smooth function on } G\}.$$

The restriction of $\pi(\delta^1 \otimes \nu^1)$ to this subspace is called $\pi^\infty(\delta^1 \otimes \nu^1)$, and this process is called smooth parabolic induction.

(b) Still more generally, suppose (γ, V_γ) is a representation of P^1 on a Hilbert space $\mathcal{H}_\gamma$; and suppose $\mathcal{H}_\gamma$ has a filtration

$$0 = V_\gamma^0 \subseteq V_\gamma^1 \subseteq \dots \subseteq V_\gamma^n = V_\gamma$$

by closed invariant subspaces, such that on each

$V_\gamma^i/V_\gamma^{i-1}$, N acts trivially, A acts by scalars, and

M acts by an admissible representation satisfying the

conditions of Theorem 0.3.21. Then we can define

$$(\pi(p^1, \gamma), \mathcal{H}_\gamma^{p^1}) = \text{Ind}_{p^1}^G(\gamma)$$

just as above. The Harish-Chandra module of $\pi(\gamma)$ is written $X_G(p^1, \gamma)$, or simply $X(\gamma)$.

Proposition 4.1.12 In the setting of Definition 4.1.11(b),

$\pi(\gamma) = \text{Ind}_{p^1}^G(\gamma)$ is an admissible representation of G

satisfying the conditions of Theorem 0.3.21. If γ is

unitary, so is $\text{Ind}_{p^1}^G(\gamma)$.

Proof. We simply imitate the proof of Theorem 4.1.4. To study the restriction of $\pi(\gamma)$ to K , we consider the space

$$\begin{aligned} \mathcal{H} &= \{f: K \rightarrow V_\gamma \mid f \text{ is measurable and square} \\ (4.1.13) \quad &\text{integrable, and for every } k \in K \text{ and} \\ &m \in M^1 \cap K, f(km) = \gamma(m^{-1})f(k)\}; \end{aligned}$$

this is the space of

$$\text{Ind}_{M^1 \cap K}^K(\gamma|_{M^1 \cap K}).$$

Obviously restriction to K defines an injection of $\mathcal{H}_\gamma$ into $\mathcal{H}$, which intertwines the action of K . To see that this map is onto, we choose a minimal parabolic subgroup $P = MAN$ contained in p^1 , and use the decomposition

$$(4.1.14) \quad G \cong K \times M^1 \cap (NA) \times N_A^1$$

(which follows from (4.1.2)). If $f_0 \in \mathcal{H}$ and $g \in G$, write

$$g = km^1 n^1 a^1$$

in accordance with (4.1.14), and define

$$f(g) = \gamma(m^1 n^1 a^1)^{-1} (a^1)^{-\rho} f_0(g).$$

Then it is easy to check that $f \in \mathcal{H}_\gamma$, and f restricted to K is f_0 . By Frobenius reciprocity, if $\mu \in \hat{K}$,

$$\begin{aligned} m(\mu, \pi(\gamma)|_K) &= m(\mu, \text{Ind}_{M^1 \cap K}^K(\gamma|_{M^1 \cap K})) \\ (4.1.15) \quad &= \sum_{\delta \in (M^1 \cap K)^\wedge} m(\delta, \gamma|_{M^1 \cap K}) m(\delta, \mu). \end{aligned}$$

Now by the hypothesis on γ , and the condition in Theorem

0.3.21, there are positive constants C and N so that

$$m(\delta, \gamma|_{M^1 \cap K}) \leq C \dim(\delta)^N.$$

If δ occurs in μ , then $\dim(\delta)$ is at most $\dim(\mu)$; so (4.1.15) gives

$$\begin{aligned} m(\mu, \pi(\gamma)|_K) &\leq \sum_{\delta \in (M^1 \cap K)^\wedge} C(\dim \delta)^N m(\delta, \mu) \\ &\leq C(\dim \mu)^{N-1} \sum_{\delta \in (M^1 \cap K)^\wedge} (\dim \delta) m(\delta, \mu) \\ &= C(\dim \mu)^{N-1} \dim \mu \\ &= C(\dim \mu)^N. \end{aligned}$$

So the K multiplicities in $\pi(\gamma)$ satisfy the condition of Theorem 0.3.21.

A representation is obviously $Z(\mathcal{O})$ -finite if it has a finite filtration, all of whose subquotients are $Z(\mathcal{O})$ -finite. So to prove that $\pi(\gamma)$ is $Z(\mathcal{O})$ -finite, it is enough to assume we are in case (a) of Definition 4.1.11 (that is,

$\gamma = \delta^1 \otimes \nu^1 \otimes 1$). Using a further filtration, we may

assume that δ^1 is quasisimple. So it is enough to establish the following lemma.

Lemma 4.1.16 Fix a Cartan subalgebra $\mathfrak{h}^1$ of $\mathfrak{m}^1$, and define

$$\mathfrak{h} = \mathfrak{h}^1 + \mathfrak{a}^1,$$

a Cartan subalgebra of $\mathfrak{m}^1 + \mathfrak{a}^1$ and of $\mathfrak{g}$. In the setting of 4.1.11(a), suppose δ^1 has infinitesimal character $\lambda^1 \in (\mathfrak{h}^1)^*$. Then $\pi(\delta^1 \otimes \nu^1)$ has infinitesimal character (λ^1, ν^1) .

The proof follows that of Lemma 4.1.9 word for word, so we omit it. The last statement of Proposition 4.1.12 is just an extremely special case of Mackey's unitary induction theory ([50], p. 372).

Q.E.D.

One general fact about parabolic induction will be used in the next section. To formulate it, we need a structural fact.

Lemma 4.1.17 ([50], pp. 81-82) Suppose $\mathfrak{p}^i = \mathfrak{m}^i \mathfrak{a}_N^i$ are Langlands decompositions of parabolic subgroups of

G ($i = 1, 2$) and $\mathfrak{p}^1 \subseteq \mathfrak{p}^2$. Then

a) $\mathfrak{m}^1 \subseteq \mathfrak{m}^2$

$$\mathfrak{a}^1 \supseteq \mathfrak{a}^2$$

$$\mathfrak{n}^1 \supseteq \mathfrak{n}^2.$$

b) $\mathfrak{p}^1 \cap \mathfrak{m}^2 = \mathfrak{m}^1 (\mathfrak{a}^1 \cap \mathfrak{m}^2) (\mathfrak{n}^1 \cap \mathfrak{m}^2)$ is the Langlands decomposition of a parabolic subgroup of $\mathfrak{m}^2$; and this defines a bijection between the set of parabolic subgroups of $\mathfrak{m}^2$, and the set of parabolic subgroups of G contained in $\mathfrak{p}^2$.

c) $\mathfrak{a}^1 = \mathfrak{a}^2 \cdot (\mathfrak{a}^1 \cap \mathfrak{m}^2)$ (direct product)
 $\mathfrak{n}^1 = \mathfrak{n}^2 \cdot (\mathfrak{n}^1 \cap \mathfrak{m}^2)$ (semidirect product, with first factor normal).

Proposition 4.1.18 (Induction by stages). Suppose $\mathfrak{p}^1 \subseteq \mathfrak{p}^2$ are parabolic subgroups of G ; we use the notation of Lemma 4.1.17, and Definition 4.1.11 (a).

Define

$$\delta^2 = \pi_{\mathfrak{m}_2}(\mathfrak{p}^1 \cap \mathfrak{m}^2, \delta^1 \otimes (\nu^1|_{\mathfrak{a}^1 \cap \mathfrak{m}^2}))$$

$$\nu^2 = \nu^1|_{\mathfrak{a}^2}.$$

Then there is a natural unitary isomorphism of Hilbert spaces

$$\mathcal{H}_{\delta^1 \otimes \nu^1}^{\mathfrak{p}^1} \cong \mathcal{H}_{\delta^2 \otimes \nu^2}^{\mathfrak{p}^2}$$

which intertwines the action of G ; so

$$\pi(\mathfrak{p}^1, \delta^1 \otimes \nu^1) \cong \pi(\mathfrak{p}^2, \delta^2 \otimes \nu^2).$$

Proof. Suppose we are given $f \in \mathcal{H}_{\delta^1 \otimes \nu^1}^{\mathfrak{p}^1}$. We want to define $\tilde{f} \in \mathcal{H}_{\delta^2 \otimes \nu^2}^{\mathfrak{p}^2}$. This latter space consists of maps from G to

V_{δ^2} = space of δ^2

$$= \{ \tilde{f}: \mathfrak{m}^2 \rightarrow V_{\delta^1} \mid \text{certain conditions hold} \}.$$

So given $g \in G$, $\tilde{f}(g)$ must be a function on $\mathfrak{m}^2$. If $m \in \mathfrak{m}^2$, we define

$$(4.1.19) \quad [\tilde{f}(g)](m) = f(gm).$$

Since f has values in V_{δ^1} , this makes sense. The first

thing to be checked is that $\tilde{f}(g)$ actually lies in V_{δ^2} . So

if $m^1 \in M^1$, $a \in A^1 \cap M^2$, and $n \in N^1 \cap M^2$, we compute

$$\begin{aligned} [\tilde{f}(g)](mm^1an) &= f(gmm^1an) \\ &= a^{-(v^1+\rho^1)} \delta^1(m^1)^{-1} f(gm) \\ &= a^{-(v^1+\rho^1)} \delta^1(m^1)^{-1} [\tilde{f}(g)](m). \end{aligned}$$

Since

$$\rho^1 \Big| \alpha^1 m^2 = \rho(\pi^1 \cap m^2),$$

this is exactly the formal condition for $\tilde{f}(g)$ to belong to V_{δ^2} . Similarly, one checks that $\tilde{f}$ transforms under p^2 on the right as required for an element of $\mathcal{H}_{\delta^2 \mathfrak{g}, 2}^{p^2}$. So the map $f \mapsto \tilde{f}$ makes sense formally. Also

$$\begin{aligned} ||\tilde{f}||^2 &= \int_K ||\tilde{f}(k)||^2 dk \\ &= \int_K \int_{M_{nK}^2} ||[\tilde{f}(k)](m)||^2 dm dk \\ &= \int_K \int_{M_{nK}^2} ||f(km)||^2 dm dk \\ &= \int_K ||f(k)||^2 dk = ||f||^2 \end{aligned}$$

by Fubini's theorem and the right invariance of dk . So the map is a unitary injection. To see that it is bijective, we restrict attention to K ; then it is just induction by stages for K (if we use the realization (4.1.13)). That it intertwines the action of G is entirely obvious. Q.E.D.

For completeness, we record here a fact which will not be used in the main exposition.

Proposition 4.1.20 (Harish-Chandra). In the setting of Definition 4.1.11(a), suppose $p^2 = M^1 A_N^1$ is another

parabolic subgroup with the same Levi factor $M^1 A^1$ as p^1 . Then $\pi(p^1, \delta^1 \mathfrak{g}_V^1)$ and $\pi(p^2, \delta^1 \mathfrak{g}_V^1)$ have equivalent composition series (Definition 0.3.12).

This is deduced from Proposition 0.3.24 by explicitly calculating the distribution characters of the two representations (which exist by Proposition 4.1.12 and Theorem 0.3.21). The calculation may be found in [53].

§2. The Langlands quotient for principal series.

In this section, we will begin a more thorough analysis of the principal series representations. The basic ideas are due to Langlands; the formulation here is due in part to D. Milićić.

Definition 4.2.1 If $v \in \sigma^*$, write

$$v = \operatorname{Re} v + i \operatorname{Im} v,$$

with $\operatorname{Re} v$ and $\operatorname{Im} v$ in σ_0^* . We say v is dominant with respect to N if for all $\alpha \in \Delta^+$ (cf. (4.1.1))

$$\langle \alpha, \operatorname{Re} v \rangle \geq 0.$$

We say v is negative if $-v$ is dominant. Fix a negative $v \in \sigma^*$, and define

$$\Delta^1 = \{\alpha \in \Delta \mid \langle \operatorname{Re} v, \alpha \rangle = 0\}$$

$$A^1 = \{a \in A \mid \alpha(a) = 1 \text{ for all } \alpha \in (\Delta^+)^1\}$$

$$p^1 = M^1 A_N^1 \supseteq P;$$

here p^1 is defined by

$$\Delta(\pi^1 + \sigma^1, \sigma v) = \Delta^1$$

or

$$M^1 A^1 = G^1 A^1.$$

We call p^1 the parabolic subgroup defined by $-\text{Re } \nu$.

(The reader may wish to assume at first that Δ^1 is empty, as is "generically" the case: then $p^1 = p$.) Put

$$\overline{p^1} = \theta p^1 = M^1 A^1 N^1;$$

here $\overline{N^1} = \theta N^1$, and $\overline{p^1}$ is the parabolic subgroup opposite to p^1 . Set

$$A^0 = A \cap M^1, N^0 = N \cap M^1.$$

so that

$$M^1 \cap p = M A^0 N^0$$

is a minimal parabolic subgroup of M^1 (cf. Lemma 4.1.17).

Put

$$\tilde{N} = N_N^{0,1}, \tilde{p} = M A \tilde{N} \subseteq p^1;$$

thus $\tilde{p}$ is another minimal parabolic subgroup of G .

For the rest of this section, we will fix $\delta \in \hat{M}$, and $\nu \in \hat{A}$ negative; so we have available the various subgroups of Definition 4.2.1. Set

$$(4.2.2) \quad \begin{aligned} \nu^0 &= \nu|_{A^0} & \nu^1 &= \nu|_{A^1} \\ (\delta^1, V_{\delta^1}) &= \text{Ind}_{M^1 \cap p}^{M^1} (\delta \otimes \nu^0 \otimes 1). \end{aligned}$$

It is easy to see that $(\sigma^0)^*$ is spanned by Δ^1 ; so by the definition of Δ^1 , ν^0 is purely imaginary. That is, ν^0 is a

unitary character of A^0 . By the last statement of Proposition 4.1.12,

(4.2.3) δ^1 is a unitary representation of M^1 .

By Proposition 4.1.18, there is a natural identification

$$(4.2.4) \quad \mathcal{H}_{\delta \otimes \nu}^p \cong \mathcal{H}_{\delta^1 \otimes \nu^1}^{p^1};$$

we will use this often.

Definition 4.2.5 Suppose $\nu \in \hat{A}$ is negative, and p^1 is the parabolic defined by $-\text{Re}(\nu)$ (Definition 4.2.1). With the notation described above, define the integral intertwining operator

$$A(p^1, \overline{p^1}): \mathcal{H}_{\delta \otimes \nu}^{\tilde{p}} \rightarrow \mathcal{H}_{\delta \otimes \nu}^p$$

-- or, using (4.2.4),

$$A(p^1, \overline{p^1}): \mathcal{H}_{\delta^1 \otimes \nu^1}^{\overline{p^1}} \rightarrow \mathcal{H}_{\delta^1 \otimes \nu^1}^{p^1} \quad --$$

by

$$(A(p^1, \overline{p^1})f)(g) = \int_{N^1} f(gn^1) \, dn^1.$$

Lemma 4.2.6 In the setting just described, $A(p^1, \overline{p^1})$ is a bounded intertwining operator from $\pi(\tilde{p}, \delta \otimes \nu)$ to $\pi(p, \delta \otimes \nu)$.

Sketch of Proof. Assume for a moment that the integral converges. The first thing to check is that $A(p^1, \overline{p^1})f$ satisfies the formal conditions for belonging to $\mathcal{H}_{\delta^1 \otimes \nu^1}^{p^1}$. If $n \in N^1$, then

$$\begin{aligned}
(A(P^1, \overline{P^1})f)(gn) &= \int_{N^1} f(gnn^1) dn^1 \\
&= \int_{N^1} f(gn^1) dn^1 \\
&= (A(P^1, \overline{P^1})f)(g)
\end{aligned}$$

by the translation invariance of Haar measure. If $a \in A^1$, then

$$\begin{aligned}
(A(P^1, \overline{P^1})f)(ga) &= \int_{N^1} f(gan^1) dn^1 \\
&= \int_{N^1} f(g(an^1a^{-1})a) dn^1.
\end{aligned}$$

Now a normalizes N^1 , and changes Haar measure by a factor of

$$\det(\text{Ad}(a)|_{N^1}) = a^{2\rho(n^1)}.$$

So the last integral can be written as

$$= \int_{N^1} f(gn^1a) a^{-2\rho(n^1)} dn^1.$$

Since f belongs to $\mathcal{H}_{\delta^1 \otimes \nu^1}^{\overline{P^1}}$,

$$f(xa) = a^{-(\nu^1 - \rho)(n^1)} f(x).$$

So the integral is

$$\begin{aligned}
&= \int_{N^1} a^{-(\nu^1 + \rho)(n^1)} f(gn^1) dn^1 \\
&= a^{-(\nu^1 + \rho)(n^1)} (A(P^1, \overline{P^1})f)(g).
\end{aligned}$$

Similarly, if $m \in M^1$, we compute

$$(A(P^1, \overline{P^1})f)(gm) = \delta^1(m)^{-1} (A(P^1, \overline{P^1})f)(g)$$

So, at least formally, the intertwining operator lands in the right space. Since G acts by left translation, the fact that the operator intertwines is obvious. So the only remaining point is to verify that the integral converges, and

defines a bounded operator. This is done in [34]. The idea is quite simple: by a trivial estimate, we can reduce to the case δ is trivial, and f is the constant function. In that case one can reduce by the method of Gindikin-Karpelevich to the case when G has rank one. There the integral has been explicitly computed by Helgason and Schiffman. (Except for the trivial estimate, this may be found in [51], Theorem 9.1.6.4.) Q.E.D.

We are going to study the reducibility of $\pi(\delta \otimes \nu)$ using this intertwining operator. When $P^1 \neq P$, however, there is a little more going on: the unitary representation δ^1 of M^1 may already be reducible. Since δ^1 is unitary, any invariant subspace of V_{δ^1} has an invariant complement (namely, its orthogonal complement); so any subquotient may be realized as a direct summand. It follows that δ^1 is a (Hilbert space) direct sum of irreducible representations;

$$(4.2.7) \quad (\delta^1, V_{\delta^1}) = \bigoplus_{i \in I} (\delta_i^1, V_{\delta_i^1}).$$

We will eventually prove that this sum is finite, a fact which is due to Bruhat ([50], Corollary 5.5.2.2). Since this is not particularly easy, we prefer to avoid using it. At any rate, (4.2.7) induces a Hilbert space direct sum

$$(4.2.8) \quad (\pi(\delta^1 \otimes \nu^1), \mathcal{H}_{\delta^1 \otimes \nu^1}) = \bigoplus_{i \in I} (\pi(\delta_i^1 \otimes \nu^1), \mathcal{H}_{\delta_i^1 \otimes \nu^1}^1),$$

where we can use either P^1 or $\overline{P^1}$ for the parabolic. Clearly

$$(4.2.9) \quad A(P^1, \overline{P^1}) (\mathcal{H}_{\delta^1 \otimes \nu^1}^{\overline{P^1}})^{P^1} \subseteq \mathcal{H}_{\delta^1 \otimes \nu^1}^{P^1}.$$

Definition 4.2.10 With notation (4.2.5), (4.2.8),

define

$$\begin{aligned}\bar{\pi}(P, \delta\emptyset\nu) &= \bar{\pi}(\delta\emptyset\nu) = \text{closure of the image of } A(P^1, P^1) \\ \bar{\pi}(\delta^1_{i\emptyset\nu^1}) &= \bar{\pi}^i(\delta\emptyset\nu) = \text{closure of the image of } A(P^1, P^1) \\ &\quad \text{restricted to } \pi(\delta^1_{i\emptyset\nu^1});\end{aligned}$$

these latter are called the Langlands subrepresentations of $\pi(P, \delta\emptyset\nu)$, or the Langlands quotients of $\pi(\tilde{P}, \delta\emptyset\nu)$. (We have confused representations and their underlying spaces in an obvious way here.)

Lemma 4.2.11 There is a Hilbert space direct sum

$$\bar{\pi}(\delta \otimes \nu) = \bigoplus_{i \in I} \bar{\pi}(\delta^1_{i\emptyset\nu^1});$$

and each $\bar{\pi}(\delta^1_{i\emptyset\nu^1})$ is non-zero.

Proof. The first assertion follows from (4.2.8) and (4.2.9). For the second, recall that $N^1 M^1 A^1 N^1$ is open in G (see [50], 1.2.4.9); so the space $C_c(N^1, V_{\delta^1_{i\emptyset\nu^1}})$ of compactly supported continuous functions on N^1 with values in $V_{\delta^1_{i\emptyset\nu^1}}$, may be regarded as a subspace of $\mathcal{A}^{P^1}_{\delta^1_{i\emptyset\nu^1} 1}$. If f lies in this space, then

$$(A(P^1, P^1) f)(1) = \int_{N^1} f(n^1) \, dn^1,$$

which is obviously non-zero for some choices of f . Q.E.D.

Theorem 4.2.12 (Langlands-Milićić-see [3]). Suppose $\nu \in \hat{A}$ is negative; let P^1 be the associated parabolic subgroup. Then the representations $\bar{\pi}(\delta^1_{i\emptyset\nu^1})$ defined in (4.2.10) are irreducible. Any subrepresentation of $\pi(P, \delta\emptyset\nu)$ meets $\bar{\pi}(\delta\emptyset\nu)$; and any subrepresentation of $\pi(P^1, \delta^1_{i\emptyset\nu^1})$ contains $\bar{\pi}(\delta^1_{i\emptyset\nu^1})$.

Proof. Clearly it suffices to establish the last two statements. The idea of the argument, which is carried out in Lemma 4.2.15, is straightforward: we start with an arbitrary non-zero functions in $\mathcal{A}^P_{\delta\emptyset\nu}$, move it around by the action of G , and show that in this way one can approximate elements of the image of the intertwining operator. The proof requires some simple integration formulas. (It could be rearranged slightly, for example, to avoid the use of 4.2.14(a). This seems to require the consideration of two Iwasawa decompositions at once, which would make the notation even worse.)

If $g \in G$, write $g = k(g)n(g)a(g)$, ($k(g) \in K$, $n(g) \in N$, $a(g) \in A$). Since $A = A^0 A^1$, $N = N^1 N^0$, we can also write

$$a(g) = a^0(g)a^1(g), \quad n(g) = n^1(g)n^0(g) \text{ with obvious notation.}$$

Write dh for a Haar measure on the Lie group H (we will be concerned only with unimodular groups). If H is compact, assume $\int_H dh = 1$. Since A and N are unipotent, one has $da = da^0 da^1$, and $dn = dn^1 dn^0$.

Lemma 4.2.13

a) If $f \in L^1(G)$, then for an appropriate normalization of dg ,

$$\int_G f(g) dg = \int_K \int_N \int_A f(kna) \, dk \, dn \, da.$$

b) If $f \in L^1(K)$, and $d\bar{n}$ is normalized so that

$$\begin{aligned}\int_{\bar{N}} a(\bar{n})^{-2\rho} d\bar{n} &= 1, \\ \int_K f(k) dk &= \int_{\bar{N}} \int_M f(mk(\bar{n})) a(\bar{n})^{-2\rho} d\bar{n} \, dm.\end{aligned}$$

The first formula is very well known; for a proof of (b), see for example, [51], p. 73. This normalization for $d\bar{n}$ naturally determines one for $dn(\bar{N} = \emptyset N)$; dn^0 is normalized in accordance

with the same convention for the group M^1 , and dn^1 so that $dn = dn^1 dn^0$. Put $2\rho^0 = \det(\text{Ad}(A)|_{\mathfrak{h}_0})$.

Lemma 4.2.14

a) If $f \in L^1(N^1)$,

$$\int_{N^1} f(n^1) (dn^1) = \int_{\bar{N}^1} f(n^1(n^1)) a(n^1) \bar{dn}^1 \cdot 2\rho^0.$$

b) If $f \in L^1(K)$,

$$\int_K f(k) dk = \int_{\bar{N}^1} \int_{K^0} f(k_0 k(n^1)) a(n^1) \bar{dn}^1 \cdot 2\rho^0 dk_0.$$

Proof. For (a), notice first that $N^1 N^0$ is the unipotent radical of the minimal parabolic subgroup $\bar{P}$ of G . So if $h \in L^1(G)$, 4.2.13(a) implies

$$\int_G h(g) dg = \int_K \int_{\bar{N}^1} \int_{N^0} \int_A f(kn^1 n^0 a) dk \bar{dn}^1 dn^0 da.$$

Writing out the Iwasawa decomposition of n^1 , and using the translation invariance of dk and da , we get

$$\begin{aligned} &= \int_K \int_{\bar{N}^1} \int_{N^0} \int_A f(kk(n^1)n^1(n^1)n^0(n^1)a(n^1)) a(n^1) n^0 a \\ &= \int_K \int_{\bar{N}^1} \int_{N^0} \int_A f(kn^1(n^1)n^0(n^1)) \bar{dn}^1 dn^0 da \\ &\quad \cdot [\text{Ad}(a(n^1)) \cdot n^0] a) dk \bar{dn}^1 dn^0 da. \end{aligned}$$

Now $dn^0 = a^{-2\rho^0} d(\text{Ad}(a) \cdot n^0)$. This fact, combined with the translation invariance of dn^0 , gives

$$= \int_K \int_{\bar{N}^1} \int_{N^0} \int_A f(kn^1(n^1)n^0 a) a(n^1) \bar{dn}^1 \cdot 2\rho^0 \cdot dk \bar{dn}^1 dn^0 da.$$

Comparing this with 4.2.13(a) (recall that $dn = dn^1 dn^0$) gives 4.2.14(a).

For (b), write $\bar{n} = \bar{n}^0 \bar{n}^1$. The Iwasawa decomposition of $\bar{n}^0$ takes place within the group M^1 ; so $k(n^0) \in K^0$, $n(n^0) = n^0(n^0) \in N^0$, $a(n^0) = a^0(n^0) \in A^0$. Put $x = n^0(n^0)a^0(n^0) \in M^1$. Then x normalizes N^1 , so that

$$\begin{aligned} \bar{n} &= k(n^0)n(n^0)a^0(n^0)\bar{n}^1 \\ &= k(n^0)[\text{Ad}(x) \cdot \bar{n}^1]x \\ &= k(n^0)k(\text{Ad}(x) \cdot \bar{n}^1)n(\text{Ad}(x) \cdot \bar{n}^1)a(\text{Ad}(x) \cdot \bar{n}^1)x \\ &= k(n^0)k(\text{Ad}(x) \cdot \bar{n}^1)n(\text{Ad}(x) \cdot \bar{n}^1)[\text{Ad}(a(\text{Ad}(x) \cdot \bar{n}^1)) \cdot n^0(n^0)] \\ &\quad \cdot a(\text{Ad}(x) \cdot \bar{n}^1)a^0(n^0). \end{aligned}$$

This shows that

$$\begin{aligned} k(n^0)\bar{n}^1 &= k(n^0)k(\text{Ad}(x) \cdot \bar{n}^1) \\ a(n^0)\bar{n}^1 &= a(\text{Ad}(x) \cdot \bar{n}^1)a^0(n^0). \end{aligned}$$

Recall that $d\bar{n} = d\bar{n}^1 dn^0$. Since $\bar{n}$ is unipotent, we can reverse these factors without changing the measure. If $f \in L^1(K)$, 4.2.13(b) gives

$$\begin{aligned} \int_K f(k) dk &= \int_{\bar{N}^1} \int_M f(mk(\bar{n})) a(\bar{n})^{-2\rho} d\bar{n} dm \\ &= \int_{\bar{N}^0} \int_{\bar{N}^1} \int_M f(mk(n^0\bar{n}^1)) a(n^0\bar{n}^1) \bar{dn}^0 \bar{dn}^1 dm \\ &= \int_{\bar{N}^0} \int_M \int_{\bar{N}^1} f(mk(\bar{n}^0)k(\text{Ad}(x) \cdot \bar{n}^1)a(n^0)\bar{dn}^0)^{-2\rho} \\ &\quad \cdot a(\text{Ad}(x) \cdot \bar{n}^1)^{-2\rho} d\bar{n}^0 dm d\bar{n}^1. \end{aligned}$$

Now $x \in M^1$, which acts on $\bar{n}^1$ with determinant one; so we can replace $\text{Ad}(x) \cdot \bar{n}^1$ by $\bar{n}^1$ in the integrand. The restriction of 2ρ to A^0 is just $2\rho^0$; so 4.2.13(b) implies that the integral over $\bar{N}^0 \times M$ is just integration over K^0 . Thus

$$\int_K f(k) dk = \int_{K^0} \int_{\bar{N}^1} f(k_0 k(n^1)) a(n^1) \bar{dn}^1 \cdot 2\rho^0 dk_0 dn^1.$$

Q.E.D.

The next lemma is essentially due to Helgason ([18], p. 129-130) when $\text{Re } \nu$ is regular. The extension to the general case here follows Helgason and Johnson ([19]).

Lemma 4.2.15 Fix an element $H \in \mathfrak{a}_0^1$ such that $\alpha(H) \geq 0$
for all $\alpha \in \Delta^+$, with equality precisely for $\alpha \in \Delta^+(\mathfrak{m}_c^1)$.
Put $a_t = \exp(tH) \in A^1$. Fix $f \in \mathcal{H}_{\delta \otimes \nu}^{p, \infty}$ (Definition 4.1.3),
 $g \in C^\infty(K)$. Define an element $h \in \mathcal{H}_{\delta \otimes \nu}^p$ (via Lemma 4.1.5)
by

$$h(k) = \int_K^0 g(kk_0^{-1}) f(k_0) dk_0.$$

Then, writing π for the action of $\pi(\delta \otimes \nu)$, we get

$$\lim_{t \rightarrow \infty} a_t^{\nu+\rho} \pi(g) \pi(a_t^{-1}) f = A(p^1, \bar{p}^1) h.$$

(Recall that $\pi(g) = \int_K g(k) \pi(k) dk$ -- Definition 0.3.20.)

Proof. First we make a few comments on the statement of the lemma. To see that $h \in \mathcal{H}_\delta$, suppose $m \in M$. Then

$$\begin{aligned} h(km) &= \int_K^0 g(kmk_0^{-1}) f(k_0) dk_0 \\ &= \int_K^0 g(kk_0^{-1}) f(k_0 m) dk_0 \\ &= \int_K^0 g(kk_0^{-1}) \delta(m^{-1}) f(k_0) dk_0 \\ &= \delta(m^{-1}) \left[\int_K^0 g(kk_0^{-1}) f(k_0) dk_0 \right] \\ &= \delta(m^{-1}) h(k). \end{aligned}$$

Set $2\tilde{\rho} = \det(\text{Ad}(A) \big|_{\mathfrak{m}_0^0 + \mathfrak{m}_0^1}) = 4\rho^0 - 2\rho$. By the identification $\mathcal{H}_\delta \approx \mathcal{H}_{\delta \otimes \nu}^{\tilde{p}}$,

$$(*) \quad h(ga\tilde{n}) = a^{-\nu+\rho-2\rho^0} h(g)$$

for all $g \in G$, $a \in A$, $\tilde{n} \in \tilde{N}$; and similarly,

$$(**) \quad f(gan) = a^{-\nu-\rho} f(g).$$

For $x_0 \in K$, we have

$$\begin{aligned} [\pi(g) \pi(a_t^{-1}) f](x_0) &= \int g(k) [\pi(k) \pi(a_t^{-1}) f](x_0) dk \\ &= \int_K g(k) f(a_t k^{-1} x_0) dk. \end{aligned}$$

Now change variables to $k' = k^{-1} x_0$, and apply Lemma 4.2.14(b).

The integral becomes

$$\int_K^0 \int_{N^1} g(x_0 k(n^1)^{-1} k_0^{-1}) f(a_t k_0 k(n^1) a(n^1)^{-1}) a(n^1)^{-2\rho} dk_0 dn^1.$$

Since $H \in \mathfrak{a}_0^1$, a_t commutes with $k_0 \in M^1$. Furthermore

$$k(n^1) = n^1 a(n^1)^{-1} n(n^1)^{-1}.$$

Applying this relation and (**) repeatedly, it is straightforward to compute

$$\begin{aligned} f(a_t k_0 k(n^1)) &= f(k_0 a_t k(n^1)) \\ &= f(k_0 a_t n^1 a(n^1)^{-1} n(n^1)^{-1}) \\ &= f(k_0 k(a_t n^1 a_t^{-1}) [a(a_t n^1 a_t^{-1}) a(a(n^1)^{-1})]^{-\nu-\rho}). \end{aligned}$$

Hence

$$\begin{aligned} [a_t^{\nu+\rho} \pi(g) \pi(a_t^{-1}) f](x_0) &= \\ \int_K^0 \int_{N^1} g(x_0 k(n^1)^{-1} k_0^{-1}) f(k_0 k(a_t n^1 a_t^{-1}) [a(a_t n^1 a_t^{-1}) a(n^1)^{-1}]^{-\nu-\rho}) \\ a(n^1)^{-2\rho} dk_0 dn^1. \end{aligned}$$

Now f and g restricted to K are bounded; so the integrand is majorized by

$$a(a_t n^1 a_t^{-1})^{-\text{Re}(\nu)-\rho} a(n^1) \text{Re}(\nu)^{-\rho}.$$

This function can be shown to be uniformly in $L^1(N^1)$ as $t \rightarrow \infty$ ([51], pp. 360-361). So we may take the limit in the lemma under the integral sign, by the dominated convergence theorem.

The limit is

$$\int_{K^0} \int_{N^1} g(x_0 k(n)^{-1} k_0^{-1}) f(k_0) a(n^1)^{-\rho} dk_0 dn^1 = \int_{N^1} h(x_0 k(n^1)^{-1}) a(n^1)^{-\rho} dn^1.$$

Now $k(n^1)^{-1} = n^1 (n^1)^0 (n^1) a(n^1)^{-1}$. If we combine this

with (*), the integral becomes

$$\int_{N^1} h(x_0 n^1 (n^1) a(n^1)^{-2\rho}) dn^1 = \int_{N^1} h(x_0 n^1) dn^1 = [A(p^1, p^1)h](x_0)$$

by Lemma 4.2.14(a).

Q.E.D.

Lemma 4.2.16 Suppose (V, μ) is a finite dimensional representation of K^0 . Then the map from $C^\infty(K) \otimes V$ to the smooth elements in $\text{Ind}_{K^0}^K(\mu)$ defined by

$$\phi(f \otimes v) = h, \quad h(k) = \int_{K^0} f(kk_0) \mu(k_0) v dk_0$$

is surjective.

Proof. There is an isomorphism

$$C^\infty(K) \otimes V \sim C^\infty(K, V)$$

defined by

$$f \otimes v \sim h, \quad h(k) = f(k) v.$$

Suppose then that h is a smooth element in the induced representation; write

$$h \sim \sum f_i \otimes v_i.$$

We claim that h is $\phi(\sum f_i \otimes v_i)$. For

$$\begin{aligned} \phi(\sum f_i \otimes v_i)(k) &= \int_{K^0} \sum f_i(kk_0) \mu(k_0) v_i dk_0 \\ &= \int_{K^0} \mu(k_0) h(kk_0) dk_0 \\ &= \int_{K^0} \mu(k_0) \mu(k_0^{-1}) h(kk_0) dk_0 \\ &= h(k). \end{aligned}$$

Q.E.D.

Definition 4.2.17 Suppose $\mathcal{H}^0 \subseteq \mathcal{H}_{\delta \otimes v}^P$ is a closed, G -invariant subspace; write $\mathcal{H}_K^0$ for the set of K -finite vectors in $\mathcal{H}^0$. Define

$$\begin{aligned} V_{K^0}^0 &= \{h \in V_{\delta^1} \mid h = f|_{M^1}, \text{ for some } f \in \mathcal{H}_K^0\} \\ V^0 &= \text{closure of } V_{K^0}^0 \text{ in } V_{\delta^1}. \end{aligned}$$

To make sense of this definition, remember first that V_{δ^1} is a space of functions from M^1 to V_δ , and $\mathcal{H}_{\delta \otimes v}^P$ is a space of functions from G to V_δ . If f is K -finite in $\mathcal{H}_{\delta \otimes v}^P$, then it is smooth as a function on G ; so the restriction to M^1 is well defined. It is easily seen to belong to V_{δ^1} .

Lemma 4.2.18 In the setting of Definition 4.2.17,

$V_{K^0}^0$ is an $(\mathcal{M}^1, K^0)$ submodule of V_{δ^1} ; and $V_{K^0}^0 = 0$ if and only if $\mathcal{H}^0 = 0$.

Proof. By Theorem 0.3.5, $\mathcal{H}_K^0$ is stable under σ_f and K . Since the actions of $\mathcal{M}^1$ and K^0 commute with restriction to M^1 , the first statement is obvious. For the second, suppose $\mathcal{H}^0 \neq 0$. Then we can find a K -finite vector $0 \neq f \in \mathcal{H}_K^0$. By Lemma 4.1.5,

f does not vanish on K ; so after replacing f by some $\pi(k)f$, we may assume $f(1) \neq 0$. But then the restriction of f to M^1 is non-zero. Q.E.D.

Definition 4.2.19 In the setting of Definition 4.2.17,

set

$$\begin{aligned} \mathcal{H}^{1, \overline{P^1}} &= \text{Ind}_{\overline{P^1}}^G (\delta^1|_{V^0} \otimes v^1 \otimes 1) \\ &\subseteq \mathcal{H}_{\delta^1 \otimes v^1}^{\overline{P^1}}. \end{aligned}$$

Define $(\pi^1, \mathcal{H}^1) \subseteq \mathcal{H}_{\delta^1 \otimes v^1}^{1, \overline{P^1}}$ to be the closure of the image of $A(P^1, \overline{P^1})$ restricted to $\mathcal{H}^{1, \overline{P^1}}$, so that

$$\pi^1 \subseteq \overline{\pi}(\delta \otimes v).$$

By Lemma 4.2.18, v^0 is an invariant subspace of V_{δ^1} , so $\mathcal{H}^{1, \overline{P^1}}$ is an invariant subspace of $\mathcal{H}_{\delta^1 \otimes v^1}^{\overline{P^1}}$. So π^1 is really a subrepresentation of $\overline{\pi}(\delta \otimes v)$.

Lemma 4.2.20 In the setting of Definitions 4.2.17 and 4.2.19

$$\mathcal{H}^0 \supseteq \mathcal{H}^1.$$

Proof. It is enough to prove this for K -finite vectors; so what we must show is that if $h \in \mathcal{H}^{1, \overline{P^1}}$ is K -finite, then

$$A(P^1, \overline{P^1})h \in \mathcal{H}^0.$$

Now we look at $\mathcal{H}^{1, \overline{P^1}}$ in terms of K alone, using Lemma 4.1.5.

By Lemma 4.2.16, we can find finitely many elements

$$f_i \in \mathcal{H}_K^0, \quad g_i \in C^\infty(K)$$

so that

$$h(k) = \int_K^0 g_i(kk_0^{-1}) f_i(k_0) dk_0.$$

By Lemma 4.2.15,

$$A(P^1, \overline{P^1})h = \sum_i \lim_{t \rightarrow \infty} a_t^{v+\rho} \pi(g_i) \pi(a_t^{-1}) f_i.$$

Since all f_i lie in the closed subspace $\mathcal{H}^0$, which is invariant under the operators $\pi(g_i)$ and $\pi(a_t)$, the left side lies in $\mathcal{H}^0$ as well. Q.E.D.

If we take into account the last assertion of Lemma 4.2.18, Lemma 4.2.20 provides a strong version of Theorem 4.2.12.

We turn now to the factorization of the intertwining operator -- this is due to Schiffman [34], with the main idea going back to Gindikin and Karpalevič. Define

$$\Delta_0^+ = \Delta(\sigma, \tilde{\nu}),$$

the system of positive roots determined by $\tilde{N}$. Set

$$m = |\Delta_0^+ \cap (-\Delta^+)|.$$

As is well known and easy to verify, we may choose a sequence $\{\Delta_i^+ \mid 0 \leq i \leq m\}$ of positive root systems with the property that for $1 \leq i \leq m$, there is a simple root α_i for Δ_i^+ , such that

$$\overline{\Delta_{i-1}^+} = (\overline{\Delta_i^+} - \{\alpha_i\}) \cup \{-\alpha_i\};$$

here, as usual, bar denotes the reduced roots. Furthermore, we can arrange

$$\Delta_m^+ = \Delta^+ = \Delta(\sigma, \nu).$$

Let $P(i)$ be the minimal parabolic subgroup of G corresponding to Δ_i^+ ; thus

$$\begin{aligned} P^{(0)} &= \tilde{P} \\ P^{(m)} &= P. \end{aligned}$$

Theorem 4.2.22 (Schiffman [34]). With notation as above

$$A(P^1, P^1) = A^{(m)} \circ A^{(m-1)} \dots \circ A^{(1)}.$$

In particular, $A(P^1, P^1)$ fails to be an isomorphism

on K-finite vectors if and only if $A^{(i)}$ does for some $i, 1 \leq i \leq m$.

Proof. The product formula is a simple application of the Fubini theorem; a detailed account may be found in [51], Theorem 9.1.6.2. (In fact, this is the main step in the proof that the integral for $A(P^1, P^1)$ converges). For the second statement, fix a representation $\mu \in \hat{K}$. Then the μ -primary subspaces of the various $\pi(P^{(i)}, \delta_{\emptyset\nu})$ are finite dimensional, of the same dimension; and they are preserved by the intertwining operators. So the statement follows from the analogous result about maps between vector spaces of the same dimension.

Q.E.D.

For $1 \leq i \leq m$ define

$$Q^{(i)} = M^{\alpha} i_P^{(i)}.$$

Then the Levi factor of $Q^{(i)}$ has Langlands decomposition $M^{\alpha} i_A^{(i)}$. We can define representations of $M^{\alpha} i_A^{(i)}$

$$\pi^{i, \varepsilon} = \pi_{M^{\alpha} i_A^{(i)}}^{(P^{(i-\varepsilon)}, \delta_{\emptyset\nu})} \quad (\varepsilon = 0, 1);$$

so

$$(4.2.23) \quad \text{Ind}_{Q^{(i)}}^G \pi^{i, \varepsilon} \otimes 1 = \pi(P^{(i-\varepsilon)}, \delta_{\emptyset\nu})$$

by induction by stages. There is a convergent intertwining integral

$$B^{(i)} : \pi^{i, 1} \rightarrow \pi^{i, 0}$$

Furthermore,

$$\{\alpha_i\} = \{\alpha \in \Delta^+ \mid \text{Re} \langle \alpha, \nu \rangle \neq 0\}.$$

Notation 4.2.21 Whenever $\alpha \in \Delta$, let

$$\sigma_0^\alpha = \{X \in \sigma_0 \mid \alpha(X) = 0\},$$

and let

$$M_A^{\alpha} = {}_G A^\alpha$$

be the Langlands decomposition of the centralizer of A^α in G . (Thus $\mathfrak{m}_0^\alpha$ is spanned by $\mathfrak{m}_0$ and the root vectors for the roots $c\alpha$, $c \neq 0$; and M_0^α is generated by M_0^α and M .)

In Section 3 we will write

$$K^\alpha = M_0^\alpha \cap K.$$

Let $N^{(\alpha)}$ be the connected group whose Lie algebra is spanned by the root vectors for the roots $c\alpha$, $c > 0$.

Since all the minimal parabolic subgroups $P^{(i)}$ are conjugate under W , Theorem 4.1.4 implies that all of the representations $\pi(P^{(i)}, \delta_{\emptyset\nu})$ have infinitesimally equivalent composition series. In particular, they are either all irreducible or all reducible. Define

$$A^{(i)} : \pi(P^{(i-1)}, \delta_{\emptyset\nu}) \rightarrow \pi(P^{(i)}, \delta_{\emptyset\nu})$$

by

$$(A^{(i)} f)(x) = \int_{N^{(\alpha_i)}} f(xn) \, dn;$$

here we realize $\pi(P^{(i)}, \delta_{\emptyset\nu})$ on $\mathcal{H}_{\delta_{\emptyset\nu}}^{P^{(i)}}$ as usual. Exactly as in the case of $A(P^1, P^1)$, one checks that $A^{(i)}$ converges absolutely, and defines an intertwining operator as claimed.

defined by the same formula as $A^{(i)}$. For formal reasons, one can easily check that

$$(4.2.24) \quad A^{(i)} = \text{Ind}_Q^G(i)(B^{(i)});$$

this makes sense because of (4.2.23) and the functoriality of induction.

Theorem 4.2.25 (B. Speh) Suppose $\delta \in \hat{M}$, $\nu \in \hat{A}$ is negative; and let $p^1 = M^1 A^1 N^1$ be the parabolic subgroup defined by $-\text{Re } \nu$ (Definition 4.2.1). Then the principal series representation $\pi(\delta \otimes \nu)$ is reducible if and only if at least one of the following conditions holds:

a) The unitarily induced representation

$$\text{Ind}_{P \cap M^1}^{M^1}(\delta \otimes \nu) \Big|_A^0 \otimes 1)$$

is reducible.

b) There is a root $\alpha \in \Delta$ such that $\text{Re} \langle \alpha, \nu \rangle \neq 0$, with $\pi_{M^1 A^1}^{\alpha, \alpha}(\delta \otimes \nu)$ reducible. (This is the case if and only if

$$\pi_{M_0^1 A^1}^{\alpha, \alpha}(\delta \Big|_{M \cap M_0^1} \otimes \nu)$$

is reducible.)

Proof. By induction by stages (Proposition 4.1.18) it is clear that either condition is sufficient for reducibility. To prove the theorem, suppose that $\pi(\delta \otimes \nu)$ is reducible but (a) fails; we must show that (b) holds. By Langlands'

Theorem 4.2.12, $\pi(\delta \otimes \nu)$ is irreducible; so it cannot be all of $\pi(P, \delta \otimes \nu)$. Since $\pi(\delta \otimes \nu)$ is the image of $A(p^1, p^1)$, Theorem

2.2.11 says that some $A^{(i)}$ is not an infinitesimal isomorphism. Because of (4.2.24) above, the corresponding $B^{(i)}$ is not either. So the image of $B^{(i)}$ gives a proper invariant subspace of $\pi_{M^1 A^1}^{\alpha, \alpha}(\delta \otimes \nu)$. To prove the parenthetical remark, recall first that M meets every connected component of M^α (Lemma 0.4.2). It follows that

$$[\pi_{M^1 A^1}^{\alpha, \alpha}(\delta \otimes \nu)] \Big|_{M_0^1 A^1} = \pi_{M_0^1 A^1}^{\alpha, \alpha}(\delta \Big|_{M \cap M_0^1} \otimes \nu).$$

By Proposition 0.4.7, $\delta \Big|_{M \cap M_0^1} = \delta^0$ is irreducible. So by Theorem 4.2.12 (writing $\alpha = \alpha_i$),

$$\begin{aligned} \pi_{M^1 A^1}^{\alpha, \alpha}(\delta \otimes \nu) \text{ is reducible} &\iff B^{(i)} \text{ has a kernel} \\ &\iff \pi_{M_0^1 A^1}^{\alpha, \alpha} \text{ is reducible.} \end{aligned}$$

Q.E.D.

One can describe explicitly when (a) and (b) hold, in terms of δ and ν ; so this theorem completely settles the question of reducibility of the principal series. For (a), the result is due to Knapp-Stein [26], and for (b) to Hirai [20], Thieliker (see [3], p. 195) and Wallach (unpublished -- see [31]). When δ is trivial, all of these results and Theorem 4.2.25 are due to Kostant [29].

The following reformulation of Theorem 4.2.12 is sometimes useful; we will need it in Section 4 of this chapter to study reducibility.

Proposition 4.2.26 Suppose $\delta \in \hat{M}$, and $\nu \in \hat{A}$ is negative with respect to P ; put $\bar{P} = \theta P$. Assume that $\pi(P, \delta \otimes \nu)$ satisfies the following condition:

(H) No $\bar{\pi}(\delta_i^1 \otimes \nu^1)$ occurs in $\pi(P, \delta\emptyset\nu)/\bar{\pi}(\delta\emptyset\nu)$;

that is, that the Langlands subrepresentations of $\pi(\delta\emptyset\nu)$

do not occur elsewhere in $\pi(\delta\emptyset\nu)$. Let m_i be the multiplicity of $\bar{\pi}(\delta_i^1 \otimes \nu^1)$ in $\pi(\delta\emptyset\nu)$. Use the notation of

(4.1.3) for Harish-Chandra modules; then

$$\dim \operatorname{Hom} \mathcal{G}_{K,K}(X(\bar{P}, \delta\emptyset\nu), X(P, \delta\emptyset\nu)) = \sum m_i^2.$$

(It is known that condition (H) is automatic -- see for example [3], Proposition IV.4.13.)

Proof. Let $P^1 = M^1 A_N^1$ be the parabolic associated to ν . The two representations

$$\pi_{M^1} = (P_0 M^1, \delta\emptyset\nu^0), \quad \pi_{M^1}(\bar{P}_0 M^1, \delta\emptyset\nu^0)$$

are unitary and have the same composition series (Theorem 4.1.4 and Proposition 4.1.12); so they are equivalent. So it is enough to prove the proposition with $\bar{P}$ replacing P . Let $\bar{X}(\delta\emptyset\nu)$ be the Harish-Chandra module of $\bar{\pi}(\delta\emptyset\nu)$. By its definition, there is a short exact sequence

$$0 \rightarrow Y \rightarrow X(\bar{P}, \delta\emptyset\nu) \rightarrow \bar{X}(\delta\emptyset\nu) \rightarrow 0.$$

Since $\pi(P, \delta\emptyset\nu)$ and $\pi(\bar{P}, \delta\emptyset\nu)$ have the same composition series, Y has the same composition factors as $\pi(P, \delta\emptyset\nu)/\bar{\pi}(P, \delta\emptyset\nu)$.

Suppose

$$f \in \operatorname{Hom} \mathcal{G}_{K,K}(X(\bar{P}, \delta\emptyset\nu), X(P, \delta\emptyset\nu)).$$

Then $f(Y)$ is a submodule of $X(P, \delta\emptyset\nu)$; and by assumption (H), it cannot meet $\bar{\pi}(P, \delta\emptyset\nu)$. By Theorem 4.2.12,

$$f(Y) = 0.$$

The short exact sequence therefore implies that

$$\operatorname{Hom} \mathcal{G}_{K,K}(X(\bar{P}, \delta\emptyset\nu), X(P, \delta\emptyset\nu)) \cong \operatorname{Hom} \mathcal{G}_{K,K}(\bar{X}(\delta\emptyset\nu), X(P, \delta\emptyset\nu)).$$

The same argument shows that any map from $\bar{X}(\delta\emptyset\nu)$ into $X(P, \delta\emptyset\nu)$ must have its image inside $\bar{X}(\delta\emptyset\nu)$; so we conclude

$$\operatorname{Hom} \mathcal{G}_{K,K}(X(\bar{P}, \delta\emptyset\nu), X(P, \delta\emptyset\nu)) \cong \operatorname{Hom} \mathcal{G}_{K,K}(\bar{X}(\delta\emptyset\nu), \bar{X}(\delta\emptyset\nu)).$$

Since $\bar{\pi}(\delta\emptyset\nu)$ is completely reducible (recall that it is the direct sum of the $\bar{\pi}(\delta_i^1 \otimes \nu^1)$) the proposition follows. Q.E.D.

§3. Fine K-types and the R group.

In this section we will describe the algebraic background we will need for the analysis of reducibility of unitary principal series representations. The results and definitions are due to Knapp ([25], [26]) and Bernstein, Gelfand and Gelfand ([2]). The goal is this: to each representation δ of M

(notation 4.1.1), we want to associate canonically a representation μ of K , which occurs with multiplicity one in the principal series representation $\pi(\delta\emptyset\nu)$. By Lemma 4.1.5, this means that δ should occur exactly once in $\mu|_M$. This problem will be solved in a special case (to which the general case will eventually be reduced). Actually, we find not one μ but several; and the non-uniqueness of μ turns out to be closely connected to the reducibility of the unitary principal series $\pi(\delta\emptyset 1)$. We begin with some examples, which the general theory simply imitates. (In fact it is very entertaining to do everything case by case; this is done in [41].)

Example 4.3.1 If δ is trivial, we take μ trivial as well; the fact that this is canonical corresponds to Kostant's

theorem that the unitary spherical principal series is irreducible.

Example 4.3.2 Let $G = \text{SL}(2, \mathbb{R})$; we use the notation of Chapter 1, especially Definition 1.3.5. Let δ be the non-trivial character of M (called -1 there). By Definition 1.3.1 (and Proposition 1.3.8), the principal series representations $\pi(\delta \otimes \nu)$ all contain the two "smallest" non-trivial representations of $K = \text{SO}(2)$, called ± 1 in Chapter 1; and these (like all K -types in this case) have multiplicity one. So there are two choices for μ ; this corresponds to the fact that $\pi(\delta \otimes 1)$ splits into two pieces (Proposition 1.3.3(e)).

Example 4.3.3 Let $G = \text{GL}(n, \mathbb{R})$, $K = \text{O}(n)$ (the real orthogonal group). Choose for $\mathbb{M}$ the group of diagonal matrices; writing d for "diagonal matrix corresponding to", we have

$$M = \{d(\pm 1, \dots, \pm 1)\} \cong (\mathbb{Z}/2\mathbb{Z})^n.$$

The character group $\hat{M}$ can be parametrized as follows: if S is a subset of $\{1, \dots, n\}$, define

$$\delta_S(d(\varepsilon_1, \dots, \varepsilon_n)) = \prod_{i \in S} \varepsilon_i.$$

Now let μ_M denote the representation of $\text{O}(n)$ on $[\wedge^m(\mathbb{R}^n)]_{\mathbb{C}}$; we give $\mathbb{R}^n$ the usual basis $\{e_i\}$. Then μ_M has a basis

$$\{e_S = \wedge_{i \in S} e_i \mid |S| = m\}.$$

Clearly this diagonalizes the action of M ; M acts on

e_S by the character of δ_S . So the representation of $\text{O}(n)$ attached to δ_S is $\mu_{|S|}$. Notice in this case how the Weyl group W acts: W is the permutation group on $\{1, \dots, n\}$, and it acts on $\hat{M}$ by acting on S . So we have actually shown

$$(\mu_{|S|})|_M \cong \bigoplus_{\delta \in W \cdot \delta_S} \delta;$$

that is, that the representation of K attached to δ_S is built out of all the representations of M in the W orbit of δ , each occurring once. The fact that the representation of $\text{O}(n)$ is canonical corresponds to the fact that the unitary principal series representations of $\text{GL}(n, \mathbb{R})$ are irreducible.

We now turn to the general theory.

Lemma 4.3.4 Let G^d denote the commutator (or derived) subgroup of G . Then G^d is a connected semisimple Lie group.

Proof. By abstract group theory, G^d is the intersection of the kernels of all one dimensional representations of G . So to prove that G^d is connected, it is enough to show that G/G_0 is abelian. But by Lemma 0.4.2, there is an abelian subgroup T^S of K (the compact part of a maximally split Cartan subgroup) such that

$$G/G_0 \cong T^S/T^S \cap G_0.$$

Finally, the Lie algebra of G^d is $[\mathfrak{g}_0', \mathfrak{g}_0']$, which is semisimple since $\mathfrak{g}_0$ is reductive. Q.E.D.

Definition 4.3.5 G is called quasisplit if $\mathfrak{m}_0 + \sigma_0$ is abelian (cf. 4.1.1).

We will assume for the rest of this section that G is quasisplit. It follows that MA is a Cartan subgroup of G , so M is abelian (by 0.1.2) (f).

Notation 4.3.6 A root $\alpha \in \bar{\Delta}$ (cf. 4.1.1) is called real if it is the restriction to σ of a real root of $(\mathfrak{m} + \sigma)$ in σ (Definition 0.2.1); otherwise α is called complex. For each real root α , we choose an injection

$$\phi_\alpha: \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{m}_0^\alpha$$

(notation (4.2.21)) satisfying

$$\begin{aligned} \phi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &\in \sigma_0 \\ \phi_\alpha \begin{pmatrix} -x & 1 \\ 1 & x \end{pmatrix} &= \theta \phi_\alpha(x) \\ \phi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &\in \alpha \text{ root space.} \end{aligned}$$

Let

$$\Phi_\alpha: \mathrm{SL}(2, \mathbb{R}) \rightarrow M^\alpha$$

be the exponentiated map. Put

$$\begin{aligned} Z_\alpha &= \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}_0^\alpha \\ \sigma_\alpha &= \Phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp \left\{ \frac{\pi}{2} Z_\alpha \right\} \in K^\alpha \\ m_\alpha &= \sigma_\alpha^2 = \Phi_\alpha \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Lemma 4.3.7 The map ϕ_α is unique up to conjugation of $\mathfrak{sl}(2, \mathbb{R})$ by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; so Z_α is well defined up to

sign, and σ_α up to inverse. Furthermore,

- $m_0^2 = 1$, and $m_0 \in M$
- $\sigma_\alpha \in M'$ (notation (4.1.1)), and the image $\overline{\sigma_\alpha}$ of σ_α in $W = M'/M$ is the reflection s_α (cf. 0.2.5).

proof. Any two choices of ϕ_α differ by an automorphism of $\mathfrak{sl}(2, \mathbb{R})$ which commutes with θ and fixes $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It is elementary to check that conjugation by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the only non-trivial automorphism with those properties. For (a) and (b), notice that $\phi_\alpha(\mathrm{SL}(2, \mathbb{R}))$ is contained in M^α , which centralizes the hyperplane σ_0^α in σ_0 . So (a) and (b) are reduced to statements about $\mathrm{SL}(2, \mathbb{R})$ itself, which are easy to check. Q.E.D.

Definition 4.3.8 A representation $\delta \in \hat{M}$ is called fine (with respect to G) if its restriction to the identity component of $G^d \cap M$ is trivial (cf. Lemma 4.3.4).

The reader should think of the case when G is semisimple and split (i.e., M_0 is trivial). Then all representations of M (which is a finite abelian group) are fine, and all roots are real; all of the main ideas appear in this case.

Definition 4.3.9 A representation $\mu \in \hat{K}$ is called fine

(with respect to G) if

- for each real root $\alpha \in \bar{\Delta}$, the eigenvalues of $\mu(iZ_\alpha)$ (notation (4.3.6)) are contained in $\{0, +1, -1\}$; and
- for each complex root $\alpha \in \bar{\Delta}$, $\mu|_{\mathfrak{k}_0^\alpha \cap \mathfrak{g}_0^\alpha}$ is trivial.

Example 4.3.10

a) Suppose $G = GL(n, \mathbb{R})$, $K = O(n)$. Then the fine representations of K are

$$\{\mu_m = (\wedge^m(\mathbb{R}^n))_{\mathbb{Q}} \mid m = 0, 1, \dots, n\}.$$

b) Suppose G is complex semisimple. Then only the trivial representation of K is fine.

c) Suppose $G = SU(n, n)$, $K = S(U(n) \times U(n))$. Then K has exactly three fine representations. If we write $\det$ for the determinant representation of $U(n)$, they are {trivial, $\det \otimes 1$, $1 \otimes \det$ }.

Definition 4.3.11 If $\delta \in \hat{M}$ is fine, we define

$$\underline{\bar{\Delta}}_{\delta} \subseteq \Delta$$

the set of good roots with respect to δ , to consist of those $\alpha \in \bar{\Delta}$ such that either

- a) α is real, and $\delta(m_{\alpha}) = 1$, or
- b) α is complex.

Lemma 4.3.12 If $\delta \in \hat{M}$ is fine, then $\bar{\Delta}_{\delta}$ is a root system, and $W(\bar{\Delta}_{\delta}) \subseteq W$ stabilizes δ (in the action of W on $\hat{M}$).

This will be proved in a moment.

Definition 4.3.13 If $\delta \in \hat{M}$ is fine, set

$$\begin{aligned} W_{\delta}^0 &= W(\bar{\Delta}_{\delta}) \\ M'_{\delta} &= \text{stabilizer of } \delta \text{ in } M' \\ W_{\delta} &= \text{stabilizer of } \delta \text{ in } W = M'_{\delta}/M \\ R_{\delta} &= W_{\delta}/W_{\delta}^0, \text{ the } R\text{-group for } \delta. \end{aligned}$$

Of course we have not yet checked that R_{δ} is a group. This follows from

Lemma 4.3.14 W_{δ}^0 is a normal subgroup of W_{δ} , and every element of R_{δ} has order 2.

This will also be proved in a moment. It follows that R_{δ} is even abelian, so that $\hat{R}_{\delta}$ is also a group. We now define the special set of representations of K attached to a (fine) representation of M .

Definition 4.3.15 If $\delta \in \hat{M}$ is fine, put

$$A(\delta) = \{\mu \in \hat{K} \mid \mu \text{ is fine, and } \delta \text{ occurs in } \mu|_M\}.$$

Theorem 4.3.16 Suppose G is quasisplit, and $\delta \in \hat{M}$ is fine (Definition 4.3.8). Then

(a) $A(\delta)$ is non-empty (Definition 4.3.15).

(b) Suppose $\mu \in A(\delta)$. Then $\mu|_M$ is the sum of the representations of M in the W orbit of δ in $\hat{M}$, each occurring with multiplicity one.

(c) There is a natural simply transitive action of $\hat{R}_{\delta}$ on $A(\delta)$. The action of any element of $\hat{R}_{\delta}$ is given by the action on $\hat{K}$ of some automorphism of G which commutes with θ , and acts trivially on MA .

The most difficult part of this theorem is (a). Perhaps the easiest proof consists simply of listing the necessary representations of K , for all possible simple G ; this is the approach used in [2] and [41] (where it is assumed that G is split). It is also possible to give an algebraic reduction

to rank 2, which is easily treated case by case. In the context of this book, however, the most efficient method is to postpone the issue until Chapter 5, when we will prove:

Theorem 4.3.17 (Chapter 5, after 5.4.13). Suppose
G is quasisplit, $\delta \in \hat{M}$ is fine and $v \in \hat{A}$ is unitary.
Then every irreducible component of $\pi(\delta \otimes v)$ contains
a fine K-type.

If $\mu \in \hat{K}$ occurs in $\pi(\delta \otimes v)$, then by Lemma 4.1.5, δ occurs in $\mu|_M$; so this proves 4.3.16(a).

We turn now to the proofs of the lemmas and the rest of the theorem.

Definition 4.3.18 Suppose $\beta \in \bar{\Lambda}$ is a real root. Let $\beta(m)$ denote the scalar by which m acts on the root vector X_β ; thus we have a character

$$\beta: M \rightarrow \{\pm 1\}.$$

Lemma 4.3.19 If $\beta \in \bar{\Lambda}$ is real, and $m \in M$, then

- a) $\beta(m) = (-\beta)(m)$
- b) $\frac{\text{Ad}(m)Z_\beta}{\beta} = \frac{\beta(m)Z_\beta}{\beta}$
- c) $\frac{m\sigma_\beta m^{-1}}{\sigma_\beta} = \frac{\sigma_\beta(m)}{\sigma_\beta}$
- d) $\frac{m\sigma_\beta m^{-1}}{\sigma_\beta} = \frac{1/2(1-\beta(m))}{m\beta}$
- e) if $\alpha \in \bar{\Lambda}$ is real, then $\beta(m_\alpha) = (-1)^{\langle \alpha, \beta \rangle}$

Proof. For (a), notice that $X_\beta = \theta X_{-\beta}$. Then (b) follows from the definition of Z_β , and (c) by exponentiating (b). Statement (d) is a rearrangement of (c), using $\sigma_\beta^2 = m_\beta$. Finally, (e) follows from the finite dimensional representation theory

of $\text{SL}(2, \mathbb{R})$, applied to the adjoint action of $\phi_\alpha(\text{SL}(2, \mathbb{R}))$ on $\mathcal{Y}$. Q.E.D.

Corollary 4.3.20 Suppose α, β and γ are real roots
in $\bar{\Lambda}$, and

$$\gamma = \overset{v}{\alpha} + \overset{v}{\beta} \quad (\text{notation (0.2.5)})$$

Then

$$m_\gamma = m_\alpha m_\beta.$$

Proof. This can be deduced from the usual interpretation of coroots as one parameter subgroup of the torus; but we will simply compute. We will need a detailed understanding of the rank two root systems (see, for example, [22], p. 44).

Case I. $\overset{v}{\langle \alpha, \beta \rangle} = -1$. In this case

$$\begin{aligned} s_\beta(\alpha) &= \alpha - (2\langle \alpha, \beta \rangle / \langle \beta, \beta \rangle) \beta \\ &= \alpha - (2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle) (\beta / \langle \beta, \beta \rangle) \langle \alpha, \alpha \rangle \\ &= \alpha + (\langle \alpha, \alpha \rangle) (\beta / \langle \beta, \beta \rangle). \end{aligned}$$

Since $s_\beta(\alpha)$ has the same length as α , it follows that

$$\begin{aligned} (s_\beta(\alpha))^v &= \frac{2}{\langle \alpha, \alpha \rangle} (s_\beta(\alpha)) \\ &= \frac{2\alpha}{\langle \alpha, \alpha \rangle} + \frac{2\beta}{\langle \beta, \beta \rangle} \\ &= \overset{v}{\alpha} + \overset{v}{\beta}. \end{aligned}$$

So $\gamma = s_\beta(\alpha)$. On the other hand, $\beta(m_\alpha) = -1$ by Lemma 4.3.19(e); so by 4.3.19(d),

$$m_\gamma = m_\alpha m_\beta.$$

Case II. $\overset{v}{\langle \beta, \alpha \rangle} = -1$. Since the result is symmetric in α and β , this follows from Case I.

Case III. $\langle \check{\alpha}, \gamma \rangle = 1$. We have

$$-\check{\alpha} + \check{\gamma} = \check{\beta};$$

so by Case I, applied to $(-\alpha, \gamma, \beta)$ instead of (α, β, γ) , we have

$$m_{-\alpha} m_{\gamma} = m_{\beta}.$$

But $m_{\alpha} = m_{-\alpha}$ (since we can choose $\phi_{-\alpha} = \theta \circ \phi_{\alpha}$), and $m_{\alpha} = m_{\alpha}^{-1}$; so this is

$$m_{\gamma} = m_{\alpha} m_{\beta}.$$

Case IV. $\langle \check{\beta}, \gamma \rangle = 1$, or $\langle \alpha, \check{\gamma} \rangle = 1$, or $\langle \beta, \check{\gamma} \rangle = 1$. This is treated in the same way as Case III.

By inspection of the rank 2 root systems, one sees that these cases exhaust all possibilities. Q.E.D.

Lemma 4.3.21 Suppose $\delta \in \hat{M}$ is fine, and $\beta \in \bar{\Delta}_{\delta}$ is a complex root. Then $s_{\beta} \in W_{\delta}$ (Definition 4.3.13).

Proof. We have seen that Theorem 4.3.16(a) follows from result in Chapter 5; so we will use it. Pick a representation (μ, V) of K belonging to $A(\delta)$. Choose a representative $\sigma_{\beta} \in (K \cap G^d)_0$ for the reflection s_{β} . Since μ is fine, $\mu(\sigma_{\beta})$ is the identity operator. So if we compute in the δ -primary subspace V_{δ} of V , we find (for $m \in M$ and $v \in V_{\delta}$)

$$\begin{aligned} (s_{\beta} \cdot \delta)(m) v &= \delta(\sigma_{\beta} m \sigma_{\beta}^{-1}) v \\ &= \mu(\sigma_{\beta}) \delta(m) \mu(\sigma_{\beta}^{-1}) v \\ &= \delta(m) v. \end{aligned}$$

Q.E.D.

Proof of Lemma 4.3.12 For the first statement, suppose α and β are in $\bar{\Delta}_{\delta}$; we must show that $s_{\beta}(\alpha)$ is as well. If α is complex, so is $s_{\beta}(\alpha)$, and we are done; so suppose α is real. Then $s_{\beta}(\alpha)$ is also real. The hypothesis means that

$$\delta(m_{\alpha}) = 1;$$

we want to conclude that

$$\delta(m_{s_{\beta}(\alpha)}) = 1$$

as well. Suppose first that β is complex. By Lemma 4.3.21,

$$\begin{aligned} \delta(m_{s_{\beta}(\alpha)}) &= \delta(\sigma_{\beta} m_{\alpha} \sigma_{\beta}^{-1}) \\ &= (s_{\beta} \cdot \delta)(m_{\alpha}) \\ &= \delta(m_{\alpha}) \\ &= 1; \end{aligned}$$

here we have chosen a representative σ_{β} for s_{β} in M' . So finally we may suppose β is real. Then by Lemma 4.3.19(d),

$$\begin{aligned} \delta(m_{s_{\beta}(\alpha)}) &= \delta(\sigma_{\beta} m_{\alpha} \sigma_{\beta}^{-1}) \\ &= \delta(m_{\alpha} m_{\beta}^{1/2(1-\beta(m_{\alpha}))}) \\ &= 1 \end{aligned}$$

since $\delta(m_{\alpha})$ and $\delta(m_{\beta})$ are both one. So $\bar{\Delta}_{\delta}$ is a root system. To prove that $W(\bar{\Delta}_{\delta})$ stabilizes δ , fix $\beta \in \bar{\Delta}_{\delta}$; we must show that s_{β} stabilizes δ . If β is complex, this is Lemma 4.3.21, so suppose β is real. Then

$$\begin{aligned} (s_{\beta} \cdot \delta)(m) &= \delta(\sigma_{\beta} m \sigma_{\beta}^{-1}) \\ &= \delta(m m_{\beta}^{1/2(1-\beta(m))}) \\ &= \delta(m) \end{aligned}$$

since $\delta(m_{\beta}) = 1$.

Q.E.D.

Lemma 4.3.22 $\bar{W}_\delta$ preserves $\bar{\Delta}_\delta^+$.

Proof. Fix $w \in W_\delta$ and $\alpha \in \bar{\Delta}_\delta$. If α is complex, so is $w \cdot \alpha$, and there is nothing to prove; so suppose α is real. Then

$$\begin{aligned}\delta(m_{w \cdot \alpha}) &= (w \cdot \delta)(m_\alpha) \\ &= \delta(m_\alpha) \quad (\text{since } w \in W_\delta) \\ &= 1 \quad (\text{since } \alpha \in \bar{\Delta}_\delta^+).\end{aligned}$$

So $w \cdot \alpha \in \bar{\Delta}_\delta^+$.

Q.E.D.

Fix a system of positive roots $\bar{\Delta}_\delta^+ \subseteq \bar{\Delta}_\delta$, and define

$$(4.3.23) \quad \rho_\delta = \rho(\bar{\Delta}_\delta^+).$$

By Lemma 3.2.4,

$$(4.3.24) \quad \langle \beta, \rho_\delta \rangle \neq 0, \text{ all } \beta \in \bar{\Delta}_\delta^+.$$

Choose a new positive root system $\bar{\Delta}_1^+$ for $\bar{\Delta}$, so that

$$(4.3.25) \quad \langle \alpha, \rho_\delta \rangle \geq 0, \text{ all } \alpha \in \bar{\Delta}_1^+.$$

(It is not enough to assume $\bar{\Delta}_\delta^+ \subseteq \bar{\Delta}_1^+$.) Recall that two roots α and β are called strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root; in particular this implies $\langle \alpha, \beta \rangle = 0$. Define

$$(4.3.26) \quad \bar{\Delta}_S = \{\alpha \in \bar{\Delta} \mid \langle \alpha, \rho_\delta \rangle = 0\}.$$

Lemma 4.3.27 $\bar{\Delta}_S$ consists of a strongly orthogonal set $\{\alpha_1, \dots, \alpha_r\}$ of simple roots for $\bar{\Delta}_1^+$ together with their negatives. Furthermore, each α_i is real, and

$$\delta(m_{\alpha_i}) = -1.$$

Proof. Since ρ_δ is dominant, $\bar{\Delta}_S \cap \bar{\Delta}_1^+$ is spanned by a set $\{\alpha_1, \dots, \alpha_r\}$ of simple roots for $\bar{\Delta}_1^+$. By (4.3.24) and (4.3.26), $\bar{\Delta}_S$ does not meet $\bar{\Delta}_\delta^+$; so every $\alpha \in \bar{\Delta}_S$ is real, and satisfies $\delta(m_\alpha) = -1$. The remaining assertion means simply that $\bar{\Delta}_S$ is of type $A_1 \times \dots \times A_1$. Suppose not. Then we can find α, β and γ in $\bar{\Delta}_S$ so that

$$\gamma = \alpha + \beta.$$

By Corollary 4.3.20,

$$\begin{aligned}-1 &= \delta(m_\gamma) \\ &= \delta(m_\alpha m_\beta) \\ &= \delta(m_\alpha) \delta(m_\beta) \\ &= (-1)^2,\end{aligned}$$

a contradiction.

Q.E.D.

Define

$$(4.3.28) \quad R_\delta^C = \{w \in W_\delta \mid w(\bar{\Delta}_\delta^+) = \bar{\Delta}_\delta^+\}.$$

Lemma 4.3.29 $W_\delta = R_\delta^C \cdot W_\delta^0$, a semidirect product with W_δ^0 normal. Furthermore $R_\delta^C \subseteq W(\bar{\Delta}_S)$.

(This will prove Lemma 4.3.14).

Proof. That W_δ^0 is normal in W_δ follows from Lemma 4.3.22. If $w \in W_\delta$, then $w^{-1}(\bar{\Delta}_\delta^+)$ is a positive root system for $\bar{\Delta}_\delta^+$; so there is a $w_1 \in W_\delta^0$ such that

$$w_1 w^{-1}(\bar{\Delta}_\delta^+) = \bar{\Delta}_\delta^+$$

Thus $w_1 w^{-1} \in R_\delta^C$ and

$$w = (w_1 w^{-1})^{-1} w_1^{-1};$$

so

$$W_\delta = R_\delta^C W_\delta^0.$$

The only element of W_δ^0 preserving Δ_δ^+ is 1, so

$$R_\delta^C \cap W_\delta^0 = \{1\}.$$

This proves the first statement. If $w \in R_\delta^C$, then $w\rho_\delta = \rho_\delta$; so by Chevalley's theorem (which is proved as part of Lemma 6.3.28 below), w belongs to $w(\overline{\Lambda}_S)$. Q.E.D.

Definition 4.3.30 Let M_δ be the group generated by the various M_α , for $\alpha \in \overline{\Lambda}_S$ (cf. (4.3.26)).

Lemma 4.3.31 M_δ satisfies (1.5.1) (that is, it is essentially a product of $SL(2, \mathbb{R})$'s). The group R_δ^+ defined for M_δ is just R_δ^C (cf. (4.3.28)). Let

$$P_1 = MAN_1$$

be the minimal parabolic subgroup corresponding to Δ_1^+ (cf. (4.3.25)), and let

$$\alpha_\delta = \{X \in \alpha \mid \alpha(X) = 0, \text{ all } \alpha \in \overline{\Lambda}_S\}.$$

Then there is a parabolic subgroup

$$P_\delta = M_\delta A_\delta N_\delta \supseteq P_1.$$

Proof. This is clear from Lemma 4.3.27. Q.E.D.

Lemma 4.3.29 reduces much of the study of the R-group to M_δ , which was studied in detail in Section 1.5. We begin now to prepare for the proof of Theorem 4.3.16.

Lemma 4.3.32 Suppose (μ, V) and (μ', V') are fine representations of K . Then

$$\text{Hom}_K(V, V') = \text{Hom}_{M'}(V, V').$$

Proof. One inclusion is obvious; so suppose

$$\phi: V \rightarrow V'$$

is an M' -module map. Since M' (and even M) meets each component of K by Lemma 0.4.2, it is enough to show that ϕ is a $\mathcal{K}$ -module map. But $\mathcal{K}$ is spanned by the various $\mathcal{K}^\alpha$; so we need only show that ϕ is a $\mathcal{K}^\alpha$ -module map. Now

$$\mathcal{K}^\alpha = (\mathcal{K}^\alpha \cap \mathfrak{g}^d) + \mathfrak{m};$$

so we must show that ϕ is a $\mathcal{K}^\alpha \cap \mathfrak{g}^d$ -module map. If α is complex, $\mathcal{K}^\alpha \cap \mathfrak{g}^d$ acts by zero on V and V' by the definition of fine, so we are done. If α is real, then Z_α spans $\mathcal{K}^\alpha \cap \mathfrak{g}^d$, so we must show that ϕ respects the eigenspaces of Z_α . By the definition of fine, the eigenvalues are 0, i, and -i. Now

$$\sigma_\alpha = \exp\left(\frac{\pi}{2} Z_\alpha\right)$$

acts by 1, i, and -i on the corresponding eigenspaces. Since ϕ respects the eigenspaces of σ_α , it respects those of Z_α as well. Q.E.D.

Corollary 4.3.33 Suppose (μ, V) is a fine representation of K in $A(\delta)$. Write (μ_δ, V_δ) for the representation of M'_δ (cf. 4.3.13) on the δ -primary component of V . Then

a) (μ, V) is an irreducible representation of M'

- b) $(\underline{\mu}_\delta, V_\delta)$ is an irreducible representation of M'_δ
 c) $\underline{\mu} \upharpoonright_{M'} = \text{Ind}_{M'_\delta}^{M'}(\underline{\mu}_\delta)$
 d) All representations of M occurring in $\underline{\mu}$ lie in the W orbit of δ , and have the same multiplicity.

Proof. Statement (a) follows from Lemma 4.3.32. Then (b) and (c) constitute a trivial case of the imprimitivity theorem ([50], 5.4.1.1), for which the reader can easily supply a proof. Part (d) is a consequence of (c). Q.E.D.

Lemma 4.3.34 Fix $\delta \in \hat{M}$ fine, and let $(M'_\delta)^1$ be the group generated by the kernel of δ in M , and the various

$$M' \cap \exp(\mathcal{K}_0^\alpha \cap \mathcal{J}_0^d), \quad \alpha \in \bar{\Delta}_\delta.$$

- a) $(M'_\delta)^1 / (M'_\delta)^1 \cap M \cong W_\delta^0$
 b) If $\mu \in A(\delta)$, and $\underline{\mu}_\delta$ is as in (4.3.33), then $\underline{\mu}_\delta$ is trivial on $(M'_\delta)^1$.

Proof. Part (a) is clear. For (b), $\underline{\mu}_\delta$ is obviously trivially on all the generators of $(M'_\delta)^1$ except perhaps the σ_α , with $\alpha \in \bar{\Delta}_\delta$ real. But, as we saw in the proof of Lemma 4.3.32, $\mu(\sigma_\alpha)$ does not have -1 as an eigenvalue; so the +1 eigenspaces of m_α and σ_α coincide. Since $\delta(m_\alpha) = 1$, $\mu_\delta(\sigma_\alpha) = 1$. Q.E.D.

Lemma 4.3.35 $M'_\delta / (M'_\delta)^1$ is abelian.

Proof. By 4.3.34(a) and Lemma 4.3.29, every coset has a representative of the form

$$\left[\prod_{\alpha \in A \subseteq \Delta_S} \sigma_\alpha \right] m,$$

with $m \in M$ and

$$\prod_{\alpha \in A} s_\alpha \in R_\delta^C.$$

We write the first factor as σ_A . Since M is abelian and the various σ_A commute, the commutativity we want will follow if we can show that

$$\sigma_A m \sigma_A^{-1} m^{-1} \in \ker \delta.$$

But since $w_A \in W_\delta$,

$$\begin{aligned} \delta(\sigma_A m \sigma_A^{-1} m^{-1}) &= \delta(\sigma_A m \sigma_A^{-1}) \delta(m^{-1}) \\ &= (w_A \cdot \delta)(m) \delta(m^{-1}) \\ &= \delta(m) \delta(m^{-1}) \\ &= 1. \end{aligned} \quad \text{Q.E.D.}$$

Combining (4.3.33) (b), (4.3.34) (b), and (4.3.35), we find

Corollary 4.3.36 In the setting of Corollary 4.3.33,

$\underline{\mu}_\delta$ is one dimensional.

Together with (4.3.33) (d), this proves Theorem 4.3.16(b).

Lemma 4.3.37 Let $(M'_\delta)^0$ be the preimage of W_δ^0 in M' . Then

- a) $\frac{M'_\delta / (M'_\delta)^0}{\cong R_\delta}$
 b) If $\mu, \mu' \in A(\delta)$, then (notation (4.3.33)),

$$\mu_\delta \upharpoonright_{(M'_\delta)^0} \cong \mu'_\delta \upharpoonright_{(M'_\delta)^0}.$$

Proof. The first statement is obvious. By (4.3.34) (a),

$$(M'_\delta)^0 = M \cdot (M'_\delta)^1;$$

so the second follows from (4.3.34) (b). Q.E.D.

Definition 4.3.38 Suppose $\mu, \mu' \in A(\delta)$. Define

$$r(\mu, \mu') = \mu_\delta \otimes (\mu'_\delta)^* \in (M'_\delta)^\wedge.$$

By Lemma 4.3.37(b), $r(\mu, \mu')$ is trivial on $(M'_\delta)^0$; so by (4.3.37)(a), we can regard it as

$$r(\mu, \mu') \in (R_\delta)^\wedge.$$

Lemma 4.3.39 Suppose μ, μ' , and μ'' are in $A(\delta)$. Then $\mu' \cong \mu''$ if and only if $r(\mu, \mu') = r(\mu, \mu'')$.

Proof. "Only if" is clear; so suppose

$$r(\mu, \mu') = r(\mu, \mu'').$$

By Definition 4.3.38,

$$\begin{aligned} \mu'_\delta &\cong \mu_\delta \otimes [r(\mu, \mu')]^* \\ &\cong \mu_\delta \otimes [r(\mu, \mu'')]^* \\ &\cong \mu''_\delta. \end{aligned}$$

By Corollary 4.3.33(c),

$$\mu'|_{M'} \cong \mu''|_{M'}.$$

The result follows from Lemma 4.3.32. Q.E.D.

Next, we construct the automorphisms of G required by Theorem 4.3.16(c).

Definition 4.3.40 Let $G_\mathbb{C}^d$ be the (complex semisimple) group of inner automorphisms of $\mathfrak{g}$, and identify $\text{Lie}(G_\mathbb{C}^d)$ with $\mathfrak{g}^d$ (cf. Lemma 4.3.4). Let $H_\mathbb{C}^d$ denote the Cartan subgroup of $G_\mathbb{C}^d$ with Lie algebra

$$\begin{aligned} h^d &= (m + \alpha) \cap \mathfrak{g}^d. \\ &= (m \cap \mathfrak{g}^d) \oplus (\alpha \cap \mathfrak{g}^d) \\ &= m^d \oplus \alpha^d. \end{aligned}$$

(Thus $H_\mathbb{C}^d = M_\mathbb{C}^d A_\mathbb{C}^d$, although this is not a direct product.)

Finally put

$$M_\theta = \{x \in A_\mathbb{C}^d \mid x^2 = 1\}.$$

Lemma 4.3.41 Suppose $x \in M_\theta$. Then (in its action on $\mathfrak{g}$) x commutes with θ , and preserves $\mathfrak{g}_0$. It exponentiates to an automorphism of G_0 ; and if we require that x centralize M , it extends uniquely to an automorphism of G . This extension preserves K , and centralizes MA .

Proof. Since $x \in A_\mathbb{C}^d$, and θ acts by -1 on α ,

$$\theta x = x^{-1};$$

but since x is an involution, it follows that $\theta x = x$. Write bar for the complex structure on $\mathfrak{g}$, and write

$$x = \exp(X)$$

with $X \in \alpha^d$. Since

$$1 = x^2 = \exp(2X).$$

X must be in $\sqrt{-1}$ times the root lattice; so since α_0 is split over $\mathbb{R}$,

$$X \in i\alpha_0, \quad \bar{X} = -X.$$

It follows that

$$\bar{x} = \exp(-X) = x^{-1} = x;$$

so x preserves $\mathcal{G}_0$. So x exponentiates to an automorphism of some linear covering group $\tilde{G}_0$ of G_0 . Since x is inner for $\mathcal{G}$, it acts trivially on the center of $\tilde{G}_0$; so it is defined on G_0 . Since $x \in H_{\mathbb{Q}}^d$, x centralizes $(M \cap G_0)A$. To extend x to G , we are forced to define

$$x \cdot (mg) = m(x \cdot g) \quad (m \in M, g \in G_0);$$

since x acts trivially on $M \cap G_0$, this is well defined. To see that it is an automorphism, take two elements

$$m_i g_i, \quad i = 1, 2, \quad m_i \in M, g_i \in G_0.$$

The actions of m_2 and x on G_0 commute with each other, since $\text{Ad}(m_2)$ belongs to $H_{\mathbb{Q}}^d$. So we compute

$$\begin{aligned} [x \cdot (m_1 g_1)] [x \cdot (m_2 g_2)] &= m_1 (x \cdot g_1) m_2 (x \cdot g_2) \\ &= m_1 m_2 (m_2^{-1} x \cdot g_1 m_2) (x \cdot g_2) \\ &= m_1 m_2 (x \cdot (m_2^{-1} g_1 m_2)) (x \cdot g_2) \\ &= m_1 m_2 (x \cdot (m_2^{-1} g_1 m_2 g_2)) \\ &= x \cdot [(m_1 m_2) (m_2^{-1} g_1 m_2) g_2] \\ &= x \cdot [(m_1 g_1) (m_2 g_2)] . \end{aligned}$$

The remaining assertions are clear.

Q.E.D.

To continue, we need the notation of Section 1.5, with the group M_δ replacing G . Thus we recall from Lemma 4.3.27 the roots

$$\{\alpha_1, \dots, \alpha_r\}.$$

If $A \subseteq \{1, \dots, r\}$, we define

$$(4.3.42) \quad \sigma_A = \prod_{i \in A} \sigma_{\alpha_i}, \quad w_A = \prod_{i \in A} s_{\alpha_i},$$

and so on. There is a map

$$(4.3.43) \quad \begin{aligned} q: M &\rightarrow (\mathbb{Z}/2\mathbb{Z})^r = \{(\varepsilon_1, \dots, \varepsilon_r) \mid \varepsilon_i = \pm 1\} \\ q(m) &= (\alpha_1(m), \alpha_2(m), \dots, \alpha_r(m)) \quad (\text{notation 4.3.18}). \end{aligned}$$

We identify

$$(4.3.44) \quad W(\bar{\Delta}_S) \cong P(r) = \text{subsets of } \{1, \dots, r\}$$

via (4.3.42); the dual group of $P(r)$ is identified with $(\mathbb{Z}/2\mathbb{Z})^n$ by the pairing

$$(4.3.45) \quad \varepsilon(A) = \prod_{i \in A} \varepsilon_i \quad (A \in P(r), \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^r).$$

Since R_δ^C is a subgroup of $W(\bar{\Delta}_S)$, we can regard $\hat{R}_\delta$ as a quotient of $(\mathbb{Z}/2\mathbb{Z})^r$.

Lemma 4.3.46 Define (with notation as above)

$$\begin{aligned} q_\theta: M_\theta &\rightarrow (\mathbb{Z}/2\mathbb{Z})^r \\ q_\theta(x) &= (\alpha_1(x), \dots, \alpha_r(x)). \end{aligned}$$

a) q_θ is surjective.

b) Suppose $\mu \in A(\delta)$, and $x \in M_\theta$. Make x an automorphism

of G and K in accordance with Lemma 4.3.41, so that x

also acts on $\hat{K}$. Put

$$\mu' = x \cdot \mu.$$

If we regard $\hat{R}_\delta$ as a quotient of $(\mathbb{Z}/2\mathbb{Z})^r$ as above, then

$$r(\mu, \mu') = \text{image of } q_\theta(x) \text{ in } \hat{R}_\delta.$$

Proof. For (a), consider the positive root system Δ_1^+ of (4.3.25). Let $\Delta^+(\sigma, \tau + \sigma)$ be the corresponding positive system for $\tau + \sigma$, and let Π be its set of simple roots. Then

$$-\theta\Pi = \Pi, \quad \{\alpha_i\} \subseteq \Pi.$$

Given $j \in \{1, \dots, r\}$, choose $X \in \mathfrak{h}_1^d$ so that

$$\begin{aligned} \alpha_j(\exp(X)) &= \sqrt{-1} \\ \beta(\exp(X)) &= 1 \quad \beta \in \Pi - \{\alpha_j\}. \end{aligned}$$

Then $-\theta X$ has the same property; so

$$\begin{aligned} \alpha_j(\exp(X - \theta X)) &= -1 \\ \beta(\exp(X - \theta X)) &= 1 \quad \beta \in \Pi - \{\alpha_j\}. \end{aligned}$$

Furthermore $X - \theta X$ lies in $\sigma\mathcal{L}^d$, so

$$m = \exp(X - \theta X) \in A_{\mathbb{Q}}^d.$$

Since m^2 is one on all simple roots, $m^2 = 1$; so m lies in M_{θ} . By construction

$$q_{\theta}(m) = (1, \dots, -1, \dots, 1) \quad (-1 \text{ in } j\text{th place}).$$

This proves (a).

For (b), suppose $A \subseteq \{1, \dots, r\}$, and w_A lies in R_{δ}^C . Then

$$\begin{aligned} r(\mu, \mu')(w_A) &= \mu_{\delta}(\sigma_A) \mu_{\delta}'(\sigma_A)^{-1} \\ &= \mu_{\delta}(\sigma_A) \mu_{\delta}(x \cdot \sigma_A)^{-1} \\ &= \mu_{\delta}(\sigma_A(x \cdot \sigma_A)^{-1}). \end{aligned}$$

Computing exactly as in the proof of Lemma 1.5.22, we find

$$\begin{aligned} &= \mu_{\delta}(\sigma_A \cdot \sigma_A^{-1}) \cdot \prod_{i \in A} m_{\alpha_i} \\ &= \delta(\prod_{i \in A} m_{\alpha_i})_{\alpha_i(x)=-1} \\ &= \prod_{i \in A} \alpha_i(x) \quad (\text{by Lemma 4.3.27}) \end{aligned}$$

$$r(\mu, \mu')(w_A) = q_{\theta}(x)(A) \quad (\text{by (4.3.45)}).$$

This is exactly the assertion in (b). Q.E.D.

Definition 4.3.47 Suppose $\mu \in A(\delta)$, and $r \in \hat{R}_{\delta}$.

Choose $x \in M_{\theta}$ so that the image of $q_{\theta}(x)$ in $\hat{R}_{\delta}$ is r (as is possible by 4.3.46(a)). Make x act on $\hat{K}$ by Lemma 4.3.41, and set

$$r \cdot \mu = x \cdot \mu.$$

By Lemma 4.3.46(b), this definition has the property that

$$(4.3.48) \quad r(\mu, r \cdot \mu) = r \quad (\text{notation 4.3.38}).$$

By Lemma 4.3.39, the action is therefore well-defined (independent of the choice of x). Formula (4.3.48) also guarantees that the action is simply transitive. This completes the proof of Theorem 4.3.16.

§4. Principal series for quasisplit groups: reducibility and equivalences

We assume throughout this section that G is quasisplit. Our purpose is to prove a series of rather technical results about principal series. In Chapters 5 and 6 we will describe powerful cohomological techniques for reducing the study of

more general representations to this case; so these results will be the basis of the classification of $\hat{G}$. For the present, however, they may seem a bit unmotivated. Notation will be as in Sections 2 and 3.

Proposition 4.4.1 Suppose G is quasisplit, $\delta \in \hat{M}$ is fine, and $\nu \in \hat{A}$ is negative. Then each Langlands subrepresentation $\pi(\delta_i^1 \otimes \nu^1)$ (Definition 4.2.10) of $\pi(\delta \otimes \nu)$ contains some fine K -type $\mu \in A(\delta)$; and each $\mu \in A(\delta)$ lies in some $\pi(\delta_i^1 \otimes \nu^1)$.

Proof. First we show that the full induced representation $\pi(\delta_i^1 \otimes \nu^1)$ must contain some fine K -type. For

$$\pi(\delta_i^1 \otimes \nu^1)|_K \cong \pi(\delta_i^1 \otimes 1)|_K;$$

and $\pi(\delta_i^1 \otimes 1)$ is a subrepresentation of the unitarily induced principal series representation $\pi(\delta \otimes (\nu^0, 1))$. Now Theorem 4.3.17 implies that $\pi(\delta_i^1 \otimes 1)|_K$ must contain some fine K -type μ . (Since μ occurs in $\pi(\delta \otimes \nu)$, $\mu|_M$ must contain δ by Frobenius reciprocity, so $\mu \in A(\delta)$.)

Next, we claim that the restriction of the intertwining operator $A(P^1, \overline{P^1})$ to the fine K -type μ of $\pi(\overline{P^1}, \delta_i^1 \otimes \nu^1)$ is nonsingular; clearly this will complete the proof of the proposition. By Theorem 4.2.22, it is enough to show that if

$$f \in \mathcal{H}_{\delta \otimes \nu}^{P(i-1)}$$

transforms according to the representation μ of K , then

$$A^{(i)} f \neq 0. \text{ By the identification (4.2.23)}$$

$$\mathcal{H}_{\delta \otimes \nu}^{P(i-1)} \cong \text{Ind}_Q^G(i) \pi^{i,1},$$

we may regard f as taking values in the representation space $\mathcal{H}^{i,1}$ of $\pi^{i,1}$. By Frobenius reciprocity, the values of $f|_K$ must transform according to representations of K_0^α occurring in $\mu|_{K_0^\alpha}$. By the definition of fine, these are fine representations of K_0^α . But, by (4.2.24), $A^{(i)}$ acts on f by making $B^{(i)}$ act on the values of f . In this way the claim is reduced to the case of the group M_0^α ; clearly we may as well pass to the semisimple part of M_0^α . So we may assume G is a quasisplit simple group of real rank one; so G is covered by $SL(2, \mathbb{Q})$, $SU(2, 1)$, or $SL(2, \mathbb{R})$. In the first two cases M is connected, so δ is trivial. For δ trivial, $A(\delta)$ consists of the trivial representation of K alone (as follows from the proof of 4.3.32, for example). Thus when δ is trivial, $f|_K$ is constant; so we must show that

$$[A(P^1, \overline{P^1})f](1) = \int_N f(n)dn \neq 0.$$

Now

$$\begin{aligned} n &= \theta(\theta n) \\ &= \theta(k(\theta n)n(\theta n)a(\theta n)) \\ &= k(\theta n)(\theta(n(\theta n))a(\theta n))^{-1}. \end{aligned}$$

Since $\theta(n(\theta n))$ lies in $\bar{N}$, and

$$f(k\bar{n}a) = f(k)a^{-\nu+\rho}$$

the integral becomes

$$\int_N a(\theta n)^{\nu-\rho} dn.$$

This integral can be computed explicitly in terms of gamma functions ([51], p. 325); and one finds that it is non-zero,

as required. (Remember that if α is a positive root,

$$\operatorname{Re}\langle \alpha, \nu \rangle < 0.)$$

We are left with the case $G = \operatorname{SL}(2, \mathbb{R})$ and δ non-trivial. In this case we can take

$$K = \left\{ \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \mid x \in \mathbb{R} \right\} = \{r(x)\}.$$

The characters of K are

$$\chi_n(r(x)) = e^{in x},$$

and the subgroup M is $\{\pm 1\}$. It is immediate from the definition of $A(\delta)$ that

$$A(\delta) = \{\chi_1, \chi_{-1}\}.$$

By Corollary 1.3.4, the intertwining operator is non-zero on these K -types. (One could also calculate the integral in this case.)

Q.E.D.

Corollary 4.4.2 Under the hypotheses of Theorem 4.4.1, the various $\bar{\pi}(\delta_1^1 \otimes \nu^1)$ (for fixed δ and ν) are all distinct.

Proof. Each fine K -type $\mu \in A(\delta)$ occurs exactly once in $\pi(\delta \otimes \nu)$, by Lemma 4.1.5 and Theorem 4.3.16(b). If two of the $\bar{\pi}(\delta_1^1 \otimes \nu^1)$ coincided, the fine K -types contained in them would occur at least twice in $\pi(\delta \otimes \nu)$, a contradiction.

Q.E.D.

Theorem 4.4.3 (Langlands, Harish-Chandra) Suppose G is quasisplit, $\nu, \bar{\nu} \in \hat{A}$ are negative, and $\delta, \bar{\delta} \in \hat{M}$ are fine. If some Langlands subrepresentations $\bar{\pi}(\delta_1^1 \otimes \nu^1)$ and $\bar{\pi}(\bar{\delta}_1^1 \otimes \bar{\nu}^1)$ of $\pi(\delta \otimes \nu)$ and $\pi(\bar{\delta} \otimes \bar{\nu})$ are infinitesimally equivalent, then (δ, ν) and $(\bar{\delta}, \bar{\nu})$ are conjugate under W .

Proof. By Lemma 4.1.9, the infinitesimal character of $\pi(\delta \otimes \nu)$ corresponds to the weight of $\mathfrak{m} + \mathfrak{a}$ which is the differential of $\delta \otimes \nu$. The hypothesis evidently implies that $\pi(\bar{\delta} \otimes \bar{\nu})$ must have this same infinitesimal character; so the differentials of $\delta \otimes \nu$ and $\bar{\delta} \otimes \bar{\nu}$ are conjugate under the complex Weyl group. Since δ and $\bar{\delta}$ are trivial on the semisimple part of G (Definition 4.3.8), it follows that ν and $\bar{\nu}$ are conjugate under the complex Weyl group. Now two elements of $\mathcal{O}_L^*$ conjugate under the complex Weyl group must also be conjugate under the real Weyl group W (compare Lemma 3.6.7). Replacing $\bar{\nu}$ by $w\bar{\nu} = \nu$ (with $w \in W$) and $\bar{\delta}$ by $w\bar{\delta}$ does not affect the hypothesis or the conclusion; so we may assume $\nu = \bar{\nu}$.

By Proposition 2.4.1, $\bar{\pi}(\delta_1^1 \otimes \nu^1) \cong \bar{\pi}(\delta_j^1 \otimes \nu^1)$ contains some fine K -type $\mu \in A(\delta)$; so the restriction of μ to M contains both δ and $\bar{\delta}$. By Theorem 4.3.16(b), δ and $\bar{\delta}$ are conjugate under W , so that $A(\delta) = A(\bar{\delta})$. We claim that $\bar{\pi}(\delta \otimes \nu)$ is infinitesimally equivalent to $\bar{\pi}(\bar{\delta} \otimes \nu)$. It is enough (by Proposition 4.4.1) to show that for each $\mu' \in A(\delta)$, the component $\bar{\pi}(\delta_S^1 \otimes \nu^1)$ of $\bar{\pi}(\delta \otimes \nu)$ containing μ' is equivalent to the corresponding component $\bar{\pi}(\delta_T^1 \otimes \nu^1)$ of $\bar{\pi}(\bar{\delta} \otimes \nu)$. By Theorem 4.3.16(c), we can choose an automorphism σ of G which preserves K , centralizes MA , and takes μ to μ' . In its action on representations of G , σ will obviously fix each principal series; so it must take $\bar{\pi}(\delta_i^1 \otimes \nu^1)$ to $\bar{\pi}(\delta_S^1 \otimes \nu^1)$, and similarly, for $\bar{\delta}$. This proves the claim.

Let Y_δ and $Y_{\bar{\delta}}$ be the spaces of K -finite vectors in $\bar{\pi}(\delta \otimes \nu)$ and $\bar{\pi}(\bar{\delta} \otimes \nu)$, respectively. We want to show that Y_δ^1 and $Y_{\bar{\delta}}^1$ are infinitesimally equivalent. To do this, we

will show how to recover them from Y_δ and $Y_{\bar{\delta}}$ (which we have just shown are equivalent). Define (in the notation of Lemma 4.2.15)

$$(Y_\delta)_0 = \{v \in Y_\delta \mid \lim_{t \rightarrow \infty} a_t^{\nu+\rho} \pi(g) \pi(a_t^{-1}) v = 0\}$$

for all K-finite $g \in C^\infty(K)$.

Notice that the limit takes place within the finite dimensional space

$$\pi(g) \cdot Y_\delta,$$

and so does not depend on the topology. Furthermore, this definition depends only on the $(\mathfrak{g}, K)$ module structure of Y_δ , as can be seen by expanding the analytic function

$$t \mapsto a_t^{\nu+\rho} \pi(g) \pi(a_t^{-1}) v$$

in a power series. So

$$(4.4.4) \quad (Y_\delta)_0 \cong (Y_{\bar{\delta}})_0.$$

Set

$$(Y_\delta)_1 = \{v \in Y_\delta \mid \text{for all } u \in U(\mathfrak{m}^1), u \cdot v \in (Y_\delta)_0\}.$$

We claim that

$$(4.4.5) \quad (Y_\delta)_1 \text{ consists precisely of those } v \in Y_\delta \text{ which vanish on } M^1.$$

Suppose first that v vanishes on M^1 ; then, of course, $u \cdot v$ does as well, for all $u \in U(\mathfrak{m}^1)$. So we only have to show that v belongs to $(Y_\delta)_0$. But Lemma 4.2.15 computes the limit in the definition of $(Y_\delta)_0$, in terms of $v|_{K^0}$. This shows that the limit is zero, proving the claim. For the converse, suppose v does not vanish on M^1 . Then $v|_{M^1}$ defines a non-zero

vector in the representation space of δ^1 . By Theorem 4.3.17, we can find $u \in U(\mathfrak{m}^1)$ so that $0 \neq u \cdot v|_{M^1}$ transforms according to fine representations of K^0 . By Lemma 4.2.16, there is a K-finite function $g \in C^\infty(K)$ such that

$$h(k) = \int_{K^0} g(kk_0) (u \cdot v)(k_0) dk_0$$

is non-zero, and transforms according to fine representations of K . By Proposition 4.4.1,

$$A(P^1, \bar{P}^1)h \neq 0.$$

By Lemma 4.2.15, it follows that

$$u \cdot v \notin (Y_\delta)_0,$$

so v does not belong to $(Y_\delta)_1$. This proves (4.4.5).

By (4.4.5), and inspection of the definition of principal series,

$$Y_\delta / (Y_\delta)_1 \cong X_M^1(\delta \otimes v|_{A^0})$$

as an $(\mathfrak{m}^1, K^0)$ module. But all of this works for $\bar{\delta}$ as well; so (4.4.4) implies that

$$X_M^1(\delta \otimes v|_{A^0}) \cong X_M^1(\bar{\delta} \otimes v|_{A^0})$$

as $(\mathfrak{m}^1, K^0)$ modules. (This says precisely that δ^1 and $\bar{\delta}^1$ are infinitesimally equivalent.) By Theorem 4.1.4, there is an element $w \in W(M^1, A^0)$ such that

$$w \cdot (v|_{A^0}) = v|_{A^0}, \quad w \cdot \delta = \bar{\delta}.$$

Since all of $W(M^1, A^0)$ fixes v^1 , we deduce that

$$w \cdot v = v, \quad w \cdot \delta = \bar{\delta}.$$

Q.E.D.

Definition 4.4.6 Suppose G is quasisplit, $\nu \in \hat{A}$, and $\delta \in \hat{M}$ is fine. If $\mu \in A(\delta)$, define

$$\bar{\pi}(\delta \otimes \nu)(\mu)$$

to be the unique irreducible subquotient of $\pi(\delta \otimes \nu)$ which contains the K -type μ .

Since μ occurs exactly once in $\pi(\delta \otimes \nu)$, this is well-defined; and Theorem 4.1.4 shows that it is independent of whatever minimal parabolic subgroup containing MA is used to define the induced representation (for all such subgroups are conjugate under W). If we choose such a parabolic P' so that ν is negative, then it makes sense to talk about the Langlands subrepresentations $\bar{\pi}(\delta_i^1 \otimes \nu^1)$ of $\pi(\delta \otimes \nu)$. Proposition 4.4.1 shows that

$$(4.4.7) \quad \{\bar{\pi}(\delta \otimes \nu)(\mu) \mid \mu \in A(\delta)\} = \{\bar{\pi}(\delta_i^1 \otimes \nu^1) \mid 1 \leq i \leq r\}.$$

Notice, however, that there may be repetitions on the left; two distinct $\mu, \mu' \in A(\delta)$ may lie in the same irreducible subquotient of $\pi(\delta \otimes \nu)$.

Theorem 4.4.8 Suppose G is quasisplit, and $\mu \in \hat{K}$ is a fine representation of K ; say $\mu \in A(\delta)$, for some fine $\delta \in \hat{M}$.

Then every irreducible $(\mathfrak{g}, K)$ module containing the K type μ is equivalent to

$$\bar{\pi}(\delta \otimes \nu)(\mu)$$

for some $\nu \in \hat{A}$. (In particular, μ must occur with multiplicity one.)

Let $W_\delta \subseteq W$ be the stabilizer of δ ; then

$$\bar{\pi}(\delta \otimes \nu)(\mu) \cong \bar{\pi}(\delta \otimes \nu')(\mu)$$

if and only if $\nu = w\nu'$ for some $w \in W_\delta$.

Proof. Fix an irreducible $(\mathfrak{g}, K)$ module X containing the K -type μ . By Harish-Chandra's subquotient theorem (Theorem 4.1.9), we can find $\delta' \in \hat{M}$, $\nu' \in \hat{A}$ so that X is a subquotient of $\pi(\delta' \otimes \nu')$. By Lemma 4.1.5, δ' must occur in $\mu|_M$; so by Theorem 4.3.16(b), there is a $w \in W$ such that

$$\delta = w\delta'.$$

Define $\nu = w \cdot \nu'$. By Theorem 4.1.4, X is also a subquotient of

$$\pi((w \cdot \delta') \otimes (w \cdot \nu')) = \pi(\delta \otimes \nu).$$

So by Definition 4.4.6, X must be $\bar{\pi}(\delta \otimes \nu)(\mu)$. The inequality statement follows from Theorem 4.4.3 and (4.4.7). Q.E.D.

Theorem 4.4.8 does not quite constitute a classification of the irreducible representations of G containing a fine K -type; for such a representation might contain several fine K -types. To describe this phenomenon, we need the R -group.

Definition 4.4.9 Suppose G is quasisplit, $\delta \in \hat{M}$ is fine, and $\nu \in \hat{A}$. With notation as in (4.3.13), put

$$W(\nu) = \{w \in W \mid w\nu = \nu\}$$

$$W_\delta(\nu) = W(\nu) \cap W_\delta = \text{stabilizer of } \delta \otimes \nu \text{ in } W$$

$$W_\delta^0(\nu) = W(\nu) \cap W_\delta^0$$

$$R_\delta(\nu) = W_\delta(\nu)/W_\delta^0(\nu) \subseteq R_\delta$$

$$R_\delta^1(\nu) = \text{annihilator of } R_\delta(\nu) \subseteq \hat{R}_\delta.$$

Theorem 4.4.10 Suppose G is quasipplit, $\delta \in \hat{M}$ is fine, and $\nu \in \hat{A}$ is negative. Two fine K -types μ and μ' in $A(\delta)$ (Definition 4.3.15) occur in the same Langlands subrepresentation of $\pi(\delta \otimes \nu)$ if and only if they lie in the same orbit of $R_\delta^\perp(\nu)$ (Definitions 4.3.47 and 4.4.9).

Before beginning the proof, we note a corollary. The original proof of this corollary involved much more analysis than was involved in Theorem 4.4.10 (but, to be fair, much less algebra).

Corollary 4.4.11 (Knapp-Stein [26]) Suppose G is quasipplit, $\delta \in \hat{M}$ is fine, and $\nu \in \hat{A}$ is unitary. Then $\pi(\delta \otimes \nu)$ is the sum of $|R_\delta(\nu)|$ inequivalent irreducible representations.

Here we have used the fact that $\hat{R}_\delta$ acts simply transitively on $A(\delta)$ (Theorem 4.3.16), and the fact that

$$|\hat{R}_\delta/R_\delta^\perp(\nu)| = |R_\delta(\nu)|.$$

We turn now to the proof of the theorem. This relies on a careful use of induction by stages (Proposition 4.1.18), so we begin by investigating how the R -group behaves under that process.

Lemma 4.4.12 Suppose $P^2 = M^2 A^2 N^2$ is a parabolic subgroup containing P . Use a superscript 2 to the left to refer to M^2 ; thus ${}^2\bar{\Lambda}_\delta$ is the set of good roots for δ in M^2 (Definition 4.3.11), ${}^2A(\delta)$ is the set of fine representations of $M^2 \cap K$ containing δ , and so on.

- a) ${}^2W_\delta^0 = W_\delta^0 \cap W(M^2)$
 b) There is a natural inclusion

$${}^2R_\delta \hookrightarrow R_\delta$$

and a natural quotient map

$$\hat{R}_\delta \rightarrow {}^2\hat{R}_\delta$$

- c) There is a natural surjection

$$A(\delta) \rightarrow {}^2A(\delta)$$

defined by $\mu \mapsto \mu^2$ when μ^2 occurs in $\mu|_{M^2 \cap K}$. This is compatible with the actions of $\hat{R}_\delta$ and ${}^2\hat{R}_\delta$ (Definition 4.3.47) and the map of (b).

Proof. The group $M^2 A^2$ is the centralizer of A^2 in G ; so by Chevalley's theorem,

$$\begin{aligned} W_\delta^0 \cap W(M^2) &= \{w \in W(\bar{\Lambda}_\delta) \mid w|_{A^2} \text{ is trivial}\} \\ &= W(\{\alpha \in \bar{\Lambda}_\delta \mid \alpha|_{A^2} = 0\}) \\ &= W(\bar{\Lambda}_\delta \cap \Delta(M^2)) \\ &= W({}^2\bar{\Lambda}_\delta) \\ &= {}^2W_\delta^0. \end{aligned}$$

This proves (a). Then (b) is an immediate consequence of Definition 4.3.13. For (c), suppose $\mu \in A(\delta)$. Since δ occurs exactly once in μ , there is a unique $M^2 \cap K$ -type μ^2 which contains δ and occurs in μ . Since μ is fine for G , μ^2 is fine for M^2 . In the notation of Corollary 4.3.33, this map has the property that

$$\mu_\delta|_{{}^2M_\delta^0} = \mu_\delta^2.$$

The compatibility with (b) now follows from Definition 4.3.38 and (4.3.48). Since the R -groups act simply transitively, the surjectivity in (c) implies surjectivity in (a) as well.

Q.E.D.

Corollary 4.4.13 In the setting of Theorem 4.4.10, use the notation of Lemma 4.4.12 for the parabolic subgroup P^1 . Suppose $\mu, \mu' \in A(\delta)$ lie in the same orbit of the annihilator $\text{Ann}(^1R_\delta)$ of $^1R_\delta$ in $\hat{R}_\delta$. Then μ and μ' occur in the same Langlands subrepresentation of $\pi(\delta \otimes \nu)$.

Proof. Recall the notation (4.2.2), (4.2.7)

$$\pi_{M^1}(\delta \otimes \nu^0) = \bigoplus_{i=1}^r \delta_i^1 = \delta^1;$$

say μ occurs in $\pi(\delta_i^1 \otimes \nu^1)$. By Proposition 4.4.1, we need only show that μ' does as well. Define μ^1 as in 4.4.12(c). By (4.1.15), μ^1 must occur in δ_i^1 . Write $\mu' = r \cdot \mu$, for some $r \in \hat{R}_\delta$. By hypothesis, r annihilates $^1R_\delta$, and so its image $\bar{r}$ is the trivial element of $^1\hat{R}_\delta$. By 4.1.12(c),

$$\begin{aligned} (\mu')^1 &= (r \cdot \mu)^1 \\ &= \bar{r} \cdot \mu^1 \\ &= \mu^1. \end{aligned}$$

So μ' contains μ^1 , and hence (by (4.1.15) again) μ' occurs in $\pi(\delta_i^1 \otimes \nu^1)$.

Q.E.D.

This result falls short of the "if" part of the theorem, since $^1R_\delta$ can be larger than $R_\delta(\nu)$; but we will later reduce to the case when they are equal.

For the "only if" part, we begin by modifying Δ^+ (and hence, P) by an element of $W(M^1)$, in such a way that $\text{Im}(\nu^0)$ becomes dominant for $\Delta^+(\mathcal{M}^1)$; this does not affect the hypotheses or conclusions of the theorem. We can then define

$$P^2 = M^2 A N^2, \quad P \subseteq P^2 \subseteq P^1$$

by

$$(4.4.14) \quad \begin{cases} \Delta(\mathcal{M}^2) = \{\alpha \in \Delta \mid \langle \alpha, \nu \rangle = 0\} \\ = \{\alpha \in \Delta(\mathcal{M}^1) \mid \langle \alpha, \nu^0 \rangle = 0\}. \end{cases}$$

(Here we have to recall the definition of $\mathcal{M}^1$, and the fact that ν^0 is purely imaginary.) By Chevalley's theorem,

$$(4.4.15) \quad W(\nu) = W(M^2).$$

In the notation of Lemma 4.4.12, therefore,

$$(4.4.16) \quad \begin{aligned} R_\delta(\nu) &= {}^2R_\delta. \\ R_\delta^\perp(\nu) &= \ker(\hat{R}_\delta \rightarrow {}^2\hat{R}_\delta) \end{aligned}$$

Lemma 4.4.17 $\pi_{M^2 A}(\delta \otimes \nu)$ is a sum of $|R_\delta(\nu)|$ distinct irreducible representations. In particular, any two distinct $M^2 \cap K$ -types in ${}^2A(\delta)$ (notation (4.4.12)), lie in distinct summands.

Proof. Since $P^2 \subseteq P^1$, ν is unitary on $M^2 \cap A$; so the representation is a direct sum of inequivalent pieces, each containing a fine $M^2 \cap K$ type. By (4.4.16) and Theorem 4.3.16(c),

$$|{}^2A(\delta)| = |R_\delta(\nu)|.$$

So it is enough to prove that the representation splits into at least $|R_\delta(v)|$ pieces. To see this, define P_δ for the group M^2 by Lemma 4.3.31. By that lemma, the R-group for M_δ is isomorphic to ${}^2R_\delta = R_\delta(v)$. By induction by stages, it is enough to show that

$$\pi_{M_\delta}(\delta \otimes v|_{A \cap M_\delta})$$

splits into $|R_\delta(v)|$ pieces. By the definition of M^2 , v is trivial on $A \cap M_\delta$ (or even on $A \cap M^2$); so the desired result follows from Proposition 1.5.9. Q.E.D.

Thus we have found all the reducibility we need just by studying $SL(2, \mathbb{R})$.

Proof of Theorem 4.4.10. We begin with the "only if". So suppose μ, μ' are in $A(\delta)$, and lie in the same Langlands subrepresentation. In the notation of Lemma 4.4.12, induction by stages implies that μ^2 and $(\mu')^2$ lie in the same irreducible component of $\pi_{M^2 A^2}(\delta \otimes v)$. By Lemma 4.4.17, this means

$$\mu^2 = (\mu')^2.$$

Write $\mu' = r \cdot \mu$, for some $r \in \hat{R}_\delta$. By Lemma 4.4.12(c), it follows that the image of r in ${}^2\hat{R}_\delta$ is trivial. By (4.4.16), $r \in R_\delta^\perp(v)$, which is what we wanted to show.

For the "if" part, set

$$\begin{aligned} \rho^2 &= \rho(\pi^2) \\ &= \frac{1}{2} \sum_{\alpha \in \Delta} (\pi^2, \alpha) \alpha. \end{aligned} \tag{4.4.18}$$

By Corollary 3.2.5, we have for $\alpha \in \Delta^+$,

$$\langle \alpha, \rho^2 \rangle \geq 0, \tag{4.4.19}$$

with equality if and only if $\alpha \in \Delta(\pi^2)$. Therefore $v - 4\rho^2$ is negative (cf. 4.2.1).

$$\{\alpha \in \Delta^+ | \langle \alpha, \text{Re}(v - 4\rho^2) \rangle = 0\}$$

$$= \{\alpha \in \Delta^+ | \langle \alpha, \text{Re}(v) \rangle = 0 \text{ and}$$

$$\langle \alpha, 4\rho^2 \rangle = 0\} \tag{4.4.20}$$

$$\begin{aligned} &= \{\alpha \in \Delta^+ (\pi^2) | \langle \alpha, 4\rho^2 \rangle = 0\} \\ &= \Delta^+ (\pi^2). \end{aligned}$$

So the parabolic subgroup defined by $-\text{Re}(v - 4\rho^2)$ is P^2 . By Corollary 4.4.13, $\pi(\delta \otimes (v - 4\rho^2))$ has at most $|{}^2R_\delta|$ Langlands subrepresentations; and by (4.4.16), this number is $|R_\delta(v)|$.

By Proposition 4.2.26

$$(4.4.21) \quad \dim \text{Hom}_{\mathfrak{g}, K}(\bar{X}_{\delta \otimes (v-4\rho^2)}^{\bar{P}}, X_{\delta \otimes (v-4\rho^2)}^P) \leq |R_\delta(v)|.$$

But now we can invoke the Jantzen-Zuckerman translation principle, in the following guise.

Lemma 4.4.22 In the setting just described,

$$\text{Hom}_{\mathfrak{g}, K}(\bar{X}_{\delta \otimes (v-4\rho^2)}^{\bar{P}}, X_{\delta \otimes (v-4\rho^2)}^P) \cong \text{Hom}_{\mathfrak{g}, K}(\bar{X}_{\delta \otimes v}^{\bar{P}}, X_{\delta \otimes v}^P).$$

This will be proved in the next section. Applying it with (4.4.21) and Proposition 4.2.26, we find that $\pi(\delta \otimes v)$ has at most $|R_\delta(v)|$ Langlands subrepresentations. In combination with the "only if" part, which we have already proved, this implies the theorem. Q.E.D.

§5. An introduction to the translation principle.

The existence of finite dimensional representations of G imposes some structure on the infinite dimensional representations. This will be explored in depth in Chapter 7, but (to avoid the appearance of circular arguments) we will describe enough of the theory here to prove Lemma 4.4.22. In its general form the theory is due to Jantzen and Zuckerman ([23], [54]); applications like Lemma 4.4.22 were first made by Wallach (see [47]).

Definition 4.5.1 A $\mathcal{O}$ module X is called strongly $Z(\mathcal{O})$ -finite if for every finite dimensional representation F of G , $X \otimes F$ is $Z(\mathcal{O})$ -finite (Definition 0.3.18). We write $\mathcal{M}_f(\mathcal{O})$ for the category of strongly $Z(\mathcal{O})$ -finite $\mathcal{O}$ modules. Similarly, we write $\mathcal{M}_f(\mathcal{O}, K)$ for the category of strongly $Z(\mathcal{O})$ -finite $(\mathcal{O}, K)$ modules.

Kostant has shown in [30] that strongly $Z(\mathcal{O})$ -finite is the same as $Z(\mathcal{O})$ -finite; but this fact would not simplify the exposition much, so we will not stop to prove it now (see Corollary 7.1.13). Our first goal is to prove that principal series representations are strongly $Z(\mathcal{O})$ -finite.

Lemma 4.5.2 In the setting of Definition 4.1.11(b), let F be a finite dimensional representation of G . Then

a) there is a natural isomorphism

$$\pi(\gamma) \otimes F \cong \pi(\gamma \otimes F).$$

b) Choose a p^l -stable filtration

$\{0\} = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = F$
of F . Then there is a G -stable filtration of $\pi(\gamma) \otimes F$, whose subquotients are isomorphic to

$$\pi(\gamma \otimes F_i / F_{i-1}).$$

Proof. Recall that $\mathcal{H}_\gamma$ is a space of functions

$$\mathcal{H}_\gamma = \{f: G \rightarrow V_\gamma \mid \text{certain conditions hold}\}.$$

The map of part (a) is defined by

$$f \otimes v \mapsto h, \quad h(g) = f(g) \otimes (g^{-1} \cdot v) \quad (f \in \mathcal{H}_\gamma, v \in F).$$

Clearly h is a map from G to $V_\gamma \otimes F$, and it is straightforward to verify that h actually belongs to $\mathcal{H}_{\gamma \otimes F}$. To see that this correspondence is an isomorphism of Hilbert spaces, one can use the realization (4.1.13) and the corresponding fact about unitarily induced representations of compact groups. That the correspondence is an intertwining operator is formal.

This proves (a). Part (b) is an immediate consequence. Q.E.D.

Notation 4.5.3 Choose a Cartan subalgebra $\mathcal{L}_0^S$ of $\mathfrak{m}_0$, and set

$$h_0^S = \mathcal{L}_0^S + \sigma_{\mathcal{L}_0}$$

$T^S =$ centralizer of $\mathcal{L}_0^S$ in M

$$H^S = T^S A.$$

Here s stands for split; this is a θ -stable maximally split Cartan subgroup of G (Definition 0.4.1). In later chapters, when $\sigma_{\mathcal{L}_0}$ will not necessarily be maximal abelian in $\mathfrak{p}_0$, we will write A^S instead of A ; so

$$H^S = T_A^{S,S}.$$

We also fix a positive root system

$$\Delta^+(m, t^S),$$

and write

$$\rho(m) = \rho(\Delta^+(m, t^S)).$$

Thus if we put

$$\Delta^+(q, h^S) = \Delta^+(m, t^S) \cup \Delta(m, h^S),$$

we get

$$\rho^S = \rho(\Delta^+(q, h^S)) = (\rho(m), \rho) \in (t^S + a^S)^*.$$

Lemma 4.5.4 A principal series (q, K) module is

strongly $Z(q)$ -finite. More precisely, suppose $\delta \in \hat{M}$, $\nu \in \hat{A}$. Let

$$\Lambda_0 \in \hat{T}^S$$

denote the highest weight of δ (Definition 0.4.3). Let

λ_0 be the differential of Λ_0 , and

$$\lambda = \lambda_0 + \rho(m) \quad (\text{notation 4.5.3})$$

Then every infinitesimal character occurring in $\pi(\delta \otimes \nu)$

$\otimes F$ is of the form (notation 0.2.9):

$$(\lambda, \nu) + \gamma, \text{ some } \gamma \in \Delta(F, h^S).$$

Proof. Fix a finite dimensional representation F of G . We can choose a P -stable filtration of F so that the subquotients F_i/F_{i-1} are irreducible representations of MA . If we apply Lemma 4.5.2(b), this implies that

(4.5.5) $\pi(\delta \otimes \nu) \otimes F$ has a filtration whose subquotients

are again principal series representations $\pi(\delta_i \otimes \nu_i)$.

Here $\delta_i \otimes \nu_i$ occurs exactly as often as it does in

$$(\delta \otimes \nu) \otimes (F|_{MA}).$$

In particular, $\pi(\delta \otimes \nu) \otimes F$ is $Z(q)$ -finite, proving the first claim. For the second, we have to get some information about a typical $\delta_i \otimes \nu_i$. By Proposition 3.2.12, $\delta_i \otimes \nu_i$ must have highest weight

$$(\lambda_0, \nu) + \gamma$$

for some weight γ of F . By Lemma 4.1.8, $\pi(\delta_i \otimes \nu_i)$ has infinitesimal character

$$[(\lambda_0, \nu) + \gamma] + \rho(m) = (\lambda, \nu) + \gamma.$$

Q.E.D.

Corollary 4.5.6 Any irreducible (q, K) module X is

strongly $Z(q)$ -finite. More precisely, if X has

infinitesimal character $\lambda \in \mathcal{H}^*$, for some Cartan sub-

algebra $\mathcal{H}$ of q , then every infinitesimal character

occurring in $X \otimes F$ is of the form

$$\lambda + \gamma, \gamma \in \Delta(F, \mathcal{H}).$$

Proof. This follows from Lemma 4.5.4 and the subquotient theorem (Theorem 4.1.9). Q.E.D.

We are ready now to define the translation functors.

The notation is complicated by the fact that we want to keep track of weights in $\mathcal{H}^*$ determining infinitesimal characters, and weights of H determining finite dimensional representations.

The first kind of weight need not exponentiate to H , and the second kind is not determined by its differential.

Definition 4.5.7 Let H^S be as in (4.5.3). Write $\Lambda^S \subseteq (\hat{H}^S)$ for the lattice of weights of finite dimensional representations of G (Definition 0.4.11). If $\lambda \in \Lambda^S$, let F_{λ} denote the finite dimensional representation of G of extremal weight λ (Proposition 0.4.12); and write

$$\bar{\lambda} \in (\mathcal{H}^S)^*$$

for the differential of λ . Suppose $\lambda_i \in \Lambda$ ($i = 1, 2$) and $\xi \in \mathcal{H}^*$. The Jantzen-Zuckerman translation functor $\psi_{\xi+\bar{\lambda}_1}^{\xi+\bar{\lambda}_2}$ is defined on $\mathcal{M}_f(\mathcal{G})$ or $\mathcal{M}_f(\mathcal{G}, K)$ (Definition 4.5.1) by

$$\psi_{\xi+\bar{\lambda}_2}^{\xi+\bar{\lambda}_1}(X) = P_{\xi+\bar{\lambda}_2}^{(F)} \lambda_2^{-\lambda_1} \otimes (P_{\xi+\bar{\lambda}_1}^{(F)} X) \quad (\text{notation 0.3.18}).$$

In this definition, $\xi + \lambda_2$ is only a formal symbol, since ξ and λ_2 do not lie in the same group.

Proposition 4.5.8 (Zuckerman [54]). In the setting of Definition 4.5.7, $\psi_{\xi+\bar{\lambda}_2}^{\xi+\bar{\lambda}_1}$ is an exact covariant functor. If X and Y lie in $\mathcal{M}_f(\mathcal{G}, K)$, then there is a natural isomorphism

$$\text{Hom}_{\mathcal{G}, K}(X, \psi_{\xi+\bar{\lambda}_2}^{\xi+\bar{\lambda}_1} Y) \cong \text{Hom}_{\mathcal{G}, K}(\psi_{\xi+\bar{\lambda}_1}^{\xi+\bar{\lambda}_2} X, Y).$$

(The same adjointness holds in $\mathcal{M}_f(\mathcal{G})$ and similar categories; we will use this as required later on, since the proof is

identical.) The proof is based on the following well-known fact, which we will use again later.

Lemma 4.5.9 Suppose X and Y are $(\mathcal{G}, K)$ modules, and F is a finite dimensional representation of G . Then there is a natural isomorphism

$$\text{Hom}_{\mathcal{G}, K}(X, Y \otimes F) \cong \text{Hom}_{\mathcal{G}, K}(X \otimes F^*, Y)$$

Proof. Suppose f is in the Hom on the left. Define $\tilde{f}$ on the right as follows: if

$$f(x) = \sum Y_i \otimes v_i \quad (Y_i \in Y, v_i \in F)$$

and $v^* \in F^*$, then

$$\tilde{f}(x \otimes v^*) = \sum v^*(v_i) Y_i$$

It is straightforward to check that this is a well-defined map of $X \otimes F^*$ into Y . If $g \in \mathcal{G}$, then

$$g \cdot [\tilde{f}(x \otimes v^*)] = \sum v^*(v_i) g \cdot Y_i.$$

On the other hand, since f is a $\mathcal{G}$ module map

$$\begin{aligned} f(g \cdot x) &= g \cdot (f(x)) \\ &= \sum (g \cdot Y_i \otimes v_i + Y_i \otimes g \cdot v_i). \end{aligned}$$

So by definition,

$$\begin{aligned} \tilde{f}(g \cdot (x \otimes v^*)) &= \tilde{f}((g \cdot x) \otimes v^*) + \tilde{f}(x \otimes g \cdot v^*) \\ &= \sum (v^*(v_i) (g \cdot Y_i) + v^*(g \cdot v_i) Y_i) \\ &\quad + \sum (g \cdot v^*) (v_i) Y_i. \end{aligned}$$

By the definition of the contragredient representation, this is

$$\begin{aligned}
 &= \Sigma(v^*(v_1)(g \cdot Y_1) - (g \cdot v^*)(v_1)Y_1) \\
 &\quad + \Sigma(g \cdot v^*)(v_1)Y_1 \\
 &= \Sigma v^*(v_1)g \cdot Y_1.
 \end{aligned}$$

This calculation shows that

$$g \cdot [\tilde{f}(x \otimes v^*)] = \tilde{f}(g \cdot (x \otimes v^*)),$$

so that $\tilde{f}$ lies in $\text{Hom}_{\mathcal{O}}(X \otimes F^*, Y)$; that $\tilde{f}$ is a K -module map is proved in the same way. The map $f \rightarrow \tilde{f}$ has an inverse: if $\tilde{f}$ is on the right side in the lemma, we define $\tilde{f}$ on the left side as follows. Fix a basis $\{v_i^j\}$ of F , and let $\{v_i^j\}$ denote the dual basis of F^* . If

$$\tilde{f}(x \otimes v_i^j) = Y_i,$$

then we set

$$\tilde{f}(x) = \Sigma Y_i \otimes v_i.$$

It is straightforward to check that f actually lies in

$\text{Hom}_{\mathcal{O}, K}(X, Y \otimes F)$; $\tilde{f} \rightarrow \tilde{f}$ is more or less obviously the inverse of $f \rightarrow \tilde{f}$.

Q.E.D.

Proof of Proposition 4.5.8. By definition, $\psi_{\xi+\lambda_2}^{\xi+\lambda_1}(Y)$ has generalized infinitesimal character $\xi + \bar{\lambda}_2$. So by Proposition 0.3.19, restriction to $P_{\xi+\bar{\lambda}_2}(X)$ defines an isomorphism

$$\text{Hom}_{\mathcal{O}, K}(X, \psi_{\xi+\lambda_2}^{\xi+\lambda_1} Y) \cong \text{Hom}_{\mathcal{O}, K}(P_{\xi+\bar{\lambda}_2} X, \psi_{\xi+\lambda_2}^{\xi+\lambda_1} Y).$$

Exactly the same argument in reverse allows us to remove

$P_{\xi+\bar{\lambda}_2}$ from the second argument on the right, giving

$$\cong \text{Hom}_{\mathcal{O}, K}(P_{\xi+\bar{\lambda}_2} X, F_{\lambda_2-\lambda_1} \otimes (P_{\xi+\bar{\lambda}_1} Y)).$$

The contragredient of $F_{\lambda_2-\lambda_1}$ obviously contains $\lambda_1 - \lambda_2$ as an extremal weight; so by Lemma 4.5.9,

$$\cong \text{Hom}_{\mathcal{O}, K}(F_{\lambda_1-\lambda_2} \otimes (P_{\xi+\bar{\lambda}_2} X), P_{\xi+\bar{\lambda}_1} Y).$$

Now reversing the first two steps gives

$$\cong \text{Hom}_{\mathcal{O}, K}(\psi_{\xi+\lambda_1}^{\xi+\lambda_2} X, Y),$$

as we wished to show.

Q.E.D.

We will not carry the general theory any further now, although the proof of Lemma 4.4.22 uses some ideas which will reappear later. Assume now that G is quasisplit, so that (in the notation of (4.5.3)),

$$H^S = \text{MA}, \quad \rho(\mathfrak{m}) = 0.$$

Put

$$\begin{aligned}
 \xi &= \text{differential of } \delta \otimes v \in (\mathfrak{h}^S)^* \\
 \lambda &= \text{weight of MA on } (\wedge^{\dim \mathfrak{n}_2} \mathfrak{n}^2) \quad (\text{cf. 4.4.14}) \\
 &= \text{determinant character of MA on } \mathfrak{n}^2.
 \end{aligned}$$

Lemma 4.5.11 The weight λ just defined lies in the weight lattice (even the root lattice) of $(\text{MA})^\wedge$ (Definition 0.4.11). The restriction of λ to $\mathfrak{M}$ takes values in $\{\pm 1\}$, and 2λ is trivial on $\mathfrak{M}$. Finally

$$\bar{\lambda} = 2\rho^2 \in \mathfrak{a}^* \quad (\text{notation (4.4.18)}, (4.5.7)).$$

Proof. Obviously λ is the sum of the roots of MA in $\mathcal{N}^2$, which proves the first claim. For the second, write

$$\lambda(x) = \det(\text{Ad}(x)|_{\mathcal{N}_0^2}) \quad (x \in \text{MA}).$$

This shows that λ is real valued; since M is compact,

$$\lambda(M) \subseteq \{\pm 1\}.$$

The statement about 2λ follows. The last claim is obvious.

Q.E.D.

Proof of Lemma 4.4.22. We will show that, in the notation of (4.5.10),

$$(4.5.12) \quad \begin{aligned} (a) \quad & \psi_{\xi-2\lambda}^{\xi}(X_{\delta\emptyset\nu}^P) \cong X_{\delta\emptyset(\nu-4\rho^2)}^P \\ (b) \quad & \psi_{\xi}^{\xi-2\lambda}(X_{\delta\emptyset(\nu-4\rho^2)}^{\bar{P}}) \cong X_{\delta\emptyset\nu}^{\bar{P}}. \end{aligned}$$

This makes Lemma 4.4.22 a special case of Proposition 4.5.8. Consider first (4.5.12) (a). Since $X_{\delta\emptyset\nu}^P$ already has infinitesimal character ξ (by (4.5.10) and Lemma 4.1.8), the term P in the definition of the translation functor has no effect.

So

$$\psi_{\xi-2\lambda}^{\xi}(X_{\delta\emptyset\nu}^P) = P_{\xi-2\lambda}(X_{\delta\emptyset\nu}^P \otimes F_{-2\lambda}).$$

By (4.5.5), the right side has a filtration whose subquotients are of the form

$$W_{\gamma} = P_{\xi-2\lambda}(X_{\delta\emptyset\nu}^P)^{\gamma} \quad \gamma \in \Delta(F_{-2\lambda}).$$

such a term occurs as often as γ is a weight of $F_{-2\lambda}$. One of these terms corresponds to $\gamma = -2\lambda$; this is

$$W_{-2\lambda} = P_{\xi-2\lambda}(X_{\delta\emptyset\nu}^P)^{-2\lambda}.$$

By (4.5.10) and Lemma 4.1.8, the principal series representation has infinitesimal character $\xi - 2\bar{\lambda}$, so the $P_{\xi-2\bar{\lambda}}$ has no effect. By Lemma 4.5.11, -2λ is trivial on M and $-4\rho^2$ on A, so

$$W_{-2\lambda} \cong X_{\delta\emptyset(\nu-4\rho^2)}^P.$$

So (4.5.12) (a) will follow if we can show that all the other subquotients in the filtration are zero; that is, if we can prove

$$(4.5.13) \quad \begin{cases} \text{if } \gamma \in \Delta(F_{-2\lambda}), \text{ and } X_{\delta\emptyset\nu}^P \text{ has} \\ \text{infinitesimal character } \xi - 2\bar{\lambda}, \text{ then} \\ \gamma = -2\lambda. \end{cases}$$

So suppose we have such a γ ; by (4.5.10), Lemma 4.1.8 and Corollary 0.2.10, this means that there is a $w \in W(g, h^s)$ such that

$$\xi + \bar{\gamma} = w(\xi - 2\bar{\lambda}).$$

By (4.4.19), 2λ is dominant; so by Proposition 0.4.9 (b), there are non-negative integers n_{α} such that

$$(4.5.14) \quad \gamma = -2\lambda + \sum_{\alpha \in \Delta^+} (g, h^s)_{\alpha} n_{\alpha}.$$

Combining the last two equations, we compute

$$\begin{aligned} 0 &= \langle \xi - 2\bar{\lambda}, \xi - 2\bar{\lambda} \rangle - \langle w(\xi - 2\bar{\lambda}), w(\xi - 2\bar{\lambda}) \rangle \\ &= \langle \xi - 2\bar{\lambda}, \xi - 2\bar{\lambda} \rangle - \langle \xi + \bar{\gamma}, \xi + \bar{\gamma} \rangle \\ &= \langle \xi - 2\bar{\lambda}, \xi - 2\bar{\lambda} \rangle - \langle \xi - 2\bar{\lambda} + \sum n_{\alpha} \alpha, \xi - 2\bar{\lambda} + \sum n_{\alpha} \alpha \rangle \\ &= \langle \xi, \xi \rangle - 4\langle \bar{\lambda}, \xi \rangle + \langle 2\bar{\lambda}, 2\bar{\lambda} \rangle - \langle \xi, \xi \rangle + 2\langle \xi, 2\bar{\lambda} - \sum n_{\alpha} \alpha \rangle \\ &\quad - \langle \bar{\gamma}, \bar{\gamma} \rangle \\ &= 2\langle -\xi, \sum n_{\alpha} \alpha \rangle + \langle 2\bar{\lambda}, 2\bar{\lambda} \rangle - \langle \bar{\gamma}, \bar{\gamma} \rangle. \end{aligned}$$

Since δ is fine, its differential is zero on all roots. So

$$0 = 2\langle -\nu, \sum n_\alpha \alpha \rangle + \langle 2\bar{\lambda}, 2\bar{\lambda} \rangle - \langle \bar{\gamma}, \bar{\gamma} \rangle.$$

The second term is non-negative by Proposition 0.4.3(b), and the first has non-negative real part since ν is negative.

So both terms must be zero. Considering only the real part of the first, we see that $\alpha \in \Delta^+(\mathcal{M}^1)$ whenever $n_\alpha \neq 0$. But Δ^+ was modified to make the imaginary part of ν positive on $\Delta^+(\mathcal{M}^1)$; so in fact

$$n_\alpha \neq 0 \implies \alpha \in \Delta^+(\mathcal{M}^1).$$

By Corollary 3.2.5, $\bar{\lambda} = 2\rho^2$ is orthogonal to the roots of $\mathcal{M}^2$; so by (4.5.14) again,

$$\begin{aligned} \langle \bar{\gamma}, \bar{\gamma} \rangle &= \langle -2\bar{\lambda} + \sum n_\alpha \alpha, -2\bar{\lambda} + \sum n_\alpha \alpha \rangle \\ &= \langle -2\bar{\lambda}, -2\bar{\lambda} \rangle + \langle \sum n_\alpha \alpha, \sum n_\alpha \alpha \rangle. \end{aligned}$$

By Proposition 0.4.3(b), it follows that

$$\sum n_\alpha \alpha = 0,$$

and thus that $\gamma = -2\lambda$. This proves (4.5.13), and therefore (4.5.12)(a). The proof of (4.5.12)(b) is similar but easier; we leave it to the reader.

Q.E.D.

Chapter 5. Cohomology of $(\mathfrak{g}, K)$ Modules

In the finite dimensional representation theory of reductive Lie algebras, one of the first points is to show that an extremal weight is annihilated by certain elements in the Lie algebra; that is, that it defines a cohomology class of degree zero. For $(\mathfrak{g}, K)$ modules, one can do something similar: an extremal K -type (in an appropriate sense) gives rise to a cohomology class (no longer in degree zero in general). This class will then provide some structural information about the $(\mathfrak{g}, K)$ module. (An argument of this type was given for Lemma 1.2.11.) The present chapter is devoted simply to finding the cohomology class. As usual, we have presented the argument in a logical form: extremal is defined in Section 4, the cohomology class is constructed in Section 3, and the fact that it is non-zero is proved in Section 2. The main results are Corollaries 5.4.7 and 5.4.9. The results are taken from [42].

§1. Representations of K

The material in this section is trivial when K is connected, so the reader may wish to omit it.

Definition 5.1.1 Fix a Cartan subalgebra $\mathcal{L}_0^C$ of $\mathfrak{k}_0$ and a positive root system

$$\Delta^+(\mathfrak{k}, \mathcal{L}_0^C).$$

The centralizer H^C of $\mathcal{L}_0^C$ in G is called a fundamental (or maximally compact - hence the c) Cartan subgroup of G ([50], Lemma 1.3.3.2). Put

$$A^C = \exp(\mathfrak{h}_0^C \cap \mathfrak{p})$$

$T^C = H^C \cap K$ = centralizer of $\mathfrak{x}_0$ in K

(which is abelian by 0.1.2(f)); we call T^C a Cartan subgroup of K . Put

$$\begin{aligned} W(K, T^C) &= (\text{normalizer of } T^C \text{ in } K) / T^C \\ &\cong W(K_0, T_0^C) = W(\mathfrak{k}, \mathfrak{t}^C), \end{aligned}$$

the Weyl group of T^C in K . Suppose X is a K -finite K module, and $\mu \in \hat{T}^C$ or $\mu \in (\mathfrak{t}^C)^*$. The μ -highest weight space of X is

$$X^\mu = \{v \in X \mid v \text{ has weight } \mu, \text{ and } v \text{ is annihilated}$$

by all root vectors for roots in $\Delta^+(\mathfrak{k}, \mathfrak{t}^C)\}$.

An irreducible representation of K is said to have

highest weight μ if its μ -highest weight space is non-zero. Put

$$A(\mu) = \bar{A}^K(\mu) = \{\pi \in \hat{K} \mid \pi \text{ has highest weight } \mu\}.$$

Notice that if K is not connected the highest weight of a K -type is not unique, and does not determine the K -type. For example, if $G = SL^\pm(2, \mathbb{R})$, and n is a positive integer, the representation μ_n of $K = O(2)$ (Lemma 1.4.2) has highest weights n and $-n$; and the two representations μ_0^+ and μ_0^- both have highest weight 0. We want to describe this situation precisely.

The reader will notice a close formal analogy with the R -group theory of Section 4.3. This is not an accident, and seems to be part of a very strong duality in the structure of $(\mathfrak{g}, K)$ modules, analogous to Conjecture 2.1.7(B) for highest weight modules.

Definition 5.1.2 Put

$$\begin{aligned} G^\# &= \{g \in G \mid \text{Ad}(g) \in \text{Ad}(G_0)\} \\ K^\# &= K \cap G^\# \\ R &= R^G = \{w \in W(K, T^C) \mid w(\Delta^+(\mathfrak{k}, \mathfrak{t}^C)) \\ &= \Delta^+(\mathfrak{k}, \mathfrak{t}^C)\}, \end{aligned}$$

the R -group for G . Suppose $\mu \in \hat{T}^C$ is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t}^C)$. Put

$$R_\mu = R_\mu^G = \{w \in R \mid w\mu = \mu\},$$

the R -group for μ .

Lemma 5.1.3 In the setting of Definition 5.1.2, we have

- $T^C \subseteq G^\#, G^\# = T^C G_0, K^\# = T^C K_0$
- $W(K^\#, T^C) \cong W(K_0, T_0^C)$.
- There is a semidirect product decomposition

$$W(K, T^C) \cong R^G \cdot W(K_0, T_0^C),$$

with the second factor normal.

- There are natural isomorphisms

$$G/G^\# \cong K/K^\# \cong R^G.$$

- R^G is a product of copies of $\mathbb{Z}/2\mathbb{Z}$.
- $K^\#$ is a maximal linear reductive subgroup of K (cf. (0.1.2)).

Proof. For (a), we use the fundamental Cartan subalgebra $\mathfrak{h}^C$ of Definition 5.1.1. Fix some ordering of $\mathfrak{t}^C$ compatible with $\Delta^+(\mathfrak{k}, \mathfrak{t})$, and use it to define a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h}^C)$; since no root vanishes on $\mathfrak{t}^C$, this is possible. Clearly $\Delta^+(\mathfrak{g}, \mathfrak{h}^C)$ will be stable by the Cartan involution θ ,

so its set $\{\alpha_i\}$ of simple roots is also. Choose a root vector X_{α_i} for each simple root. Now fix $t \in T^C$. We want to show that $\text{Ad}(t) \in \text{Ad}(G_0)$. Obviously it is enough to find $t_0 \in T_0^C$ such that

$$(5.1.4) \quad \text{Ad}(t) \cdot X_{\alpha_i} = \text{Ad}(t_0) X_{\alpha_i}$$

for all i . Write

$$(5.1.5) (a) \quad \text{Ad}(t) X_{\alpha_i} = \lambda_i X_{\alpha_i};$$

since T^C is compact, λ_i has absolute value one. So we can find a real number x_i such that

$$(5.1.5) (b) \quad \lambda_i = \exp(\sqrt{-1} x_i).$$

Since the simple roots are linearly independent, there is an element z_0 of $\mathcal{H}^C$ such that

$$(5.1.5) (c) \quad \alpha_i(z_0) = \sqrt{-1} x_i.$$

Since $\theta t = t$, if $\theta \alpha_i = \alpha_j$, $\lambda_i = \lambda_j$; so we may assume $x_i = x_j$. It follows that

$$\begin{aligned} \alpha_i(\theta z_0) &= (\theta \alpha_i)(z_0) \\ &= \sqrt{-1} x_j \\ &= \sqrt{-1} x_i. \end{aligned}$$

So

$$\alpha_i\left(\frac{1}{2}(z_0 + \theta z_0)\right) = \sqrt{-1} x_i.$$

Replacing z_0 by $\frac{1}{2}(z_0 + \theta z_0)$, we may therefore assume that $z_0 \in \mathcal{H}^C$. The roots take purely imaginary values on $\mathcal{H}_0^C$ by Lemma 0.2.2(b); so a similar argument, replacing z_0 by

$1/2(z_0 + \overline{z_0})$, shows that we may assume z_0 lies in $\mathcal{H}_0^C$. Now define

$$t_0 = \exp(z_0);$$

by (5.1.5) (a) - (c), we get

$$\begin{aligned} (5.1.6) \quad \text{Ad}(t_0) \cdot X_{\alpha_i} &= \exp(\sqrt{-1} x_i) X_{\alpha_i} \\ &= \lambda_i \cdot X_{\alpha_i} \\ &= \text{Ad}(t) X_{\alpha_i}, \end{aligned}$$

proving (5.1.4), and hence the first assertion of (a). For the others, suppose $g \in G^\#$ (respectively, $K^\#$); we want to multiply g by an element of G_0 (or K_0) to get an element of T^C . In the first case, $g(K_0)g^{-1}$ is a maximal compact subgroup of G_0 , and hence conjugate to K_0 by an element of G_0 . Modifying g by this element, we can assume g normalizes K_0 . (This is automatic if $g \in K^\#$.) Similarly, we can assume g normalizes T_0^C , and hence its centralizer H^C in G . So g defines an element

$$w \in W(G, H^C).$$

Since g is assumed to lie in $G^\#$, its action on $\mathcal{H}^C$ is also given by that of some element x of G_0 . As was pointed out in Definition 0.2.6, we may assume x lies in K_0 . Replacing g by gx^{-1} , we may assume that g lies in H^C . But

$$H^C = (H^C \cap K)A^C = T_A^C,$$

and A^C is contained in G_0 ; so we are done.

Part (b) follows from the last assertion of (a). Part (c) is easy; a formally identical fact is proved in Lemma 4.3.29.

For (d), (c) allows us to regard R^G as

$$\begin{aligned} R^G &\cong W(K, T^C) / W(K^\#, T^C) \\ &\cong (\text{normalizer in } K \text{ of } T^C) / \\ &\quad (\text{normalizer in } K^\# \text{ of } T^C). \end{aligned}$$

This gives a natural injection

$$R^G \hookrightarrow K/K^\#.$$

Since t_0^C is unique up to conjugacy in K_0 , every component of K meets the normalizer of T^C ; so this map is onto. The first isomorphism in (d) follows from (0.1.2)(d).

The proof of (e) is exactly like that of Lemma 4.3.14.

Choose a positive root system $\Delta^+(g, h^C)$ making $\rho(\Delta^+(h, t))$ dominant, and set

$$(5.1.7) \text{ (a) } \Delta_S = \{\alpha \in \Delta(g, h^C) \mid \langle \alpha, \rho(\Delta^+(h, t)) \rangle = 0\}.$$

Define subalgebras $\mathfrak{l}^S$ and $\mathfrak{u}^S$ as the $\text{ad}(h^C)$ -stable subspaces with

$$(5.1.7) \text{ (b) } \begin{aligned} \mathfrak{l}^S &\supseteq \mathfrak{h}^C, & \Delta(\mathfrak{l}^S, \mathfrak{h}^C) &= \Delta_S \\ \Delta(\mathfrak{u}^S, \mathfrak{h}^C) &= \Delta^+(g, h^C) - \Delta_S \cap \Delta^+(g, h^C). \end{aligned}$$

Then

$$(5.1.7) \text{ (c) } \mathfrak{g}^S = \mathfrak{l}^S + \mathfrak{u}^S$$

is the Levi decomposition of a parabolic subalgebra of $\mathfrak{g}$. Since $\rho(\Delta^+(h, t))$ takes purely imaginary values on t_0^C , we

have

$$(5.1.7) \text{ (d) } \mathfrak{g}^S \cap \overline{\mathfrak{g}^S} = \mathfrak{l}^S.$$

Thus the normalizer of $\mathfrak{g}^S$ in G is a real reductive group L^S with complexified Lie algebra $\mathfrak{l}^S$. Clearly

$$L^S \supseteq H^C.$$

By Lemma 3.2.4, Δ_S does not meet the roots of t^C in $\mathfrak{k}$; so by (5.1.7)(b),

$$(5.1.7) \text{ (e) } \Delta(\mathfrak{l}^S \cap \mathfrak{k}, t^C) = \emptyset.$$

It follows (by the classification of real simple Lie algebras, or for many better reasons) that $\mathfrak{l}^S$ is a product of copies of $\mathfrak{sl}(2, \mathbb{R})$ and center. By definition, each element w of R^G normalizes $\Delta^+(h, t^C)$, so

$$w \cdot \rho(\Delta^+(h, t)) = \rho(\Delta^+(h, t)) \quad (w \in R^G).$$

It follows that the representatives of w in G lie in L^S ; so

$$(5.1.8) \quad R^G = W(L^S, H^C).$$

Since $\mathfrak{l}^S$ is essentially a product of $\mathfrak{sl}(2)$'s, this Weyl group is contained in

$$W(\mathfrak{l}^S, \mathfrak{h}^C) \cong (\mathbb{Z}/2\mathbb{Z})^r.$$

For (f), notice that T^C is a maximally split Cartan subgroup of K (since K is compact); so if K^1 is any reductive linear subgroup of K containing K_0 , Lemma 0.4.2(b) gives

$$W(K^1, T^C \cap K^1) \cong W(K_0, T_0^C).$$

By (e) and (d), if $K^1 \supseteq K^\#$,

$$W(K^1, T^C) \cong K^1/K^\# \cdot W(K_0, T_0^C).$$

These two formulas force $K^1 = K^\#$. So we need only check that $K^\#$ is a linear reductive group. The only non-trivial property in (0.1.2) to check is (0.1.2)(b). For T^C this was already verified in (a), since we showed

$$\text{Ad}(T^C) \subseteq \text{Ad}(T_0^C);$$

so the result we want follows from the last assertion of (a).

Q.E.D.

Proposition 5.1.9 Suppose $\mu \in \hat{T}^C$ is dominant for $\Delta^+(\mathcal{A}, \mathcal{L}^C)$. Fix an element (π, E) of $A(\mu)$ (Def. 5.1.1).

- a) The set of highest weights of (π, E) is $R^G \cdot \mu$.
- b) Every highest weight of (π, E) has multiplicity one.
- c) Identify $(\hat{R})$ with the set of representations of K trivial on $K^\#$ (Lemma 5.1.3(d)). If $r \in (\hat{R})$ and $\pi \in A(\mu)$, set

$$r \cdot \pi = r \otimes \pi.$$

This is a transitive action of $\hat{R}^G$ on $A(\mu)$, with isotropy group equal to the annihilator $R_\mu^\perp$ of R_μ in $\hat{R}^G$.

- d) $|A(\mu)| = |R_\mu|$. More precisely, (c) defines a natural simply transitive action of $\hat{R}_\mu$ on $A(\mu)$.
- e) In the identification $K/K^\# \cong R^G$, let $K \supseteq K_\mu \supseteq K^\# \leftrightarrow R_\mu^G \supseteq R_\mu \supseteq \{1\}$. Then $\text{Ind}_{K_\mu}^K$ defines a bijection from $K_\mu^\mu(\mu)$ to $K^\mu(\mu)$.

Proof. Statement (a) is obvious. For (b), we use the group

$L^S \cap K \supseteq T^C$ defined by (5.1.7). Suppose (π_i, E_i) have highest weight μ ($i = 1, 2$); let

$$E_1^0 = (L^S \cap K) \cdot E_1^\mu$$

(notation 5.1.1). Using (5.1.8), one can easily adapt the proof of the Cartan-Weyl highest weight theorem (Proposition 0.4.3) to show

$$\text{Hom}_K(E_1, E_2) \cong \text{Hom}_{L^S \cap K}(E_1^0, E_2^0).$$

An immediate consequence is that E^0 is irreducible; so it is enough to prove (b) with (K, E) replaced by $(L^S \cap K, E^0)$. Since L^S is linear, we can find an irreducible finite dimensional representation F of L^S whose restriction to $L^S \cap K$ contains E^0 . By Proposition 0.4.7, the restriction of F to L_0^S is still irreducible; so by the finite dimensional representation theory of $\text{SL}(2, \mathbb{R})$, all the T_0^C weights of F have multiplicity one.

The rest of the lemma is formal — it is a trivial case of the "Mackey machine", and (b) says that an appropriate cocycle is trivial. The reader can supply his own proofs, or imitate those in Section 4.3. Q.E.D.

Corollary 5.1.10 Suppose L is a reductive subgroup of G containing T^C . Let $\mu \in \hat{T}^C$ be a weight which is dominant for $\Delta^+(\mathcal{A}, \mathcal{L})$. Suppose that every element of R_μ (cf. (5.1.2)) has a representative in L . Then there is a natural bijection

$$A^{L \cap K}(\mu) \cong A^K(\mu),$$

uniquely specified by the property that if $\pi \in A^K(\mu)$, then the corresponding element $\pi^0 \in A^{L \cap K}(\mu)$ occurs in

$$\pi|_{L \cap K}.$$

§2. The Hochschild-Serre spectral sequence.

Definition 5.2.1 By a θ -stable parabolic subalgebra of $\mathfrak{g}$ we will mean a subalgebra constructed in the following way. Fix $\mathcal{L}_0^C \subseteq \mathcal{L}_0$ and $\Delta^+(\mathcal{L}, \mathcal{L}_0^C)$ as in Definition 5.1.1; and fix an element

$$x \in i(\mathcal{L}_0^C)^*,$$

a linear functional on $\mathcal{L}_0^C$ taking purely imaginary values. (Often x will be dominant for $\Delta^+(\mathcal{L}, \mathcal{L}_0^C)$.) Define $\text{ad}(\mathcal{L}_0^C)$ -stable subalgebras $\mathfrak{l}, \mathfrak{u}$, and $\mathfrak{g}$ by

$$\begin{aligned} \text{(a)} \quad \Delta(\mathfrak{l}, \mathcal{L}_0^C) &= \{\alpha \in \Delta(\mathfrak{g}, \mathcal{L}_0^C) \mid \langle \alpha, x \rangle = 0\} \\ \Delta(\mathfrak{u}, \mathcal{L}_0^C) &= \{\alpha \in \Delta(\mathfrak{g}, \mathcal{L}_0^C) \mid \langle \alpha, x \rangle > 0\} \\ \Delta(\mathfrak{g}, \mathcal{L}_0^C) &= \{\alpha \in \Delta(\mathfrak{g}, \mathcal{L}_0^C) \mid \langle \alpha, x \rangle \geq 0\}. \end{aligned}$$

Then $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra, and

$$\begin{aligned} \text{(b)} \quad \theta \mathfrak{g} &= \mathfrak{g}, \quad \theta \mathfrak{l} = \mathfrak{l}, \quad \theta \mathfrak{u} = \mathfrak{u} \\ \mathfrak{g} &= \mathfrak{g} \cap \mathcal{L} + \mathfrak{g} \cap \mathfrak{p}, \text{ etc.} \end{aligned}$$

Since the roots take purely imaginary values on $\mathcal{L}_0^C$, we have

$$\text{(c)} \quad \Delta(\mathfrak{u}, \mathcal{L}) = \{\alpha \in \Delta(\mathfrak{g}, \mathcal{L}) \mid \langle \alpha, x \rangle < 0\}$$

$$\bar{\mathfrak{l}} = \mathfrak{l}$$

$$\mathfrak{g} \cap \mathfrak{g} = \mathfrak{l}.$$

Put

$$L = \text{normalizer of } \mathfrak{g} \text{ in } G.$$

It will often be convenient to define

$$R = \dim(\mathfrak{u} \cap \mathfrak{p}), \quad S = \dim(\mathfrak{u} \cap \mathcal{L}).$$

Clearly, L is a closed, θ -stable reductive subgroup of G , containing H^C (cf. 5.1.1); and the complexified Lie algebra of L is $\mathfrak{l}$. It is not true that any parabolic subalgebra of $\mathfrak{g}$ which is preserved by θ is of the form in 5.1.1 (although this is true up to conjugacy by $\text{Ad}(\mathcal{L}) \subseteq \text{Ad}(\mathfrak{g})$.) The property which characterizes these parabolic subalgebras up to conjugation by G is the last part of (5.2.1)(c), namely that $\mathfrak{g}$ and $\theta \mathfrak{g}$ should be in opposite position. If we insist on this property and (5.2.1)(b), then all such $\mathfrak{g}$ are obtained by the procedure described (if we vary $\mathcal{L}_0^C$ and x). We will make no use of this fact, however.

Theorem 5.2.2 (Hochschild-Serre) Suppose $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra of $\mathfrak{g}$ (Definition 5.2.1). Then there is a direct sum decomposition

$$\Lambda^* \mathfrak{u} \cap \mathfrak{p} = \bigoplus_{a=0}^N V_a$$

as $L \cap K$ modules, with the following properties:

- a) $V_a \subseteq \Lambda^a \mathfrak{r}(\mathfrak{a})(\mathfrak{u} \cap \mathfrak{p})$; and $a \leq a'$ implies $r(\mathfrak{a}) \leq r(\mathfrak{a}')$.
- b) $V_0 = \Lambda^0(\mathfrak{u} \cap \mathfrak{p}) = \mathbb{C}$, and $V_N = \Lambda^R(\mathfrak{u} \cap \mathfrak{p})$.
- c) Suppose X is a $(\mathfrak{g}, L \cap K)$ module (which is defined like a $(\mathfrak{g}, K)$ module -- cf. (0.3.8) and (6.1.1)).

Then there is a spectral sequence

$$E_t^{ab} \Rightarrow H^{a+b}(u, X),$$

with

$$E_1^{ab} \cong H^{a+b-r}(a)(u \cap k, X) \otimes V_a^*.$$

The differential d_t has bidegree $(t, 1-t)$ and is an $L \cap K$ module map.

Before proving this, we make an observation on the structure of the spectral sequence. Since

$$d_t: E_t^{a,b} \rightarrow E_t^{a+t, b+1-t}$$

and

$$E_t^{N+t, b+1-t} = 0 \quad (t \geq 1),$$

d_t must be zero on $E_t^{N,b}$. So

$$E_{t+1}^{N,b} = E_t^{N,b} / (\text{Im } d_t \cap E_t^{N,b}),$$

and there is a natural surjective map

$$E_t^{N,b} \rightarrow E_{t+1}^{N,b}.$$

If we compose a finite number of these, we obtain a map

$$E_1^{N,b} \rightarrow E_\infty^{N,b}.$$

These groups are identified by the theorem; so we make

Definition 5.2.3 The edge homomorphisms

$$\pi_i: H^i(u \cap k, X) \otimes [\Lambda^R(u \cap k)]^* \rightarrow H^{i+R}(u, X)$$

are the $L \cap K$ -module maps just defined. If $\mu \in \hat{T}^C$, we also put

$$\pi_i^\mu: H^i(u \cap k, X)^\mu \otimes [\Lambda^R(u \cap k)]^* \rightarrow H^{i+R}(u, X)^{\mu-2\rho(u \cap k)} \quad (\text{notation (5.1.1)}).$$

These maps have a simple explicit description on the cochain level, which is given in [42].

Proof of Theorem 5.2.2 Recall from Definition 5.2.1 the element x which was used to define σ_f . Choose an element $\tilde{x} \in i\mathcal{I}_f$ dual to x , in the sense that

$$\langle \lambda, x \rangle = \lambda(\tilde{x})$$

for any $\lambda \in \mathcal{I}^*$. Then $\tilde{x}$ centralizes $\mathcal{L}$, and $\text{ad}(\tilde{x})$ has positive eigenvalues on $\mathcal{U}$. By integration over $L \cap K$, we can assure that $\tilde{x}$ is fixed by $\text{Ad}(L \cap K)$. If we decompose $\Lambda^*(u \cap k)$ according to the eigenvalues of $\text{ad}(\tilde{x})$, we can arrange (a) and (b) of the theorem, and also

$$b)' \quad \text{If } Z \in u \cap k \text{ and } P \in V_a, \text{ then}$$

$$Z \cdot P \subseteq \langle V_0, \dots, V_{a-1} \rangle.$$

We now regard

$$A^n = \text{Hom}(\Lambda^n \mathcal{U}, X)$$

as a graded differential $L \cap K$ module. We filter it by

$$A_a^n = \{f \in A^n \mid f(Q \wedge P) = 0 \text{ whenever } Q \in \Lambda^*(u \cap k), \\ P \in V_a, \text{ for some } a' < a, \text{ and } \deg Q + \deg P = n\}.$$

Thus $A_0 = A$, A_1 consists of functions vanishing $\Lambda^*(u \cap k)$, and so on; in particular

$$A_0^n \supseteq A_1^n \supseteq A_2^n \supseteq \dots$$

There is a natural isomorphism of $L \cap K$ modules

$$\text{Hom}(\wedge^{n-r}(a) (u \cap k), X) \otimes V_a^* \cong \text{Hom}((\wedge^{n-r}(a) (u \cap k)) \otimes V_a, X);$$

so restriction of functions gives a natural $L \cap K$ module map

$$\tau_a^n: A_a^n \rightarrow \text{Hom}(\wedge^{n-r}(a) (u \cap k), X) \otimes V_a^*.$$

By the definition of the filtration, we get

$$(1) \quad \text{if } f \in A_a^n, \text{ then } f \in A_{a+1}^n \text{ if and only if } \tau_a^n(f) = 0.$$

Furthermore

$$(2) \quad \text{if } f \in A_a^n, \text{ then}$$

$$\tau_a^{n+1}(df) = (d(u \cap k) \otimes 1)(\tau_a^n(f)); \text{ and}$$

$$\tau_{a'}^{n+1}(df) = 0 \text{ for } a' < a.$$

To prove this, fix $a' \leq a$, $Q \in \wedge^{n-r}(a')+1(u \cap k)$, $P \in V_{a'}$, and $f \in A_a^n$. We can regard $\tau_a^{n+1}(df)$ as a map from $(n-r(a')+1)(u \cap k) \otimes V_a$ to X , and we will verify (2) after evaluation of this map at $Q \wedge P$. We may as well take

$$Q = q_0 \wedge \dots \wedge q_{n-r}(a')$$

to be decomposable. V_a may not be spanned by decomposable elements, but we will assume

$$P = p_{n-r}(a')+1 \wedge \dots \wedge p_n$$

to simplify the notation. Exactly the same argument would apply if we took P to be a sum of such terms. Applying (3.1.1.1) we find

$$\begin{aligned} \tau_a^{n+1}(df)(Q \otimes P) &= df(Q \wedge P) = \\ (a) \quad &\sum_{i=0}^{n-r(a')} (-1)^i q_i \cdot f(q_0 \wedge \dots \wedge \hat{q}_i \wedge \dots \wedge q_{n-r(a')} \wedge P) \\ (b) + &\sum_{i=n-r(a')+1}^n (-1)^i p_i \cdot f(q_0 \wedge p_{n-r(a')+1} \wedge \dots \wedge \hat{p}_i \wedge \dots \wedge p_n) \\ (c) + &\sum_{i < j \leq n-r(a')} (-1)^{i+j} f([q_i, q_j] \wedge q_0 \wedge \dots \wedge \hat{q}_i \wedge \dots \wedge \hat{q}_j \wedge \dots \\ &\quad \wedge p_{n-r(a')+1} \wedge \dots \wedge \hat{p}_j \wedge \dots \wedge p_n) \\ (d) + &\sum_{i=1}^{n-r(a')} (-1)^i \sum_{j=n-r(a')+1}^n (-1)^{j+n-r(a')} q_{n-r(a')} \wedge p_j \\ (e) + &\sum_{n-r(a')+1 \leq i < j} (-1)^{i+j} f([p_i, p_j] \wedge Q \wedge p_{n-r(a')} \wedge \dots \\ &\quad \wedge \hat{p}_i \wedge \dots \wedge \hat{p}_j \wedge \dots \wedge p_n). \end{aligned}$$

Since $f \in A_a^n$, $f(X \wedge Y) = 0$ if X lies in $\wedge^m(u \cap k)$ for some $m > n-r(a)$. Since $a' \leq a$, $r(a') \leq r(a)$; so every term in lines (b) and (c) is of this form, and they are zero. Line (d) may be written

$$\sum_{i=1}^{n-r(a')} (-1)^{i+n-r(a')} f(q_0 \wedge \dots \wedge \hat{q}_i \wedge \dots \wedge q_{n-r(a')} \wedge (q_i \cdot P)).$$

By (b)' above, $q_i \cdot P$ lies in the span

$$\langle V_0, \dots, V_{a'-1} \rangle \subseteq \langle V_0, \dots, V_{a-1} \rangle.$$

By the definition of A_a^n again, line (d) is therefore zero.

If $a' < a$, then

$$P \in \langle V_0, \dots, V_{a-1} \rangle,$$

so lines (a) and (c) are also zero. This proves the second part of (2) above, and also verifies

$$(3) \quad dA_a \subseteq A_a.$$

Finally, if $a = a'$, lines (a) and (c) are nothing but

$$(d \mathbf{u} \cap \mathbf{k} \otimes 1)(\tau_a^n(f))(Q \otimes P),$$

which proves the first part of (2).

Now (1), (2), and (3) imply

$$(4) \quad \tau_a^n \text{ induces a map from } A_a^n / A_{a+1}^n \text{ to}$$

$$\text{Hom}(\Lambda^{n-r}(a) (\mathbf{u} \cap \mathbf{k}), X) \otimes V_a^*$$

which is an isomorphism of differential $L \cap K$ modules. The spectral sequence of the filtration $\{A_a\}$ gives (c). Q.E.D.

Corollary 5.2.4 In the setting of Theorem 5.2.2, if X is a $(\mathcal{G}, K)$ module, then $H^*(\mathbf{u}, X)$ is an $(\ell, L \cap K)$ module.

If X is admissible (cf. (0.3.8)), then so is $H^*(\mathbf{u}, X)$.

If X is $Z(\mathcal{G})$ -finite, then $H^*(\mathbf{u}, X)$ is $Z(\ell)$ -finite.

Proof. The cochain complex

$$\text{Hom}(\Lambda^* \mathbf{u}, X)$$

is already an $(\ell, L \cap K)$ module, so $H^*(\mathbf{u}, X)$ is also. Suppose X is admissible. To show that $H^*(\mathbf{u}, X)$ is admissible, it is enough to show that any representation μ of $(L \cap K)_0$ occurs in E_1^{ab} (see Theorem 5.2.2) with finite multiplicity. There are only finitely many representations μ' of $(L \cap K)_0$ such that μ occurs in $\mu' \otimes V_a^*$ (namely, the constituents of $\mu \otimes V_a$). So it is enough to show that each μ' has finite multiplicity in $H^*(\mathbf{u} \cap \mathbf{k}, X)$. Now (notation (0.3.4))

$$H^*(\mathbf{u} \cap \mathbf{k}, X) \cong \bigoplus_{\delta \in \hat{K}_0} H^*(\mathbf{u} \cap \mathbf{k}, X(\delta));$$

and Kostant's theorem 3.2.3 shows that there is at most one

$\delta \in \hat{K}_0$ such that μ' occurs in $H^*(\mathbf{u} \cap \mathbf{k}, X(\delta))$. Since $X(\delta)$ is finite dimensional, this shows that μ' occurs only finitely often in $H^*(\mathbf{u} \cap \mathbf{k}, X)$, as we wished to prove. The last assertion is Corollary 3.1.6. Q.E.D.

Definition 5.2.5 Suppose $\mu \in \hat{T}^C$ is dominant for $\Delta^+(\mathbf{k}, \mathcal{G})$ (notation (5.1.1)), and X is a $(\mathcal{G}, K)$ module.

We say that μ is strongly $\mathbf{u}$ -minimal in X if whenever

$$\delta \in \hat{K}, (J, J') \neq (0, R) \text{ (notation (5.2.1))}, \text{ and}$$

$$(H^J(\mathbf{u} \cap \mathbf{k}, \delta) \otimes [\Lambda^{J'}(\mathbf{u} \cap \mathbf{k})]^*)^{\mu - 2\rho(\mathbf{u} \cap \mathbf{k})} \neq 0$$

$$\text{(notation (5.1.1))}, \text{ then } X(\delta) = 0.$$

Thus the assumption is that certain representations of K do not occur in X . Because $H^0(\mathbf{u} \cap \mathbf{k}, X)$ consists of the elements of X annihilated by $\mathbf{u} \cap \mathbf{k}$, there is an obvious identification

$$H^0(\mathbf{u} \cap \mathbf{k}, X)^\mu \cong X^\mu.$$

Using this identification, we can state

Theorem 5.2.6 Suppose X is a $(\mathcal{G}, K)$ module and $\mu \in \hat{T}^C$ is strongly $\mathbf{u}$ -minimal in X . Then

$$\pi_0^\mu: X^\mu \otimes [\Lambda^R(\mathbf{u} \cap \mathbf{k})]^* \rightarrow H^R(\mathbf{u}, X)^{\mu - 2\rho(\mathbf{u} \cap \mathbf{k})}$$

is bijective; and no representation of $L \cap K$ of

highest weight $\mu - 2\rho(\mathbf{u} \cap \mathbf{k})$ occurs in $H^i(\mathbf{u}, X)$

for $i \neq R$.

Proof. In terms of the spectral sequence in Theorem 5.2.2, the assumption that μ is strongly $\mathbf{u}$ -minimal means that

$$(E_1^{\text{ab}})^{\mu - 2\rho(\mathbf{u} \cap \mathbf{k})} = 0$$

if $(a,b) \neq (N,R-N)$. So the spectral sequence degenerates, and the edge homomorphism π_0^μ defines an isomorphism

$$(E_1^{N,R-N})^{\mu-2\rho}(u\eta p) \cong \bigoplus_{a,b} (E_\infty^{ab})^{\mu-2\rho}(u\eta p) .$$

The right side here is (up to a filtration)

$$H^*(u,X)^{\mu-2\rho}(u\eta p) ,$$

so this is the assertion of the theorem. Q.E.D.

§3. Construction of parabolic subalgebras.

In order to apply Theorem 5.2.6, we need to choose parabolic subalgebras which are well matched with particular representations of K . The philosophy is this. If $\mathfrak{g}$ is a fairly small, θ -stable parabolic subalgebra, then the non-vanishing of $H^*(u,X)$ is a fairly strong constraint on X .

We know from (for example) Theorem 4.4.3 that there are many representations containing K -types which are "small"; so we cannot hope to attach a small parabolic subalgebra to a small K -type, and still get non-vanishing cohomology. On the other hand, representations which contain a "generic" representation of K extremally are rare (compare Proposition 1.2.14); so we can hope to attach a strong invariant, like $H^*(u,X)$ for small $\mathfrak{g}$, to them.

So what we want is a construction which attaches a small parabolic subalgebra (even a Borel subalgebra) to a generic representation of K , and a large parabolic subalgebra (even $\mathfrak{g}$ itself) to a small representation. For $SL(2, \mathbb{R})$, $\mathfrak{g}$ will be attached to 0 and ± 1 , and a Borel subalgebra to all other

K -types (cf. Proposition 1.2.14). In terms of the construction of parabolics explained in Definition 5.2.1, we should attach a regular weight $\lambda \in (\mathcal{L}^C)^*$ to a generic representation, and a singular weight to a small representation; then the parabolic will be the one defined by λ . The most obvious candidate for λ is simply a highest weight of the representation of K . For complex groups this is the right choice; but the example just given shows that it fails for $SL(2, \mathbb{R})$. (To the K -type 1 we must attach not 1 but 0 as a weight in $(\mathcal{L}^C)^*$.) The construction given here for λ (in Proposition 5.3.3) is almost the simplest one imaginable which works for both complex groups and $SL(2, \mathbb{R})$. (The proof of Proposition 5.3.3, however, can make no such claims of naturalness; it evolved from a case-by-case argument, and its dismal ancestry is still obvious.)

We begin with some general structural remarks, using the Cartan subalgebra defined at (5.1.1). Fix a noncompact imaginary root $\beta \in \Delta(\mathfrak{g}, \mathcal{L}^C)$; recall from Definition 0.2.1 that this means that the root vector X_β for β centralizes $\mathfrak{sc}$, and lies in $\mathfrak{p}$. Write $\bar{}$ for the complex structure on $\mathfrak{g}$; since β takes purely imaginary values on $\mathcal{L}_0^C, \bar{X}_\beta$ must be a root vector for the root $-\beta$. Put

$$Z_\beta = X_\beta + \bar{X}_\beta \in \mathfrak{p}_0$$

$\mathfrak{g}_0^\beta =$ centralizer of Z_β in $\mathfrak{g}_0$.

Since $\theta Z_\beta = -Z_\beta$, θ preserves $\mathfrak{g}_0^\beta$. As an element of $\mathfrak{p}_0$, Z_β is reductive in $\mathfrak{g}_0$, so $\mathfrak{g}_0^\beta$ is reductive. Accordingly, the subgroup G_0^β of G with Lie algebra $\mathfrak{g}_0^\beta$ again satisfies (0.1.2);

we can use the restriction of θ as the Cartan involution.

Define

$$\begin{aligned} (\mathfrak{h}_0^C)^\perp &= \{x \in \mathfrak{h}_0^C \mid \beta(x) = 0\} \\ &= (\mathfrak{t}_0^C)^\perp \oplus \mathfrak{a}_0^C \\ \mathfrak{h}_0^\beta &= (\mathfrak{h}_0^C)^\perp \oplus \langle \mathfrak{z}_\beta \rangle. \end{aligned}$$

Obviously $(\mathfrak{h}_0^C)^\perp$ and $\mathfrak{z}_\beta$ commute; so since $\mathfrak{z}_\beta$ is reductive, $\mathfrak{h}_0^\beta$ is a Cartan subalgebra of $\mathfrak{g}_0^\beta$ and of $\mathfrak{g}_0$.

Proposition 5.3.1 (Schmid [36]). Suppose β is a

noncompact imaginary root in $\Delta(\mathfrak{g}, \mathfrak{h}^C)$. With

notation as above, $\mathfrak{h}_0^\beta$ is a fundamental Cartan

subalgebra of $\mathfrak{g}_0^\beta$. The root system $\Delta(\mathfrak{g}_0^\beta, \mathfrak{h}_0^\beta)$

may be identified with the set

$$\beta^\perp \cap \Delta(\mathfrak{g}, \mathfrak{h}^C) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C) \mid \langle \alpha, \beta \rangle = 0\}$$

of roots of $\mathfrak{h}^C$ in $\mathfrak{g}$ orthogonal to β . Under this

identification, the notions of complex and imaginary

are preserved. A compact imaginary root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C)$

orthogonal to β corresponds to a compact imaginary

root $\tilde{\alpha} \in \Delta(\mathfrak{g}_0^\beta, \mathfrak{h}_0^\beta)$ if and only if α and β are

strongly orthogonal (i.e., $\alpha \pm \beta$ is not a root);

otherwise $\tilde{\alpha}$ is noncompact. Similarly a noncompact

imaginary root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C)$ corresponds to a non-

compact $\tilde{\alpha} \in \Delta(\mathfrak{g}_0^\beta, \mathfrak{h}_0^\beta)$ if and only if α and β are

strongly orthogonal.

Proof. Let $\mathfrak{g}_1$ be the three dimensional subalgebra of $\mathfrak{g}$ generated by x_β and $\bar{x}_\beta$; then we can choose an isomorphism of $\mathfrak{g}_1$ with $\mathfrak{sl}(2, \mathbb{C})$, in such a way that

$$x_\beta \mapsto \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \bar{x}_\beta \mapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix},$$

with a and b non-zero. (We could even arrange $a = b = 1$,

but this is not important.) The element corresponding to

$\mathfrak{z}_\beta$ is therefore conjugate under $\mathrm{SL}(2, \mathbb{C})$ to

$$\begin{pmatrix} \sqrt{ab} & 0 \\ 0 & -\sqrt{ab} \end{pmatrix}$$

a regular element of a Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. By

the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{z}_\beta$ therefore annihilates

exactly a one-dimensional subspace of every odd dimensional

representation of $\mathfrak{g}_1$, and only zero in an even dimensional

representation. To compute the centralizer $\mathfrak{g}_1^\beta$ of $\mathfrak{z}_\beta$, we

therefore decompose $\mathfrak{g}$ according to the adjoint action of

$\mathfrak{g}_1$ (cf. [22], 8.4). The pieces are $\mathfrak{g}_1$ itself, which is

three dimensional; $(\mathfrak{h}^C)^\perp$, on which $\mathfrak{g}_1$ acts trivially;

and the spaces spanned by β -strings of root vectors. (If α

is a root, the β -string through α is the maximal set of roots

of the form

$$\{\alpha - r\beta, \alpha - r\beta + \beta, \dots, \alpha + s\beta\}.$$

The corresponding root vectors (if $\alpha \neq \pm\beta$) span an irreducible

$\mathfrak{g}_1$ module. A string has length 1, 2, 3, or 4.) The con-

tribution of $\mathfrak{g}_1$ to $\mathfrak{g}_1^\beta$ is just $\mathfrak{z}_\beta$; so $\mathfrak{g}_1$ and $(\mathfrak{h}^C)^\perp$

together give $\mathfrak{h}^\beta$. For the rest of $\mathfrak{g}_1^\beta$, we must look at

root strings of odd length. If $\{\alpha\}$ is a β -string of length

1, then $\alpha \pm \beta$ are not roots, so α and β are strongly

orthogonal. Also σ_1 centralizes X_α , so X_α lies in σ_1^β . It is a weight vector for $\mathcal{h}^\beta$, of weight $\alpha|_{(\mathcal{h}_C)^\perp}$ on $(\mathcal{h}_C)^\perp$ and 0 on Z_β . If $\{\alpha-\beta, \alpha, \alpha+\beta\}$ is a string of length 3, then α is orthogonal to β but not strongly orthogonal. Both

$$[X_\beta, X_{\alpha-\beta}] \text{ and } [\bar{X}_\beta, X_{\alpha+\beta}]$$

are non-zero multiples of X_α ; so for an appropriate $c \neq 0$,

$$\tilde{X}_\alpha = X_{\alpha-\beta} + cX_{\alpha+\beta}$$

centralizes Z_β . It is a weight vector for $\mathcal{h}^\beta$, of weight $\alpha|_{(\mathcal{h}_C)^\perp}$ on $(\mathcal{h}_C)^\perp$ and 0 on Z_β .

This construction establishes the identification of $\Delta(\sigma_1^\beta, \mathcal{h}^\beta)$ with $\beta^\perp \cap \Delta(\sigma_1, \mathcal{h}^C)$; for a β -string of even length contains no roots orthogonal to β . In particular, no root of $\mathcal{h}^\beta$ in σ_1^β is supported on

$$\mathcal{h}^\beta \cap \mathfrak{p} = \sigma^C + \langle Z_\beta \rangle.$$

By [50], Proposition 1.3.3.4, this implies that $\mathcal{h}_0^\beta$ is a fundamental Cartan subalgebra of σ_1^β . Finally, suppose $\alpha \in \Delta(\sigma_1, \mathcal{h}^C) \cap \beta^\perp$ is compact imaginary; let $\tilde{\alpha}$ be the corresponding root of $\mathcal{h}^\beta$ in σ_1^β , and $X_{\tilde{\alpha}}$ a root vector. If α and β are strongly orthogonal, we have seen that

$$X_{\tilde{\alpha}} = X_\alpha \in \mathcal{h},$$

so $\tilde{\alpha}$ is compact. Otherwise,

$$\begin{aligned} X_{\alpha+\beta} &= c_1 [X_\alpha, X_\beta] \in \mathfrak{p} \\ X_{\alpha-\beta} &= c_2 [X_\alpha, \bar{X}_\beta] \in \mathfrak{p}, \end{aligned}$$

since $X_\alpha \in \mathcal{h}$ and $X_\beta \in \mathfrak{p}$; so

$$X_{\tilde{\alpha}} = X_{\alpha-\beta} + cX_{\alpha+\beta} \in \mathfrak{p},$$

and $\tilde{\alpha}$ is noncompact. An identical argument proves the last statement. Q.E.D.

We fix now a $\Delta^+(\mathcal{h}, \mathcal{t}^C)$ -dominant weight $\mu \in \hat{\Gamma}^C$. We will also write $\mu \in (\mathcal{t}^C)^*$ for the differential of μ . Define

$$(5.3.2) \quad 2\rho_C = 2\rho(\Delta^+(\mathcal{h})) \in (\mathcal{t}^C)^*.$$

We will be interested in θ -invariant positive root systems $\Delta^+(\sigma_1, \mathcal{h}^C)$ for $\mathcal{h}^C$ in σ_1 , with the property that $\mu+2\rho_C$ is dominant. Such systems always exist. To see this, choose an element γ of $i\mathcal{t}_0^C$, on which every root is non-zero; this is possible since no root vanishes on $\mathcal{t}^C$. Set

$$\Delta^+(\sigma_1, \mathcal{h}^C) = \{\alpha \in \Delta(\sigma_1, \mathcal{h}^C) \mid \langle \alpha, \mu+2\rho_C \rangle > 0, \text{ or } \langle \alpha, \mu+2\rho_C \rangle = 0 \text{ and } \alpha(\gamma) > 0\}.$$

Obviously this choice of $\Delta^+(\sigma_1, \mathcal{h}^C)$ is θ -invariant, and makes $\mu+2\rho_C$ dominant.

The statement and proof of the next result follow ideas of Schmid in [36].

Proposition 5.3.3 For each $\Delta^+(\mathcal{h}, \mathcal{t}^C)$ -dominant weight $\mu \in \hat{\Gamma}^C$, there is a unique element $\lambda \in (\mathcal{t}^C)^*$ having the following properties: fix a θ -invariant positive root system $\Delta^+(\sigma_1, \mathcal{h}^C)$ for $\mathcal{h}^C$ in σ_1 , making $\mu+2\rho_C$ dominant; and write $\rho = \rho(\Delta^+(\sigma_1, \mathcal{h}^C))$. Then λ is dominant for $\Delta^+(\sigma_1, \mathcal{h}^C)$, and there is a set $\{\beta_1, \dots, \beta_r\} \subseteq \Delta^+(\sigma_1, \mathcal{h}^C)$ of imaginary roots, satisfying

a) if we put

$$c_i = -2\langle \beta_i, \mu + 2\rho_C - \rho \rangle / \langle \beta_i, \beta_i \rangle,$$

$$v = \sum c_i \beta_i,$$

then

$$0 \leq c_i \leq 1,$$

and

$$\lambda = \mu + 2\rho_C - \rho + \frac{1}{2}v.$$

b) If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C)$ is imaginary, and $\langle \alpha, \lambda \rangle = 0$, then $\langle \alpha, \beta_i \rangle \neq 0$ for some i .

c) The root β_1 is noncompact; and either it is simple, or there is a complex simple root α of

$$\Delta^+(\mathfrak{g}, \mathfrak{h}^C) \text{ such that } \beta_1 = \alpha + \theta\alpha.$$

d) Write

$$\mathfrak{g}^1 = \mathfrak{g}^{\beta_1}, \mathfrak{h}^1 = \mathfrak{h}^{\beta_1}$$

as in Proposition 5.3.1. Then the positive system

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) \cap \beta_1^\perp \text{ and its subset } \{\beta_2, \dots, \beta_r\} \text{ for } \Delta(\mathfrak{g}^1, \mathfrak{h}^1) \text{ satisfy these same conditions for } \mathfrak{g}^1$$

$$\text{and the weight } \mu|_{\mathfrak{g}^1 \cap \mathfrak{t}}.$$

e) If $c_i \neq 0$ and $c_j = 0$, then $i < j$.

It should be pointed out that the set $\{\beta_1, \dots, \beta_r\}$ is not unique in general; the simplest example is for $G = \mathrm{SU}(2, 1)$ and the trivial representation of K .

Proof. We proceed by induction on $\dim \mathfrak{t}^C$. Existence of λ is proved first: we will choose β_1 , choose $\{\beta_2, \dots, \beta_r\}$ by induction, and then define λ by (a). We will then verify

that this choice of λ satisfies the proposition. The only hard part is to prove that λ is dominant.

By definition, ρ lies in $(\mathfrak{h}^C)^*$. But $\theta(\Delta^+(\mathfrak{g}, \mathfrak{h})) = \Delta^+(\mathfrak{g}, \mathfrak{h})$, so $\theta\rho = \rho$. So ρ can be regarded as an element of $(\mathfrak{t}^C)^*$. Define

$$\tilde{\lambda} = \mu + 2\rho_C - \rho.$$

If $\tilde{\lambda}$ is dominant, and $\langle \alpha, \tilde{\lambda} \rangle \neq 0$ whenever $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C)$ is an imaginary root, then we set $\{\beta_i\} = \emptyset$, $\tilde{\lambda} = \lambda$; it is clear that the properties specified in the proposition are satisfied. So suppose this is not the case; then either $\tilde{\lambda}$ is not dominant, or it is dominant, but

$$\langle \alpha, \tilde{\lambda} \rangle = 0$$

for some imaginary root α . Suppose first that $\tilde{\lambda}$ is not dominant. Then we can find a simple root $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^C)$ such that

$$\langle \alpha, \tilde{\lambda} \rangle < 0.$$

Write $\gamma = \alpha|_{\mathfrak{t}^C}$, $\delta = \alpha|_{\mathfrak{a}^C}$. Notice that

$$\theta\alpha = (\gamma, -\delta) \in (\mathfrak{t}^C)^* \oplus (\mathfrak{a}^C)^* \cong (\mathfrak{h}^C)^*$$

is also a simple positive root, since $\Delta^+(\mathfrak{g}, \mathfrak{h}^C)$ is assumed to be θ -stable. We treat several cases separately; some of the calculations will be needed again later.

Case I. α is complex, and $\langle \alpha, \theta\alpha \rangle = 0$. This is clearly equivalent to

$$\langle \gamma, \gamma \rangle = \frac{1}{2} \langle \alpha, \alpha \rangle.$$

Now let X_α be a root vector for α ; put

$$X_{\theta\alpha} = \theta(X_\alpha),$$

which is clearly a root vector for $\theta\alpha$. Since α is complex, $\theta\alpha \neq \alpha$, so X_α and $X_{\theta\alpha}$ are linearly independent. So

$$Y_\gamma = X_\alpha + X_{\theta\alpha} \in \mathcal{K}$$

is a non-zero vector of weight γ for $\mathcal{K}^C$. So γ may be regarded as a root of $\mathcal{K}^C$ in $\mathcal{K}$. Because

$$\frac{1}{2}\langle \gamma, \mu+2\rho_C \rangle = \langle \alpha, \mu+2\rho_C \rangle + \langle \theta\alpha, \mu+2\rho_C \rangle \geq 0$$

(since $\mu+2\rho_C$ is dominant for $\Delta^+(\mathfrak{g}, \mathcal{K}^C)$), γ is necessarily positive. By Lemma 3.2.4,

$$\langle \gamma, \mu+2\rho_C \rangle = \langle \gamma, \mu \rangle + 2\langle \gamma, \rho_C \rangle \geq \langle \gamma, \gamma \rangle.$$

Again by Lemma 3.2.4

$$\langle \gamma, \rho \rangle = \langle \alpha, \rho \rangle = \frac{1}{2}\langle \alpha, \alpha \rangle = \langle \gamma, \gamma \rangle;$$

here we have used the fact that $\rho \in (\mathcal{K}^C)^*$. So

$$\begin{aligned} \langle \alpha, \tilde{\lambda} \rangle &= \langle \gamma, \mu+2\rho_C - \rho \rangle \\ &\geq \langle \gamma, \gamma \rangle - \langle \gamma, \gamma \rangle = 0. \end{aligned}$$

This contradicts the choice of α .

Case II. α is complex, and $\langle \alpha, \theta\alpha \rangle \neq 0$. Since α and $\theta\alpha$ are adjacent simple roots of the same length, they generate a subsystem of type A_2 . Define

$$\beta_1 = \alpha + \theta\alpha;$$

then β_1 is an imaginary root, of the same length as α and $\theta\alpha$. We claim that it is noncompact; for let

$X = x + y$
be a root vector for α , with $x \in \mathcal{K}$ and $y \in \mathfrak{p}$. Then

$$\theta X = x - y$$

is a root vector for $\theta\alpha$, so

$$[X, \theta X] = [x+y, x-y] = -2[x, y] \in \mathfrak{p}$$

is a root vector for $\alpha + \theta\alpha$.

Finally, we claim that

$$2\langle \beta_1, \tilde{\lambda} \rangle / \langle \beta_1, \beta_1 \rangle \geq -1.$$

To see this, notice first that, just as in Case I, γ may be regarded as a root of $\mathcal{K}^C$ in $\mathcal{K}$, so

$$\langle \gamma, \mu+2\rho_C \rangle \geq \langle \gamma, \gamma \rangle.$$

Now γ is just the orthogonal projection of α in the β_1 direction; so by a calculation in A_2 ,

$$\langle \gamma, \gamma \rangle = \frac{1}{4}\langle \alpha, \alpha \rangle.$$

So

$$\langle \gamma, \rho \rangle = \langle \alpha, \rho \rangle = \frac{1}{2}\langle \alpha, \alpha \rangle = 2\langle \gamma, \gamma \rangle.$$

Thus

$$\langle \gamma, \tilde{\lambda} \rangle = \langle \gamma, \mu+2\rho_C - \rho \rangle \geq -\langle \gamma, \gamma \rangle;$$

and since $\beta_1 = 2\gamma$,

$$2\langle \beta_1, \tilde{\lambda} \rangle / \langle \beta_1, \beta_1 \rangle \geq -4\langle \gamma, \gamma \rangle / 4\langle \gamma, \gamma \rangle = -1.$$

By the choice of α , we also have

$$\langle \beta_1, \tilde{\lambda} \rangle < 0.$$

Case III. α is imaginary. By Lemma 3.2.4,

$$\frac{2\langle\alpha, \rho\rangle}{\langle\alpha, \alpha\rangle} = 1.$$

If α is compact,

$$2\langle\alpha, \mu+2\rho_C\rangle/\langle\alpha, \alpha\rangle \geq 2,$$

so

$$2\langle\alpha, \lambda\rangle/\langle\alpha, \alpha\rangle \geq 1,$$

a contradiction; so α is noncompact. Put $\beta_1 = \alpha$. By the choice of $\Delta^+(g, h^C)$,

$$2\langle\beta_1, \mu+2\rho_C\rangle/\langle\beta_1, \beta_1\rangle \geq 0,$$

so

$$2\langle\beta_1, \lambda\rangle/\langle\beta_1, \beta_1\rangle \geq -1;$$

and $\langle\beta_1, \lambda\rangle$ is negative by the choice of α .

In all cases, we have therefore chosen an imaginary root β_1 satisfying (c) of the proposition; and if we put

$$c_1 = -2\langle\beta_1, \tilde{\lambda}\rangle/\langle\beta_1, \beta_1\rangle,$$

then

$$0 < c_1 \leq 1,$$

(in accordance with (a) of the proposition).

We turn next to the choice of β_1 when $\tilde{\lambda}$ is dominant, but is singular with respect to some imaginary root. Put

$$\begin{aligned}\Delta_1 &= \{\alpha \in \Delta(g, h^C) \mid \langle\alpha, \tilde{\lambda}\rangle = 0\} \\ \Delta_1^+ &= \Delta_1 \cap \Delta^+.\end{aligned}$$

Then Δ_1 is a root system, stable under θ ; the simple roots of Δ_1^+ are simple for Δ^+ . The hypothesis is that Δ_1^+ contains an imaginary root. If there is a simple imaginary root α , then the considerations of Case III show that α must be non-compact; we take $\beta_1 = \alpha$. So assume that all simple roots of Δ_1^+ are complex. If α is such a root, and α is adjacent to $\theta\alpha$, Case II above shows that we may take

$$\beta_1 = \alpha + \theta\alpha;$$

this is a noncompact imaginary root. So finally assume that α is never adjacent to $\theta\alpha$. Let Γ denote the Dynkin diagram of Δ_1^+ . Then θ defines an automorphism of Γ of order 2, without fixed points, such that $\theta\alpha$ is never adjacent to α . By inspection, one verifies that a simple Dynkin diagram cannot admit such an automorphism; so θ does not fix any component of Γ . We can therefore write

$$\Gamma = \Gamma_0 \cup \Gamma_1,$$

with Γ_0 and Γ_1 disjoint subdiagrams, and

$$\theta: \Gamma_i \rightarrow \Gamma_{1-i} \quad (i = 0, 1)$$

an isomorphism. So Δ_1 is the union of two disjoint subsystems, interchanged by θ ; so Δ_1 contains no imaginary (i.e., θ -fixed)

a) if we put

$$c_i = -2\langle \beta_i, \mu + 2\rho_C - \rho \rangle / \langle \beta_i, \beta_i \rangle,$$

$$v = \sum c_i \beta_i,$$

then

$$0 \leq c_i \leq 1,$$

and

$$\lambda = \mu + 2\rho_C - \rho + \frac{1}{2}v.$$

b) If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C)$ is imaginary, and $\langle \alpha, \lambda \rangle = 0$, then $\langle \alpha, \beta_i \rangle \neq 0$ for some i .

c) The root β_1 is noncompact; and either it is simple, or there is a complex simple root α of

$$\Delta^+(\mathfrak{g}, \mathfrak{h}^C) \text{ such that } \beta_1 = \alpha + \theta\alpha.$$

d) Write

$$\mathfrak{g}^1 = \mathfrak{g}^{\beta_1}, \quad \mathfrak{h}^1 = \mathfrak{h}^{\beta_1}$$

as in Proposition 5.3.1. Then the positive system

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) \cap \beta_1^\perp \text{ and its subset } \{\beta_2, \dots, \beta_r\} \text{ for}$$

$$\Delta(\mathfrak{g}^1, \mathfrak{h}^1) \text{ satisfy these same conditions for } \mathfrak{g}^1$$

$$\text{and the weight } \mu|_{\mathfrak{g}^1 \cap \mathfrak{t}}.$$

e) If $c_i \neq 0$ and $c_j = 0$, then $i < j$.

It should be pointed out that the set $\{\beta_1, \dots, \beta_r\}$ is not unique in general; the simplest example is for $G = \mathrm{SU}(2, 1)$ and the trivial representation of K .

Proof. We proceed by induction on $\dim \mathfrak{t}^C$. Existence of λ is proved first: we will choose β_1 , choose $\{\beta_2, \dots, \beta_r\}$ by induction, and then define λ by (a). We will then verify

that this choice of λ satisfies the proposition. The only hard part is to prove that λ is dominant.

By definition, ρ lies in $(\mathfrak{h}^C)^*$. But $\theta(\Delta^+(\mathfrak{g}, \mathfrak{h})) = \Delta^+(\mathfrak{g}, \mathfrak{h})$, so $\theta\rho = \rho$. So ρ can be regarded as an element of $(\mathfrak{t}^C)^*$. Define

$$\tilde{\lambda} = \mu + 2\rho_C - \rho.$$

If $\tilde{\lambda}$ is dominant, and $\langle \alpha, \tilde{\lambda} \rangle \neq 0$ whenever $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C)$ is an imaginary root, then we set $\{\beta_i\} = \emptyset$, $\tilde{\lambda} = \lambda$; it is clear that the properties specified in the proposition are satisfied. So suppose this is not the case; then either $\tilde{\lambda}$ is not dominant, or it is dominant, but

$$\langle \alpha, \tilde{\lambda} \rangle = 0$$

for some imaginary root α . Suppose first that $\tilde{\lambda}$ is not dominant. Then we can find a simple root $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}^C)$ such that

$$\langle \alpha, \tilde{\lambda} \rangle < 0.$$

Write $\gamma = \alpha|_{\mathfrak{t}^C}$, $\delta = \alpha|_{\mathfrak{a}^C}$. Notice that

$$\theta\alpha = (\gamma, -\delta) \in (\mathfrak{t}^C)^* \oplus (\mathfrak{a}^C)^* \cong (\mathfrak{h}^C)^*$$

is also a simple positive root, since $\Delta^+(\mathfrak{g}, \mathfrak{h}^C)$ is assumed to be θ -stable. We treat several cases separately; some of the calculations will be needed again later.

Case I. α is complex, and $\langle \alpha, \theta\alpha \rangle = 0$. This is clearly equivalent to

$$\langle \gamma, \gamma \rangle = \frac{1}{2}\langle \alpha, \alpha \rangle.$$

Now let X_α be a root vector for α ; put

$$X_{\theta\alpha} = \theta(X_\alpha),$$

which is clearly a root vector for $\theta\alpha$. Since α is complex, $\theta\alpha \neq \alpha$, so X_α and $X_{\theta\alpha}$ are linearly independent. So

$$Y_\gamma = X_\alpha + X_{\theta\alpha} \in \mathcal{K}$$

is a non-zero vector of weight γ for $\mathcal{K}^C$. So γ may be regarded as a root of $\mathcal{K}^C$ in $\mathcal{K}$. Because

$$\frac{1}{2}\langle \gamma, \mu+2\rho_C \rangle = \langle \alpha, \mu+2\rho_C \rangle + \langle \theta\alpha, \mu+2\rho_C \rangle \geq 0$$

(since $\mu+2\rho_C$ is dominant for $\Delta^+(\mathfrak{g}, \mathcal{K}^C)$), γ is necessarily positive. By Lemma 3.2.4,

$$\langle \gamma, \mu+2\rho_C \rangle = \langle \gamma, \mu \rangle + 2\langle \gamma, \rho_C \rangle \geq \langle \gamma, \gamma \rangle.$$

Again by Lemma 3.2.4

$$\langle \gamma, \rho \rangle = \langle \alpha, \rho \rangle = \frac{1}{2}\langle \alpha, \alpha \rangle = \langle \gamma, \gamma \rangle;$$

here we have used the fact that $\rho \in (\mathcal{K}^C)^*$. So

$$\begin{aligned} \langle \alpha, \tilde{\lambda} \rangle &= \langle \gamma, \mu+2\rho_C - \rho \rangle \\ &\geq \langle \gamma, \gamma \rangle - \langle \gamma, \gamma \rangle = 0. \end{aligned}$$

This contradicts the choice of α .

Case II. α is complex, and $\langle \alpha, \theta\alpha \rangle \neq 0$. Since α and $\theta\alpha$ are adjacent simple roots of the same length, they generate a subsystem of type A_2 . Define

$$\beta_1 = \alpha + \theta\alpha;$$

then β_1 is an imaginary root, of the same length as α and $\theta\alpha$. We claim that it is noncompact; for let

$$X = x + y$$

be a root vector for α , with $x \in \mathcal{K}$ and $y \in \mathcal{P}$. Then

$$\theta X = x - y$$

is a root vector for $\theta\alpha$, so

$$[X, \theta X] = [x+y, x-y] = -2[x, y] \in \mathcal{P}$$

is a root vector for $\alpha + \theta\alpha$.

Finally, we claim that

$$2\langle \beta_1, \tilde{\lambda} \rangle / \langle \beta_1, \beta_1 \rangle \geq -1.$$

To see this, notice first that, just as in Case I, γ may be regarded as a root of $\mathcal{K}^C$ in $\mathcal{K}$, so

$$\langle \gamma, \mu+2\rho_C \rangle \geq \langle \gamma, \gamma \rangle.$$

Now γ is just the orthogonal projection of α in the β_1 direction; so by a calculation in A_2 ,

$$\langle \gamma, \gamma \rangle = \frac{1}{4}\langle \alpha, \alpha \rangle.$$

So

$$\langle \gamma, \rho \rangle = \langle \alpha, \rho \rangle = \frac{1}{2}\langle \alpha, \alpha \rangle = 2\langle \gamma, \gamma \rangle.$$

Thus

$$\langle \gamma, \tilde{\lambda} \rangle = \langle \gamma, \mu+2\rho_C - \rho \rangle \geq -\langle \gamma, \gamma \rangle;$$

and since $\beta_1 = 2\gamma$,

$$2\langle \beta_1, \tilde{\lambda} \rangle / \langle \beta_1, \beta_1 \rangle \geq -4\langle \gamma, \gamma \rangle / 4\langle \gamma, \gamma \rangle = -1.$$

If the choice of α , we also have

$$\langle \beta_1, \tilde{\lambda} \rangle < 0.$$

Case III. α is imaginary. By Lemma 3.2.4,

$$\frac{2\langle\alpha, \rho\rangle}{\langle\alpha, \alpha\rangle} = 1.$$

If α is compact,

$$2\langle\alpha, \mu+2\rho_C\rangle/\langle\alpha, \alpha\rangle \geq 2,$$

so

$$2\langle\alpha, \lambda\rangle/\langle\alpha, \alpha\rangle \geq 1,$$

a contradiction; so α is noncompact. Put $\beta_1 = \alpha$. By the choice of $\Delta^+(\mathfrak{g}, \mathfrak{h}^C)$,

$$2\langle\beta_1, \mu+2\rho_C\rangle/\langle\beta_1, \beta_1\rangle \geq 0,$$

so

$$2\langle\beta_1, \lambda\rangle/\langle\beta_1, \beta_1\rangle \geq -1;$$

and $\langle\beta_1, \lambda\rangle$ is negative by the choice of α .

In all cases, we have therefore chosen an imaginary root β_1 satisfying (c) of the proposition; and if we put

$$c_1 = -2\langle\beta_1, \tilde{\lambda}\rangle/\langle\beta_1, \beta_1\rangle,$$

then

$$0 < c_1 \leq 1,$$

(in accordance with (a) of the proposition).

We turn next to the choice of β_1 when $\tilde{\lambda}$ is dominant, but is singular with respect to some imaginary root. Put

$$\begin{aligned}\Delta_1 &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^C) \mid \langle\alpha, \tilde{\lambda}\rangle = 0\} \\ \Delta_1^+ &= \Delta_1 \cap \Delta^+.\end{aligned}$$

Then Δ_1 is a root system, stable under θ ; the simple roots of Δ_1^+ are simple for Δ^+ . The hypothesis is that Δ_1^+ contains an imaginary root. If there is a simple imaginary root α , then the considerations of Case III show that α must be non-compact; we take $\beta_1 = \alpha$. So assume that all simple roots of Δ_1^+ are complex. If α is such a root, and α is adjacent to $\theta\alpha$, Case II above shows that we may take

$$\beta_1 = \alpha + \theta\alpha;$$

this is a noncompact imaginary root. So finally assume that α is never adjacent to $\theta\alpha$. Let Γ denote the Dynkin diagram of Δ_1^+ . Then θ defines an automorphism of Γ of order 2, without fixed points, such that $\theta\alpha$ is never adjacent to α . By inspection, one verifies that a simple Dynkin diagram cannot admit such an automorphism; so θ does not fix any component of Γ . We can therefore write

$$\Gamma = \Gamma_0 \cup \Gamma_1,$$

with Γ_0 and Γ_1 disjoint subdiagrams, and

$$\theta: \Gamma_i \rightarrow \Gamma_{1-i} \quad (i = 0, 1)$$

an isomorphism. So Δ_1 is the union of two disjoint subsystems, interchanged by θ ; so Δ_1 contains no imaginary (i.e., θ -fixed)

roots, a contradiction. So in all cases we can find a non-compact imaginary root β_1 satisfying (c) of the proposition, such that if

$$c_1 = -2\langle \beta_1, \tilde{\lambda} \rangle / \langle \beta_1, \beta_1 \rangle,$$

then

$$0 \leq c_1 \leq 1.$$

Now define $\mathfrak{g}_1^1$ as in (d) of the proposition. Put

$$\begin{aligned} \mathfrak{k}^1 &= \mathfrak{g}_1^1 \cap \mathfrak{k} \\ \mathfrak{x}^1 &= \mathfrak{k}^1 \cap \mathfrak{x}^c = \beta_1^1 \cap \mathfrak{x}^c \\ \mu^1 &= \mu|_{\mathfrak{x}^1}. \end{aligned}$$

Let K_0^1 be the subgroup of G with Lie algebra $\mathfrak{k}_0^1$; we use analogous notation for other subgroups. First, we claim that μ^1 is a dominant weight for T_0^1 . Since μ^1 is the restriction to $\mathfrak{x}^1$ of μ , which is the differential of a character, μ^1 is the differential of a character of T_0^1 . So we need only show that μ^1 is dominant. Suppose γ is a positive root of $\mathfrak{x}^1$ in $\mathfrak{k}^1$; say

$$\gamma = \alpha|_{\mathfrak{x}^1}, \quad \alpha \in \Delta^+(\mathfrak{g}_1^1, \mathfrak{k}^1).$$

If α is complex, then α is identified with a positive complex root $\tilde{\alpha}$ of $\mathfrak{h}^c$ in $\mathfrak{g}$ (Proposition 3.2.1), and

$$\langle \gamma, \mu^1 \rangle = \langle \alpha, \mu^1 \rangle = \langle \tilde{\alpha}, \mu \rangle.$$

But, as we saw in Cases I and II earlier, $\tilde{\alpha}|_{\mathfrak{x}^c}$ is a positive root of $\mathfrak{x}^c$ in $\mathfrak{k}$, so

$$\langle \tilde{\alpha}, \mu \rangle \geq 0.$$

So μ^1 is dominant with respect to such α . If α is imaginary, then α is identified with a positive imaginary root $\tilde{\alpha}$ of $\mathfrak{h}$ in $\mathfrak{g}$, and

$$\langle \alpha, \mu^1 \rangle = \langle \tilde{\alpha}, \mu \rangle.$$

If $\tilde{\alpha}$ is compact, this is non-negative. If $\tilde{\alpha}$ is noncompact, then by Proposition 3.2.1.1, $\tilde{\alpha} \pm \beta_1$ are compact roots. Now $\tilde{\alpha}$ is positive, and β_1 was chosen to be simple, at least in the subsystem of imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$. So $\tilde{\alpha} \pm \beta_1$ are positive. So

$$2\langle \tilde{\alpha}, \mu \rangle = \langle \tilde{\alpha} + \beta_1, \mu \rangle + \langle \tilde{\alpha} - \beta_1, \mu \rangle \geq 0,$$

as desired.

Recall that we are using

$$\Delta^+(\mathfrak{g}_1^1) = \Delta^+(\mathfrak{g}) \cap \beta_1^1;$$

write

$$\rho^1 = \rho(\Delta^+(\mathfrak{g}_1^1)), \quad \rho_C^1 = \rho(\Delta^+(\mathfrak{k}^1, \mathfrak{x}^1)).$$

Lemma 5.3.4 (Schmid [36]). With notation as

above,

$$2\rho_C^1 - \rho^1 = (2\rho_C - \rho)|_{\mathfrak{x}^1}.$$

Proof. Assume first that all roots are imaginary. List $\mathfrak{x}^+(\mathfrak{g}, \mathfrak{k}^c)$ according to β_1 strings as

$$\begin{aligned} \Delta^+(\mathfrak{g}, \mathfrak{k}^c) &= \{\beta_1\} \cup \{\alpha_0, \alpha_0 + \beta_1, \dots, \alpha_0 + m_0\beta_1\} \cup \dots \\ &\quad \cup \{\alpha_r, \alpha_r + \beta_1, \dots, \alpha_r + m_r\beta_1\}. \end{aligned}$$

The roots in each string are alternately compact and noncompact.

Define ε_α to be +1 if α is compact and -1 if α is noncompact; set $\varepsilon_i = \varepsilon_{\alpha_i}$. Then

$$2\rho_C - \rho = -\frac{1}{2}\beta_1 + \frac{1}{2}\sum_{i=0}^r \varepsilon_i \sum_{j=0}^{m_i} (-1)^j (\alpha_i + m_j \beta_1)$$

by inspection. The strings of even length contribute a multiple of β_1 to this sum, which is zero on $\mathcal{L}^1$. Recalling that all strings have length 1, 2, 3, or 4, we therefore compute

$$(*) \quad 2\rho_C - \rho \Big|_{\mathcal{L}^1} = \frac{1}{2} \sum_{m_i=0}^r \varepsilon_i \alpha_i + \frac{1}{2} \sum_{m_i=2}^r \varepsilon_i (\alpha_i + \beta_1).$$

The roots of $\mathcal{L}^1$ in $\mathfrak{g}^1$ are identified with the roots of $\mathcal{L}^C$ in $\mathfrak{g}$ orthogonal to β_1^1 ; suppose $\tilde{\alpha} \in \Delta(\mathfrak{g}^1, \mathcal{L}^1)$ corresponds to $\alpha \in \Delta(\mathfrak{g}, \mathcal{L}^C)$. Clearly

$$(\beta_1^1)^\perp \cap \Delta^+(\mathfrak{g}, \mathcal{L}^C) = \{\alpha_i | m_i = 0\} \cup \{\alpha_i + \beta_1 | m_i = 2\};$$

and $\varepsilon_{\tilde{\alpha}} = \varepsilon_\alpha$ for α in the first set, and $-\varepsilon_\alpha$ for α in the second by Proposition 5.3.1. Since

$$\varepsilon_{(\alpha_i + \beta_1)} = -\varepsilon_{\alpha_i} = -\varepsilon_i,$$

we can write (*) as

$$\begin{aligned} 2\rho_C - \rho \Big|_{\mathcal{L}^1} &= \frac{1}{2} \sum_{\tilde{\alpha} \in \Delta^+(\mathfrak{g}^1, \mathcal{L}^1)} \varepsilon_{\tilde{\alpha}} \tilde{\alpha} \\ &= 2\rho_C^1 - \rho^1. \end{aligned}$$

The general case will follow from the next lemma, which is useful in its own right.

Lemma 5.3.5 Let $\tilde{\mathfrak{g}} = \mathfrak{g}^{\mathcal{L}^C}$, the centralizer of α^C in $\mathfrak{g}$; put

$$\rho(\Delta^+(\tilde{\mathfrak{g}}, \mathcal{L}^C)) = \rho(\tilde{\mathfrak{g}}),$$

etc. (Thus $\Delta^+(\tilde{\mathfrak{g}}, \mathcal{L}^C)$ is the system of imaginary roots of $\mathcal{L}^C$ in $\mathfrak{g}$.) Then

$$2\rho_C - \rho = 2\rho_C(\tilde{\mathfrak{g}}) - \rho(\tilde{\mathfrak{g}}).$$

Proof. The complex roots of $\mathcal{L}^C$ in $\mathfrak{g}$ come in pairs $(\alpha, \theta\alpha)$; we must show that such a pair contributes zero to $2\rho_C - \rho$.

But we have seen (for example, in Case I above) that

$$\alpha \Big|_{\mathcal{L}} = \frac{1}{2}(\alpha + \theta\alpha)$$

is a root of $\mathcal{L}$ in $\mathcal{L}$; so the pair $(\alpha, \theta\alpha)$ contributes $\frac{1}{2}(\alpha + \theta\alpha)$ to $2\rho_C$. But obviously it contributes $-\frac{1}{2}(\alpha + \theta\alpha)$ to $-\rho$, so the net contribution is zero. Q.E.D.

Now define

$$\tilde{\lambda}^1 = \mu^1 + 2\rho_C^1 - \rho^1 = \tilde{\lambda} \Big|_{\mathcal{L}^1}.$$

We want to apply the inductive hypothesis to $\mathfrak{g}^1$ and the $\mathcal{L}^1$ -type μ^1 , using the positive root system $\Delta^+(\mathfrak{g}^1, \mathcal{L}^1)$ coming from $\Delta^+(\mathfrak{g}, \mathcal{L}^C)$. To do this, we must check that $\mu^1 + 2\rho_C^1$ is dominant for $\Delta^+(\mathfrak{g}^1, \mathcal{L}^1)$. This will be deduced from the following lemma.

Lemma 5.3.6 Suppose $\beta \in \Delta^+(\mathfrak{g}, \mathcal{L}^C)$ is imaginary, and either β is simple, or there is a simple complex root $\tilde{\beta}$ such that $\beta = \tilde{\beta} + \theta\tilde{\beta}$. Let α be an imaginary root orthogonal to β , which is a simple root in the

positive system $\Delta^+(\mathfrak{g}, \mathfrak{h}^C) \cap \beta^\perp$ for $\Delta(\mathfrak{g}, \mathfrak{h}) \cap \beta^\perp$.

Then

$$2\langle \alpha, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle \geq -1.$$

Proof. (H. Hecht). Using Lemma 5.3.5, we can immediately reduce to the case when there are no complex roots; in particular β is simple. If γ is any positive root in $\Delta(\mathfrak{g}, \mathfrak{h}^C)$ and

$$\gamma = \sum n_i \alpha_i$$

is its expression in terms of simple roots, we put

$$\ell(\gamma) = \sum n_i.$$

We will proceed by induction on $\ell(\alpha)$. Of course we may also assume $\mathfrak{g}$ is simple. The case of type G_2 may be treated in a very straightforward and tedious way by listing all the possibilities for β and α ; this is left to the reader. So we may assume that there are at most two root lengths, differing by a factor of two. Any two roots will span a subsystem of type $A_1 \times A_1$, A_2 , or B_2 ; and we will assume a thorough familiarity with these (see [22], 9.3-9.4).

If $\ell(\alpha) = 1$, α is simple; so

$$\begin{aligned} 2\langle \alpha, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle &= 2\langle \alpha, \tilde{\lambda} + \rho \rangle / \langle \alpha, \alpha \rangle - 2\langle \alpha, \rho \rangle / \langle \alpha, \alpha \rangle \\ (5.3.7) \quad &= 2\langle \alpha, \mu + 2\rho_C \rangle / \langle \alpha, \alpha \rangle - 1 \\ 2\langle \alpha, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle &\geq -1 \end{aligned}$$

since $\mu + 2\rho_C$ is dominant. (This holds whenever α is simple.)

If $\ell(\alpha) = 2$, we write

$$\alpha = \gamma + \delta,$$

with γ and δ simple. Say $\langle \gamma, \gamma \rangle \geq \langle \delta, \delta \rangle$; then

$$\langle \alpha, \alpha \rangle = \langle \delta, \delta \rangle$$

(by inspection in rank 2). If γ is compact, then by the argument given in Case III earlier,

$$2\langle \gamma, \tilde{\lambda} \rangle / \langle \gamma, \gamma \rangle \geq 1;$$

so

$$\begin{aligned} 2\langle \alpha, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle &= 2\langle \gamma, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle + 2\langle \delta, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle \\ &= (2\langle \gamma, \tilde{\lambda} \rangle / \langle \gamma, \gamma \rangle) (\langle \gamma, \gamma \rangle / \langle \delta, \delta \rangle) \\ &\quad + (2\langle \delta, \tilde{\lambda} \rangle / \langle \delta, \delta \rangle) \end{aligned}$$

$$\frac{2\langle \alpha, \tilde{\lambda} \rangle}{\langle \alpha, \alpha \rangle} \geq \frac{\langle \gamma, \gamma \rangle}{\langle \delta, \delta \rangle} + \frac{2\langle \delta, \tilde{\lambda} \rangle}{\langle \delta, \delta \rangle} \geq \frac{\langle \gamma, \gamma \rangle}{\langle \delta, \delta \rangle} - 1 \geq 0;$$

here we have used the fact that α and δ have the same length, (5.3.7) applied to δ , and the preceding inequality for γ . If δ is compact, then since

$$\langle \gamma, \gamma \rangle \leq 2\langle \delta, \delta \rangle,$$

the same argument leads to

$$\begin{aligned} 2\langle \alpha, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle &= 2\langle \gamma, \tilde{\lambda} \rangle / \langle \delta, \delta \rangle + 2\langle \delta, \tilde{\lambda} \rangle / \langle \delta, \delta \rangle \\ &\geq 4\langle \gamma, \tilde{\lambda} \rangle / \langle \gamma, \gamma \rangle + 2\langle \delta, \tilde{\lambda} \rangle / \langle \delta, \delta \rangle \\ &\geq -2 + 1 = -1. \end{aligned}$$

(To be precise, the first inequality may hold only if $\langle \gamma, \tilde{\lambda} \rangle$ is non-positive; but if $\langle \gamma, \tilde{\lambda} \rangle$ is positive, then so is $\langle \alpha, \tilde{\lambda} \rangle$, and the final result still holds.)

If δ and γ are both noncompact, then $\alpha + \delta$ is compact,

so

$$2\langle\alpha, \mu+2\rho_C\rangle/\langle\delta, \delta\rangle \geq 2.$$

Combining Lemma 1.3.6 with

$$\langle\gamma, \gamma\rangle \leq 2\langle\delta, \delta\rangle,$$

we find

$$\begin{aligned} 2\langle\alpha, \rho\rangle/\langle\delta, \delta\rangle &= (2\langle\gamma, \rho\rangle/\langle\gamma, \gamma\rangle)(\langle\gamma, \gamma\rangle/\langle\delta, \delta\rangle) \\ &\quad + 2\langle\delta, \rho\rangle/\langle\delta, \delta\rangle \\ &\leq 1 \cdot 2 + 1 = 3. \end{aligned}$$

Hence,

$$\begin{aligned} 2\langle\alpha, \tilde{\lambda}\rangle/\langle\delta, \delta\rangle &= 2\langle\alpha, \mu+2\rho_C\rangle/\langle\delta, \delta\rangle - 2\langle\alpha, \rho\rangle/\langle\delta, \delta\rangle \\ &\geq 2 - 3 = -1. \end{aligned}$$

This completes the case $\ell(\alpha) = 2$.

These arguments in fact prove

(5.3.8) If γ and δ are adjacent simple (imaginary) roots of the same length, then $\langle\gamma+\delta, \tilde{\lambda}\rangle \geq 0$.

For we have not used the assumption that β is orthogonal to $\gamma+\delta$; we simply repeat the arguments, using

$$\langle\gamma, \gamma\rangle = \langle\delta, \delta\rangle$$

instead of the inequality

$$\langle\gamma, \gamma\rangle \leq 2\langle\delta, \delta\rangle.$$

Suppose now that $\ell(\alpha) > 2$. Choose a simple root γ for

$\Delta^+(\alpha, h^C)$, such that

$$2\langle\alpha, \gamma\rangle/\langle\alpha, \alpha\rangle > 0;$$

then $\alpha-\gamma$ is a root. If γ were orthogonal to β , then

$$\alpha = \gamma + (\alpha-\gamma)$$

would be an expression for α as a sum of positive roots in $\beta^\perp$. Since α is assumed to be simple there, this is a contradiction; so

$$(5.3.9) \quad \langle\beta, \gamma\rangle < 0.$$

We will construct a root $\tilde{\alpha}$ orthogonal to γ , such that

$$\ell(\tilde{\alpha}) < \ell(\alpha).$$

The inductive hypothesis will be applied to γ and $\tilde{\alpha}$, and the resulting inequality applied to get information about α . We separate two cases.

a) $2\langle\alpha, \gamma\rangle/\langle\gamma, \gamma\rangle = 2$. In this case $\alpha-\gamma$ is clearly orthogonal to γ , and

$$\ell(\alpha-\gamma) = \ell(\alpha) - 1.$$

We will apply the inductive hypothesis to $\alpha-\gamma$ and the simple root γ ; so we must show that $\alpha-\gamma$ is simple in $\gamma^\perp$. Normalize $\langle, \rangle$ so that α has length 2; then γ and $\alpha-\gamma$ have length 1. If β has length two, then (5.3.9) implies

$$2\langle\beta, \gamma\rangle/\langle\gamma, \gamma\rangle = -2;$$

so $\beta+\gamma$ has length 1, and

$$\begin{aligned} \langle\beta+\gamma, \alpha-\gamma\rangle &= \langle\beta, \alpha\rangle - \langle\beta, \gamma\rangle + \langle\gamma, \alpha-\gamma\rangle \\ &= -\langle\beta, \gamma\rangle = 1 \end{aligned}$$

since α is orthogonal to β and γ to $\alpha-\gamma$. Since $\alpha-\gamma$ also

has length 1, the Cauchy-Schwarz inequality forces

$$\alpha - \gamma = \beta + \gamma,$$

which is certainly simple in $\gamma^\perp$. So we may assume β has length 1. Suppose that $\alpha - \gamma$ is not simple in $\gamma^\perp$; then we can write

$$\alpha - \gamma = \delta_1 + \delta_2,$$

with δ_1 a positive root orthogonal to γ . So

$$\begin{aligned} \langle \delta_1 + \delta_2, \alpha \rangle &= \langle \delta_1 + \delta_2, \alpha - \gamma \rangle \\ &= \langle \alpha - \gamma, \alpha - \gamma \rangle = 1; \end{aligned}$$

so α has positive inner product with (say) δ_1 . Thus

$$\alpha - \delta_1 = \gamma + \delta_2$$

is a positive root. Since γ is orthogonal to δ_2 , this implies that $\gamma + \delta_2$ has length 2. Now

$$\alpha = \delta_1 + (\gamma + \delta_2),$$

a sum of two positive roots. By the simplicity of α in $\beta^\perp$, these cannot be orthogonal to β ; so for some $\epsilon = \pm 1$,

$$(\gamma + \delta_2) + \epsilon\beta, \delta_1 - \epsilon\beta$$

are also roots; they are positive since β is simple. (Since γ is orthogonal to δ_1 and β is not, $\delta_1 \neq \beta$; and $\gamma + \delta_2$ is not simple, so it cannot be equal to β either.) But $\gamma + \delta_2$ is long and β is short; so $\gamma + \delta_2 + \epsilon\beta$ is orthogonal to β , and the expression

$$\alpha = [\delta_1 - \epsilon\beta] + [\gamma + \delta_2 + \epsilon\beta]$$

contradicts the simplicity of α in $\beta^\perp$. So $\alpha - \gamma$ is in fact

simple in $\gamma^\perp$. By induction and (5.3.7),

$$\begin{aligned} 2\langle \alpha - \gamma, \tilde{\lambda} \rangle / \langle \alpha - \gamma, \alpha - \gamma \rangle &\geq -1 \\ 2\langle \gamma, \tilde{\lambda} \rangle / \langle \gamma, \gamma \rangle &\geq -1 \end{aligned}$$

If we add these, and use the fact that

$$\langle \alpha, \alpha \rangle = 2\langle \gamma, \gamma \rangle = 2\langle \alpha - \gamma, \alpha - \gamma \rangle$$

we obtain

$$2\langle \alpha, \tilde{\lambda} \rangle / \langle \alpha, \alpha \rangle \geq -1,$$

as desired. This completes case (a).

b) $2\langle \alpha, \gamma \rangle / \langle \gamma, \gamma \rangle = 1$. Since α is orthogonal to β , (5.3.9) implies that

$$\langle \beta, \alpha - \gamma \rangle > 0;$$

so $\alpha - \gamma - \beta$ is either zero or a root. Since $\ell(\alpha) > 2$, it cannot be zero, so it is a root. Furthermore

$$\begin{aligned} 2\langle \gamma, \alpha - \gamma - \beta \rangle / \langle \gamma, \gamma \rangle &= 1 - 2 - 2\langle \gamma, \beta \rangle / \langle \gamma, \gamma \rangle \\ (5.3.10) \qquad \qquad \qquad &= -(1 + 2\langle \gamma, \beta \rangle / \langle \gamma, \gamma \rangle). \end{aligned}$$

We claim that

$$\tilde{\alpha} = \alpha - \gamma - \beta$$

is orthogonal to γ ; by 5.3.10, this is the same as showing

$$(5.3.11) \qquad 2\langle \gamma, \beta \rangle / \langle \gamma, \gamma \rangle = -1.$$

Suppose not; then (5.3.9) implies that β is long, γ is short, and $2\gamma + \beta$ is a root orthogonal to β . Since

$$\langle \alpha, 2\gamma + \beta \rangle = 2\langle \alpha, \gamma \rangle < 0,$$

$\alpha - (2\gamma + \beta)$ is either 0 or a root orthogonal to β . Since $\ell(\alpha) > 2$, the second case would contradict the simplicity of α in $\beta^\perp$; so

$$\alpha = 2\gamma + \beta.$$

So α is a long root. But γ is short; so this contradicts the hypothesis

$$(5.3.12) \quad 2\langle \alpha, \gamma \rangle / \langle \gamma, \gamma \rangle = 1.$$

So $\tilde{\alpha}$ is orthogonal to γ .

Next we claim that γ and β have the same length. Suppose not. By (5.3.11), γ is long and β is short; so $\gamma + \beta$ is orthogonal to β , and

$$\alpha = (\alpha - \gamma - \beta) + (\gamma + \beta)$$

contradicts the simplicity of α in $\beta^\perp$. So γ and β , and hence also $\gamma + \beta$, have the same length. Finally (5.3.12) implies that α is not longer than γ , so it is not longer than $\gamma + \beta$; so

$$(5.3.13) \quad \langle \alpha, \alpha \rangle = \langle \alpha - \gamma - \beta, \alpha - \gamma - \beta \rangle = \langle \tilde{\alpha}, \tilde{\alpha} \rangle.$$

We want to apply the inductive hypothesis to $\tilde{\alpha}$ and γ , so we need to show that $\tilde{\alpha}$ is simple in $\gamma^\perp$. Suppose not; write

$$\tilde{\alpha} = \delta_1 + \delta_2,$$

with δ_i a positive root orthogonal to γ . Now

$$(5.3.14) \quad \begin{aligned} 2\langle \gamma + \beta, \delta_1 + \delta_2 \rangle / \langle \gamma + \beta, \gamma + \beta \rangle &= 2\langle \gamma + \beta, \alpha \rangle / \langle \gamma + \beta, \gamma + \beta \rangle - 2 \\ &= -1 \end{aligned}$$

by (5.3.12) and the fact that γ and $\gamma + \beta$ have the same length.

So one of the δ_i , say δ_2 , has negative inner product with $\gamma + \beta$. So $\gamma + \beta + \delta_2$ is a positive root; and

$$\alpha = \delta_1 + (\gamma + \beta + \delta_2).$$

By the simplicity of α in $\beta^\perp$,

$$\begin{aligned} 0 \neq 2\langle \gamma + \beta + \delta_2, \beta \rangle &= 2\langle \gamma + \beta, \beta \rangle + 2\langle \delta_2, \beta \rangle \\ &= \langle \beta, \beta \rangle \left(1 + \frac{2\langle \delta_2, \beta \rangle}{\langle \beta, \beta \rangle} \right), \end{aligned}$$

since γ and β span an A_2 . Since δ_2 has negative inner product with $\gamma + \beta$, and is orthogonal to γ , the second term on the right is negative. Since the sum is non-zero,

$$\frac{2\langle \delta_2, \beta \rangle}{\langle \beta, \beta \rangle} = -2.$$

By (5.3.14) and the fact that the δ_i are orthogonal to γ , this gives

$$\frac{2\langle \delta_1, \beta \rangle}{\langle \beta, \beta \rangle} = 1.$$

It follows that $\tilde{\delta}_1 = \delta_1 - (\gamma + \beta)$ and $\tilde{\delta}_2 = \delta_2 + 2(\gamma + \beta)$ are non-zero roots orthogonal to β . The second is obviously positive, and $\tilde{\delta}_1$ is positive unless δ_1 is γ or β , which is impossible since neither of these is orthogonal to γ . So

$$\alpha = \tilde{\delta}_1 + \tilde{\delta}_2$$

contradicts the simplicity of α in $\beta^\perp$. So $\tilde{\alpha}$ is simple in $\gamma^\perp$.

By induction,

$$\frac{2\langle \tilde{\alpha}, \tilde{\lambda} \rangle}{\langle \tilde{\alpha}, \tilde{\alpha} \rangle} \geq -1.$$

By (5.3.8), and the observation above that γ and β have the same length,

$$\langle \gamma + \beta, \tilde{\lambda} \rangle \geq 0.$$

Finally, we have seen that α and $\tilde{\alpha}$ have the same length; so summing the last two inequalities gives

$$2\langle\tilde{\alpha}+\gamma+\beta, \tilde{\lambda}\rangle/\langle\alpha, \alpha\rangle \geq -1.$$

Since $\alpha = \tilde{\alpha}+\gamma+\beta$, this completes case (b) and proves the lemma.

Q.E.D.

Returning to the notation used before the lemma, we wanted to show that $\mu^1 + 2\rho_C^1$ is dominant for $\Delta^+(\sigma_j^1, h^1)$. So let α be a simple root of $\Delta^+(\sigma_j^1, h^1)$. If α is complex, then $\alpha|_{h^1}$ can be regarded as a positive root of $\mathcal{L}^1$ in h^1 ; so

$$\langle\alpha, \mu^1 + 2\rho_C^1\rangle > 0$$

since μ^1 is dominant for $\Delta^+(h^1, \mathcal{L}^1)$. So suppose α is imaginary. By Proposition 5.3.1, we identify α with a root of h^C in σ_j orthogonal to β . Then by Lemma 5.3.4,

$$\begin{aligned} 2\langle\alpha, \mu^1 + 2\rho_C^1\rangle/\langle\alpha, \alpha\rangle &= 2\langle\alpha, \mu^1 + 2\rho_C^1 - \rho^1\rangle/\langle\alpha, \alpha\rangle + 1 \\ &= 2\langle\alpha, \tilde{\lambda}^1\rangle/\langle\alpha, \alpha\rangle + 1 \\ &= 2\langle\alpha, \tilde{\lambda}\rangle/\langle\alpha, \alpha\rangle + 1 \\ &\geq -1 + 1 = 0. \end{aligned}$$

Here the last equality is Lemma 5.3.4, and the inequality is Lemma 5.3.6. So $\mu^1 + 2\rho_C^1$ is dominant for $\Delta^+(\sigma_j^1, h^1)$.

By inductive hypothesis, we can choose an orthogonal set $\{\beta_2, \dots, \beta_r\} \subseteq \Delta^+(\sigma_j^1, h^1)$, with the properties specified in the proposition. We also regard the β_i as roots of h in σ_j , orthogonal to β_1 ; so $\{\beta_1, \dots, \beta_r\}$ is an orthogonal set of imaginary roots. If c_i is defined as in (a) of the proposition, then we have already shown that

$$0 \leq c_i \leq 1$$

when $i = 1$; for $i > 1$ it follows by induction. We define

$$(5.3.15) \quad \lambda = \mu + 2\rho_C - \rho + \frac{1}{2} \sum c_i \beta_i = \tilde{\lambda} + \frac{1}{2} \sum c_i \beta_i;$$

then (a) holds. For (b) of the proposition, suppose α is imaginary, and

$$\langle\alpha, \tilde{\lambda}\rangle = 0.$$

If α is not orthogonal to β_1 , then the conclusion of (b) is immediate. So suppose it is. By our definitions,

$$\lambda|_{\mathcal{L}^1} = \lambda^1;$$

here λ^1 means the weight associated to μ^1 by the proposition. So α may be regarded as an imaginary root in $\Delta(\sigma_j^1, h^1)$, with

$$\langle\alpha, \lambda^1\rangle = 0.$$

So the conclusion of (b) follows by induction. We have chosen β_1 to satisfy (c), and (d) holds by definition. For (e), suppose $c_i \neq 0$, $c_j = 0$, and $i > j$; we want to get a contradiction. By induction, we must have $j = 1$. By our choice of β_1 , c_1 is zero only when $\tilde{\lambda}$ is actually dominant. Thus

$$c_i = -2\langle\beta_i, \tilde{\lambda}\rangle/\langle\beta_i, \beta_i\rangle \leq 0;$$

since $0 \leq c_i \leq 1$, this forces $c_i = 0$, a contradiction. To complete the existence half of the proof, it remains only to show that λ (as defined by (5.3.15)) is dominant.

Let α be a simple root of $\Delta^+(\sigma_j, h)$; we must show

$$\langle\alpha, \lambda\rangle \geq 0.$$

If α is orthogonal to β_1 , this follows from the inductive

hypothesis. If $\alpha = \beta_1$, it is clear from the definition that

$$(5.3.16) \quad \langle \beta_1, \lambda \rangle = 0.$$

So we may suppose that α and β_1 span a simple subsystem of rank 2. Suppose there is a complex simple root δ such that

$$\beta_1 = \delta + \theta\delta.$$

Since $\langle \alpha, \beta_1 \rangle \neq 0$, α is adjacent to one of δ and $\theta\delta$, say to δ , without loss of generality; and δ is adjacent to $\theta\delta$, since $\delta + \theta\delta$ is a root. Since θ is an automorphism, $\theta\alpha$ is adjacent to $\theta\delta$. Since a Dynkin diagram can contain no loops, we must have

$$\alpha \neq \theta\alpha,$$

and the piece of the Dynkin diagram containing these four simple roots looks like

$$\alpha \quad \delta \quad \theta\delta \quad \theta\alpha \\ \cdot \quad \cdot \quad \cdot \quad \cdot$$

(conceivably with some double bonds). But δ and $\theta\delta$ have the same length, as do α and $\theta\alpha$; so by the classification, we see that the four roots generate an A_4 . Therefore

$$\gamma = \alpha + \delta + \theta\delta + \theta\alpha$$

is a root, orthogonal to β_1 ; and

$$\begin{aligned} \langle \alpha, \lambda \rangle &= \frac{1}{2} \langle \alpha + \theta\alpha, \lambda \rangle \\ &= \frac{1}{2} \langle \alpha + \theta\alpha + \beta_1, \lambda \rangle \\ &= \frac{1}{2} \langle \gamma, \lambda \rangle \\ &\geq 0; \end{aligned}$$

the second equality uses (5.3.16), and the final inequality follows from the inductive hypothesis. So we may assume that β_1 is itself simple. If $\theta\alpha \neq \alpha$, a very similar argument shows that the Dynkin diagram contains

$$\alpha \quad \beta_1 \quad \theta\alpha \\ \cdot \quad \cdot \quad \cdot$$

a subsystem of type A_3 ; so

$$\gamma = \alpha + \beta_1 + \theta\alpha$$

is a root, orthogonal to β_1 . Arguing exactly as above, we deduce that

$$\langle \alpha, \lambda \rangle \geq 0.$$

So we may assume that α is imaginary. Lemma 5.3.5 allows us to pass to $\alpha\bar{\gamma}$ without changing λ ; so for the rest of the proof that λ is dominant, we may assume that there are no complex roots. If the span of α and β_1 is of type B_2 or G_2 , there are positive integers s and t such that $s\alpha + t\beta_1$ is a positive root orthogonal to β_1 ; so

$$\langle \alpha, \lambda \rangle = \frac{1}{s} \langle s\alpha + t\beta_1, \lambda \rangle \geq 0$$

by (5.3.16) and inductive hypothesis. So we may as well assume that α and β_1 generate a subalgebra of type A_2 . For convenience we normalize $\langle \cdot, \cdot \rangle$ so that α and β_1 have length 2. Then

$$(5.3.17) \quad \langle \alpha, \beta_1 \rangle = -1,$$

so the orthogonal projection of α on β_1 has (squared) length $1/2$; we express this by saying that one-fourth of the length

Suppose there are three β_{i_j} . Now $\alpha + \beta_1$ is a root since α and β_1 span an A_2 . Since $\{\beta_{i_j}\}$ is an orthogonal set,

$$\langle \beta_{i_2}, \alpha + \beta_1 \rangle = \langle \beta_{i_2}, \alpha \rangle < 0;$$

so $\alpha + \beta_1 + \beta_{i_2}$ is a root; and similarly (adding first β_{i_3} , then α)

$$\gamma = 2\alpha + \beta_1 + \beta_{i_2} + \beta_{i_3}$$

is a root. Since α and β_1 span an A_2 , one checks immediately that γ is orthogonal to β_1 ; so

$$\langle \gamma, \lambda \rangle \geq 0$$

by induction. Since λ is orthogonal to all β_{i_j} , it follows that

$$\langle \alpha, \lambda \rangle \geq 0.$$

So we may assume that there are at most two β_{i_j} . By (5.3.18) and the definition of $\tilde{\lambda}$,

$$\begin{aligned} \langle \alpha, \lambda \rangle &= \langle \alpha, \tilde{\lambda} \rangle + \frac{1}{2} \sum_i c_i \langle \alpha, \beta_i \rangle \\ &\geq \langle \alpha, \tilde{\lambda} \rangle + \frac{1}{2} \sum_j c_{i_j} \langle \alpha, \beta_{i_j} \rangle. \end{aligned}$$

We have normalized $\langle \cdot, \cdot \rangle$ so that the β_{i_j} have length 2; so (5.3.17) and the definition of c_i give

$$\begin{aligned} \langle \alpha, \lambda \rangle &\geq \langle \alpha, \tilde{\lambda} \rangle + \frac{1}{2} \sum_j \langle \beta_{i_j}, \tilde{\lambda} \rangle, \\ (5.3.19) \quad &-1 \leq \langle \beta_{i_j}, \tilde{\lambda} \rangle \leq 0. \end{aligned}$$

We now separate three possibilities.

- a) α compact. From Case III at the beginning of the proof, we know that $\langle \alpha, \tilde{\lambda} \rangle \geq 1$;

so, using the fact that the sum in (5.3.19) has

of α lies in the β_1 direction. Define

$$(5.3.18) \quad \{\beta_{i_j}\}_{i=1}^t = \{\beta_i \mid \langle \alpha, \beta_i \rangle < 0\},$$

with

$$1 = i_1 < i_2 < \dots < i_t.$$

Fix j , with $1 < j \leq t$.

If α and β_{i_j} span a G_2 , then α is contained in the span of β_1 and β_{i_j} (since G_2 is not contained in any larger root system). Since λ is orthogonal to β_1 and β_{i_j} , it follows that

$$\langle \alpha, \lambda \rangle = 0.$$

So we may assume that this never happens. If α and β_{i_j} span a B_2 , then for $s = 1$ or $s = 2$,

$$\beta_{i_j} + s\alpha$$

is a root of length different from the length of α and β_1 . So for some positive integers s' and t' ,

$$s'(\beta_{i_j} + s\alpha) + t'\beta_1$$

is a positive root orthogonal to β_1 . Just as in the case when α and β_1 span a B_2 or G_2 , this implies that

$$\langle \alpha, \lambda \rangle \geq 0$$

by inductive hypothesis. So we may as well assume that α and β_{i_j} span an A_2 . So one-fourth of the length of α lies in the β_{i_j} direction. Since the β_{i_j} are orthogonal, there are therefore at most four of them. If there are four, then α lies in their span, which is orthogonal to λ , so $\langle \alpha, \lambda \rangle = 0$.

at most two terms, we deduce immediately that

$$\langle \alpha, \lambda \rangle \geq 1 + \frac{1}{2}(-1 + -1) = 0.$$

b) α noncompact, $\{\beta_{i,j}\} = \{\beta_1\}$. By (5.3.8),

$$\langle \alpha + \beta_1, \tilde{\lambda} \rangle \geq 0.$$

Since also

$$\langle \beta_1, \tilde{\lambda} \rangle \leq 0,$$

we compute from (5.3.19) that

$$\begin{aligned} \langle \alpha, \lambda \rangle &\geq \langle \alpha + \frac{1}{2} \beta_1, \tilde{\lambda} \rangle \\ &\geq \langle \alpha + \beta_1, \tilde{\lambda} \rangle \geq 0. \end{aligned}$$

c) α noncompact, $\{\beta_{i,j}\} = \{\beta_1, \beta_t\}$. Here (5.3.19) may be written as

$$\begin{aligned} \langle \alpha, \lambda \rangle &= \langle \alpha, \tilde{\lambda} \rangle + \frac{1}{2} \langle \beta_1, \tilde{\lambda} \rangle + \frac{1}{2} \langle \beta_t, \tilde{\lambda} \rangle \\ &= \frac{1}{2} \langle \alpha + \beta_1, \tilde{\lambda} \rangle + \frac{1}{2} \langle \alpha + \beta_t, \tilde{\lambda} \rangle. \end{aligned}$$

By (5.3.8), the first term is non-negative; it will suffice to show that the second is also. If

$$\langle \beta_1, \tilde{\lambda} \rangle = 0,$$

then by the construction of β_1 , $\lambda = \tilde{\lambda}$ is dominant, and we are done. So we may assume

$$\langle \beta_1, \tilde{\lambda} \rangle \neq 0;$$

by the second inequality of (5.3.19),

$$-1 \leq \varepsilon = \langle \beta_1, \tilde{\lambda} \rangle < 0.$$

Now G is assumed to be linear, so μ is integral with respect

to all (imaginary) roots; so

$$\tilde{\lambda} = \mu + 2\rho_C - \rho$$

is also, since $2\rho_C$ is a sum of roots and ρ is integral. So ε must be an integer (recall that β_1 has length 2); so

$$\langle \beta_1, \tilde{\lambda} \rangle = -1.$$

Now (5.3.8) implies that

$$\langle \alpha, \tilde{\lambda} \rangle \geq 1;$$

so by the second inequality of (5.3.19),

$$\langle \alpha + \beta_t, \tilde{\lambda} \rangle \geq 0,$$

as we wished to prove. (The use of the linearity of G in this argument can be removed without much difficulty - see [42], p. 27).

This completes the proof that λ is dominant, and thus the "existence" part of the proof of Proposition 5.3.3. For the uniqueness, we will use the following lemma.

Lemma 5.3.20 Suppose λ satisfies 5.3.3 (a) - (e), for some $\{\beta_i\}$. Suppose that there are no complex roots of $\mathcal{h}$ in $\mathcal{O}_f$. Let β' be a simple (imaginary) root of $\mathcal{h}$ in $\mathcal{O}_f$ such that

$$\langle \beta', \tilde{\lambda} \rangle < 0.$$

Let $(\mathcal{O}_f^1)' = \mathcal{O}_f^{\beta'}$ be as in Proposition 5.3.1; define $(\mathcal{O}_f^1)' = \mathcal{O}_f^{\beta'}$, $(\mu^1)' = \mu|_{(\mathcal{O}_f^1)'}$, etc. Then the restriction of λ to $(\mathcal{O}_f^1)'$ satisfies 5.3.3 (a) - (e) for the algebra $(\mathcal{O}_f^1)'$ and the weight $(\mu^1)'$.

Proof. Notice first that by the calculations under "Case III" at the beginning of the proof of the proposition, β' is necessarily noncompact; so the statements of the proposition make sense. We proceed by induction on $\dim \mathcal{L}^C$. Since $\langle \beta', \tilde{\lambda} \rangle$ is negative, $\tilde{\lambda}$ is not dominant, so $\tilde{\lambda} \neq \lambda$, the set $\{\beta_i\}$ is non-empty, and some c_i is non-zero. By 5.3.3(e), $c_1 \neq 0$, i.e.,

$$\langle \beta_1, \tilde{\lambda} \rangle < 0.$$

Suppose first that β_1 and β' are orthogonal. Then β' may be identified with a simple imaginary root of $\mathcal{h}^1$ in $\mathcal{g}^1$; and thanks to Lemma 5.3.4, we can apply Lemma 5.3.20 with $\mathcal{g}$ replaced by $\mathcal{g}^1$. The conclusion is that $\lambda|_{\mathcal{L}_n^1(\mathcal{L}^1)}$ satisfies 5.3.3 (a) - (e) for $\mathcal{g}_n^1(\mathcal{g}^1)$. By the inductive constructions of the "existence" proof already given, this implies that $\lambda|_{(\mathcal{L}^1)}$ satisfies 5.3.3 (a) - (e), as claimed. If $\beta_1 = \beta'$, the lemma is obvious.

So we may assume that β_1 and β' are distinct but adjacent. As we showed at the end of the existence proof, the linearity of G implies that

$$c_1 = -2\langle \beta_1, \tilde{\lambda} \rangle / \langle \beta, \beta \rangle$$

is an integer; since it is positive and bounded by 1, it is 1. Similarly

$$c' = -2\langle \beta', \tilde{\lambda} \rangle / \langle \beta', \beta' \rangle$$

is a positive integer; the same proof given for β_1 (see Case III at the beginning of the proof) shows that c' is bounded by 1, so c' is 1. By Lemma 1.3.6,

$$\mu + 2\rho_C = \tilde{\lambda} + \rho$$

is orthogonal to β_1 and β' . Put

$$\alpha = \beta_1 + \beta';$$

this is a compact positive root since β_1 and β' are adjacent noncompact roots. So

$$\langle \alpha, \mu + 2\rho_C \rangle \geq \langle \alpha, 2\rho_C \rangle > 0,$$

contradicting our previous calculation that β_1 and β' are orthogonal to $\mu + 2\rho_C$. This contradiction completes the proof. Q.E.D.

Again, the use of linearity can be eliminated with a little work. For the proof that λ is unique, let (Δ^+) be another θ -stable positive system for $\mathcal{h}^C$ in $\mathcal{g}$, such that $\mu + 2\rho_C$ is dominant for (Δ^+) ; and let $\lambda' \in (\mathcal{L}^C)^*$ be another element satisfying 5.3.3 (a) - (e), say for a set $\{\beta_i'\}$ of imaginary roots. We want to show $\lambda = \lambda'$. Using Lemma 5.3.5, we see that λ and λ' both satisfy the proposition for $\mathcal{g}$; so we may as well assume that all roots are imaginary.

If $\tilde{\lambda}$ is dominant, then

$$\mu + 2\rho_C = \tilde{\lambda} + \rho$$

is regular, so $\Delta^+ = (\Delta^+)$; and

$$\lambda = \tilde{\lambda} = \lambda',$$

and we are done. Otherwise c_1 and c_1' are both non-zero. Let β' be either β_1' or $-\beta_1'$, whichever lies in Δ^+ . We claim that β' is actually simple in Δ^+ . Arguing as in the proof of 5.3.20, we find that

$$2\langle \beta'_1, \tilde{\lambda}' \rangle / \langle \beta'_1, \beta'_1 \rangle = -1,$$

so that by Lemma 3.2.4,

$$(5.3.21) \text{ (a)} \quad \langle \beta', \mu + 2\rho_C \rangle = 0.$$

If β' is not simple, then

$$\beta' = \gamma_1 + \gamma_2$$

with γ_1 in Δ^+ ; and (5.3.21) (a), together with the fact that $\mu + 2\rho_C$ is dominant for Δ^+ , implies that

$$\langle \gamma_1, \mu + 2\rho_C \rangle = 0.$$

So β', γ_1 , and γ_2 must all be noncompact. But the sum of two noncompact roots is compact, so this is impossible. So β' is simple. Now apply Lemma 5.3.20; we conclude that $\lambda \Big|_{(\mathcal{L}^1)}$ satisfies 5.3.3 (a) - (e). But, by 5.3.3 (d), $\lambda' \Big|_{(\mathcal{L}^1)}$, also satisfies 5.3.3 (a) - (e). So by an application of the uniqueness theorem to $(\mathcal{L}^1)'$ (which is available by induction), we conclude that

$$\lambda \Big|_{(\mathcal{L}^1)'} = \lambda' \Big|_{(\mathcal{L}^1)'}$$

Since $(\mathcal{L}^C)^*$ is spanned by $(\mathcal{L}^1)^*$ and β' , it is enough to prove

$$(5.3.21) \text{ (b)} \quad \langle \lambda, \beta' \rangle = \langle \lambda', \beta' \rangle;$$

this will show that $\lambda = \lambda'$. The right side is zero, so we must show that the left is also. Suppose not; then since λ is dominant,

$$(5.3.21) \text{ (c)} \quad \langle \lambda, \beta' \rangle > 0.$$

All the β_i are orthogonal to λ , so

$$\langle \lambda, \beta' - \beta_i \rangle > 0;$$

so $\beta' - \beta_i$ is non-zero, and is positive if it is a root. Since β' is simple, $\beta' - \beta_i$ cannot be a positive root; so it is neither zero nor a root, and

$$\langle \beta', \beta_i \rangle \leq 0,$$

for all i . Hence

$$\begin{aligned} \langle \beta', \lambda \rangle &= \langle \beta', \tilde{\lambda} \rangle + \frac{1}{2} \sum c_i \langle \beta', \beta_i \rangle \\ &\leq \langle \beta', \tilde{\lambda} \rangle < 0 \end{aligned}$$

(by (5.3.21) (a)), which contradicts (5.3.21) (c). This proves (5.3.21) (b), and hence the uniqueness. Q.E.D.

Definition 5.3.22 If $\mu \in \hat{T}^C$ is dominant, let $\lambda \in (\mathcal{L}^C)^*$ be the element associated to μ by Proposition 5.3.3.

Let

$$\mathcal{O}f = \ell + \mathcal{U}$$

be the parabolic subalgebra defined by λ , as in (5.2.1).

We call $\mathcal{O}f$ the parabolic associated to μ . If μ is a highest weight of a K-type π , we also call $\mathcal{O}f$ a parabolic associated to π .

Lemma 5.3.23 Suppose $\mu \in \hat{T}^C$ is ${}^+_{\Delta}(\mathcal{K}, \mathcal{L}^C)$ -dominant; let $\mathcal{O}f = \ell + \mathcal{U}$ be the parabolic subalgebra associated to μ . Then ℓ_0 is quasisplit, and the parabolic subalgebra

of $\mathfrak{l}$ associated to the weight $\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})$ is $\mathfrak{l}$ itself.

Proof. Choose $\{\beta_1\}$ as in Proposition 5.3.3. Since β_1 is orthogonal to λ , it is a root of $\mathfrak{h}^C$ in $\mathfrak{l}$; so the new Cartan subalgebra $\mathfrak{h}_0^1$, constructed from β_1 in Proposition 3.2.1, is contained in $\mathfrak{l}$. Since $\lambda \in (\mathfrak{t}^C)^*$ is zero on β_1 , we can identify λ with a linear functional $\lambda^1 \in (\mathfrak{t}^1)^* \subseteq (\mathfrak{h}^1)^*$. It is easy to verify that

$$\begin{aligned}\Delta(\mathfrak{g}, \mathfrak{h}^1) &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^1) \mid \langle \alpha, \lambda^1 \rangle \geq 0\} \\ \Delta(\mathfrak{l}, \mathfrak{h}^1) &= \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^1) \mid \langle \alpha, \lambda^1 \rangle = 0\}.\end{aligned}$$

We now iterate this procedure: β_2 may be regarded as a non-compact imaginary root of $\mathfrak{h}_0^1$ in $\mathfrak{g}_0^1$ (the centralizer of $Z\beta_1$ in $\mathfrak{g}_0^1$), and we can therefore use it to define a more split Cartan subalgebra $\mathfrak{h}_0^2$ of $\mathfrak{g}_0^1$. Again $\mathfrak{h}^2 \subseteq \mathfrak{l}$, and λ^1 may be identified with $\lambda^2 \in (\mathfrak{t}^2)^* \subseteq \mathfrak{h}^2$. Continuing in this way, we finally obtain a Cartan subalgebra $\mathfrak{h}_0^r \subseteq \mathfrak{l}_0$, such that

$$(5.3.23) \quad \left\{ \begin{array}{ll} \text{(a)} & \mathfrak{h}_0^r = \mathfrak{t}_0^r \oplus \mathfrak{o}_0^r \quad (\mathfrak{t}_0^r \subseteq \mathfrak{t}_0^C, \mathfrak{o}_0^r \subseteq \mathfrak{p}) \\ \text{(b)} & \lambda^r \in (\mathfrak{t}^r)^* \text{ corresponds to } \lambda. \\ \text{(c)} & \mathfrak{o}_0^r = \mathfrak{o}_0^C \oplus \langle Z\beta_1 \rangle \oplus \dots \oplus \langle Z\beta_r \rangle, \text{ an orthogonal decomposition.} \\ \text{(d)} & \text{if } \mathfrak{g}^r = \mathfrak{g}^{\langle Z\beta_1, \dots, Z\beta_r \rangle}, \text{ then the root system of } \mathfrak{h}^r \text{ in } \mathfrak{g}^r \text{ can be identified (by restricting roots to } \mathfrak{t}^r + \mathfrak{o}_0^C \text{) with the roots of } \mathfrak{h} \text{ in } \mathfrak{g} \text{ orthogonal to } \langle \beta_1, \dots, \beta_r \rangle. \end{array} \right.$$

Because of b), it is clear that

$$(5.3.23) \quad \left\{ \begin{array}{ll} \text{(e)} & \Delta(\mathfrak{g}, \mathfrak{h}^r) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^r) \mid \langle \alpha, \lambda^r \rangle \geq 0\} \\ & \Delta(\mathfrak{l}, \mathfrak{h}^r) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^r) \mid \langle \alpha, \lambda^r \rangle = 0\}. \end{array} \right.$$

We claim that $\mathfrak{h}_0^r$ is a maximally split Cartan subalgebra of $\mathfrak{l}_0$, i.e., that $\mathfrak{o}_0^r$ is a maximal abelian subalgebra of $\mathfrak{l}_0 \cap \mathfrak{p}_0$. So let $\mathfrak{s}_0$ denote the centralizer of $\mathfrak{o}_0^r$ in $\mathfrak{l}_0$; we must show that $\mathfrak{s}_0 \cap \mathfrak{p}_0 = \mathfrak{o}_0^r$. Because of (d), the root system

$$\Delta(\mathfrak{s}, \mathfrak{h}^r)$$

can be identified with

$$\{\alpha \in \Delta(\mathfrak{l}, \mathfrak{h}) \mid \alpha \text{ is imaginary and } \langle \alpha, \beta_i \rangle = 0 \text{ for all } i\}.$$

By (e), then, this is

$$\{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \alpha \text{ imaginary, } \langle \alpha, \lambda \rangle = 0, \text{ and } \langle \alpha, \beta_i \rangle = 0, \text{ all } i\}.$$

Because of 5.3.3(b), this set is empty, i.e.,

$$\mathfrak{s} = \mathfrak{l}_0^r = \mathfrak{h}^r.$$

In particular $\mathfrak{o}_0^r$ is in fact maximal abelian in $\mathfrak{l}_0 \cap \mathfrak{p}_0$, which proves the claim. But we have proved even more; since $\mathfrak{h}_0^r$ is maximally split in $\mathfrak{l}_0$, $\mathfrak{s}_0$ is the Levi factor of a minimal parabolic subalgebra of $\mathfrak{l}_0$. So this minimal parabolic is a Borel subalgebra of $\mathfrak{l}_0$, so $\mathfrak{l}_0$ is quasisplit.

Finally, we compute the parabolic subalgebra of $\mathfrak{l}$ associated to the weight $\mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})$. It is clear from Proposition 5.3.3 that this parabolic will not change if we tensor the weight with a one-dimensional representation of L ; for this adds to the highest weight a term orthogonal to all the roots of L . So we tensor with $\Lambda^{\dim \mathfrak{u}}(\mathfrak{u})$, i.e., we add $2\rho(\mathfrak{u})$

to the weight. So we consider instead the weight $\mu + 2\rho(\alpha, h)$. To find the corresponding λ , we must add to this the sum of the positive roots of $\mathfrak{t}^C$ in $\mathfrak{l} \cap \mathfrak{k}$; clearly the sum is $\mu + 2\rho_C$. Next we must choose a positive root system for $\mathfrak{h}^C$ in $\mathfrak{l}$, making $\mu + 2\rho_C$ dominant. We take

$$\Delta^+(\mathfrak{g}, \mathfrak{h}^C) \cap \Delta(\mathfrak{l}, \mathfrak{h}^C),$$

with $\Delta^+(\mathfrak{g}, \mathfrak{h}^C)$ the positive system used to get the β_i mentioned above. We claim that the element associated to $\mu + 2\rho(\alpha \cap \mathfrak{k})$ is

$$\lambda(\mathfrak{l}) = \mu + 2\rho_C - \rho(\Delta^+(\mathfrak{l}, \mathfrak{h})) + \frac{1}{2} \sum c_i \beta_i,$$

with the same β_i and c_i as appear for μ . To see this, notice first that the β_i are all roots of $\mathfrak{h}$ in $\mathfrak{l}$, since they are orthogonal to λ . That they satisfy 5.3.3 (a) - (e) is obvious from the corresponding fact for $\mathfrak{g}$ and μ , if we recall that

$$\langle \alpha, \rho \rangle = \langle \alpha, \rho(\Delta^+(\mathfrak{l}, \mathfrak{h})) \rangle$$

for all $\alpha \in \Delta(\mathfrak{l})$ (as follows from Lemma 3.2.4). Finally, this same observation shows that $\lambda(\mathfrak{l})$ is orthogonal to all the roots of $\mathfrak{l}$ (since λ is); so the parabolic subalgebra of $\mathfrak{l}$ defined by $\lambda(\mathfrak{l})$ is $\mathfrak{l}$ itself.

Q.E.D.

Definition 5.3.24 A $\Delta^+(\mathfrak{k})$ -dominant weight μ is called small (with respect to G) if the parabolic subalgebra of $\mathfrak{g}$ associated to μ by (5.3.22) is all of $\mathfrak{g}$. A representation $\pi \in \hat{K}$ is called small if it has a small highest weight.

Corollary 5.3.25 In the setting of Lemma 5.3.23, $\mu - 2\rho(\alpha \cap \mathfrak{k})$ is small with respect to L .

This is just a restatement of part of the lemma. We will show in the next section that a K type is fine if and only if it is small (cf. Definition 4.3.9). We can prove one implication now, namely

Proposition 5.3.26 Suppose $\pi \in \hat{K}$ is small with respect to G . Then G is quasisplit, and π is fine.

Proof. Fix a highest weight μ of π . By Lemma 5.3.23, the Levi factor of the parabolic subalgebra associated to μ is quasisplit; but by definition, this is all of $\mathfrak{g}_0$. So G is quasisplit. We use the notation of Section 4.3. Now π is fine if and only if its restriction to each subalgebra $\mathfrak{k}_0^\alpha$ of $\mathfrak{k}_0$ (corresponding to a root of α in $\mathfrak{g}$) is fine, by Definition 4.3.9. Roughly speaking, we will prove a certain condition on the weights of π ; prove that this condition is inherited by the subalgebras $\mathfrak{k}_0^\alpha$; and finally show that the condition implies fineness in rank one, i.e., for $\mathfrak{k}_0^\alpha$. So we need three lemmas. It is convenient to give the condition a name.

Definition 5.3.27 A representation π of K is said to satisfy condition S if the following property holds. Fix an element $x \in i\mathfrak{t}_0 \cap [\mathfrak{g}, \mathfrak{g}]$ which is dominant for $\Delta^+(\mathfrak{k})$; and a θ -stable positive root system $\tilde{\Delta}^+$ for $\mathfrak{h}$ in $\mathfrak{g}$, such that $\beta(x) \geq 0$ for $\beta \in \tilde{\Delta}^+$. Write $\tilde{\rho} = \rho(\tilde{\Delta}^+)$.

Then for any weight γ of π , and any x as above,

$$(S) \quad \gamma(x) \leq \tilde{\rho}(x) - 2\rho_C(x).$$

Lemma 5.3.28 A small K -type π satisfies condition S.

Proof. Since the element x in the definition of condition S is assumed to lie in the semisimple part of $\mathfrak{g}$, we may as well assume that $\mathfrak{g}$ is semisimple. Fix a highest weight μ of π , and let λ be associated to μ by Proposition 5.3.3, using a positive root system Δ^+ and roots $\{\beta_i\}$. Since the parabolic defined by λ is assumed to be $\mathfrak{g}$, which is semisimple, λ must be zero; thus

$$(*) \quad 0 = \mu + 2\rho_C - \rho + \sum c_i \beta_i.$$

Fix x as in Definition 5.3.27, and choose an element

$\sigma \in W(\mathfrak{g}, \mathcal{K}^C)$ such that

$$\sigma^{-1}(\Delta^+) = \tilde{\Delta}^+;$$

then

$$\sigma^{-1}\rho = \tilde{\rho}.$$

Since $\mu + 2\rho_C$ is dominant for Δ^+ , and x is dominant for $\tilde{\Delta}^+$,

$$(\mu + 2\rho_C)(x) \leq (\mu + 2\rho_C)(\sigma x).$$

Combining this with (*), we get

$$\begin{aligned} \mu(x) &= (\mu + 2\rho_C)(x) - 2\rho_C(x) \\ &\leq (\mu + 2\rho_C)(\sigma x) - 2\rho_C(x) \\ &= (\rho - \sum c_i \beta_i)(\sigma x) - 2\rho_C(x) \\ &= \rho(\sigma x) - \sum c_i \beta_i(\sigma x) - 2\rho_C(x) \\ &= \tilde{\rho}(x) - 2\rho_C(x) - \sum c_i \beta_i(\sigma x). \end{aligned}$$

Now c_i is non-negative, σx is dominant for Δ^+ , and $\beta_i \in \Delta^+$.

So

$$c_i \beta_i(\sigma x) \geq 0,$$

and therefore

$$\mu(x) \leq \tilde{\rho}(x) - 2\rho_C(x).$$

If γ is any weight of the K -type π , then $\gamma = \mu - p$, with μ a highest weight, and p a sum of positive roots of $\mathcal{K}^C$ in $\mathcal{K}$. We are assuming x to be dominant for these roots, so $p(x)$ is non-negative. Hence

$$\gamma(x) = \mu(x) - p(x)$$

$$\leq \mu(x)$$

$$\leq \tilde{\rho}(x) - 2\rho_C(x).$$

This is condition S.

Q.E.D.

Lemma 5.3.29 Let $\tilde{\mathcal{K}}_0 = \tilde{\mathcal{K}}_0 + \tilde{\alpha}_0$ be a θ -stable Cartan subalgebra of $\mathfrak{g}_0$. Assume that $\tilde{\mathcal{K}}_0 \subseteq \mathcal{K}_0^C$.

Write

$$\begin{aligned} \lambda_0 &= \mathfrak{g}_0^{\tilde{\mathcal{K}}_0} \\ \tilde{\mathfrak{g}}_0 &= \mathfrak{g}_0^{\tilde{\alpha}_0}. \end{aligned}$$

Fix a positive root system $\tilde{\Delta}^+$ for $\tilde{\mathcal{K}}$ in $\tilde{\mathfrak{g}}$. Then there is a θ -invariant positive root system Δ^+ for

$\mathcal{K}^C$ in $\mathfrak{g}$ such that

- a) the simple roots of $\mathcal{K}^C$ in $\mathfrak{g}$ are simple in $\mathfrak{g}$
 b) $\Delta^+|_{\tilde{\mathcal{K}}} \supseteq \tilde{\Delta}^+|_{\tilde{\mathcal{K}}}$

For such a choice of Δ^+ , we have

$$2\rho_C - \rho|_{\tilde{\mathcal{K}}} = 2\rho_C(\tilde{\mathfrak{g}}) - \rho(\tilde{\mathfrak{g}}).$$

Proof. Notice that $_{\ell}\tilde{\alpha} = \alpha_{\ell}^{\tilde{I}} + \tilde{\alpha} = \alpha_{\ell}^{\tilde{I}} = \tilde{h}$; so

$$_{\ell}\tilde{\alpha} \cap \mathfrak{p} = \tilde{\alpha},$$

so $\tilde{\alpha}_0$ is maximal abelian in $_{\ell_0} \cap \mathfrak{p}_0$. All maximal abelian subalgebras of $_{\ell_0} \cap \mathfrak{p}_0$ are conjugate under $L \cap K$; and we can replace $\tilde{h}_0$ by any conjugate under $L \cap K$ without affecting the hypotheses or the conclusion. So it is enough to prove the lemma with $\tilde{\alpha}_0$ replaced by any maximal abelian subalgebra of $_{\ell_0} \cap \mathfrak{p}_0$.

To choose Δ^+ , choose an element $x \in i\tilde{X}_0$ which is dominant for $\tilde{\Delta}^+$, and such that

$$\Delta(\ell, \mathfrak{h}^C) = \{\alpha \in \Delta(\alpha_{\ell}, \mathfrak{h}^C) \mid \alpha(x) = 0\};$$

this is certainly possible. Next, choose an arbitrary θ -invariant positive root system $\Delta^+(\ell, \mathfrak{h}^C)$ for $\mathfrak{h}^C$ in ℓ , and put

$$\Delta^+(\alpha_{\ell}, \mathfrak{h}^C) = \{\alpha \in \Delta(\alpha_{\ell}, \mathfrak{h}^C) \mid \alpha(x) > 0\} \cup \Delta^+(\ell, \mathfrak{h}^C).$$

It is straightforward to check that this satisfies (a) and (b).

To verify the last assertion, we proceed by induction on $\dim \mathfrak{t}^C$. After replacing $\tilde{\alpha}_0$ by another maximal abelian subalgebra of $_{\ell_0} \cap \mathfrak{p}_0$, we may assume that

$$\tilde{\alpha}_0 \supseteq \alpha_0^C$$

the split part of $\mathfrak{h}$. In that case $\tilde{\alpha}$ is contained in the centralizer $\tilde{\alpha}$ of α_0^C in α . Because of Lemma 5.3.5, it is enough to prove the lemma for $\tilde{\alpha}$; in other words, we may assume that all the roots of $\mathfrak{h}$ in α are imaginary.

If $\tilde{\alpha} = \alpha^C$, then since all roots of $\mathfrak{h}$ in α are imaginary,

$$\tilde{\alpha} = \alpha_{\alpha}^{\tilde{\alpha}} = \alpha;$$

so the conclusion of the lemma is trivial.

So we may assume $\tilde{\alpha} \neq \alpha^C$. So $\ell \cap \mathfrak{p} \neq \alpha^C$; so there must be a noncompact root of $\mathfrak{h}^C$ in ℓ . Since the sum of two compact roots is compact if it is a root, there must be a simple noncompact root β . Define $Z_{\beta} \in \ell_0 \cap \mathfrak{p}_0$ as for Proposition 5.3.1. After replacing $\tilde{\alpha}_0$ by another maximal abelian subalgebra of $_{\ell_0} \cap \mathfrak{p}_0$, we may assume that $Z_{\beta} \in \tilde{\alpha}_0$. Accordingly the subalgebra $\alpha_{\alpha}^{\tilde{\alpha}}$ considered in Proposition 5.3.1 contains $\tilde{\alpha}_0$; and by Lemma 5.3.4, it is enough to prove the lemma with $\alpha_{\alpha}^{\tilde{\alpha}}$ replacing α_0 . The fundamental Cartan subalgebra

$$\mathfrak{h}_0^{\beta} = \mathfrak{t}_0^{\beta} + \alpha_0^{\beta}$$

of $\alpha_{\alpha}^{\tilde{\alpha}}$ satisfies

$$\dim \mathfrak{t}_0^{\beta} = \dim \mathfrak{t}_0^C - 1;$$

so the lemma is known for $\alpha_{\alpha}^{\tilde{\alpha}}$ by inductive hypothesis.

Q.E.D.

Lemma 5.3.30 Suppose G is quasisplit, and the representation π of K satisfies condition S. Let

$$\mathfrak{h}_0^S = \mathfrak{t}_0^S + \alpha_0^S$$

be a maximally split Cartan subalgebra of α_0 , and

$$\alpha \in \Delta(\alpha_0, \alpha_0^S)$$

a restricted root. Define

$$\begin{aligned} \sigma_0^\alpha &= \{x \in \sigma_0^s \mid \alpha(x) = 0\} \\ \text{of } (\sigma_0^\alpha) &= m_0^\alpha + \sigma_0^\alpha \end{aligned}$$

as in (4.2.21). Then the restriction of μ to

$m_0^\alpha \cap k_0$ satisfies condition S.

Proof. Extend $\mathcal{T}_0^s$ to a Cartan subalgebra $\mathcal{T}_0^1$ of $m_0^\alpha \cap k_0$. After replacing $\mathcal{T}_0^c$ by a conjugate under K , we may assume $\mathcal{T}_0^1 \subseteq \mathcal{T}_0^c$. Fix a positive root system $\Delta^+(m_0^\alpha \cap k, \mathcal{T}_0^1)$, and an element

$$x \in i\mathcal{T}_0^1 \cap [m_0^\alpha, m_0^\alpha]$$

which is dominant for $\Delta^+(m_0^\alpha \cap k, \mathcal{T}_0^1)$. Put

$$\begin{aligned} \sigma_0^1 &= (m_0^\alpha \cap \mathfrak{p}_0) \mathcal{T}_0^1, \\ \sigma_0^1 &= \sigma_0^1 \oplus \sigma_0^\alpha; \end{aligned}$$

then $k_0^1 = \mathcal{T}_0^1 + \sigma_0^1$ is a fundamental Cartan subalgebra of

$m_0^\alpha + \sigma_0^\alpha$, and a θ -stable Cartan subalgebra of $\mathfrak{g}_0$. Choose

a θ -invariant positive root system

$$\Delta^+(m_0^\alpha + \sigma_0^\alpha, k_0^1)$$

for which x is dominant, and write

$$\rho(\Delta^+(m_0^\alpha + \sigma_0^\alpha, k_0^1)) = \rho^1,$$

etc. What we want to show is that if γ is any weight of μ , then

$$\gamma(x) \leq \rho^1(x) - 2\rho_C^1(x).$$

We will apply Lemma 5.3.29 with $k^1 = k$. Clearly we can choose $\Delta^+(\mathfrak{g}, k)$ in such a way as to make x dominant, and

to satisfy 5.3.29(a) and (b). Combining Lemma 5.3.29 with condition S for μ gives

$$\gamma(x) \leq (\rho(\tilde{\mathfrak{g}}) - 2\rho_C(\tilde{\mathfrak{g}}))(x),$$

with

$$\tilde{\mathfrak{g}}_0 = \sigma_0^1.$$

So we must show that

$$(*) \quad \rho(\tilde{\mathfrak{g}}) - 2\rho_C(\tilde{\mathfrak{g}}) = \rho^1 - 2\rho_C^1.$$

If $\sigma_0^1 = \{0\}$, then $\sigma_0^1 = \sigma_0^2$, and $\tilde{\mathfrak{g}} = m_0^\alpha + \sigma_0^\alpha$; so (*) holds by definition. So we may suppose $\sigma_0^1 \neq 0$. Now

$$\begin{aligned} \dim \sigma_0^\alpha &= \dim \sigma_0^s - 1, \\ \mathcal{T}_0^1 &\supseteq \mathcal{T}_0^s, \sigma_0^1 = \sigma_0^\alpha + \tilde{\sigma}_0^1 \\ \dim \mathcal{T}_0^1 + \dim \sigma_0^1 &= \dim \mathcal{T}_0^s + \dim \sigma_0^s \end{aligned}$$

(since both sides are the dimensions of Cartan subalgebras of $\mathfrak{g}_0$). Together these force

$$\begin{aligned} \mathcal{T}_0^1 &= \mathcal{T}_0^s \\ \dim \sigma_0^1 &= 1. \end{aligned}$$

Since σ_0^1 is the centralizer of $\mathcal{T}_0^1$ in $m_0^\alpha \cap \mathfrak{p}_0$, this gives

$$\sigma_0^s = \sigma_0^\alpha + \tilde{\sigma}_0^1 = \sigma_0^1$$

So

$$\tilde{\mathfrak{g}}_0 = \sigma_0^s = h_0^s$$

since $\mathfrak{g}_0$ is quasisplit. So the left side of (*) is zero.

Also

$$(m_0^\alpha) \sigma_0^1 + \sigma_0^\alpha = (\mathfrak{g}_0) \sigma_0^1 + \sigma_0^s = h_0^s;$$

so there are no imaginary roots of $\mathcal{L}_0^1$ in $\mathfrak{m}_0^\alpha$. By Lemma 5.3.5, this means that the right side of (*) is zero. Q.E.D.

Lemma 5.3.31 Suppose G is a simple quasisplit group of real rank one, and $\pi \in \hat{K}$ satisfies condition S. Then π is fine.

Proof. By the classification of real groups, G is (covered by) $SL(2, \mathbb{R})$, $SL(2, \mathbb{C})$ or $SU(2, 1)$. We treat these cases separately.

Case I. $G = SL(2, \mathbb{R})$. Define Z_α as in Section 2.3; we want to show that $\mu(iZ_\alpha)$ is 0, 1 or -1. Since $\exp(2\pi Z_\alpha)$ is the identity, $\mu(iZ_\alpha)$ is an integer. There are no compact roots, so $2\rho_{\mathbb{C}} = 0$; and if Δ^+ is chosen to make iZ_α dominant, a computation gives

$$\rho(iZ_\alpha) = 1.$$

So by condition S, $\mu(iZ_\alpha) \leq 1$. By symmetry (more precisely, by applying condition S to $-iZ_\alpha$), $\mu(iZ_\alpha) \geq -1$.

Case II. $G = SL(2, \mathbb{C})$. Here there are no imaginary roots of $\mathcal{K}^{\mathbb{C}}$ in $\mathfrak{g}$, so the restriction of any root to $\mathcal{K}^{\mathbb{C}}$ may be regarded as a root of $\mathcal{K}^{\mathbb{C}}$ in $\mathcal{K}$. So given $\Delta^+(\mathcal{K}, \mathcal{K}^{\mathbb{C}})$, there is a unique θ -invariant positive system $\Delta^+(\mathfrak{g}, \mathcal{K}^{\mathbb{C}})$ compatible with it; and $2\rho_{\mathbb{C}} - \rho = 0$ by Lemma 5.3.5. So if $x \in i\mathcal{H}_0^{\mathbb{C}}$ is dominant and γ is a weight of π , condition S says

$$\gamma(x) \leq 0.$$

Since -1 is in the Weyl group of $\mathcal{K}^{\mathbb{C}}$ in $\mathcal{K}$, $-\gamma$ is also a weight; so

$$-\gamma(x) \leq 0,$$

and so $\gamma(x) = 0$. Since x was an arbitrary dominant element, $\gamma = 0$; so π is trivial, and therefore fine.

Case III. $G = SU(2, 1)$. We can choose a positive system Δ^+ for $\mathcal{K}^{\mathbb{C}}$ in $\mathfrak{g}$ so that both simple roots β_1 and β_2 are non-compact. The only other root is $\beta_1 + \beta_2$, which is compact; so

$$\rho = \frac{1}{2}(\beta_1 + \beta_2 + (\beta_1 + \beta_2)) = \beta_1 + \beta_2 = 2\rho_{\mathbb{C}};$$

so $\rho - 2\rho_{\mathbb{C}} = 0$. So if x is dominant for Δ^+ , condition S says

$$\gamma(x) \leq 0$$

for any weight γ of π . Let w_0 be the non-trivial element of $W(\mathcal{K}, \mathcal{K}^{\mathbb{C}})$, i.e., $s_{\beta_1 + \beta_2}$; then $w_0\gamma$ is also a weight, and $-w_0x$ is dominant for Δ^+ . So

$$-\gamma(x) = w_0\gamma(-w_0x) \leq 0.$$

So $\gamma(x) = 0$. Since x was an arbitrary element dominant for Δ^+ , $\gamma = 0$; so π is trivial, and therefore fine. Q.E.D.

Because of the remarks made after its statement, Proposition 5.3.26 follows from Lemmas 5.3.28, 5.3.30, and 5.3.31.

Q.E.D.

§4. Construction of cohomology.

Definition 5.4.1 Suppose $\mu \in \hat{T}^C$ is dominant for $\Delta^+(\mathcal{h}, \mathcal{t}^C)$. Define $\lambda \in (\mathcal{t}^C)^*$ by Proposition 5.3.3, and set

$$||\mu||_{\text{lambda}} = \langle \lambda, \lambda \rangle.$$

If $\pi \in \hat{K}$ has μ as a highest weight, set

$$||\pi||_{\text{lambda}} = ||\mu||_{\text{lambda}}.$$

Finally, if X is a non-zero $(\mathfrak{g}, K)$ module, define the set of lambda-lowest K-types of X to be

$$\{\pi \in \hat{K}[X(\pi)] \neq 0 \text{ (notation (0.3.4))}, \text{ and } ||\pi||_{\text{lambda}} \text{ is minimal}\}.$$

If π is such a K-type, put

$$||X||_{\text{lambda}} = ||\pi||_{\text{lambda}}$$

Since T^C is compact, the differentials of weights form a lattice in $(\mathcal{t}^C)^*$. Since the weight λ attached to μ by Proposition 5.3.3 satisfies

$$\langle \lambda - \mu, \lambda - \mu \rangle \leq C_G$$

for some constant C_G depending only on G , it follows that the set of lambda-lowest K-types of X is finite and non-empty.

Proposition 5.4.2 Let X be a $(\mathfrak{g}, K)$ module, and π a lambda-lowest K-type of X , of highest weight $\mu \in \hat{T}^C$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{u}$ be the θ -stable parabolic subalgebra of $\mathfrak{g}$ associated to μ (Definition 5.3.22). Then

- a) μ is strongly $\mathfrak{u}$ -minimal in X (Definition 5.2.5).
 b) $R_{\mu}^G = R_{\mu}^L$ (Definition 5.1.2).

In conjunction with Theorem 5.2.6, this proves the non-vanishing of $H^R(\mathfrak{u}, X)$. We will discuss this more carefully in Corollary 5.4.9. (Assertion (b) plays a minor technical part in the theory; we will need it to recover information about X from $H^R(\mathfrak{u}, X)$, using Corollary 5.1.10.)

Proof. We begin with (b). Give $w \in R_{\mu}^G$, choose a representative $n \in G$; n normalizes T^C , preserves $\Delta^+(\mathcal{h}, \mathcal{t}^C)$, and fixes μ . Let $\lambda \in \mathcal{t}^*$ be the weight associated to μ by Proposition 5.3.3. Clearly $n \cdot \lambda$ also satisfies the conditions of the proposition; so by the uniqueness statement, n fixes λ . By the definition of $\mathfrak{g}$ (Definition 5.3.22), n normalizes $\mathfrak{g}$; so by Definition 5.2.1, $n \in L$. So w has a representative in L , and therefore

$$R_{\mu}^G \subseteq R_{\mu}^L.$$

The other inclusion is formal.

To prove (a), we may as well assume G is connected; for the statements to be proved are essentially unchanged if we pass from G to G_0 , and regard X as a $(\mathfrak{g}, K_0)$ module by restriction. So for the rest of this section, we assume that G (and therefore also K, T, L and $L \cap K$) is connected. Fix a θ -stable positive root system

$$\Delta^+ = \Delta^+(\mathfrak{g}, \mathcal{h}^C)$$

so that $\mu + 2\rho_G$ is dominant, and choose $\{(\beta_i, c_i)\}$ as in

Proposition 5.3.3. Fix $\delta \in \hat{K}$, and suppose for some $(J, J'') \neq (0, R)$,

$$[H^J(u \cap \mathcal{K}, \delta) \otimes (\Lambda^{J''}(u \cap \mathcal{P}))^*]^{J'-2\rho(u \cap \mathcal{P})} \neq 0$$

(cf. Definition 5.2.5). We want to conclude that

$$X(\delta) = 0;$$

by Definition 5.4.1, we must show that

$$||\delta||_{\text{lambda}} < ||\mu||_{\text{lambda}}.$$

By Proposition 3.2.12, we can find an Lnk-type of highest weight ϕ , occurring in $H^J(u \cap \mathcal{K}, \delta)$, and a weight $-\psi$ of

$$\Lambda^{J''}(u \cap \mathcal{P})^*$$

such that

$$(5.4.3) \text{ (a)} \qquad \phi - \psi = \mu - 2\rho(u \cap \mathcal{P}).$$

Let W_K denote the Weyl group of $\mathcal{K}^C$ in $\mathcal{K}$, and write μ' for the highest weight of δ . By Theorem 3.2.3, there is an element σ of W_K^1 of length J , such that

$$(5.4.3) \text{ (b)} \qquad \phi = \sigma(\mu' + \rho_C) - \rho_C.$$

Let $J' = R - J''$ (recall that $R = \dim u \cap \mathcal{P}$). Then

$$(J, J') \neq (0, 0).$$

Since $-\psi$ is a weight of

$$\Lambda^{J''}(u \cap \mathcal{P}) \cong [\Lambda^{R-J''}(u \cap \mathcal{P})] \otimes \Lambda^R(u \cap \mathcal{P})^*,$$

we can find a subset

$$\{\gamma_i\} \subseteq \Delta(u \cap \mathcal{P}, \mathcal{K}^C)$$

so that

$$(5.4.3) \text{ (c)} \qquad \psi = 2\rho(u \cap \mathcal{P}) - \sum_{i=1}^{J'} \gamma_i.$$

Now (5.4.3) gives

$$(5.4.4) \text{ (a)} \qquad \sigma(\mu' + \rho_C) - \rho_C = \mu - \sum_{i=1}^{J'} \gamma_i.$$

Choose another θ -stable positive root system (Δ^+) for $\mathcal{K}^C$ in $\mathcal{K}$, making $\mu' + 2\rho_C$ dominant, and let λ' be the weight attached to μ' by Proposition 5.3.3. Then we can write (with notation as in that proposition)

$$(5.4.4) \text{ (b)} \qquad \begin{aligned} \lambda &= \mu + 2\rho_C - \rho + \frac{1}{2} \sum c_i \beta_i \\ \lambda' &= \mu' + 2\rho_C - \rho' + \frac{1}{2} \sum c_i' \beta_i'. \end{aligned}$$

Write

$$\rho_n = \rho(\Delta^+(\mathcal{P}, \mathcal{K}^C)), \quad \rho_n' = \rho((\Delta^+)'(\mathcal{P}, \mathcal{K}^C)).$$

Then (5.4.4) (a) - (b) give

$$(5.4.4) \text{ (c)}$$

$$\sigma^{-1}\lambda = \lambda' + (\rho_n' - \frac{1}{2} \sum c_i' \beta_i') - \sigma^{-1}(\rho_n - \sum \gamma_i - \frac{1}{2} \sum c_i \beta_i).$$

We are trying to show that

$$(*) \qquad \langle \lambda', \lambda' \rangle < \langle \lambda, \lambda \rangle$$

To study this inequality, we use

Lemma 5.4.5 (Schmid) Let ρ_n be half the sum of a set $\Delta^+(\mathcal{P})$ of positive roots of $\mathcal{K}^C$ in $\mathcal{P}$. For each $\beta \in \Delta^+(\mathcal{P})$, suppose c_β is a real number between 0

and 1. Put

$$Q = \rho_n - \sum_{\beta} c_{\beta} \beta,$$

and let S be the set of weights of this form. Then S is independent of the choice of $\Delta^+(\mathfrak{p})$; and in particular, it is invariant under $w(K, T^C)$.

We will prove this in a moment. Using it, we find numbers

$$\{\alpha_{\gamma} | \gamma \in (\Delta^+)'(\mathfrak{p})\}$$

between 0 and 1, with

$$\sigma^{-1}(\rho_n - \sum \gamma_i - \frac{1}{2} \sum c_i \beta_i) = \rho_n' - \sum a_{\gamma} \gamma.$$

So (5.4.4)(c) becomes

$$(5.4.6) \quad \sigma^{-1} \lambda = \lambda' + \sum_{\gamma \in (\Delta^+)'}(\mathfrak{p}) a_{\gamma} \gamma - \frac{1}{2} \sum c_i' \beta_i'.$$

Since β_i' is orthogonal to λ' , and λ' is dominant for $(\Delta^+)'$,

$$\langle \lambda', \sum a_{\gamma} \gamma - \frac{1}{2} \sum c_i' \beta_i' \rangle \geq 0.$$

In combination with (5.4.6), this implies that

$$\begin{aligned} \langle \lambda, \lambda \rangle &= \langle \sigma^{-1} \lambda, \sigma^{-1} \lambda \rangle \\ &\geq \langle \lambda', \lambda' \rangle, \end{aligned}$$

with equality only if

$$\sum a_{\gamma} \gamma - \frac{1}{2} \sum c_i' \beta_i' = 0.$$

If the inequality is strict, this is (*) and we are done.

So suppose it is not; then by the condition for equality,

(5.4.6) reduces to

$$\sigma^{-1} \lambda = \lambda'.$$

Since λ and λ' are both dominant for $\Delta^+(\mathfrak{p}, \mathfrak{t}^C)$, and

$\sigma \in W_K^1$, this means $\sigma = 1$, and $\lambda = \lambda'$. Since λ is dominant for Δ^+ and λ' for $(\Delta^+)'$, these two positive systems differ only in roots orthogonal to λ . So by (5.4.4)(c),

$$\lambda = \lambda - \sum \gamma_i + (\text{terms orthogonal to } \lambda).$$

Now take the inner product of each side with λ . Since each

γ_i is a root in $\Delta(u \cap \mathfrak{p}, \mathfrak{t}^C)$, it has a positive inner product with λ . So we conclude that $\{\gamma_i\}$ is empty. So

$$J = \mathfrak{L}(\sigma) = 0$$

$$J' = |\{\gamma_i\}| = 0.$$

But by hypothesis $(J, J') \neq (0, 0)$; so this is a contradiction.

Q.E.D.

Proof of Lemma 5.4.5 We write the weights Q in a different way, which makes the invariance obvious. Let $A \subseteq \Delta(\mathfrak{p}, \mathfrak{t}^C)$ be a set satisfying

- 1) $|A| = \frac{1}{2} |\Delta(\mathfrak{p}, \mathfrak{t}^C)|$
- 2) $A \cup (-A) = \Delta(\mathfrak{p}, \mathfrak{t}^C)$.

For each $\beta \in A$, suppose e_{β} is a real number between $-\frac{1}{2}$ and $\frac{1}{2}$. Put

$$S(A, e) = \sum_{\beta \in A} e_{\beta} \beta.$$

We claim that these are exactly the weights described in the lemma. To see this, fix Q as in the lemma, and put

$$A = \{\beta \in \Delta^+(\mathfrak{p}, \mathfrak{t}^C) | c_{\beta} \leq \frac{1}{2}\} \cup \{-\beta | \beta \in \Delta^+(\mathfrak{p}), c_{\beta} > \frac{1}{2}\}$$

$$e_{\beta} = \begin{cases} \frac{1}{2} - c_{\beta} & \text{if } \beta \in \Delta^+(\mathcal{P}) \\ \frac{1}{2} - \frac{1}{c_{\beta}} & \text{if } \beta \notin \Delta^+(\mathcal{P}). \end{cases}$$

Since

$$\rho_n = \frac{1}{2} \sum_{\beta \in \Delta^+(\mathcal{P})} \beta,$$

it is clear that

$$Q = S(A, e).$$

The other direction is similar.

Q.E.D.

Corollary 5.4.7 In the setting of Proposition 5.4.2, let $\tilde{\pi}^0$ be the unique representation of $L \cap K$ of highest weight μ , which occurs in $\pi|_{L \cap K}$; and let

$$\pi^0 = \tilde{\pi}^0 \otimes [\Lambda^R(\mathcal{U} \cap \mathcal{P})]^* \in (L \cap K)^{\wedge}.$$

Then

- π^0 is a fine representation of $L \cap K$ (with respect to L -- cf. Definition 4.3.9).
- π^0 occurs in $H^i(\mathcal{U}, X)$ only if $i = R$.
- The multiplicity of π^0 in $H^R(\mathcal{U}, X)$ is equal to the multiplicity of π in X .

Proof. By Corollary 5.1.10, and Proposition 5.4.2(b), $\tilde{\pi}^0$ really exists as claimed. Part (a) is Corollary 5.3.25 and Proposition 5.3.26. Parts (b) and (c) are Theorem 5.2.6 and Proposition 5.4.2(a).

Q.E.D.

Definition 5.4.8 Suppose X is an admissible $(\mathcal{G}, K)$ -module, and $\pi \in \hat{K}$ is a lambda-lowest K -type of X , having highest weight $\mu \in \hat{\mathfrak{t}}^C$ (Definition 5.4.1). Let

$\mathcal{G} = \ell + \mathcal{U}$ be the θ -stable parabolic attached to

μ (Definition 5.3.22). Define $\pi^0 \in (L \cap K)^{\wedge}$ as in

Corollary 5.4.7. A canonical cohomology class of X is an irreducible $(\ell, L \cap K)$ module Y which is a sub-quotient of $H^R(\mathcal{U}, X)$ and contains the $L \cap K$ -type π^0 .

A set of θ -stable data corresponding to X is a quadruple

$(\mathcal{G}, H, \delta, \nu)$ such that

a) $\mathcal{G}$ is as above.

b) $H = TA$ is a θ -stable maximally split Cartan subgroup of L ($T = H \cap K$, $A = \exp(-\mathcal{H}_0 \cap \mathcal{P}_0)$).

c) $\delta \in \hat{\mathfrak{t}}$, $\nu \in \hat{A}$, and δ is fine with respect to L .

d) With π^0 as above, δ occurs in $\pi^0|_T$, so that $\pi^0 \in A_L(\delta)$ (notation (4.3.15)).

e) The representation $\bar{\pi}_L(\delta \otimes \nu)(\pi^0)$ (Definition 4.4.6)

is a canonical cohomology class of X (that is, it occurs in $H^R(\mathcal{U}, X)$).

Corollary 5.4.9 Suppose X is an admissible $(\mathcal{G}, K)$ module. Then there is a set $(\mathcal{G}, H, \delta, \nu)$ of θ -stable data corresponding to X . More precisely, fix a lambda-lowest K -type π of X , of highest weight μ , and let $\mathcal{G} = \ell + \mathcal{U}$ be the θ -stable parabolic associated to μ (Definition 5.3.22). Define $\pi^0 \in (L \cap K)^{\wedge}$ as in Corollary 5.4.7. Let $H = TA$ be a θ -stable maximally split Cartan subgroup of L , and suppose

$\delta \in \hat{T}$ occurs in $\pi^0|_T$. Let Y be any irreducible subquotient of $H^R(\mathcal{U}, X)$ containing π^0 , so that

$$Y \cong \bar{\pi}_L(\delta \otimes \nu)(\pi^0)$$

for some $\nu \in \hat{A}$ (by Theorem 4.4.8). Then $(\mathcal{U}, H, \delta, \nu)$ is a set of θ -stable data corresponding to X , and all such sets arise in this way.

Stated in this way, the corollary proves itself. We will show in Chapter 6 (Corollary 6.5.15) that if X is irreducible, then the set of θ -stable data is essentially unique and determines X .

Corollary 5.4.10 In the setting of Corollary 5.4.9,

let $\lambda \in (\mathcal{X}^C)^*$ be the weight attached to μ by

Proposition 5.3.3. Since λ is zero on the roots

of $\mathcal{H}^C$ in $\mathfrak{l}$ (by definition of $\mathfrak{l}$), we can regard λ

as a weight of a one-dimensional representation

of $\mathfrak{l}$, and hence as a weight of any Cartan subalgebra

of $\mathfrak{l}$. If we consider λ as an element of $\mathcal{H}^*$ in this way, then

a) λ vanishes on $\sigma\mathcal{U}^*$; so we write $\lambda \in \mathcal{X}^*$.

b) The differential of δ is $\lambda - \rho(\mathcal{U})$.

c) If X has an infinitesimal character, it corresponds to the weight

$$(\lambda, \nu) \in \mathcal{X}^* + \sigma\mathcal{U}^* = \mathcal{H}^*.$$

Proof. We may as well choose for $\mathcal{H}$ the Cartan subalgebra

$\mathcal{H}^X$ constructed in Lemma 5.3.23. Then (a) is (5.3.23)(b).

For (b), write

$$\mathcal{Y} = (\text{center of } \mathfrak{l}) \cap \mathcal{K} \subseteq \mathcal{X}^X \subseteq \mathcal{X}^C,$$

and write

$$\text{differential of } \delta = \xi \in (\mathcal{X}^X)^*.$$

By Definition 4.3.8, ξ is supported on $\mathcal{Y}$; and clearly $\lambda - \rho(\mathcal{U})$ is as well. Now π^0 must act by scalars on $\mathcal{Y}$, so

$$\xi = \mu - 2\rho(\mathcal{U} \cap \mathcal{P})|_{\mathcal{Y}}.$$

So (b) will follow if we can show that

$$(*) \quad \mu - 2\rho(\mathcal{U} \cap \mathcal{P})|_{\mathcal{Y}} = \lambda - \rho(\mathcal{U})|_{\mathcal{Y}}$$

Computing as in the proof of Lemma 5.3.23, we find that the right side is

$$\begin{aligned} \lambda - \rho(\mathcal{U})|_{\mathcal{Y}} &= \mu + 2\rho_C - \rho - \rho(\mathcal{U})|_{\mathcal{Y}} \\ &= \mu + 2\rho(\mathcal{U} \cap \mathcal{K}) - 2\rho(\mathcal{U}) + \\ &\quad 2\rho(\Delta^+(\mathfrak{l} \cap \mathcal{K}) - \rho(\Delta^+(\mathfrak{l})))|_{\mathcal{Y}} \\ &= \mu - 2\rho(\mathcal{U} \cap \mathcal{P})|_{\mathcal{Y}} \end{aligned}$$

(since the β_i are orthogonal to $\mathcal{X}^X$, and hence to $\mathcal{Y}$)

since the remaining terms are combinations of roots in $\mathfrak{l}$. This proves (*), and hence (b).

For (c), Lemma 4.1.8 and (b) show that $\bar{\pi}_L(\delta \otimes \nu)(\pi^0)$ has infinitesimal character

$$(\lambda, \nu) + \rho(\mathcal{U}).$$

By definition, this representation occurs in $H^*(\mathcal{U}, X)$; so the result follows from Corollary 3.1.6. Q.E.D.

To use this corollary, a definition is helpful.

Definition 5.4.11 Suppose $h_0 = t_0 + \sigma_0$ is a θ -stable Cartan subalgebra of $\mathfrak{g}_0$, and

$$\gamma = (\lambda, \nu) \in t^* + \sigma^* = h^*.$$

Define the canonical real and imaginary parts of γ as follows. Write

$$\gamma = \gamma_1 + \gamma_2 = (\lambda_1, \nu_1) + i(\lambda_2, \nu_2),$$

so that λ_1 takes purely imaginary values on t_0 , and ν_1 takes purely real values on σ_0 ; and put

$$\text{RE}(\gamma) = \gamma_1, \quad \text{IM}(\gamma) = \gamma_2.$$

If $\xi_\gamma: Z(\mathfrak{g}) \rightarrow \mathbb{C}$ is the corresponding infinitesimal character, (Definition 0.2.9). The canonical real and imaginary parts of ξ_γ are defined by

$$\text{RE}(\xi_\gamma) = \xi_{\text{RE}(\gamma)}, \quad \text{IM}(\xi_\gamma) = \xi_{\text{IM}(\gamma)}$$

To see that this last definition is independent of choices, we need

Lemma 5.4.12 Suppose

$$h_0^j = t_0^j + \sigma_0^j \quad (j = 1, 2)$$

are two θ -stable Cartan subalgebras of $\mathfrak{g}_0$. Let x be any element of $\text{Ad}(\mathfrak{g})$ such that

$$\text{Ad}(x)(h^1) = h^2.$$

Then

$$\text{Ad}(x)(it_0^1 + \sigma_0^1) = (it_0^2 + \sigma_0^2).$$

Proof. $\text{Ad}(x)$ acts trivially on the center of $\mathfrak{g}$; so we may divide by it and assume $\mathfrak{g}$ is semisimple. By Lemma 0.2.2, $it_0^j + \sigma_0^j$ can be characterized as the subspace of h^j on which the roots take real values. Since $\text{Ad}(x)$ takes roots of h^1 to roots of h^2 , the lemma follows. Q.E.D.

Corollary 5.4.13 In the setting of Corollary 5.4.10(c), the canonical real and imaginary parts of the infinitesimal character of X (Definition 5.4.11) correspond to

$$(\lambda, \text{Re}(\nu)) \quad \text{and} \quad (0, \text{Im}(\nu)),$$

respectively.

This is obvious. We turn now to the proof of Theorem 4.3.17, which was postponed from Chapter 4; so suppose G is quasi-split, $\delta \in \hat{M}$ is fine, and $\nu \in \hat{A}$ is unitary. Let X be an irreducible component of $\pi(\delta \otimes \nu)$, and π a λ -lowest $\bar{K}$ -type of X . We want to show that π is fine. Clearly we may assume G is semisimple (by restricting everything to the derived group). Then the differential of δ is zero by Definition 4.3.8; so by Lemma 4.1.8, X has infinitesimal character

$$(0, \nu) \in (\pi\lambda + \sigma\lambda)^*.$$

Since ν is unitary, this has real part zero. By Corollary 5.4.13, the weight λ attached to π by Proposition 5.3.3 is also zero; so the associated θ -stable parabolic is $\mathfrak{g}$ itself. By Proposition 5.3.26, π is fine. Q.E.D.

This argument essentially proves the converse of Proposition 5.3.26:

Corollary 5.4.14 A representation $\pi \in \hat{K}$ is small
(Definition 5.3.24) if and only if it is fine
(Definition 4.3.8).

Proof. "Only if" is Proposition 5.3.26; so suppose π is fine. We use the notation of Section 4.3. Choose $\delta \in \hat{M}$ occurring in $\pi|_{M'}$ and a unitary $\nu \in \hat{A}$. Let X be the irreducible component of $\pi(\delta \otimes \nu)$ containing the K -type π , and let π' be a λ -lowest K -type of X . By the proof just given, π' is fine; so $\pi' \in A(\delta)$ (Definition 4.3.15). By Theorem 4.3.16(c), π' and π differ by an automorphism coming from $\text{Ad}(\mathfrak{g})$; so obviously

$$||\pi||_{\lambda} = ||\pi'||_{\lambda}$$

(Definition 5.4.1). So π is also a λ -lowest K -type of X ; so by the preceding proof again, π' is small. Q.E.D.

Corollary 5.4.15 Fix an infinitesimal character

$\xi: Z(\mathfrak{g}) \rightarrow \mathbb{C}$. Then there is a finite set $S \subseteq \hat{K}$, depending on ξ , such that if X is a $(\mathfrak{g}, K)$ module with infinitesimal character ξ , then the λ -lowest K -types of X belong to S .

Proof. Let π be a λ -lowest K -type of a $(\mathfrak{g}, K)$ module with infinitesimal character ξ . Fix a highest weight μ of π , and define λ by Proposition 5.3.3. By Corollary 5.4.13,

$$\langle \lambda, \lambda \rangle \leq \langle \text{RE}(\xi), \text{RE}(\xi) \rangle.$$

Since λ is close to μ , this forces μ to lie in a certain ball determined by ξ . The corollary follows. Q.E.D.

Corollary 5.4.16 (Harish-Chandra) Let X be a $(\mathfrak{g}, K)$ module. The following conditions are equivalent:

- a) X has finite length.
- b) X is finitely generated and $Z(\mathfrak{g})$ -finite.
- c) X is admissible and $Z(\mathfrak{g})$ -finite.
- d) X is admissible and finitely generated.

(Recall that X is admissible if and only if $\dim X(\pi) < \infty$ for all $\pi \in \hat{K}$.)

Proof. That a) $\implies$ b) is obvious by induction on the length. By Theorem 2.2.1.1 of [50] (which is Harish-Chandra's basic result on $(\mathfrak{g}, K)$ modules) a finitely generated $Z(\mathfrak{g})$ -finite $(\mathfrak{g}, K)$ module is admissible; so b) $\implies$ c), b) $\implies$ d).

Suppose d) holds. Let V be a generating subspace of X . We can choose a finite set $S \subseteq \hat{K}$ so that if (notation 0.3.4))

$$X(S) = \bigoplus_{\pi \in S} X(\pi),$$

then $V \subseteq X(S)$. Since X is admissible, $X(S)$ is finite dimensional; and, of course, it generates X . Clearly, it is invariant under $Z(\mathfrak{g})$, so it is annihilated by an ideal $I \subseteq Z(\mathfrak{g})$ of finite codimension. Since $X(S)$ generates X , I must annihilate all of X . So X is $Z(\mathfrak{g})$ -finite. So d) $\implies$ b), d) $\implies$ c).

Finally suppose c) holds; we want to show that X has finite length. By an obvious induction, we may assume that $Z(\mathfrak{g})$ acts by scalars on X . Choose a finite set $S \subseteq \hat{K}$ as in Corollary 5.4.15, and define $X(S)$ as above. Then every irreducible subquotient of X meets $X(S)$; so obviously the

number of irreducible subquotients of X , counted with multiplicity, is at most $\dim X(S)$. This proves (a). Q.E.D.

Corollary 5.4.17 (Harish-Chandra) The number of inequivalent irreducible $(\mathfrak{g}, K)$ modules with a fixed infinitesimal character is finite.

Proof. By Lemma 4.1.8, there are only finitely many principal series representations with a fixed infinitesimal character. By Corollary 5.4.16 ((c) $\Rightarrow$ (a)), these have finite length. By Theorem 4.1.9, every irreducible $(\mathfrak{g}, K)$ module with this infinitesimal character occurs among their (finitely many) composition factors. Q.E.D.

It is possible to give a much simpler definition of lowest K -type.

Definition 5.4.18 If $\gamma \in (\mathcal{L}^C)^*$, then define (notation (5.3.2))

$$||\gamma|| = \langle \gamma + 2\rho_C, \gamma + 2\rho_C \rangle.$$

If $\pi \in \hat{K}$ has highest weight γ , set

$$||\pi|| = ||\gamma||$$

If X is a non-zero $(\mathfrak{g}, K)$ module, the set of lowest K -types of X is

$\{\pi \in \hat{K} | X(\pi) \neq 0, \text{ and } ||\pi|| \text{ is minimal with this property}\}.$

Since the form $\langle, \rangle$ is positive definite on the weight lattice in $(\mathcal{L}^C)^*$, the set of lowest K -types of a (non-zero) $(\mathfrak{g}, K)$ module is non-empty and finite. It can be shown that if X is irreducible, the sets of lambda-lowest and lowest K -types coincide ([42], Lemma 8.8). They do not coincide for arbitrary $(\mathfrak{g}, K)$ modules. To see this, take $G = \mathrm{SL}(2, \mathbb{R})$, and suppose X is the sum of the trivial representation (whose only K -type is χ_0 , in the notation of Lemma 1.4.2) and the two-dimensional representation (whose K -types are $\chi_{\pm 1}$). Then

$$\begin{aligned} ||\chi_0|| &= 0, \quad ||\chi_{\pm 1}|| = 1 \\ ||\chi_0||_{\text{lambda}} &= ||\chi_{\pm 1}||_{\text{lambda}} = 0. \end{aligned}$$

So χ_0 is the unique lowest K -type, but all the K -types are lambda-lowest.

The simplicity of Definition 5.4.18 is appealing, but Definition 5.4.1 is far easier to deal with for theoretical purposes. (The interested reader may compare the proof of Theorem 8.1.6 given in this book with the one in [46]. No new ideas are used here, but the computations are noticeably shorter.) So we will ignore Definition 5.4.18 entirely; but it could be inserted in place of lambda-lowest in most theorems, using Lemma 8.8 of [42] or ideas from its proof.

Chapter 6. Zuckerman's Construction and the Classification of Irreducible $(\mathfrak{g}, \bar{K})$ modules.

In the theory of finite dimensional representations of $\mathfrak{g}$, as developed for example in [22], the second point (after the existence of highest weight vectors) is the construction of a module which is universal with respect to possessing a highest weight vector; this is a Verma module (called $Z(\lambda)$ in [22]). The properties of this Verma module then show that the highest weight vector is unique, and give some information about the finite dimensional representation (for example, Proposition 0.4.2(g)). Here we do the same thing for $(\mathfrak{g}, K)$ modules. We have available from Chapter 5 some canonical cohomology classes for a $(\mathfrak{g}, K)$ module; these are the analogues of highest weight vectors. In this chapter (in Section 5) we will construct the corresponding universal objects, and deduce the uniqueness of the canonical cohomology class (for irreducibles), as well as some weak structural information. In hindsight, it seems rather obvious how to do all of this; but the program existed for about twenty years before it was completely carried out. The key ideas in the particular approach to be described here are due to Zuckerman. They require a little background in homological algebra. Taking myself as a model for what a representation theorist knows about homological algebra, I have decided to include a sketchy account of this in Section 1; the reader with a better background will be able to skip most of it, as well as some proofs later on. It is due to a number of people; we refer to [3] for details and references.

§1. The category $\mathcal{M}(\mathfrak{h}, B)$.

Definition 6.1.1 Suppose we are given a complex Lie algebra $\mathfrak{h}$ and a compact Lie group B (possibly disconnected), subject to the following conditions:

- a) The complexified Lie algebra $\mathfrak{b}$ of B is contained in $\mathfrak{h}$.
- b) There is an action (which we call Ad) of B on $\mathfrak{h}$, by Lie algebra automorphisms, whose differential is $\text{ad}|_{\mathfrak{h}_0}$, and which extends the adjoint action of B on $\mathfrak{b}$.

An $(\mathfrak{h}, B)$ module is a complex vector space V carrying representations of $\mathfrak{h}$ and B , such that

- i) V is completely reducible as a B module; that is

$$V = \bigoplus_{\delta \in B} V(\delta),$$

with $V(\delta)$ a direct sum of (possibly infinitely many) copies of δ .

- ii) The differential of the action of B (which exists by (i)) is the restriction of the $\mathfrak{h}$ action to $\mathfrak{b}$.
- iii) If $x \in \mathfrak{h}$, $b \in B$, and $v \in V$, then

$$(\text{Ad}(b)x) \cdot v = b \cdot (x \cdot (b^{-1}v)).$$

These are just the conditions 0.3.7(a) - (c) for $(\mathfrak{g}, \bar{K})$ modules. In the applications, $\mathfrak{h}$ will usually be a parabolic subalgebra of $\mathfrak{g}$, and B the compact part of the Levi factor of a (possibly smaller) parabolic subalgebra.

Definition 6.1.2 Suppose X and Y are $(\mathfrak{h}, B)$ modules.

Let

$$\text{Hom}_{\mathcal{H}, B}(X, Y)$$

be the complex vector space of linear maps from X to Y which preserve both the $\mathcal{H}$ -module and the B -module structures; these are called $(\mathcal{H}, B)$ -module maps. We let $\mathcal{M}(\mathcal{H}, B)$ denote the category whose objects are $(\mathcal{H}, B)$ modules, and whose maps are $(\mathcal{H}, B)$ module maps.

Many of the standard constructions for $\mathcal{H}$ modules (like image and kernel) work in the category $\mathcal{M}(\mathcal{H}, B)$ as well. We will use the trivial ones without comment.

Lemma 6.1.1.3 Suppose X and Y are $(\mathcal{H}, B)$ modules, and

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

is a short exact sequence of $(\mathcal{H}, B)$ modules. Then there are natural exact sequences (of vector spaces):

$$0 \rightarrow \text{Hom}_{\mathcal{H}, B}(X, V_1) \rightarrow \text{Hom}_{\mathcal{H}, B}(X, V_2) \rightarrow \text{Hom}_{\mathcal{H}, B}(X, V_3)$$

$$0 \rightarrow \text{Hom}_{\mathcal{H}, B}(V_3, Y) \rightarrow \text{Hom}_{\mathcal{H}, B}(V_2, Y) \rightarrow \text{Hom}_{\mathcal{H}, B}(V_1, Y).$$

Thus $\text{Hom}_{\mathcal{H}, B}(X, *)$ is a left exact covariant functor, and $\text{Hom}_{\mathcal{H}, B}(*, Y)$ is a left exact contravariant functor.

The trivial proof is left to the reader; the last statements are (by definition) just a restatement of the first part.

Definition 6.1.1.4 The $(\mathcal{H}, B)$ module X is called projective if the functor $\text{Hom}_{\mathcal{H}, B}(X, *)$ is exact.

Equivalently, we require that if there is a surjection

$$V_2 \xrightarrow{f} V_3 \rightarrow 0$$

and any map

$$\begin{array}{c} \bar{f} \\ X \rightarrow V_3 \end{array},$$

then we can find a map

$$\begin{array}{c} f \\ X \rightarrow V_2 \end{array}$$

such that the diagram

$$\begin{array}{ccc} V_2 & \xrightarrow{f} & V_3 \rightarrow 0 \\ & \searrow f & \uparrow \bar{f} \\ & & X \end{array}$$

commutes. Similarly, the $(\mathcal{H}, B)$ module Y is called injective if the functor $\text{Hom}_{\mathcal{H}, B}(*, Y)$ is exact; this can be characterized in the same way as projective, with "surjection" replaced by "injection", and all arrows reversed.

Our goal is to define the derived functors $\text{Ext}_{\mathcal{H}, B}^i$. Just as in the case of modules over a ring, this requires the construction of projective or injective resolutions; so we need a way to construct lots of projective or injective modules. It is convenient to give a slightly more general construction at this point. Suppose $\mathcal{O}_{\mathcal{H}}$ is a complex subalgebra of $\mathcal{H}$, containing $\mathcal{C}$ and invariant under $\text{Ad}(B)$; thus $\mathcal{M}(\mathcal{O}_{\mathcal{H}}, B)$ is defined.

Definition 6.1.1.5 In the setting just described, suppose X is a $(\mathcal{O}_{\mathcal{H}}, B)$ module. Define

$$Y = \text{ind}_{\mathcal{O}_{\mathcal{H}}, B}^{\mathcal{H}, B}(X) = \text{ind}(X) = U(\mathcal{H}) \otimes_{\mathcal{O}_{\mathcal{H}}} X,$$

which is an $\mathcal{h}$ module by the left action of $\mathcal{h}$ on $U(\mathcal{h})$ by multiplication. Make Y a B module as follows: B acts on $\mathcal{h}$ by automorphisms (this was one of our assumptions on the pair $(\mathcal{h}, B)$) and hence, on the enveloping algebra $U(\mathcal{h})$. B also acts on X , and therefore on the tensor product

$$\tilde{Y} = U(\mathcal{h}) \otimes X.$$

$\tilde{Y}$ is the quotient of $\tilde{Y}$ by the subspace generated by

$$\{(uq \otimes x - u \otimes qx) \mid u \in U(\mathcal{h}), q \in \mathcal{q}, x \in X\},$$

which is easily seen to be invariant under B ; so Y is a B module. Finally, if Z is an $(\mathcal{h}, B)$ module, define

$$\text{res}_{\mathcal{h}, B}'(Z) = \text{res}(Z)$$

to be Z regarded as a $(\mathcal{q}, B)$ module.

Lemma 6.1.6 (algebraic Frobenius reciprocity). In the setting of Definition 6.1.5

$$\text{ind}_{\mathcal{q}, B}^{\mathcal{h}, B}(X)$$

is an $(\mathcal{h}, B)$ module. If

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

is a short exact sequence of $(\mathcal{q}, B)$ modules, then there is a natural short exact sequence

$$0 \rightarrow \text{ind}(V_1) \rightarrow \text{ind}(V_2) \rightarrow \text{ind}(V_3) \rightarrow 0$$

of $(\mathcal{h}, B)$ modules. Thus

$\text{ind: } \mathcal{M}(\mathcal{q}, B) \rightarrow \mathcal{M}(\mathcal{h}, B)$
is an exact functor. If X is a $(\mathcal{q}, B)$ module and Z is an $(\mathcal{h}, B)$ module, then there is a natural isomorphism

$$(6.1.7) \quad \text{Hom}_{\mathcal{h}, B}(\text{ind}(X), Z) \cong \text{Hom}_{\mathcal{q}, B}(X, \text{res}(Z)).$$

The easy proof is left to the reader. (Notice that res is obviously an exact functor.)

Corollary 6.1.8 Let X be any $(\mathcal{h}, B)$ module. Then

$$\text{ind}_{(\mathcal{h}, B)}^{(\mathcal{h}, B)}(X)$$

is a projective $(\mathcal{h}, B)$ module.

Proof. The functor

$$\text{Hom}_{\mathcal{h}, B}(\text{ind}(X), *)$$

is equivalent to

$$\text{Hom}_{\mathcal{h}, B}(X, \text{res}(*))$$

by (6.1.7). This second functor is exact because of standard facts about representations of compact groups -- we leave the verification to the reader. But the exactness of the first functor means that $\text{ind}(X)$ is projective. Q.E.D.

Definition 6.1.9 Let X be an $(\mathcal{h}, B)$ module. A projective resolution of X is a long exact sequence

$$\dots P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

of $(\mathcal{h}, B)$ modules, with P_i projective. We write

$$P_{\star} \rightarrow X \rightarrow 0$$

for short. An injective resolution of X is a long exact sequence

$$0 \rightarrow X \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

with I_i injective; we write this as

$$0 \rightarrow X \rightarrow I_{\star}.$$

Corollary 6.1.10 Any $(\mathcal{A}, B)$ module X has a projective resolution.

Proof. Define

$$P_0 = \text{ind}_{(\mathcal{L}, B)}^{(\mathcal{A}, B)} \text{res}_{(\mathcal{A}, B)}^{(\mathcal{L}, B)}(X).$$

Then by (4.1.7)

$$\text{Hom}_{\mathcal{A}, B}(P_0, X) \cong \text{Hom}_{\mathcal{A}, B}(\text{res}(X), \text{res}(X)).$$

Let

$$d_0: P_0 \rightarrow X$$

be the map corresponding to the identity map on $\text{res}(X)$ in this identification. We claim that d_0 is surjective. This is obvious if one traces through the construction: the map d_0 is just

$$U(\mathcal{A}) \otimes_B X \rightarrow X, u \otimes x \rightarrow u \cdot x.$$

(A purely formal proof could also be given: let V be the image of d_0 , an $(\mathcal{A}, B)$ submodule of X . Then by the naturality of (6.1.7), the image of the identity map on $\text{res}(X)$ is contained in $\text{res}(V)$, so $\text{res}(V) = \text{res}(X)$; so $V = X$.)

Let X_1 be the kernel of d_0 . By repeating the construction just given, we can find a projective module P_1 and a surjection

$$d_1: P_1 \rightarrow X_1.$$

Composing this with the inclusion of X_1 in P_0 gives

$$P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0;$$

and the image of d_1 is the kernel of d_0 . An obvious induction completes the proof. Q.E.D.

Definition 6.1.11 Suppose X and Y are $(\mathcal{A}, B)$ modules, and $P_{\star} \rightarrow X \rightarrow 0$ is a projective resolution of X . Then

$$0 \rightarrow \text{Hom}_{\mathcal{A}, B}(P_0, Y) \rightarrow \text{Hom}_{\mathcal{A}, B}(P_1, Y) \rightarrow \text{Hom}_{\mathcal{A}, B}(P_2, Y) \rightarrow \dots$$

is a cochain complex; its i^{th} cohomology group is called

$$\underline{\text{Ext}}_{\mathcal{A}, B}^i(X, Y).$$

Proposition 6.1.12 Up to natural isomorphism, $\underline{\text{Ext}}_{\mathcal{A}, B}^i(X, Y)$ is independent of the choice of projective resolution of X . Suppose $i > 0$, P is projective, and I is injective. Then

$$\underline{\text{Ext}}_{\mathcal{A}, B}^i(P, Y) = 0 = \underline{\text{Ext}}_{\mathcal{A}, B}^i(X, I).$$

If

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

is a short exact sequence of $(\mathcal{A}, B)$ modules, then

there are natural long exact sequences

$$\begin{aligned} \dots &\rightarrow \text{Ext}^i(X, V_1) \rightarrow \text{Ext}^i(X, V_2) \rightarrow \text{Ext}^i(X, V_3) \rightarrow \text{Ext}^{i+1}(X, V_1) \rightarrow \\ &\dots \rightarrow \text{Ext}^i(V_3, Y) \rightarrow \text{Ext}^i(V_2, Y) \rightarrow \text{Ext}^i(V_1, Y) \rightarrow \text{Ext}^{i+1}(V_3, Y) \rightarrow \end{aligned}$$

Finally, there is a natural isomorphism

$$\text{Ext}_{\mathcal{H}, B}^0(X, Y) \cong \text{Hom}_{\mathcal{H}, B}(X, Y).$$

This is proved exactly as for modules over rings, and we leave it to the reader (cf. [3]). This construction is summarized by saying that $\text{Ext}_{\mathcal{H}, B}^i(X, *)$ is the i^{th} right derived functor of $\text{Hom}_{\mathcal{H}, B}(X, *)$.

Definition 6.1.13 Suppose X is an $(\mathcal{H}, B)$ module.

Write $\mathbb{Q}$ for the trivial one-dimensional $(\mathcal{H}, B)$ module. The i^{th} $(\mathcal{H}, B)$ cohomology of X is by definition

$$H^i(\mathcal{H}, B; X) = \text{Ext}_{\mathcal{H}, B}^i(\mathbb{Q}, X);$$

in particular,

$$H^0(\mathcal{H}, B; X) = \text{Hom}_{\mathcal{H}, B}(\mathbb{Q}, X).$$

The cohomology of $\mathcal{H}$ with coefficients in X is

$$H^i(\mathcal{H}, X) = H^i(\mathcal{H}, \{1\}; X).$$

Thus we are implicitly claiming that the Lie algebra cohomology groups defined by the complex (3.1.1) are the right derived functors of

$$\text{Hom}_{\mathcal{H}}(\mathbb{Q}, *).$$

We will prove this by exhibiting an explicit complex for

computing $H^*(\mathcal{H}, B; X)$; when B is trivial, this will reduce to 3.1.1.1).

Choose a B -stable complement $\mathcal{P}$ to $\mathcal{L}$ in $\mathcal{H}$. Let I be the ideal in $\Lambda^* \mathcal{H}$ (as an algebra under exterior multiplication) generated by $\mathcal{L}$; put

$$I^i = I \cap \Lambda^i \mathcal{H}.$$

Then

$$(6.1.14) \quad \Lambda^i \mathcal{P} \cong (\Lambda^i \mathcal{H}) / I^i,$$

an identification which preserves the adjoint action of B .

Lemma 6.1.15 With notation as above, suppose X is an $(\mathcal{H}, B)$ module. Make $\text{Hom}_{\mathbb{Q}}(\Lambda^* \mathcal{H}, X)$ a complex using the differential (3.1.1). Then

$$\text{Hom}_B(\Lambda^i \mathcal{P}, X) \cong \text{Hom}_B(\Lambda^i(\mathcal{H}) / I^i, X)$$

is a subcomplex. That is, if

$$\omega \in \text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{H}, X)$$

is zero on I and is a B -module map, then $d\omega$ has the same properties.

Proof. If $b \in B$ and $\omega \in \text{Hom}_{\mathbb{Q}}(\Lambda^i \mathcal{H}, X)$, put

$$[\theta(b) \cdot \omega](X_1 \wedge \dots \wedge X_i) = b \cdot (\omega(\text{Ad}(b^{-1})X_1 \wedge \dots \wedge \text{Ad}(b^{-1})X_i)).$$

Then ω is a B -module map if and only if

$$\theta(b)\omega = \omega, \quad \text{all } b \in B.$$

By a straightforward calculation, $d\theta = \theta d$; so $d\omega$ is a B-module map if ω is. So suppose ω is a B-module map and vanishes on I^i ; we want to show that $d\omega$ is zero on I^{i+1} . Now I^{i+1} is spanned by elements of the form

$$X_0 \wedge \dots \wedge X_i,$$

with

$$\begin{array}{l} X_j \in \mathcal{L}, \\ X_j \in \mathcal{P}, \end{array} \quad \begin{array}{l} 0 \leq j \leq r \\ j > r, \end{array}$$

and $r \geq 0$. Let us compute $d\omega(X_0 \wedge \dots \wedge X_i)$. If $r \geq 1$, every argument of ω on the right side of (3.1.1) has a factor in $\mathcal{L}$; so they are zero since ω is zero on I^i . So we may suppose $r = 0$. After the terms involving ω on I^i (i.e., those with an X_0 factor in the argument) are dropped from (3.1.1), we are left with

$$\begin{aligned} d\omega(X_0 \wedge \dots \wedge X_i) &= X_0 \cdot \omega(X_1 \wedge \dots \wedge X_i) - \\ &\quad \sum_{k=1}^i (-1)^k \omega([X_0, X_k] \wedge \dots \wedge \hat{X}_k \wedge \dots) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (\theta(\exp t X_0) \omega - \omega) \\ &= 0 \end{aligned}$$

since ω is a B-module map.

Q.E.D.

Proposition 6.1.16 Suppose X is an $(\mathcal{L}, B)$ module.

Then

$$H^i(\mathcal{L}, B; X)$$

is the i^{th} cohomology group of the cochain complex described in Lemma 6.1.15.

Proof. Define

$$P_i = U(\mathcal{L}) \otimes_{\mathcal{L}}^i \mathcal{P} = \text{ind}_{\mathcal{L}, B}^{\mathcal{L}, B} ({}^i \mathcal{P}).$$

Then P_i is a projective $(\mathcal{L}, B)$ module. We make P_* into a chain complex by defining

$$d: P_{i+1} \rightarrow P_i$$

$$\begin{aligned} (6.1.17) \quad d(u \otimes (X_0 \wedge \dots \wedge X_i)) &= \sum_{j=0}^i (-1)^j u X_j \otimes \\ &\quad + \sum_{j < k} (-1)^{j+k} u \otimes (\text{pr}[X_j, X_k] \wedge X_0 \wedge \dots \wedge \hat{X}_j \dots \wedge \hat{X}_k \dots \wedge X_i). \end{aligned}$$

Here $\text{pr}: \mathcal{L} \rightarrow \mathcal{P} \cong \mathcal{L}/\mathcal{L}$ is the natural projection map. An argument exactly like the proof of the lemma just given shows that this differential really makes P_* into a chain complex. P_0 is just

$$U(\mathcal{L}) \otimes_{\mathcal{L}} \mathcal{P} \cong U(\mathcal{L})/U(\mathcal{L})\mathcal{L};$$

so there is a natural map

$$d_0: P_0 \cong U(\mathcal{L})/U(\mathcal{L})\mathcal{L} \rightarrow U(\mathcal{L})/U(\mathcal{L})\mathcal{L} \cong \mathbb{C}$$

from P_0 to the trivial $(\mathcal{L}, B)$ module. It is trivial to check that d_0 is zero on the image of d ; so we get a chain complex

$$P_* \rightarrow \mathbb{C} \rightarrow 0.$$

We claim that this is a projective resolution of $\mathbb{C}$. We need only show that it is exact. To see this, set

$$U_n(\mathcal{L}) = \text{span of products } X_1 \dots X_r, \quad r \leq n, \quad X_i \in \mathcal{L};$$

this is the standard filtration of $U(\mathcal{h})$. Let P_i^n be the image of $U_{n-i}(\mathcal{h}) \otimes \wedge^i \mathcal{p}$ in P_i ; then one verifies immediately that

$$d: P_{i+1}^n \rightarrow P_i^n.$$

Let $S^n(\mathcal{p})$ denote the n^{th} symmetric power of $\mathcal{p}$. Then

$$P_i^n / P_{i-1}^{n-1} \cong S^{n-i}(\mathcal{p}) \otimes \wedge^i \mathcal{p}$$

as one easily checks; and d induces the map

$$\bar{d}: S^{n-i-1}(\mathcal{p}) \otimes \wedge^{i+1} \mathcal{p} \rightarrow S^{n-i}(\mathcal{p}) \otimes \wedge^i \mathcal{p},$$

$$\bar{d}(u \otimes (X_0 \wedge \dots \wedge X_{i-1})) = \sum_{j=0}^i (-1)^j u X_j \otimes (X_0 \wedge \dots \wedge \hat{X}_j \wedge \dots \wedge X_i).$$

It is well known (for example, by Poincaré's lemma) that this complex is exact except for $n = 0$, $i = 0$, when its cohomology is $\mathbb{C}$. One can now show that

$$\dots \rightarrow P_i^n \rightarrow P_{i-1}^n \rightarrow \dots \rightarrow P_0^n \rightarrow \mathbb{C} \rightarrow 0$$

is exact by induction on n (or by using the spectral sequence of the filtration); details are left to the reader. Since

$$U P_i^n = P_i,$$

P_* is in fact a resolution of $\mathbb{C}$.

It follows that $H^i(\mathcal{h}, B; X)$ is the cohomology of the complex

$$\text{Hom}_{\mathcal{h}, B}(P_i, X) \cong \text{Hom}_{\mathcal{h}, B}(\wedge^i \mathcal{p}, X),$$

this latter isomorphism being (6.1.7). It is easy to check that the differential induced by (6.1.17) in this complex is (3.1.1).

Q.E.D.

Corollary 6.1.18 Suppose X is an $(\mathcal{h}, B)$ module. Then $H^i(\mathcal{h}, B; X)$ is zero unless

$$\text{Hom}_B(\wedge^i(\mathcal{h}/\mathcal{L}), X) \neq 0;$$

in particular, $H^i(\mathcal{h}, B; X)$ vanishes for $i > \dim \mathcal{h}/\mathcal{L}$.

Definition 6.1.19 Suppose X and Y are $(\mathcal{h}, B)$ modules. We make $\text{Hom}_{\mathbb{C}}(X, Y)$ into an $\mathcal{h}$ module and a B module as follows: if

$$f \in \text{Hom}_{\mathbb{C}}(X, Y), \quad x \in X, \quad g \in B, \quad y \in \mathcal{h},$$

then we define

$$(g \cdot f)(x) = g \cdot (f(g^{-1} \cdot x))$$

$$(y \cdot f)(x) = y \cdot (f(x)) - f(y \cdot x).$$

Define

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(X, Y)_{B\text{-finite}} &= \{f \in \text{Hom}_{\mathbb{C}}(X, Y) \mid \text{the span of} \\ &\quad \text{all } g \cdot f \text{ for } g \in B \text{ is finite} \\ &\quad \text{dimensional}\}. \end{aligned}$$

Lemma 6.1.20 $\text{Hom}_{\mathbb{C}}(X, Y)_{B\text{-finite}}$ is an $(\mathcal{h}, B)$ module (in the setting of Definition 6.1.19). The functor

$\text{Hom}_{\mathbb{C}}(X, *)_{B\text{-finite}}$ is covariant and exact; the functor $\text{Hom}_{\mathbb{C}}(*, Y)_{B\text{-finite}}$ is contravariant and exact.

As usual we leave the proof to the reader. Using this definition, we can define "produced" modules. For a variety of reasons (for example, because producing is a closer algebraic model of group theoretic induction than ind) it would be more

sensible to interchange the notations pro and ind. Advocates of contrahomology are free to do this.

Definition 6.1.21 In the setting of Definition 6.1.1.5, suppose X is a $(\mathcal{O}_f, B)$ module. Define

$$Y = \text{pro}_{\mathcal{O}_f, B}^{\mathcal{h}, B}(X) = \text{Hom}_{\mathcal{O}_f}(U(\mathcal{h}), X)_{B\text{-finite}}.$$

Here $U(\mathcal{h})$ is regarded as $\mathcal{O}_f$ module by left multiplication to define $\text{Hom}_{\mathcal{O}_f}$, and as an $\mathcal{h}$ module on the right to make the Hom an $\mathcal{h}$ module. The B module structure of Y is defined by regarding $U(\mathcal{h})$ as a B module under the adjoint action, X as a B module by restriction, and using 6.1.19 for $(\mathcal{L}, B)$ modules. More explicitly, suppose f lies in

$$\text{Hom}_{\mathbb{Q}}(U(\mathcal{h}), X).$$

Then f lies in

$$\text{Hom}_{\mathcal{O}_f}(U(\mathcal{h}), X)_{B\text{-finite}}$$

if and only if the following two conditions are satisfied:

a) For all $q \in U(\mathcal{O}_f)$, $h \in U(\mathcal{h})$

$$f(qh) = q \cdot f(h);$$

and

b) The span of the functions $\{b \cdot f | b \in B\}$ is finite dimensional. Here if $h \in U(\mathcal{h})$, then

$$(b \cdot f)(h) = b \cdot (f(\text{Ad}(b^{-1})h)).$$

If f satisfies these properties, then we make B act on it as in (b), and $U(\mathcal{h})$ as follows: if $h, h_0 \in U(\mathcal{h})$, then

$$c) \quad (h_0 \cdot f)(h) = f(hh_0).$$

Lemma 6.1.22 (algebraic Frobenius reciprocity). In the setting of Definition 6.1.21,

$$\text{pro}_{\mathcal{O}_f, B}^{\mathcal{h}, B}(X)$$

is an $(\mathcal{h}, B)$ module. Also

$$\text{pro}: \mathcal{M}(\mathcal{O}_f, B) \rightarrow \mathcal{M}(\mathcal{h}, B)$$

is an exact functor; and if Z is any $(\mathcal{h}, B)$ module,

$$(6.1.23) \quad \text{Hom}_{\mathcal{h}, B}(Z, \text{pro}(X)) \cong \text{Hom}_{\mathcal{O}_f, B}(\text{res}(Z), X)$$

in a natural way.

Corollary 6.1.24 Let X be any $(\mathcal{L}, B)$ module. Then

$$\text{pro}_{(\mathcal{L}, B)}^{(\mathcal{h}, B)}(X)$$

is an injective $(\mathcal{h}, B)$ module. If X and Y are $(\mathcal{h}, B)$ modules, then we can find an injective resolution

$$0 \rightarrow Y \rightarrow I_{\star}^i \text{ of } Y. \text{ Then } \text{Ext}_{\mathcal{h}, B}^i(X, Y) \text{ is the } i^{\text{th}}$$

cohomology group of the complex

$$\text{Hom}_{\mathcal{h}, B}(X, I_{\star}).$$

This is proved in exactly the same way as the corresponding results for ind and projective modules.

There is another useful consequence of Definition 6.1.19.

Lemma 6.1.25 Suppose X, Y and Z are $(\mathcal{H}, B)$ modules. Then there is a natural isomorphism

$$\text{Hom}_{\mathcal{H}, B} \left[X, \text{Hom}_{\mathbb{Q}}(Y, Z)_{B\text{-finite}} \right] \cong \text{Hom}_{\mathcal{H}, B} (X \otimes_{\mathbb{Q}} Y, Z)$$

Proof. A map f taking x to f_x on the left corresponds to the map $\tilde{f}$ defined by

$$\tilde{f}(x \otimes y) = f_x(y).$$

The necessary verifications are left to the reader. Q.E.D.

Corollary 6.1.26 If X is an $(\mathcal{H}, B)$ module and I is an injective $(\mathcal{H}, B)$ module, then

$$J = \text{Hom}_{\mathbb{Q}}(X, I)_{B\text{-finite}}$$

is an injective $(\mathcal{H}, B)$ module. If Y and Z are also $(\mathcal{H}, B)$ modules, then there is a natural isomorphism

$$\text{Ext}_{\mathcal{H}, B}^i(X, \text{Hom}_{\mathbb{Q}}(Y, Z)_{B\text{-finite}}) \cong \text{Ext}_{\mathcal{H}, B}^i(X \otimes_{\mathbb{Q}} Y, Z).$$

In particular, taking $X = \mathbb{Q}$,

$$\text{Ext}_{\mathcal{H}, B}^i(Y, Z) \cong H^i(\mathcal{H}, B; \text{Hom}_{\mathbb{Q}}(Y, Z)_{B\text{-finite}})$$

Proof. The functor $\text{Hom}_{\mathcal{H}, B}(*, \text{Hom}_{\mathbb{Q}}(X, I)_{B\text{-finite}})$ is equivalent to

$$\text{Hom}_{\mathcal{H}, B}((\otimes_{\mathbb{Q}} X), I).$$

Tensoring over $\mathbb{Q}$ is exact, and Hom into I is exact; so this functor is exact, and J is injective. For the rest of the corollary, take an injective resolution $Z \rightarrow I_*$ of Z , and use it to compute the Ext groups. Q.E.D.

Proposition 6.1.27 (Shapiro's lemma). In the setting of Lemma 6.1.22, let X be a $(\mathcal{G}, B)$ module and Z an $(\mathcal{H}, B)$ module. Then there are natural isomorphisms

$$\text{Ext}_{\mathcal{H}, B}^i(Z, \text{pro}(X)) \cong \text{Ext}_{\mathcal{G}, B}^i(\text{res}(Z), X),$$

$$\text{Ext}_{\mathcal{H}, B}^i(\text{ind}(X), Z) \cong \text{Ext}_{\mathcal{G}, B}^i(X, \text{res}(Z)).$$

proof. Let $0 \rightarrow X \rightarrow I_*$ be an injective resolution of X as

a $(\mathcal{G}, B)$ module. Using (6.1.23) it is easy to check that $\text{pro}(I_*)$ is an injective $(\mathcal{H}, B)$ module; so since pro is exact, $0 \rightarrow \text{pro}(X) \rightarrow \text{pro}(I_*)$ is an injective resolution of $\text{pro}(X)$. By Corollary 6.1.24, $\text{Ext}_{\mathcal{H}, B}^i(Z, \text{pro}(X))$ is the cohomology of the complex

$$0 \rightarrow \text{Hom}_{\mathcal{H}, B}(Z, \text{pro}(I_0)) \rightarrow \text{Hom}_{\mathcal{H}, B}(Z, \text{pro}(I_1)) \rightarrow \dots$$

But this complex may be identified with

$$0 \rightarrow \text{Hom}_{\mathcal{G}, B}(\text{res}(Z), I_0) \rightarrow \text{Hom}_{\mathcal{G}, B}(\text{res}(Z), I_1) \rightarrow \dots$$

by (6.1.23), and the cohomology of this complex is $\text{Ext}_{\mathcal{G}, B}^i(\text{res}(Z), X)$. The second statement is proved in exactly the same way. Q.E.D.

We consider next a situation which will arise in at least two guises. Suppose

$$(6.1.28) \quad \begin{array}{ll} \text{a) } \mathcal{G} = \mathcal{L} + \mathcal{U}, \text{ a direct sum of subalgebras} \\ \text{b) } \mathcal{L} \subseteq \mathcal{L}, \text{ and } \mathcal{U} \text{ is an ideal invariant by } \text{Ad}(B). \end{array}$$

If Y is a $(\mathcal{G}, B)$ module, we write

$$\text{res}_{\mathcal{G}, B}^{\mathcal{U}}(Y), \text{ res}_{\mathcal{G}, B}^{\mathcal{L}}(Y)$$

for the underlying $\mathcal{U}$ module and (ℓ, B) -module. Our aim is the Hochschild-Serre spectral sequence, which relates the Ext functors in the three categories involved here. To begin with, we have already noticed (in Section 3.1) that if Y is a $\mathcal{G}$ module, then $H^*(\mathcal{U}, Y)$ may be regarded as an ℓ module via the cochain complex (3.1.1); and that this action of ℓ is natural (i.e., that it commutes with the maps in the long exact sequence of $\mathcal{U}$ cohomology for $\mathcal{G}$ modules). It is very easy to check that $H^*(\mathcal{U}, Y)$ can in fact be regarded as an (ℓ, B) module when Y is a $(\mathcal{G}, B)$ module; for, by hypothesis (6.1.28) (b), $\text{Ad}(B)$ acts on $\mathcal{U}$, and hence on the complex (3.1.1). Details are left to the reader. With these conventions, we can state

Proposition 6.1.29 (Hochschild-Serre spectral sequence - cf. [3]). In the setting (6.1.28)

suppose X and Y are $(\mathcal{G}, B)$ modules, and that $\mathcal{U}$ acts by zero on X . Then there is a first quadrant spectral sequence

$$E_n^{p,q} \Rightarrow \text{Ext}_{\mathcal{G}, B}^{p+q}(X, Y)$$

with E_2 term

$$E_2^{p,q} = \text{Ext}_{\ell, B}^p(\text{res}_{\mathcal{G}, B}^{\ell, B}(X), H^q(\mathcal{U}, Y)).$$

Here $H^*(\mathcal{U}, Y)$ is regarded as an (ℓ, B) module as outlined above. The differential has bidegree $(n, 1-n)$.

Corollary 6.1.30 In the setting (6.1.28), let Y be a $(\mathcal{G}, B)$ module. Then there is a spectral sequence

$$H^p(\ell, B; H^q(\mathcal{U}, Y)) \Rightarrow H^{p+q}(\mathcal{G}, B; Y).$$

For we simply take $X = \mathbb{Q}$ in the proposition.

Partly because of our choice of definitions, we will also need a slightly less pretty version of Proposition 6.1.29, namely

Proposition 6.1.31 In the setting (6.1.28), suppose

X and Y are $(\mathcal{G}, B)$ modules, and that $\mathcal{U}$ acts by zero

on Y . Put $m = \dim \mathcal{U}$, and let D be the one-dimensional

$(\mathcal{G}, B)$ module

$$D = (\wedge^m \mathcal{U}).$$

Then there is a first quadrant spectral sequence

$$\text{Ext}_{\ell, B}^p(H^{m-q}(\mathcal{U}, X \otimes D), \text{res}_{\mathcal{G}, B}^{\ell, B}(Y)) \Rightarrow \text{Ext}_{\mathcal{G}, B}^{p+q}(X, Y).$$

The differential has bidegree $(1-n, n)$.

We will prove Proposition 6.1.29 in some detail, and then sketch the modifications needed to get (6.1.31).

Lemma 6.1.32 Suppose T is a $(\mathcal{G}, B)$ module. Then if we define

$$I = \text{pro}_{\mathcal{G}, B}^{\mathcal{G}, B}(T) = \text{Hom}_{\mathcal{G}}(U(\mathcal{G}), T)_{B\text{-finite}},$$

there is a $\mathcal{U}$ -module isomorphism

$$I \cong \text{Hom}_{\mathbb{Q}} \left[U(\mathcal{U}), \left[\text{Hom}_{\mathcal{G}}(U(\ell), T)_{B\text{-finite}} \right] \right].$$

Also

$$\begin{aligned} H^0(\mathcal{U}, I) &\cong \text{Hom}_{\mathcal{G}}(U(\ell), T)_{B\text{-finite}} \\ &= \text{pro}_{\ell, B}^{\ell, B}(T) \end{aligned}$$

as (ℓ, B) modules.

Proof. Because of 6.1.28(a), we have

$$U(\mathcal{G}) \cong U(\mathcal{U}) \otimes_{\mathbb{C}} U(\ell)$$

and therefore

$$\text{Hom}_{\mathcal{L}}(U(\mathcal{G}), T) \cong \text{Hom}_{\mathbb{C}}(U(\mathcal{U}), \text{Hom}_{\mathcal{L}}(U(\ell), T))$$

(as is well known and easy to check). The first isomorphism follows by passage to B-finite elements.

Now $H^0(\mathcal{U}, I)$ consists of the $\mathcal{U}$ invariant elements of I , namely,

$$\text{Hom}_{\mathcal{U}} \left(U(\mathcal{U}), \left[\text{Hom}_{\mathcal{L}}(U(\ell), T)_{B\text{-finite}} \right] \right).$$

If S is any $\mathcal{U}$ module, then

$$\text{Hom}_{\mathcal{U}}(U(\mathcal{U}), S) \cong S,$$

so the last assertions follow.

Q.E.D.

Lemma 6.1.33 Suppose J is an injective $(\mathcal{G}, B)$ module. Then J is injective as a $\mathcal{U}$ module, and $H^0(\mathcal{U}, J)$ is injective as an (ℓ, B) module.

Proof. Let T be the underlying (ℓ, B) module of J . By Corollary 6.1.24,

$$I = \text{pro}_{\mathcal{L}, B}^{\mathcal{G}, B}(T) \cong \text{Hom}_{\mathcal{L}}(U(\mathcal{G}), T)_{B\text{-finite}}$$

is an injective $(\mathcal{G}, B)$ module, and there is an embedding

$$\begin{array}{c} \phi \\ 0 \rightarrow J \rightarrow I. \end{array}$$

Since J is injective, the functor $\text{Hom}_{\mathcal{G}, B}(*, J)$ is exact. So the map

$$\text{Hom}_{\mathcal{G}, B}(I, J) \xrightarrow{\phi^*} \text{Hom}_{\mathcal{G}, B}(J, J)$$

induced by ϕ is surjective (since ϕ is injective). Choose a preimage ψ of the identity map from J to J . Thus

$$\psi: I \rightarrow J, \quad \psi \circ \phi = \text{Id}_J.$$

Let K be the kernel of ψ ; then it is easy to check that

$$I = J \oplus K$$

(where we identify J and $\phi(J)$). So it is enough to check the conclusions of the lemma with J replaced by I ; this amounts to proving them for both J and K . But then they follow from Lemma 6.1.32 and Corollary 6.1.24. Q.E.D.

Proof of Proposition 6.1.29 Regard X as an (ℓ, B) -module, and choose a projective resolution

$$P_* \rightarrow \text{res}_{\mathcal{G}, B}^{\ell, B}(X) \rightarrow 0.$$

We can and do regard each P_i as a $(\mathcal{G}, B)$ module in which $\mathcal{U}$ acts by zero. Next choose an injective resolution

$$0 \rightarrow Y \rightarrow I_*$$

of Y as a $(\mathcal{G}, B)$ module. Consider the double complex

$$C_{ij} = \text{Hom}_{\mathcal{G}, B}(P_i, I_{*j}) \quad 0 \leq i, j < \infty.$$

We have differentials

$$\begin{array}{ll} d_1^{ij}: & C_{ij} \rightarrow C_{i+1, j} \\ d_2^{ij}: & C_{ij} \rightarrow C_{i, j+1} \end{array}$$

induced by the complexes P_* and I_* , respectively; and total differential

$$\partial^{ij} = d_1^{ij} + (-1)^{i+j} d_2^{ij}.$$

The cohomology of the total complex can be computed by two different spectral sequences, with E_2 terms

$$H^*(d_2, (H^*(d_1, C_{ij})))$$

$$H^*(d_1, (H^*(d_2, C_{ij})))$$

(cf. [5], Chapter XV). Consider the first of these. We must compute the cohomology of the complex C_{ij} with respect to the differential coming from the complex P_* . This complex is exact except at P_0 , where its cohomology is X . Since I_j is injective, this exactness is preserved under $\text{Hom}_{\mathcal{G}, B}$ into I_j ; so

$$H^i(d_1, C_{*, j}) = \begin{cases} 0 & i > 0 \\ \text{Hom}_{\mathcal{G}, B}(X, I_j), & i = 0. \end{cases}$$

So the cohomology of this complex with respect to d_2 is nothing but

$$\text{Ext}_{\mathcal{G}, B}^j(X, Y).$$

Since the E_2^{ij} term vanishes for $i > 0$, the spectral sequence degenerates; so the cohomology of the total complex is just this Ext group. The other spectral sequence for the double complex will be the one we want; we need only show that the E_2 terms are as specified in the proposition. Since $\mathcal{U}$ acts trivially on P_i , it is clear that

$$C_{ij} \cong \text{Hom}_{\ell, B}(P_i, H^0(\mathcal{U}, I_j)).$$

Since P_i is a projective (ℓ, B) module, the cohomology of this complex with respect to d_2 is just

$$\text{Hom}_{\ell, B}(P_i, (\text{cohomology of } H^0(\mathcal{U}, I_j) \text{ complex})).$$

By Lemma 6.1.33, I_* is an injective resolution of Y as a $\mathcal{U}$ module; so the complex

$$H^0(\mathcal{U}, I_j) \cong \text{Hom}_{\mathcal{U}}(\mathbb{C}, I_j)$$

has cohomology

$$\text{Ext}_{\mathcal{U}}^j(\mathbb{C}, Y) \cong H^j(\mathcal{U}, Y).$$

So the cohomology of C_{ij} with respect to d_2 is

$$\text{Hom}_{\ell, B}(P_i, H^j(\mathcal{U}, Y)).$$

Since P_i is a projective resolution of X as an (ℓ, B) module, the cohomology of this with respect to d_1 is

$$\text{Ext}_{\ell, B}^i(X, H^j(\mathcal{U}, Y))$$

as claimed. Q.E.D.

The proof of Proposition 6.1.31 is similar. One needs to know that a projective $(\mathcal{G}, B)$ module is a projective $\mathcal{U}$ module; and that if X is a projective $\mathcal{U}$ module, then

$$H^{m-q}(\mathcal{U}, X \otimes D) \cong \begin{cases} X/u \cdot X & q = 0 \\ 0 & q > 0. \end{cases}$$

(The first of these is easily deduced from (3.1.1)). The point is that one should really define $\text{Tor}_{\mathcal{H}, B}^i(X, Y)$ (the derived functors of $X \otimes_{\mathcal{H}, B} Y$), and Lie algebra homology as

$$H_i(\mathcal{H}, B; X) = \text{Tor}_{\mathcal{H}, B}^i(\mathbb{C}, X).$$

Then, as is proved in [5], p. 288,

$$H^{m-q}(\mathcal{U}, X \otimes D) \cong H_q(\mathcal{U}, X).$$

I have retained a formulation entirely in terms of cohomology essentially for my own convenience; apologies are extended to the reader for this.

For one or two very minor technical reasons, we will need very special cases of a result of D. Wigner. For the reader's entertainment, we will prove it in some generality. (The proof here is due, independently, to Springer.)

Proposition 6.1.34 (D. Wigner) Suppose X and Y are $(\mathcal{H}, B)$ modules, and $z \in Z(\mathcal{H})^B$ (the B invariants in the center of the universal enveloping algebra of $\mathcal{H}$). Thus the action of z defines $(\mathcal{H}, B)$ module maps

$$\phi: X \rightarrow X, \quad \psi: Y \rightarrow Y.$$

Let

$$\begin{aligned} \psi^i: \operatorname{Ext}_{\mathcal{H}, B}^i(X, Y) &\rightarrow \operatorname{Ext}_{\mathcal{H}, B}^i(X, Y) \\ \phi_i: \operatorname{Ext}_{\mathcal{H}, B}^i(X, Y) &\rightarrow \operatorname{Ext}_{\mathcal{H}, B}^i(X, Y) \end{aligned}$$

be the induced maps on Ext . Then $\phi_i = \psi^i$. In particular, suppose $\mathcal{A}_X$ and $\mathcal{A}_Y$ are maximal ideals in $Z(\mathcal{H})^B$, $(\mathcal{A}_X)^n$ annihilates X , and $(\mathcal{A}_Y)^n$ annihilates Y . If $\mathcal{A}_X \neq \mathcal{A}_Y$ then

$$\operatorname{Ext}_{\mathcal{H}, B}^i(X, Y) = 0.$$

Proof. We proceed by induction on i . If $i = 0$, then

$$\operatorname{Ext}_{\mathcal{H}, B}^0(X, Y) = \operatorname{Hom}_{\mathcal{H}, B}(X, Y).$$

If ω is in this space, then

$$\begin{aligned} [\psi^0(z) \cdot \omega](x) &= z \cdot \omega(x) \\ [\phi_0(z) \cdot \omega](x) &= \omega(z \cdot x). \end{aligned}$$

These are equal since ω is an $\mathcal{H}$ -module map. So suppose $i > 0$, and the result is known for $i - 1$. Choose an injective $(\mathcal{H}, B)$ module I containing Y :

$$0 \rightarrow Y \rightarrow I \rightarrow Q \rightarrow 0.$$

Since I is injective, Proposition 6.1.12 says

$$\operatorname{Ext}_{\mathcal{H}, B}^i(X, I) = 0;$$

so the long exact sequence of (6.1.12) gives

$$\operatorname{Ext}_{\mathcal{H}, B}^{i-1}(X, Q) \xrightarrow{\delta} \operatorname{Ext}_{\mathcal{H}, B}^i(X, Y) \rightarrow 0.$$

Fix

$$\omega \in \operatorname{Ext}_{\mathcal{H}, B}^i(X, Y),$$

and choose a preimage

$$\tilde{\omega} \in \operatorname{Ext}_{\mathcal{H}, B}^{i-1}(X, Q), \quad \delta(\tilde{\omega}) = \omega.$$

By the naturality of the connecting homomorphism δ ,

$$\begin{aligned} \phi_i(\omega) &= \phi_i(\delta(\tilde{\omega})) \\ &= \delta(\phi_{i-1}(\tilde{\omega})) \\ &= \delta(\psi^{i-1}(\tilde{\omega})), \end{aligned}$$

the last equality being the inductive hypothesis. To see that we can move the ψ^{i-1} outside and make it a ψ^i , one has to recall the construction of δ . It involves a diagram chase, in which a number of choices are necessary. At each step,

if some element x is chosen for $\tilde{\omega}$, it makes sense to choose $z \cdot x$ for $\psi^{i-1}(\tilde{\omega})$. If this is done, one gets

$$\delta \circ \psi^{i-1}(\tilde{\omega}) = \psi^i \circ \delta(\tilde{\omega})$$

by inspection. Since the right side is $\psi^i(\omega)$, this proves the first claim. (A similar argument is given in the proof of Proposition 3.1.2).

For the second, choose $z_0 \in Z(\mathcal{L})^B$ so that

$$z_0 \in \mathcal{A}_X, \quad 1 - z_0 \in \mathcal{A}_Y.$$

Then clearly

$$z_0^n \in (\mathcal{A}_X)^n, \quad 1 - z_0^n \in \mathcal{A}_Y;$$

and so

$$1 - (1 - z_0^n)^n \in (\mathcal{A}_X)^n, \quad (1 - z_0^n)^n \in (\mathcal{A}_Y)^n.$$

Put

$$z = 1 - (1 - z_0^n)^n$$

then z annihilates X , and acts by 1 on Y . By the first part of the proposition, the induced maps on Ext are both zero and the identity, so the Ext groups vanish. Q.E.D.

The proof of this result bears a striking resemblance to that of the Casselman-Osborne theorem (3.1.5); and in fact it is easy to formulate a common generalization of them. We leave this to the interested reader.

§2. Zuckerman's functors

The results in this section and the next were proved by Zuckerman in 1977, and are taken largely from his lectures at the Institute for Advanced Study in Princeton in early 1978 (see [55]).

Throughout this section, we fixed a closed subgroup

$$(6.2.1) \quad B \subseteq K$$

of G . Set

$$(6.2.2) \quad K_1 = BK_0.$$

Definition 6.2.3 The forgetful functor from $\mathcal{M}(\mathcal{G}, K)$ to $\mathcal{M}(\mathcal{G}, B)$,

$$F_{\mathcal{G}, K}^{\mathcal{G}, B} = F$$

is the functor taking $(\mathcal{G}, K)$ modules and maps to the underlying $(\mathcal{G}, B)$ modules and maps.

Definition 6.2.4 Suppose V is a $(\mathcal{G}, B)$ module.

Write

$$\tilde{\Gamma}_0 V = \{v \in V \mid \dim U(\mathcal{L}) \cdot v < \infty\}$$

$\Gamma_0 V = \{v \in \tilde{\Gamma}_0 V \mid \text{the representation of } \mathcal{K}_0 \text{ on } U(\mathcal{L}) \cdot v \text{ is the differential of a representation of } K_0\};$

these are called the $\mathcal{K}$ -finite and K_0 -finite vectors in V , respectively.

Lemma 6.2.5 Suppose V is a $(\mathcal{G}, B)$ module. Then $\tilde{\Gamma}_0 V$ and $\Gamma_0 V$ are invariant under $\mathcal{G}$, and $\Gamma_0 V$ is a $(\mathcal{G}, K_0)$ module.

Proof. That $\tilde{\Gamma}V$ is $\mathcal{G}$ -invariant is an elementary observation, first made by Harish-Chandra in [12]; we leave it to the reader. Let $\tilde{K}_0$ be the universal covering group of K_0 . Then the representation of K_0 on $\tilde{\Gamma}_0 V$ exponentiates to $\tilde{K}_0$; and if $Z \subseteq \tilde{K}_0$ is the kernel of the covering map $\tilde{K}_0 \rightarrow K_0$, then

$$\Gamma_0 V = (\tilde{\Gamma}_0 V)^Z.$$

Since Z acts trivially in the adjoint action, this shows that $\Gamma_0 V$ is $\mathcal{G}$ -invariant. The last assertion is trivial.

Definition 6.2.6 Suppose V is a $(\mathcal{G}, B)$ module;

let π denote the action of B on V , and ρ the action of K_0 on $\Gamma_0 V$. Put

$$\Gamma_1 V = \{v \in \Gamma_0 V \mid \pi(b_0)v = \rho(b_0)v, \text{ all } b_0 \in B \cap K_0\},$$

the K_1 -finite vectors of V .

Lemma 6.2.7 Suppose V is a $(\mathcal{G}, B)$ module. There is a unique action ρ_1 of K_1 in $\Gamma_1 V$, which extends both π and ρ (notation 6.2.6). This action makes $\Gamma_1 V$ a $(\mathcal{G}, K_1)$ module.

Proof. Using the compatibility condition 6.1.1(iii), it is straightforward to check that for $b \in B$, $k \in K_0$, and $v \in \Gamma_0 V$,

$$(6.2.8) \quad \rho(b)\pi(k)v = \pi(bkb^{-1})\rho(b)v;$$

the compatibility is just the differentiated version of this. A trivial computation from (6.2.8) shows that $\Gamma_1 V$ is stable under $\pi(B)$ and $\rho(K_0)$; and the compatibility condition can be used directly to show that $\Gamma_1 V$ is $\mathcal{G}$ -invariant. If $k \in K_1$, write

$$k = bk_0 \quad (b \in B, k_0 \in K_0);$$

then, if $v \in \Gamma_1 V$, we define

$$\rho_1(k)v = \pi(b)\rho(k_0)v.$$

This is well defined almost by the definition of $\Gamma_1 V$; and it lies in $\Gamma_1 V$ by the observations above. That ρ_1 is a representation follows from (6.2.8) by a simple computation. Finally, the compatibility condition which has to be verified to show that $\Gamma_1 V$ is a $(\mathcal{G}, K_1)$ module can be reduced to the one for V . Q.E.D.

Definition 6.2.9 Suppose V is a $(\mathcal{G}, B)$ module.

Put

$$\Gamma V = \text{Ind}_{\mathcal{G}, K_1}^{\mathcal{G}, K} (\Gamma_1 V)$$

(cf. (6.2.7) and (0.3.25)); we call this $(\mathcal{G}, K)$ module the space of K -finite vectors in V (although it is not in general a submodule of V). We make Γ a functor (in other words, apply it to maps) as follows. Suppose

$$f: V \rightarrow W$$

is a map of $(\mathcal{G}, B)$ modules. Then f takes $\Gamma_1 V$ into $\Gamma_1 W$ (clearly), and we write $\Gamma_1 f$ for the restriction (which is a $(\mathcal{G}, K_1)$ module map). Finally, let Γf be the map from ΓV to ΓW induced by $\Gamma_1 f$. For

precision, we may occasionally write

$$\Gamma \mathcal{G}_B^K \text{ or } \Gamma_B^K$$

instead of Γ .

Lemma 6.2.10 The functor Γ is left exact. If X is a $(\mathcal{G}, K)$ module and V is a $(\mathcal{G}, B)$ module, then there is a natural isomorphism (cf. (6.2.3))

$$\text{Hom}_{\mathcal{G}, K}(X, \Gamma V) \cong \text{Hom}_{\mathcal{G}_B}(\mathcal{G} X, V).$$

Proof. By the exactness of Ind and Lemma 0.3.26, it is enough to prove these statements with Γ_1, K_1 replacing Γ, K . Let

$$0 \rightarrow V_1 \xrightarrow{i} V_2 \xrightarrow{s} V_3 \rightarrow 0$$

be a short exact sequence of $(\mathcal{G}, B)$ modules. Then we have maps

$$(*) \quad \Gamma_1 V_1 \xrightarrow{\Gamma_1(i)} \Gamma_1 V_2 \xrightarrow{\Gamma_1(s)} \Gamma_1 V_3.$$

Since $\Gamma_1(i)$ is a restriction of i , it is still injective. The kernel of $\Gamma_1(s)$ is obviously

$$i(V_1) \cap \Gamma_1(V_2),$$

the K_1 -finite elements of V_2 lying in $i(V_1)$. This is the same as the set of K_1 -finite elements of $i(V_1)$, i.e.,

$$\Gamma_1(i)(\Gamma_1 V_1).$$

So the kernel of $\Gamma_1(s)$ is the image of $\Gamma_1(i)$, proving that $(*)$ is left exact. The last statement of the lemma simply

says that a $(\mathcal{G}, B)$ module map of a $(\mathcal{G}, K_1)$ module into V must have its image in the K_1 -finite vectors; but this is clear. Q.E.D.

Definition 6.2.11 The Zuckerman functors $\{\Gamma^i\}_{i \geq 0}$ (sometimes written $\{(\Gamma_B^K)^i\}$) from $\mathcal{M}(\mathcal{G}, B)$ to $\mathcal{M}(\mathcal{G}, K)$ are the right derived functors of Γ . More explicitly, let V be a $(\mathcal{G}, B)$ module, and

$$0 \rightarrow V \rightarrow I_*$$

an injective resolution of V (Definition 6.1.9). Then $\Gamma^i V$ is the i th cohomology group of the complex

$$0 \rightarrow \Gamma I_0 \rightarrow \Gamma I_1 \rightarrow \dots$$

of $(\mathcal{G}, K)$ modules.

That $\Gamma^i V$ is independent of the choice of resolution is proved by a standard argument ([5], Chapter V); but since these ideas are rather new to representation theory, we sketch it. Suppose we have two resolutions

$$0 \rightarrow V \rightarrow I_0 \rightarrow I_1 \rightarrow \dots$$

$$0 \rightarrow V \rightarrow J_0 \rightarrow J_1 \rightarrow \dots$$

and a $(\mathcal{G}, B)$ module map ϕ from V to V . Construct maps

$$\phi_i: I_i \rightarrow J_i$$

so that the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & V & \rightarrow & I_0 & \rightarrow & I_1 \rightarrow \dots \\ & & \downarrow \phi & & \downarrow \phi_0 & & \downarrow \phi_1 \\ 0 & \rightarrow & V & \rightarrow & J_0 & \rightarrow & J_1 \rightarrow \dots \end{array}$$

commutes. This is done by induction on i : suppose $i \geq 0$, and that ϕ_{i-1} is already constructed. (We define ϕ_{-1} to be $\phi \cdot$.) Composition of the differential for J_* with ϕ_{i-1} gives a map ψ_i from I_{i-1} to J_i . If $v \in I_{i-1}$ lies in the kernel of the differential d^I for I_* , then there is a $w \in I_{i-2}$ with

$$v = d^I(w).$$

(Here we define $I_{-1} = V$, $I_{-2} = \{0\}$.) Then

$$\begin{aligned}\psi_i(v) &= d^J(\phi_{i-1}(v)) \\ &= d^J(\phi_{i-1}(d^J(w))) \\ &= d^J(d^J(\phi_{i-2}(w))) \\ &= 0\end{aligned}$$

since J_* is a complex. So

$$\psi_i: I_{i-1}/\ker d^I \rightarrow J_i.$$

By the injectivity of I_i , we can find

$$\phi_i: I_i \rightarrow J_i$$

with the property that

$$\phi_i \circ d^I = \psi_i = d^J \circ \phi_{i-1}.$$

So we get maps $\Gamma\phi_i$ from ΓI_i to ΓJ_i , and (since these commute with d)

$$\overline{\phi_i}: H^i(\Gamma I_*) \rightarrow H^i(\Gamma J_*)$$

between the cohomology groups. We claim that these maps depend only on ϕ (and not on the choices of ϕ_i). It is enough

to show that if $\phi = 0$, then all $\overline{\phi_i}$ are zero. So suppose $\phi = 0$. We will construct maps

$$\xi_i: I_i \rightarrow J_{i-1} \quad (i = 0, 1, 2, \dots)$$

with the property that

$$(*) \quad d^J \xi_i + \xi_{i+1} d^I = \phi_i \quad (i = 0, 1, \dots).$$

Suppose for a moment that this has been done; we claim that $\overline{\phi_i}$ is zero. For fix $\overline{v} \in H^i(\Gamma I_*)$, and a representative v of $\overline{v}$ in ΓI_* . Then $(\Gamma d^I)v = 0$; so a representative of $\overline{\phi_i}(\overline{v})$ is

$$\begin{aligned}(\Gamma\phi_i)v &= [(\Gamma d^J)(\Gamma\xi_i) + (\Gamma\xi_{i+1})(\Gamma d^I)]v \\ &= (\Gamma d^J)(\Gamma\xi_i)v;\end{aligned}$$

so $(\Gamma\phi_i)v$ is a coboundary in ΓJ_* , and $\overline{\phi_i}(\overline{v})$ is zero.

To prove $(*)$, we proceed by induction on i . Set $\xi_0 = 0$, and suppose ξ_i has been defined. We want to define ξ_{i+1} so as to make $(*)$ true. This means that we must "extend"

$\phi_i - d^J \xi_i$ from I_i to I_{i+1} . Since I_{i+1} is injective, this will be possible if $\phi_i - d^J \xi_i$ is zero on the kernel of d^I , which is the same as the image of d^I ; so we must prove that

$$\phi_i d^I - d^J \xi_i d^I = 0.$$

By the choice of ϕ_i , this is the same as

$$\begin{aligned}0 &= d^J \phi_{i-1} d^J \xi_i d^I \\ &= d^J (\phi_{i-1} d^I \xi_i) .\end{aligned}$$

If $i = 0$, this is clear since $\phi_{-1} = \phi$ and ξ_0 are both zero. If $i > 0$, this is the same as

$$d^J(d^J \xi_{i-1})$$

by (*) and the inductive hypothesis; but this expression is zero since $(d^J)^2 = 0$. So $\overline{\phi_i}$ depends only on the resolutions I_* and J_* . It follows easily that if ψ is another map from V to V , and

$$0 \rightarrow V \rightarrow K_*$$

is a third resolution of V , then the maps $\overline{\psi_i}$ from $H^i(\Gamma J_*)$ to $H^i(\Gamma K_*)$ satisfy

$$(\overline{\psi\phi})_i = \overline{\psi_i} \overline{\phi_i}.$$

Taking ϕ and ψ to be the identity, and $K_* = I_*$, we find that $\overline{\phi_i}$ is an isomorphism from $H^i(\Gamma I_*)$ to $H^i(\Gamma J_*)$, with inverse $\overline{\psi_i}$. So $H^i(\Gamma I_*)$ and $H^i(\Gamma J_*)$ are canonically isomorphic; so $\Gamma^i V$ is well-defined. Similar ideas prove the following result.

Lemma 6.2.12 The Zuckerman functors

$$\Gamma^i: \mathcal{M}(\mathcal{O}, B) \rightarrow \mathcal{M}(\mathcal{O}, K)$$

are functors; that is, if

$$f \in \text{Hom}_{\mathcal{O}, B}(V_1, V_2),$$

then there are natural maps

$$(\Gamma^i f) \in \text{Hom}_{\mathcal{O}, K}(\Gamma^i V_1, \Gamma^i V_2).$$

If

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

is a short exact sequence of $(\mathcal{O}, B)$ modules, then there is a natural long exact sequence

$$\begin{aligned} 0 \rightarrow \Gamma^0 V_1 \rightarrow \Gamma^0 V_2 \rightarrow \Gamma^0 V_3 \rightarrow \Gamma^1 V_1 \rightarrow \Gamma^1 V_2 \rightarrow \Gamma^1 V_3 \rightarrow \dots \\ \dots \rightarrow \Gamma^i V_1 \rightarrow \Gamma^i V_2 \rightarrow \Gamma^i V_3 \rightarrow \Gamma^{i+1} V_1 \rightarrow \dots \end{aligned}$$

Here the maps which do not increase degree are those mentioned above. We have

$$\Gamma^0 V = \Gamma V = \{\text{space of } K\text{-finite vectors in } V\}.$$

If I is an injective $(\mathcal{O}, B)$ module, then

$$\Gamma^i I = 0 \quad (i > 0).$$

Again, these are general results about right derived functors.

We will want to keep track of the action of $Z(\mathcal{O})$. This is very easy.

Lemma 6.2.13 Let V be a $(\mathcal{O}, B)$ module, and sup-

pose $z \in Z(\mathcal{O})$. Regard z as an element h_z of

$$\text{Hom}_{\mathcal{O}, B}(V, V) \text{ by}$$

$$v \mapsto z \cdot v.$$

Then z acts on $\Gamma^i V$ by $\Gamma^i(h_z)$. In particular, if

V is annihilated by some ideal $\mathcal{I} \subseteq Z(\mathcal{O})$, so is $\Gamma^i V$; if V is $Z(\mathcal{O})$ -finite, so is $\Gamma^i V$; and if V has infinitesimal character χ , so does $\Gamma^i V$.

Proof. Obviously we need only prove the first statement.

The maps $\Gamma^i(h_z)$ are constructed from h_z by the procedure outlined after Definition 6.2.11; that is, we first choose

an injective resolution $0 \rightarrow V \rightarrow I_\star$, and then fix maps ϕ_i from I_i to I_i so that

$$\begin{array}{ccccccc} 0 & \rightarrow & V & \rightarrow & I_0 & \rightarrow & I_1 & \rightarrow & I_2 & \rightarrow & \dots \\ & & \downarrow h_z & & \downarrow \phi_0 & & \downarrow \phi_1 & & \downarrow \phi_2 & & \\ 0 & \rightarrow & V & \rightarrow & I_0 & \rightarrow & I_1 & \rightarrow & I_2 & \rightarrow & \dots \end{array}$$

commutes. Clearly we can choose ϕ_i to be the action of z on I_i . Then we get a map of complexes $\Gamma(\phi)$ from $\Gamma I_\star$ to $\Gamma I_\star$; obviously this is also given by the action of z . The maps $\Gamma^i(h_z)$ are the induced maps on the cohomology of $\Gamma I_\star$, and it is clear that these too are just the action of z . Q.E.D.

The next result lies at the heart of all the results on the structure of Γ^*V . It is a special case of a general homological algebra result, and is both a generalization and a simple consequence of Lemma 6.2.10.

Theorem 6.2.14 (Frobenius reciprocity spectral

sequence). Suppose X is a $(\mathcal{O}, K)$ module, and V is a $(\mathcal{O}, B)$ module. Then there is a first quadrant spectral sequence

$$E_n^{p,q} \Rightarrow \text{Ext}_{\mathcal{G}, B}^{p+q}(\mathcal{H}X, V),$$

with E_2 term

$$E_2^{p,q} \cong \text{Ext}_{\mathcal{G}, K}^p(X, \Gamma^q V).$$

The differential d_n has bidegree $(n, 1-n)$.

Proof. This is proved in exactly the same way as Proposition 6.1.31. We choose an injective resolution

$$0 \rightarrow V \rightarrow I_\star$$

of V as a $(\mathcal{O}, B)$ module, and a projective resolution

$$P_\star \rightarrow X \rightarrow 0$$

of X as a $(\mathcal{O}, K)$ module; and we consider the double complex

$$\begin{aligned} C_{pq} &= \text{Hom}_{\mathcal{G}, B}(\mathcal{H}P_p, I_q) \\ &\cong \text{Hom}_{\mathcal{G}, K}(P_p, \Gamma I_q) \end{aligned}$$

(by Lemma 6.2.10). $C_\star$ has a differential d_1 coming from the differential in $P_\star$, and a differential d_2 coming from that in $I_\star$; the total differential is

$$\partial = d_1 + (-1)^{p+q} d_2.$$

The cohomology of $C_\star$ with respect to ∂ can be computed using two different spectral sequences, having E_2 terms

$$H^*(d_2, H^*(d_1, C_\star))$$

$$H^*(d_1, H^*(d_2, C_\star)).$$

The complex $\{\mathcal{H}P_p\}$ is exact except at zero, where its cohomology is $\mathcal{H}X$. Since I_q is an injective $(\mathcal{O}, B)$ module, it follows that the cohomology of $C_\star$ with respect to d_1 is just

$$\begin{cases} \text{Hom}_{\mathcal{G}, B}(\mathcal{H}X, I_q) & (p = 0) \\ 0 & (p \neq 0) \end{cases}$$

The cohomology of this complex with respect to d_2 is

$$\begin{cases} \text{Ext}_{\mathcal{G}, B}^q(\mathcal{H}X, V) & (p = 0) \\ 0 & (p \neq 0) \end{cases}$$

So the first spectral sequence degenerates, and we get

$$H^q(\partial, C_*) \cong \text{Ext}_{\mathcal{G}, B}^q(\mathcal{F} X, V).$$

To prove the theorem, it therefore suffices to show that the E_2 term of the other spectral sequence is

$$\text{Ext}_{\mathcal{G}, K}^p(X, \Gamma^q V).$$

To compute $H^*(d_2, C_*)$, we use the second characterization of C_{pq} . The cohomology of $\{\Gamma^q\}$ is $\Gamma^q V$. Since P_p is a projective $(\mathcal{G}, K)$ module, the exactness of $\text{Hom}_{\mathcal{G}, K}(P_p, *)$ gives

$$H^q(d_2, C_{p,*}) \cong \text{Hom}_{\mathcal{G}, K}(P_p, \Gamma^q V).$$

The cohomology of this with respect to d_1 is the desired Ext group by definition. Q.E.D.

Corollary 6.2.15 Suppose $\delta \in \hat{K}$; let Z be an irreducible $(\mathcal{K}, K)$ module of type δ . Regard Z as a $(\mathcal{K}, B)$ module via the corresponding forgetful functor $\mathcal{F}$. Let V be a $(\mathcal{G}, B)$ module. Then the multiplicity of δ in $\Gamma^q V$ is

$$\dim \text{Ext}_{\mathcal{K}, B}^q(\mathcal{F} Z, \text{res}_{\mathcal{G}, B}^{\mathcal{K}, B}(V)).$$

A more reasonable (but slightly less precise) way to write this is

$$\dim \text{Ext}_{\mathcal{K}, B}^q(Z, V),$$

with the various restrictions understood.

Proof. Define

$$X = \text{ind}_{\mathcal{K}, K}^{\mathcal{G}, K}(Z) = U(\mathcal{G}) \otimes_{\mathcal{K}} Z.$$

By Corollary 6.1.8, X is a projective $(\mathcal{G}, K)$ module; so by Proposition 6.1.12,

$$\text{Ext}_{\mathcal{G}, K}^p(X, \Gamma^q V) = 0, \quad p > 0.$$

So the spectral sequence (6.2.14) collapses, and we get

$$\text{Hom}_{\mathcal{G}, K}(X, \Gamma^q V) \cong \text{Ext}_{\mathcal{G}, B}^q(\mathcal{F} X, V).$$

By (6.1.7), the left side is isomorphic to

$$\text{Hom}_{\mathcal{K}, K}(Z, \Gamma^q V),$$

the dimension of which is the multiplicity of δ in $\Gamma^q V$. So we need only identify the right side with the Ext group specified in the corollary. It is essentially obvious that

$$\mathcal{F} X \cong \text{ind}_{\mathcal{K}, B}^{\mathcal{G}, B}(\mathcal{F} Z).$$

So by Shapiro's lemma (6.1.19),

$$\text{Ext}_{\mathcal{G}, B}^q(\mathcal{F} X, V) \cong \text{Ext}_{\mathcal{K}, B}^q(\mathcal{F} Z, \text{res}_{\mathcal{G}, B}^{\mathcal{K}, B}(V)),$$

which is what we want. Q.E.D

Using Corollary 6.1.18 and Corollary 6.1.26 to compute the Ext group, one finds that if A and B are $(\mathcal{K}, B)$ modules, then

$$\text{Ext}_{\mathcal{K}, B}^q(A, B) = 0 \quad (q > \dim \mathcal{K}/\mathcal{L}).$$

It follows that if $q > \dim \mathcal{K}/\mathcal{L}$, then $\Gamma^q V$ has no K -types (if V is a $(\mathcal{G}, B)$ module); so $\Gamma^q V = 0$ for such q .

Corollary 6.2.16 Suppose V is a $(\mathcal{G}, B)$ module.

Let q_0 be the least q for which $\Gamma^{q_0} V \neq 0$. Then q_0 is also the least q for which there is a

$(\mathcal{G}, K)$ module X_0 such that

$$\text{Ext}_{\mathcal{G}, B}^q(\mathcal{F}X_0, V) \neq 0.$$

If X is any $(\mathcal{G}, K)$ module, then

$$\text{Hom}_{\mathcal{G}, K}(X, \Gamma^{q_0} V) \cong \text{Ext}_{\mathcal{G}, B}^{q_0}(\mathcal{F}X, V).$$

With fairly obvious conventions, we can allow the possibility that $\Gamma^{q_0} V$ is zero for all q .

Proof. The assumption implies that

$$\text{Ext}_{\mathcal{G}, K}^p(X, \Gamma^{q_0} V) = 0 \quad (q < q_0)$$

for any $(\mathcal{G}, K)$ module X ; so in the spectral sequence (6.2.14),

$$(*) \quad E_n^{p, q} = 0 \quad (n \geq 2, \quad q < q_0)$$

as one sees by induction on n . In particular

$$(\text{Im } d_n) \cap E_n^{0, q_0} = d_n(E_n^{-n, q_0+n-1}) = 0 \quad (n \geq 2)$$

since this is a first quadrant spectral sequence; and

$$\begin{aligned} (\ker d_n) \cap E_n^{0, q_0} &= \ker(d_n: E_n^{0, q_0} \rightarrow E_n^{n, q_0-(n-1)}) \\ &= E_n^{0, q_0} \end{aligned} \quad (n \geq 2)$$

since the target space $E_n^{n, q_0-(n-1)}$ is zero. So

$$\begin{aligned} E_{n+1}^{0, q_0} &= (\ker d_n) \cap E_n^{0, q_0} / (\text{Im } d_n) \cap E_n^{0, q_0} \\ &\cong E_n^{0, q_0}. \end{aligned}$$

By induction on n ,

$$E_n^{0, q_0} \cong E_2^{0, q_0} \cong \text{Hom}_{\mathcal{G}, K}(X, \Gamma^{q_0} V).$$

By $(*)$,

$$E_n^{t, q_0-t} = 0 \quad (t \geq 1, \quad n \geq 2);$$

so

$$\bigoplus_{p+q=q_0} E_n^{p, q} \cong \text{Hom}_{\mathcal{G}, K}(X, \Gamma^{q_0} V) \quad (n \geq 2)$$

and by $(*)$,

$$\bigoplus_{p+q < q_0} E_n^{p, q} = 0 \quad (n \geq 2).$$

So both these statements apply to the limit of the spectral sequence:

$$\text{Ext}_{\mathcal{G}, B}^q(\mathcal{F}X, V) \cong \begin{cases} \text{Hom}_{\mathcal{G}, K}(X, \Gamma^{q_0} V) & (q = q_0) \\ 0 & (q < q_0) \end{cases}$$

Taking $X = \Gamma^{q_0} V$, we see that we can arrange for the first of these to be non-zero. Q.E.D.

The next result will be needed only much later, in Chapter 8.

Proposition 6.2.17 (Induction-by-stages spectral sequence). Suppose $C \subseteq B \subseteq K$ are closed subgroups, and V is a $(\mathcal{G}, C)$ module. Then there is a first quadrant spectral sequence in $\mathcal{M}(\mathcal{G}, K)$

$$(\Gamma_B^K)^p(\Gamma_C^B)^{q_0} V \implies (\Gamma_C^K)^{p+q} V.$$

Proof. The underlying homological algebra in this result is a little complicated, so we will not give it in detail. At some point we will simply refer to a result in [5] (composition of functors spectral sequence). That result is stated only for the category of modules over a ring, but the proof

can be copied almost word for word, using Corollary 6.1.24 (existence of injective resolutions in $\mathcal{M}(\mathcal{A}, B)$) in place of [5], Chapter V, Proposition 1.2a.

Choose an injective resolution

$$(6.2.18) \text{ (a)} \quad 0 \rightarrow V \rightarrow I_*$$

of V and let

$$(6.2.18) \text{ (b)} \quad Y_* = \Gamma_C^B I_*,$$

a complex in $\mathcal{M}(\mathcal{O}, B)$. By [5], Chapter XVII, §7, there are two spectral sequences in $\mathcal{M}(\mathcal{O}, K)$ with a common limit Z_{p+q} , and E_2 terms

$$(6.2.18) \text{ (c)} \quad I_2^{p,q} = H^p((\Gamma_B^K)^q Y_*) \quad (\text{the } p^{\text{th}} \text{ cohomology group of the complex } (\Gamma_B^K)^q Y_*)$$

$$(6.2.18) \text{ (d)} \quad II_2^{p,q} = (\Gamma_B^K)^p (\Gamma_C^B)^q V.$$

We will use the first of these to compute Z_{p+q} ; then the second will be the spectral sequence we want. This uses

Lemma 6.2.19 Suppose I is an injective $(\mathcal{O}, C)$ module. Then $\Gamma_C^B I$ is an injective $(\mathcal{O}, B)$ module.

Proof. Let W be any $(\mathcal{O}, B)$ module. By Lemma 6.2.10,

$$\text{Hom}_{\mathcal{O}, B}(W, \Gamma_C^B I) \cong \text{Hom}_{\mathcal{O}, C}(\mathcal{F} W, I).$$

The right side is an exact functor in W by the injectivity of I , so the left side is as well. But this means $\Gamma_C^B I$ is injective, by definition.

Q.E.D.

Lemma 6.2.20 Let V be a $(\mathcal{O}, C)$ module. Then

$$\Gamma_B^K \Gamma_C^B V = \Gamma_C^K V.$$

Proof. This says that the K -finite vectors in the B -finite vectors are just the K -finite vectors; this sounds very good, and is not hard to check from the definitions. Alternatively, if X is any $(\mathcal{O}, K)$ module, then

$$\begin{aligned} \text{Hom}_{\mathcal{O}, K}(X, \Gamma_B^K \Gamma_C^B V) &\cong \text{Hom}_{\mathcal{O}, B}(\mathcal{F} X, \Gamma_C^B V) \\ &\cong \text{Hom}_{\mathcal{O}, C}(\mathcal{F} \mathcal{F} X, V) \end{aligned}$$

by two applications of Lemma 6.2.10. But obviously the composition of the two forgetful functors is the forgetful functor from $\mathcal{M}(\mathcal{O}, K)$ to $\mathcal{M}(\mathcal{O}, C)$; so

$$\begin{aligned} &\cong \text{Hom}_{\mathcal{O}, C}(\mathcal{F} X, V) \\ &\cong \text{Hom}_{\mathcal{O}, K}(X, \Gamma_C^K V). \end{aligned}$$

The lemma follows by a formal argument. Q.E.D.

Consider now (6.2.18) (c). By Lemma 6.2.19, and (6.2.18) (b), Y_i is injective; so

$$(\Gamma_B^K)^q Y_* = 0 \quad (q > 0).$$

By Lemma 6.2.20,

$$\begin{aligned} (\Gamma_B^K)^0 Y_* &= \Gamma_B^K (\Gamma_C^B I_*) \\ &\cong \Gamma_C^K I_* . \end{aligned}$$

The cohomology of this complex is by definition

$$(\Gamma_C^K)^p V.$$

So the spectral sequence (6.2.18) (c) collapses to

$$(\Gamma_C^K)^{p+q} V \cong Z_{p+q}.$$

So the spectral sequence we want is (6.2.18) (d). Q.E.D.

The next result allows us to compute Zuckerman's functors using a wider variety of resolutions.

Proposition 6.2.21 Suppose $\mathcal{A}$ is a subalgebra of $\mathcal{O}_Y$ containing $\mathcal{B}$, and that $\text{Ad}(\mathcal{B})$ normalizes $\mathcal{A}$.

Let C be a closed subgroup of B , and V a $(\mathcal{O}_Y, C)$ module.

$$\text{a) } \text{res}_{\mathcal{O}_Y, C}^{\mathcal{A}, B} ((\Gamma_{\mathcal{O}_Y, C}^{\mathcal{A}, B})^i V) = (\Gamma_{\mathcal{A}, C}^{\mathcal{A}, B})^i (\text{res}_{\mathcal{O}_Y, C}^{\mathcal{A}, B} V)$$

b) Suppose

$$0 \rightarrow V \rightarrow X_*$$

is an exact complex of $(\mathcal{O}_Y, C)$ modules, and that each X_i is injective as an $(\mathcal{A}, C)$ module. Then the i^{th} cohomology group of the complex

$$\Gamma_{\mathcal{O}_Y, C}^{\mathcal{A}, B} X_*$$

is isomorphic to $(\Gamma_{\mathcal{A}, C}^{\mathcal{A}, B})^i V$.

Proof. Choose an injective resolution in $\mathcal{M}(\mathcal{O}_Y, C)$

$$0 \rightarrow V \rightarrow I_*$$

We claim that $\text{res } I_i$ is an injective $(\mathcal{A}, C)$ module. For if W is any $(\mathcal{A}, C)$ module, the (6.1.12) gives

$$\text{Hom}_{\mathcal{A}, C}(W, \text{res } I_i) \cong \text{Hom}_{\mathcal{O}_Y, C}(\text{ind}_{\mathcal{A}, C}^{\mathcal{O}_Y, C} W, I_i).$$

The right side is exact in W by the exactness of ind and the injectivity of I_i ; so the left side is also, and $\text{res } I_i$ is injective. So

$$0 \rightarrow \text{res } V \rightarrow \text{res } I_*$$

is an injective resolution of $\text{res } V$. So

$$(*) \quad (\Gamma_{\mathcal{A}, C}^{\mathcal{A}, B})^i (\text{res } V) = H^i(\Gamma_{\mathcal{A}, C}^{\mathcal{A}, B} \text{res } V).$$

The formula

$$(\text{res}_{\mathcal{O}_Y, B}^{\mathcal{A}, B} \Gamma_{\mathcal{O}_Y, C}^{\mathcal{A}, B})(V) = \Gamma_{\mathcal{A}, C}^{\mathcal{A}, B} \text{res}_{\mathcal{O}_Y, C}^{\mathcal{A}, B} V$$

is obvious; and taking cohomology groups of a complex commutes with a forgetful functor. So $(*)$ becomes

$$\begin{aligned} \Gamma^i \text{res } V &= \text{res } H^i(\Gamma V) \\ &= \text{res } \Gamma^i V \end{aligned}$$

by definition of Γ^i . This is (a). For (b), we can use the injectivity of I_* to find vertical maps making a commutative diagram in $\mathcal{M}(\mathcal{O}_Y, C)$

$$\begin{array}{ccccccc} 0 & \rightarrow & V & \rightarrow & X_0 & \rightarrow & X_1 \rightarrow \dots \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & V & \rightarrow & I_0 & \rightarrow & I_1 \rightarrow \dots \end{array}$$

This induces maps on cohomology

$$H^i(\Gamma X_*) \rightarrow H^i(\Gamma I_*) \cong \Gamma^i V.$$

If we apply $\text{res}_{\mathcal{O}_Y, B}^{\mathcal{A}, B}$ to everything, these become $(\mathcal{A}, B)$ maps, which are isomorphisms since any $(\mathcal{A}, C)$ injective resolution of $\text{res } V$ computes $\Gamma^i(\text{res } V)$. But res does not affect the property of being an isomorphism, so the $(\mathcal{O}_Y, B)$ maps were isomorphisms as well. Q.E.D.

Finally, we record a technical result about the case of disconnected K .

Proposition 6.2.22 Recall the group $K_1 = BK_0$, and the functor

$$\Gamma_1 = \Gamma_B^{K_1}.$$

For all $q \geq 0$, we have

$$(\Gamma_B^K)^q = \text{Ind}_{\mathcal{G}, K_1}^{\mathcal{G}, K} \circ (\Gamma_B^{K_1})^q.$$

Proof. For $q = 0$ this is the definition; and we also have

$$\Gamma_{K_1}^K = \text{Ind}_{\mathcal{G}, K_1}^{\mathcal{G}, K}$$

by definition. This functor is exact, so its higher derived functors vanish:

$$(\Gamma_{K_1}^K)^p = 0, \quad p > 0.$$

The proposition now follows from Proposition 6.2.17 (with (K_1, B) replacing (B, C)).

§3. Parabolic induction.

We emphasize again that the results of this section are due to Zuckerman.

Definition 6.3.1 Let $\mathcal{G} \subseteq \mathcal{G}$ be a parabolic subalgebra, and

$$\mathcal{G} = \mathfrak{l} + \mathfrak{u}$$

a Levi decomposition. Assume that $\mathfrak{l}$ is a θ -stable real Levi factor of $\mathcal{G}$ (Definition 3.2.13), and define $\mathbb{L}$ as in that definition. The i^{th} (cohomological) parabolic induction functor from $\mathcal{M}(\mathfrak{l}, \text{LnK})$ to $\mathcal{M}(\mathcal{G}, K)$ is defined by

$$\begin{aligned} \mathcal{R}_{\mathcal{G}}^i &= \mathcal{R}_{\mathcal{G}}^i = (\mathcal{R}_{\mathcal{G}}^{\mathcal{G}})^i \\ &= (\Gamma_{\mathcal{G}, \text{LnK}}^{\mathcal{G}, K})^i \circ \text{pro}_{\mathcal{G}, \text{LnK}}^{\mathcal{G}, \text{LnK}} \left(* \otimes_{\mathbb{C}} (\wedge^{\dim \mathfrak{u}} \mathfrak{u}) \right) \end{aligned}$$

(cf. (6.1.21), (6.2.11)). Here we must first regard an $(\mathfrak{l}, \text{LnK})$ module as a $(\mathcal{G}, \text{LnK})$ module on which $\mathfrak{u}$ acts trivially. (Occasionally we will use $(\mathcal{G}, \text{LnK})$ modules on which $\mathfrak{u}$ does not act trivially from which to induce).

We will use these functors as a way of going from $(\mathfrak{l}, \text{LnK})$ modules to $(\mathcal{G}, K)$ modules. The parabolic subalgebra $\mathcal{G}$ will often be θ -stable (Definition 5.2.1); if (in the opposite case) $\mathcal{G}$ is real, then we will see that the definition reduces to ordinary parabolic induction, and this will sometimes be a useful perspective. The definition has not been made in such generality without reason, however. Although very little is known about the functors $\mathcal{R}^i$ in the general case, it seems likely that they appear in trying to understand the geometric version of the Kazhdan-Lusztig conjecture (Conjecture 2.3.11); so they should be studied. We will not do so here, however.

Some historical remarks about the case when $\mathcal{G}$ is θ -stable are in order here. Enright has also given a method for constructing $(\mathcal{G}, K)$ modules from $(\mathfrak{l}, \text{LnK})$ modules, based on his work with Varadarajan on an algebraic construction of the discrete series. This has not yet been published, but the results are described in [10], and a special case is treated in [11]. Experimental evidence suggests that Enright's construction is very closely related to $\mathcal{R}^S$, with $s = \frac{1}{2} \dim(\mathfrak{k}/\mathfrak{l} \cap \mathfrak{k}) = \dim \mathfrak{u} \cap \mathfrak{k}$. Some of the results we will prove using Zuckerman's construction can also be proved using

Enright's. (There is no known analogue of Theorem 6.2.14, which will play a critical role in the Kazhdan-Lusztig conjectures, for Enright's construction; but this could undoubtedly be remedied by a little work.) My main reason for choosing Zuckerman's construction is that it rests on very well-understood ideas in homological algebra -- things which can be proved by hard work and cleverness in Enright's setting are handed to us on a silver platter for $\mathcal{R}^i$.

The functors $\mathcal{R}^i$ can be regarded in the following way (which seems to have been Zuckerman's motivation). The parabolic $\mathcal{g}$ defines a complex structure on G/L ; and a representation π of L gives rise to a holomorphic vector bundle $\mathcal{V}_\pi$ over G/L . The fibers are infinite dimensional if π is. If G/L were a Stein manifold, we could look at the representation of G on holomorphic sections of $\mathcal{V}_\pi$; this is the representation holomorphically induced from L to G . If G/L is not a Stein manifold, however, (in our case it never is unless $L \cap K = K$), $\mathcal{V}_\pi$ is not likely to have any holomorphic sections. The standard way to remedy this is to consider the sheaf $\mathcal{S}_\pi$ of germs of holomorphic sections of $\mathcal{V}_\pi$. Then the cohomology groups

$$(*) \quad H^i(G/L, \mathcal{S}_\pi)$$

are representations of G . These were studied in a special case by Schmid in his Berkeley thesis [35]. The problems are formidable. The cohomology can be computed using a Dolbeault complex, but there is no obvious reason for the $\bar{\partial}$ operator to have closed range. Thus it is not even clear how to make (*) into a topological vector space.

What Zuckerman has done is to define some of these problems out of existence. If W is the Harish-Chandra module of π , then $\text{pro}_{\mathcal{g}, L \cap K} \mathcal{H}^{L \cap K}(W)$ can be thought of as the space of germs of holomorphic sections at the identity in G/L . A section is declared to be global if it is K -finite; thus Γ is a "global sections of" functor. Formally, this leads to the identification of $\mathcal{R}^{iW}$ with the K -finite part of (*). It would be interesting to make this precise; but meanwhile we can use $\mathcal{R}^i$ instead of (*). At least one of the arguments given below (the proof of the generalized Blattner formula, Theorem 6.3.12) is a direct translation of arguments of Schmid in [35] into the language of $\mathcal{R}^i$; the analogy is a very helpful one to bear in mind.

Proposition 6.3.2 (Frobenius reciprocity) In the setting of Definition 6.3.1, suppose W is an $(\mathcal{g}, L \cap K)$ module and X a $(\mathcal{g}, K)$ module. Then there are two first quadrant spectral sequences having a common limit -- namely

$$\begin{aligned} & \text{Ext}_{\mathcal{g}, L \cap K}^{p+q}(\mathcal{H}^X, \text{pro}_{\mathcal{g}, L \cap K} \mathcal{H}^{L \cap K}(W \otimes (\wedge^{\dim W} \mathcal{U}))) \\ & \text{and } E_2 \text{ terms} \end{aligned}$$

$$I_2^{p,q} = \text{Ext}_{\mathcal{g}, K}^p(X, \mathcal{R}^{qW})$$

$$II_2^{p,q} = \text{Ext}_{\mathcal{g}, L \cap K}^p(H^{\dim \mathcal{U}-q}(\mathcal{U}, \text{res } X), W).$$

Here res denotes restriction from $(\mathcal{g}, K)$ to $(\mathcal{g}, L \cap K)$.

Proof. The first spectral sequence is Theorem 6.2.14. For the second, Shapiro's lemma (Proposition 6.1.27) gives

$$\begin{aligned} & \text{Ext}_{\mathcal{G}, \text{LnK}}^i(\mathcal{F}X, \text{pro}(W \otimes (\wedge^{\dim \mathcal{U}} \mathcal{U}))) \\ & \cong \text{Ext}_{\mathcal{G}, \text{LnK}}^i(\text{res } X, W \otimes (\wedge^{\dim \mathcal{U}} \mathcal{U})). \end{aligned}$$

So the second is Proposition 6.1.31.

Q.E.D.

Corollary 6.3.4 Suppose W is an (ℓ, LnK) module, and $\mathcal{R}^q(W)$ vanishes for $q \neq q_0$. Then there is a (possibly non-first quadrant) spectral sequence

$$\begin{aligned} & \text{Ext}_{\ell, \text{LnK}}^p (H^{(\dim \mathcal{U} - q_0) - q}(\mathcal{U}, \text{res } X), W) \\ & \implies \text{Ext}_{\mathcal{G}, K}^{p+q}(X, \mathcal{R}^{q_0} W). \end{aligned}$$

Proof. By the hypothesis, the first spectral sequence of (6.3.2) collapses to

$$\text{Ext}_{\mathcal{G}, K}^p(X, \mathcal{R}^{q_0} W) \cong \text{Ext}_{\mathcal{G}, \text{LnK}}^{p+q_0}(\mathcal{F}X, \text{pro}(W \otimes (\wedge^{\dim \mathcal{U}} \mathcal{U}))).$$

So the spectral sequence we want is the second one in (6.3.2), with a shift in the indices.

Q.E.D.

Proposition 6.3.5 Suppose $P = \text{MAN}$ is the Langlands decomposition of a parabolic subgroup P of G . In the setting of Definition 6.3.1, take $\mathcal{G}$ to be the complexified Lie algebra of P , and

$$\ell = \mathfrak{m} + \sigma, \quad \ell \cap K = M \cap K, \quad \mathcal{U} = \mathcal{N}$$

Suppose W is the Harish-Chandra module of an admissible representation (Y, V_Y) of MA .

$$D = \wedge^{\dim \mathcal{N}} \mathcal{N} \quad (\text{as a representation of } P)$$

$$\Phi_\rho = \text{positive square root of } |D|;$$

this latter corresponds to the character called ρ^1 for p^1 in (4.1.11).

- a) $\mathcal{R}^{i,W} = 0$ for $i > 0$.
 b) $\mathcal{R}^0_W$ is the Harish-Chandra module of

$$\text{Ind}_P^G(\gamma \otimes D \otimes \Phi_{-\rho})$$

(notation 4.1.11(b)).

- c) Write $X(\gamma)$ for the Harish-Chandra module of $\text{Ind}_P^G(\gamma)$. Then if Y is any $(\mathcal{G}, K)$ module, there is a spectral sequence

$$\begin{aligned} & \text{Ext}_{\mathfrak{g}, K}^p (\text{Hom}_{\mathfrak{m} + \sigma, \text{MnK}}^{(\dim \mathcal{N} - q)}(\mathcal{N}, Y), W \otimes D^* \otimes \Phi_\rho) \\ & \implies \text{Ext}_{\mathfrak{g}, K}^{p+q}(Y, X(\gamma)). \end{aligned}$$

In particular,

$$\begin{aligned} \text{Hom}_{\mathfrak{g}, K}(Y, X(\gamma)) & \cong \text{Hom}_{\mathfrak{m} + \sigma, \text{MnK}}^{(\dim \mathcal{N} - q)}(\mathcal{N}, Y), \quad W \otimes D^* \otimes \Phi_\rho \\ & \cong \text{Hom}_{\mathfrak{m} + \sigma, \text{MnK}}^{(Y/\mathcal{N}Y, W \otimes \Phi_\rho)}. \end{aligned}$$

(Parts (b) and (c) here were well known before Zuckerman's construction.)

Proof. To compute $\mathcal{R}^{i,W}$, we must first consider

$$\begin{aligned} Z &= \text{pro}_{\mathcal{G}, \text{MnK}}^{\mathcal{G}, \text{MnK}}(W \otimes D) \\ &= \text{Hom}_{\mathcal{G}}^{(U(\mathcal{G}), W \otimes D)}_{\text{MnK-finite}}. \end{aligned}$$

Since

$$\mathcal{G} = \mathcal{K} + \mathcal{G}, \quad \mathcal{G} \cap \mathcal{K} = \mathfrak{m} \cap \mathcal{K},$$

the Poincaré-Birkhoff-Witt theorem provides an isomorphism of $(\mathcal{K}, \text{MnK})$ modules

$$\begin{aligned} \text{res } Z & \cong \text{Hom}_{\mathfrak{m} \cap \mathcal{K}}(\text{U}(\mathcal{K}), \text{res}(W \otimes D))_{\text{MnK-finite}} \\ & = \text{pro}_{\mathfrak{m} \cap \mathcal{K}, \text{MnK}}^{\mathcal{K}, \text{MnK}}(\text{res}(W \otimes D)). \end{aligned}$$

By Corollary 6.1.24, $\text{res}(Z)$ is an injective $(\mathcal{K}, \text{MnK})$ module. So by Proposition 6.2.21(a),

$$\begin{aligned} \text{res}_{\mathcal{G}, K}^{\mathcal{K}, K}(\mathcal{R}^{i_W}) &= (\Gamma_{\mathcal{K}, \text{MnK}}^{\mathcal{K}, K})^i(\text{res } Z) \\ &= 0 \text{ if } i > 0. \end{aligned}$$

This proves (a). In fact we get a little more; for if $\delta \in \hat{K}$, then Corollary 6.2.15 and the computation above show that

$$m(\delta, \mathcal{R}^0_W) = \dim \text{Hom}_{\mathcal{K}, \text{MnK}}(\delta, \text{pro}(W \otimes D))$$

$$= \dim \text{Hom}_{\mathcal{M} \cap \mathcal{K}, \text{MnK}}(\text{res } \delta, W \otimes D),$$

the second equality being (6.1.23). By (4.1.15), this is also the multiplicity of δ in $\text{Ind}_P^G(\gamma \otimes D \otimes \mathbb{C}_{-\rho})$. So to prove (b), it is enough to exhibit an injective $(\mathcal{G}, K)$ module map from the Harish-Chandra module to $\mathcal{R}^0_W$. By Lemma 6.2.10, we only have to find an injective $(\mathcal{G}, \text{MnK})$ module map ϕ from the Harish-Chandra module to Z . To do this, recall that the induced representation lives on a space of functions from G to $V_\gamma \otimes D$; the K -finite vectors will take values in $W \otimes D$. If f is in the Harish-Chandra module, define $\phi(f)$ in Z as follows. We have to make $\phi(f)$ a map from $U(\mathcal{G})$ to $W \otimes D$; so if $u \in U(\mathcal{G})$, define

$$\phi(f)(u) = (u \cdot f)(1).$$

That $\phi(f)$ is a $\mathcal{G}$ -module map follows from the formal conditions in the definition of the induced representation; and it is MnK -finite since f is. So $\phi(f)$ belongs to Z . To see that ϕ is a $\mathcal{G}$ module map, suppose $u_1 \in U(\mathcal{G})$. Then

$$\begin{aligned} [u_1 \cdot \phi(f)](u) &= \phi(f)(uu_1) \quad [\text{Definition 6.1.21(c)}] \\ &= (uu_1) \cdot f(1) \\ &= u \cdot (u_1 f)(1) \\ &= [\phi(u_1 \cdot f)](u). \end{aligned}$$

Similarly, one checks that ϕ is an MnK -module map. Finally, we must show that ϕ is injective. Suppose $\phi(f) = 0$. This means that f vanishes to infinite order at 1. Since f takes values in a finite dimensional subspace of $W \otimes D$, and is K -finite, it can be regarded as an analytic function on K ; so it must vanish identically on K_0 . By the transformation law under P , it also vanishes on K_0^P , which is all of G ; so $f = 0$. This proves (b). The spectral sequence in (c) is just Corollary 6.3.4. The first equality after it is a corner of that spectral sequence and the second is an identification of $H^{\dim \mathcal{N}}$ which (as we have already observed) follows from (3.1.1). Q.E.D.

Proposition 6.3.6 (Induction by stages). Suppose we are given two parabolic subalgebras

$$\mathcal{G}_f^1 = \mathfrak{g}^1 + \mathfrak{u}^1 \quad (i = 1, 2)$$

as in Definition 6.3.1, and that

$$\mathcal{G}_f^1 \subseteq \mathcal{G}_f^2, \quad \mathfrak{g}^1 \subseteq \mathfrak{g}^2, \quad \mathfrak{u}^1 \supseteq \mathfrak{u}^2, \quad L^1 \cap K \subseteq L^2 \cap K.$$

Define

$$\mathfrak{u}^0 = \mathfrak{u}^1 \cap \mathfrak{g}^2, \quad \mathcal{G}_f^0 = \mathfrak{g}^1 + \mathfrak{u}^0 \subseteq \mathfrak{g}^2,$$

a parabolic subalgebra of $\mathfrak{g}^2$. Fix an $(\mathfrak{g}^1, L^1 \cap K)$ module

W. Then there is a first quadrant spectral sequence

$$\left[R_{\mathcal{G}^2}^{\mathcal{G}} \right]^p \left[\left[R_{\mathcal{G}^0}^{\mathcal{G}^2} \right]^q \right]_W \implies \left[R_{\mathcal{G}^1}^{\mathcal{G}} \right]^{p+q}_W.$$

Proof. Define

$$V = \text{pro}_{\mathcal{G}^1, L^1_{nK}}^{\mathcal{G}, L^1_{nK}} \left[W \otimes (\wedge^{\dim \mathcal{U}^1} \mathcal{U}^1) \right].$$

We will apply Proposition 6.2.17 to V , with $B = L^2 \cap K$ and $C = L^1 \cap K$. By definition, the limit of the spectral sequence of (6.2.17) is

$$\left[R_{\mathcal{G}^1}^{\mathcal{G}} \right]^{p+q}_W,$$

so we only have to identify the E_2 terms. So we first compute

$$(6.3.7) \quad \left[\Gamma_{L^1_{nK}}^{L^2_{nK}} \right]^q \left(\text{pro}_{\mathcal{G}^1, L^1_{nK}}^{\mathcal{G}, L^1_{nK}} \left[W \otimes (\wedge^{\dim \mathcal{U}^1} \mathcal{U}^1) \right] \right).$$

Now

$$\mathcal{U}^1 = \mathcal{U}^0 \oplus \mathcal{U}^2;$$

so if we write

$$D^i = (\wedge^{\dim \mathcal{U}^i} \mathcal{U}^i),$$

then

$$D^1 = D^0 \otimes D^2$$

as $(\mathcal{G}^1, L^1_{nK})$ modules. So

$$\begin{aligned} V &= \text{pro}_{\mathcal{G}^1, L^1_{nK}}^{\mathcal{G}, L^1_{nK}} (W \otimes D^1) \\ &\cong \text{Hom}_{\mathcal{G}^1(U(\mathcal{G}^1), (W \otimes D^0) \otimes D^2)}^{L^1_{nK}\text{-finite}} \\ &\cong \text{Hom}_{\mathcal{G}^2(U(\mathcal{G}^1), \otimes D^2)}^{\mathcal{G}^1, L^1_{nK}} \left(\text{Hom}_{\mathcal{G}^1(U(\mathcal{G}^2), W \otimes D^0)}^{\mathcal{G}^1, L^1_{nK}} \right) \end{aligned}$$

by a simple formal argument (algebraic induction by stages).

Now

$$\mathcal{G}^2 = \mathcal{U}^2 \oplus \mathcal{U}^2, \quad \mathcal{G}^1 = \mathcal{G}^0 \oplus \mathcal{U}^2$$

and $\mathcal{U}^2$ acts trivially on $W \otimes D^0$. So

$$\text{Hom}_{\mathcal{G}^1(U(\mathcal{G}^2), W \otimes D^0)}^{L^1_{nK}\text{-finite}} \cong \text{Hom}_{\mathcal{G}^0(U(\mathcal{U}^2), W \otimes D^0)}^{L^1_{nK}\text{-finite}}$$

as $(\mathcal{U}^2, L^1_{nK})$ modules. So

$$(6.3.8) \quad \begin{aligned} V &\cong \text{Hom}_{\mathcal{G}^2}^2 \left(U(\mathcal{G}^1), \text{Hom}_{\mathcal{G}^0}^0 (U(\mathcal{U}^2), W \otimes D^0) \right)_{L^1_{nK}\text{-finite}} \\ &\quad \otimes D^2 \Big]_{L^1_{nK}\text{-finite}} \end{aligned}$$

$$= \text{pro}_{\mathcal{G}^2, L^1_{nK}}^{\mathcal{G}, L^1_{nK}} \left[\left(\text{pro}_{\mathcal{G}^0, L^1_{nK}}^{\mathcal{U}^2, L^1_{nK}} [W \otimes D^0] \right) \otimes D^2 \right].$$

(Here the inner pro is regarded as a $\mathcal{G}^2$ -module by making $\mathcal{U}^2$ act trivially.) We want to apply $\Gamma_{L^1_{nK}}^{L^2_{nK}}$ to this. We need

Lemma 6.3.9 Suppose X is an $(\mathcal{U}^2, L^1_{nK})$ -module.

- $\Gamma_{L^1_{nK}}^{L^2_{nK}} \circ \text{pro}_{\mathcal{G}^1, L^1_{nK}}^{\mathcal{G}, L^1_{nK}}(X) = \text{pro} \circ \Gamma(X).$
- $\left(\Gamma_{\mathcal{G}^1, L^1_{nK}}^{\mathcal{G}, L^2_{nK}} \right)^q \circ \text{pro}(X) = \text{pro} \circ \left(\Gamma_{\mathcal{U}^2, L^1_{nK}}^{\mathcal{U}^2, L^2_{nK}} \right)^q X.$

Proof. For (a) one can check through the definitions, but it is easier to proceed formally. Let Y be any $(\mathcal{G}^1, L^2_{nK})$ module; write $\mathcal{H}$ for passing from $(*, L^2_{nK})$ to $(*, L^1_{nK})$, and res for going from $(\mathcal{G}^1, *)$ to $(\mathcal{G}^1, *)$. Then

$$\begin{aligned} \text{Hom}_{\mathcal{G}^1, L^2_{nK}}(Y, \Gamma \circ \text{pro}(X)) &\cong \text{Hom}_{\mathcal{G}^1, L^1_{nK}}(\mathcal{H} Y, \text{pro}(X)) \\ &\cong \text{Hom}_{\mathcal{G}^2}^2 (L^1_{nK})^{\text{res} \circ \mathcal{H} Y, X} \end{aligned}$$

by (6.2.10) and (6.1.23). Obviously

$$\text{res}_{\mathcal{G}^1, L^1_{nK}}^{\mathcal{G}^2, L^1_{nK}} \circ \mathcal{H}_{\mathcal{G}^1, L^2_{nK}}^{\mathcal{G}^1, L^1_{nK}} = \mathcal{H}_{\mathcal{G}^1, L^2_{nK}}^{\mathcal{G}^1, L^1_{nK}} \circ \text{res}_{\mathcal{G}^1, L^1_{nK}}^{\mathcal{G}^2, L^2_{nK}};$$

so the calculation above can be continued to read

$$\begin{aligned}
\text{Hom}_{\mathcal{Y}, L^2_{\text{nk}}}(Y, \Gamma \circ \text{pro}(X)) &\cong \text{Hom}_{\mathcal{Y}^2, L^1_{\text{nk}}}(\text{res} \circ \mathcal{F}^{(Y)}, X) \\
&\cong \text{Hom}_{\mathcal{Y}^2, L^1_{\text{nk}}}(\mathcal{F} \circ \text{res}(Y), X) \\
&\cong \text{Hom}_{\mathcal{Y}^2, L^2_{\text{nk}}}((\text{res } Y), \Gamma X) \\
&\cong \text{Hom}_{\mathcal{Y}, L^2_{\text{nk}}}(Y, \text{pro} \circ \Gamma(X)).
\end{aligned}$$

Now (a) follows formally. For (b), choose an injective resolution

$$0 \rightarrow X \rightarrow I_{\star}$$

of X as a $(\mathcal{Y}^2, L^1_{\text{nk}})$ module. By (6.1.23), $\text{pro}(I_i)$ is injective; so

$$0 \rightarrow \text{pro}(X) \rightarrow \text{pro}(I_{\star})$$

is an injective resolution of $\text{pro}(X)$. By definition,

$$\begin{aligned}
\Gamma^q \circ \text{pro}(X) &= H^q(\Gamma \circ \text{pro}(I_{\star})) \\
&\cong H^q(\text{pro} \circ \Gamma(I_{\star})) \quad (\text{by (a)}) \\
&\cong \text{pro}(H^q(\Gamma(I_{\star}))) \quad (\text{since pro is exact}) \\
&\cong \text{pro}\left(\Gamma_{\mathcal{Y}^2, L^1_{\text{nk}}}^2 \Gamma_{\mathcal{Y}^2, L^1_{\text{nk}}}^q X\right) \quad (\text{by definition}) \\
&\cong \text{pro}\left(\Gamma_{\mathcal{Y}^2, L^1_{\text{nk}}}^2 \Gamma_{\mathcal{Y}^2, L^1_{\text{nk}}}^q X\right) \quad (\text{by (6.2.21)}).
\end{aligned}$$

This is (b).

Q.E.D.

From Lemma 6.3.9(b) and (6.3.8), we compute

$$\begin{aligned}
\left[\Gamma_{L^1_{\text{nk}}}^2\right]_V^q &\cong \text{pro}_{\mathcal{Y}^2, L^2_{\text{nk}}}^{\mathcal{Y}^2, L^2_{\text{nk}}} \left[\left[\Gamma^q \circ \text{pro}_{\mathcal{Y}^2, L^1_{\text{nk}}}^{\mathcal{Y}^2, L^1_{\text{nk}}} (W \otimes D^0) \right] \otimes D^2 \right] \\
&\cong \text{pro} \left[\left[\left(\mathcal{R}_{\mathcal{Y}^2, L^1_{\text{nk}}}^2 \right)_W^q \right] \otimes D^2 \right].
\end{aligned}$$

is

$$\begin{aligned}
&\left[\Gamma_{L^2_{\text{nk}}}^K\right]^P \left[\Gamma_{L^1_{\text{nk}}}^2\right]_V^q \\
&\cong \left[\Gamma_{L^2_{\text{nk}}}^K\right]^P \circ \text{pro} \left[\left[\left(\mathcal{R}_{\mathcal{Y}^2, L^1_{\text{nk}}}^2 \right)_W^q \right] \otimes D^2 \right] \\
&\cong \left[\mathcal{R}_{\mathcal{Y}^2}^2\right]^P \circ \left[\mathcal{R}_{\mathcal{Y}^2, L^1_{\text{nk}}}^2\right]_W^q.
\end{aligned}$$

so the spectral sequence we want is exactly Proposition 6.2.17.
Q.E.D.

Corollary 6.3.10 In the setting of Proposition 6.3.6, suppose that

$$\left[\mathcal{R}_{\mathcal{Y}^2, L^1_{\text{nk}}}^2\right]_W^q = 0, \quad q \neq q_0.$$

Then

$$\left[\mathcal{R}_{\mathcal{Y}^2}^2\right]^P \left[\left[\mathcal{R}_{\mathcal{Y}^2, L^1_{\text{nk}}}^2\right]_{q_0}^q\right]_W \cong \left[\mathcal{R}_{\mathcal{Y}^2}^2\right]^{P+q_0}_1(W).$$

This is clear.

Proposition 6.3.11 In the setting of Definition

6.3.1, fix a Cartan subalgebra $\mathcal{H} \subseteq \mathfrak{g}$, and an $(\mathfrak{g}, L_{\text{nk}})$ module W having infinitesimal character $\lambda \in \mathcal{H}^*$. Then

$$\mathcal{R}_{\mathcal{H}}^i W$$

has infinitesimal character $\lambda + \rho(\mathcal{H})$.

Proof. By Lemma 6.2.13, it is enough to prove this with $\mathcal{R}_{\mathcal{H}}^i W$ replaced by

$$\begin{aligned}
&\text{pro}_{\mathcal{Y}^2, L_{\text{nk}}}^{\mathcal{Y}^2, L_{\text{nk}}} (W \otimes (\wedge^{\dim \mathcal{H}} \mathcal{U})) \\
&\cong \text{Hom}_{\mathcal{Y}^2}(U(\mathcal{Y}), W \otimes (\wedge^{\dim \mathcal{H}} \mathcal{U}))_{L_{\text{nk}}\text{-finite}}.
\end{aligned}$$

So suppose $z \in Z(\mathcal{G})$. Write

$$z = \gamma(z) + r, \quad \gamma(z) \in Z(\mathfrak{g}), \quad r \in \mathcal{W}U(\mathcal{G}),$$

and write $\xi^\lambda, \xi^{\mathcal{G}}$ for the Harish-Chandra homomorphisms (0.2.7). Then by (3.1.7) (applied to the parabolic opposite to $\mathcal{G}$),

$$(*) \quad \xi^{\mathcal{G}} = T_{-\rho}(\mathcal{W}) \circ \xi^\lambda \circ \gamma.$$

Now suppose ϕ lies in the produced module, and $u \in U(\mathcal{G})$. By Definition 6.1.21(c), and the fact that z is central in $U(\mathcal{G})$,

$$\begin{aligned} (z \cdot \phi)(u) &= \phi(uz) \\ &= \phi(zu) \\ &= \phi(\gamma(z)u) + \phi(ru). \end{aligned}$$

Since ϕ is a $\mathcal{G}$ -module map, and $\mathcal{W}$ annihilates the image, the second term is zero; and the first is

$$\gamma(z) \cdot [\phi(u)] = [(\xi^\lambda \circ \gamma)(z)](\lambda + 2\rho(\mathcal{W}))\phi(u)$$

by the assumption on the infinitesimal character of W . (The shift by $2\rho(\mathcal{W})$ is the weight of $\wedge^{\dim \mathcal{W}} \mathcal{W}$.) So z acts by

$$\begin{aligned} & [(\xi^\lambda \circ \gamma)(z)](\lambda + 2\rho(\mathcal{W})) \\ &= [T_{-\rho}(\mathcal{W}) \circ \xi^\lambda \circ \gamma(z)](\lambda + \rho(\mathcal{W})) \\ &= \xi^{\mathcal{G}}(z)(\lambda + \rho(\mathcal{W})) \end{aligned}$$

by (*).

Q.E.D.

It is possible to show that $\mathcal{R}^i W$ has finite length whenever W does; but the special case of θ -stable parabolics is

already sufficiently complicated, so we will confine our attention to it henceforth.

Theorem 6.3.12 (Generalized Blattner formula).

In the setting (6.3.1), suppose $\mathcal{G}$ is a θ -stable

parabolic subalgebra (Definition 5.2.1), and W

is an $(\mathfrak{g}, L\mathfrak{g}K)$ module. Choose $\Delta^+(\mathfrak{k}, t^C)$ compatible

with $\mathcal{G} \cap \mathfrak{k}$. Let (π, Z) be an irreducible repre-

sentation of K . Suppose π occurs in $\mathcal{R}^{S-i}(W)$

(recall that $S = \dim(\mathfrak{u} \cap \mathfrak{k})$). Define

$$W_K^1 \subseteq W(\mathfrak{k}, t)$$

with respect to $\mathfrak{g} \cap \mathfrak{k}$ by (3.2.2). Then there are

- a) an element $w \in W_K^1$ of length i ;
- b) non-negative integers n_β , for $\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}, t^C)$;
- c) a highest weight γ of an $L\mathfrak{g}K$ -type occurring in W ;
- d) a highest weight δ of π such that

$$\gamma = w(\delta + \rho_C) - \rho_C - 2\rho(\mathfrak{u} \cap \mathfrak{p}) - \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}, t^C)} n_\beta \beta.$$

The multiplicity of π is bounded by the number of such expressions, counted with the multiplicity of γ as a highest weight. Here

$$\rho_C = \rho(\Delta^+(\mathfrak{k}, t))$$

as usual. Furthermore, if W has finite length, then

$$\begin{aligned} & \sum_i (-1)^i m(\pi, \mathcal{R}^{S-i} W) \\ &= \sum_j (-1)^j \sum_{m=0}^{\infty} \dim \operatorname{Hom}_{L\mathfrak{g}K}(H^j(\mathfrak{u} \cap \mathfrak{k}, Z), \\ & \quad S^m(\mathfrak{u} \cap \mathfrak{p}) \otimes W \otimes \wedge^{\dim \mathfrak{u} \cap \mathfrak{p}} \mathfrak{u} \cap \mathfrak{p}) \end{aligned}$$

and the inner sum on the right is finite.

The proof of this theorem is a little complicated and indirect. To some extent this is inevitable, because some of the simple things for which one might hope are false. For example, despite the fact that the assertions in the theorem involve only the structure of W as an $L\cap K$ -module, it is false that $(\mathcal{R}^i W)|_K$ depends only on $W|_{L\cap K}$. (This is also a departure from the behavior of ordinary parabolic induction.)

Proof. Define

$$\begin{aligned} V &= \text{pro}_{\mathcal{G}, L\cap K}^{\mathcal{G}, L\cap K} (W \otimes_{\Lambda} \dim \mathcal{U}_{\mathcal{U}}) \\ &= \text{Hom}_{\mathcal{G}}(U(\mathcal{G}), W \otimes_{\Lambda} \dim \mathcal{U}_{\mathcal{U}})_{L\cap K\text{-finite}} \end{aligned}$$

Choose an element y such that

- (6.3.13)
- a) $y \in i \mathcal{I}_0^C \cap (\text{center of } \mathcal{L})$
- b) y commutes with $L \cap K$
- c) $\text{ad}(y)$ has eigenvalues ≥ 1 on $\mathcal{U}$;

this is possible by Definition 5.2.1. Then W is the direct sum of the eigenspaces of y (as an $(\mathcal{L}, L\cap K)$ -module); so we may prove the theorem under the additional assumption

(6.3.13)

d) y acts by the scalar C on W .

Define

$$\begin{aligned} U_{(n)}(\mathcal{G}) &= \text{span in } U(\mathcal{G}) \text{ of products } u \cdot X_1 \cdots X_r, \\ &\text{with } r \leq n, X_i \in \mathfrak{p}, u \in U(\mathcal{L}) \end{aligned}$$

$$V_{(n)} = \{ \phi \in V \mid \phi|_{U_{(n)}(\mathcal{G})} = 0 \},$$

the order of vanishing filtration of V . Obviously $V_{(n)}$ is a $(\mathcal{L}, L\cap K)$ submodule of V . (In the analogy with sheaves on $\mathbb{A}^1$ mentioned earlier, $V_{(n)}$ plays the role of germs of sections vanishing to order n along $K/L\cap K$.) By convention

$$U_{(-1)}(\mathcal{G}) = \{0\}, \quad V_{(-1)} = V.$$

The triangular decomposition

$$U(\mathcal{G}) \cong U(\overline{\mathcal{U}}) \otimes U(\mathcal{G})$$

gives rise to an $(\mathcal{L}, L\cap K)$ module isomorphism

$$V \cong \text{Hom}_{\mathbb{Q}}(U(\overline{\mathcal{U}}), W \otimes_{\Lambda} \dim \mathcal{U}_{\mathcal{U}})_{L\cap K\text{-finite}};$$

here $U(\overline{\mathcal{U}})$ is an $(\mathcal{L}, L\cap K)$ module by the adjoint action.

If we define $U_{(n)}(\overline{\mathcal{U}})$ like $U_{(n)}(\mathcal{G})$, with $\overline{\mathcal{U}} \cap \mathfrak{L}$ and $\overline{\mathcal{U}} \cap \mathfrak{p}$ replacing $\mathcal{L}$ and $\mathfrak{p}$, then we get an $(\mathcal{L} \cap \mathcal{L}, L\cap K)$ -module isomorphism

$$V_{(n)} \cong \text{Hom}_{\mathbb{Q}}(U(\overline{\mathcal{U}})/U_{(n)}(\overline{\mathcal{U}}), W \otimes_{\Lambda} \dim \mathcal{U}_{\mathcal{U}})_{L\cap K\text{-finite}}.$$

The quotient

$$U(\overline{\mathcal{U}})/U_{(n)}(\overline{\mathcal{U}})$$

is spanned by the images of products

$$X_1 \cdots X_m, \quad m > n, X_i \in \mathcal{U} \cap \mathfrak{p}.$$

By (6.3.13) (c) and (d), it follows that

(6.3.14)

the eigenvalues of y on $V_{(n)}$ are greater than or equal to $C + n + 1$.

By Corollary 6.2.15,

$$\begin{aligned}
 m(\pi, \mathcal{R}^{S-i} W) &= \dim \operatorname{Ext}_{\mathcal{K}, \operatorname{L} \cap K}^{S-i}(Z, V) \\
 (6.3.15) \quad &= \dim H^{S-i}(\mathcal{K}, \operatorname{L} \cap K, \operatorname{Hom}_{\mathbb{Q}}(Z, V));
 \end{aligned}$$

the second isomorphism is Corollary 6.1.26. Let C' be the largest eigenvalue of y in Z ; then the eigenvalues of y in $\operatorname{Hom}_{\mathbb{Q}}(Z, V_{(n)})$ are at least $C + n + 1 - C'$. So if n is large enough,

$$\operatorname{Hom}_{\operatorname{L} \cap K}(\Lambda^*(\mathcal{K}/\ell \cap \mathcal{K}), \operatorname{Hom}_{\mathbb{Q}}(Z, V_{(n)})) = 0.$$

Fix such a large n . By Corollary 6.1.18,

$$(6.3.16) \quad H^*(\mathcal{K}, \operatorname{L} \cap K, \operatorname{Hom}_{\mathbb{Q}}(Z, V_{(n)})) = 0.$$

By the long exact sequence in cohomology, (6.3.15) and

(6.3.16) give (still for our fixed large n)

$$(6.3.17) \text{ (a) } \operatorname{Ext}_{\mathcal{K}, \operatorname{L} \cap K}^{S-i}(Z, V) \cong \operatorname{Ext}_{\mathcal{K}, \operatorname{L} \cap K}^{S-i}(Z, V/V_{(n)}).$$

We will compute the right side using the spectral sequence of the filtration

$$\begin{aligned}
 (6.3.17) \text{ (b) } V/V_{(n)} &= V_{(-1)}/V_{(n)} \supseteq V_{(0)}/V_{(n)} \supseteq \dots \supseteq V_{(n)}/V_{(n)} \\
 &= \{0\}.
 \end{aligned}$$

$$(V_{(m-1)}/V_{(n)})/(V_{(m)}/V_{(n)}) \cong V_{(m-1)}/V_{(m)}.$$

To study these, notice first that the Poincaré-Birkhoff-Witt theorem gives

$$U_{(m)}/U_{(m-1)} \cong U(\mathcal{K}) \otimes S^m(\mathcal{K}).$$

By a formal calculation (in which we write D for $\wedge^{\dim \mathcal{U}} \mathcal{U}$),

$$\begin{aligned}
 \cdots (m-1)/V_{(m)} &\cong \operatorname{Hom}_{\mathcal{U} \cap \mathcal{K}}(U(\mathcal{K}) \otimes S^m(\mathcal{K}/\ell \cap \mathcal{K}), W \otimes D)_{\operatorname{L} \cap K\text{-finite}} \\
 &\cong \operatorname{Hom}_{\mathcal{U} \cap \mathcal{K}}(U(\mathcal{K}), \operatorname{Hom}_{\mathbb{Q}}(S^m(\mathcal{K}/\ell \cap \mathcal{K}), W \otimes D))_{\operatorname{L} \cap K\text{-finite}} \\
 &= \operatorname{pro}_{\mathcal{U} \cap \mathcal{K}, \operatorname{L} \cap K} \left(\operatorname{Hom}_{\mathbb{Q}}(S^m(\mathcal{K}/\ell \cap \mathcal{K}), W \otimes D) \right).
 \end{aligned}$$

Since $\mathcal{U} \cap \mathcal{K}$ is nonsingularly paired with $\mathcal{K}/\ell \cap \mathcal{K}$ by $\langle, \rangle$, this last module can be written

$$(6.3.17) \text{ (c) } V_{(m-1)}/V_{(m)} \cong \operatorname{pro}_{\mathbb{Q}}(S^m(\mathcal{U} \cap \mathcal{K}) \otimes (W \otimes D)).$$

The spectral sequence of the filtration described by (6.3.17), and Shapiro's lemma (6.1.27) give a spectral sequence

$$\begin{aligned}
 (6.3.18) \quad &\bigoplus_{m=0}^n \operatorname{Ext}_{\mathcal{U} \cap \mathcal{K}, \operatorname{L} \cap K}^p(Z, S^m(\mathcal{U} \cap \mathcal{K}) \otimes W \otimes D) \\
 &\implies \operatorname{Ext}_{\mathcal{K}, \operatorname{L} \cap K}^p(Z, V).
 \end{aligned}$$

If we filter $S^m(\mathcal{U} \cap \mathcal{K})$ by the eigenvalues of y , then $\mathcal{U} \cap \mathcal{K}$ acts trivially in the associated graded module; so we get a new spectral sequence which looks just like (6.3.18), except that now $\mathcal{U} \cap \mathcal{K}$ acts trivially on $S^m(\mathcal{U} \cap \mathcal{K}) \otimes W \otimes D$. We can therefore apply the Hochschild-Serre spectral sequence (6.1.31). Here $\ell \cap \mathcal{K}$ plays the role of $\mathcal{L}$, and $\operatorname{L} \cap K$ that of B ; so

$$\operatorname{Ext}_{\mathcal{L}, B}^p = 0, \quad p > 0,$$

and the spectral sequence collapses to

$$\begin{aligned}
 \operatorname{Ext}_{\mathcal{U} \cap \mathcal{K}, \operatorname{L} \cap K}^p(Z, S^m(\mathcal{U} \cap \mathcal{K}) \otimes W \otimes D) &\cong \operatorname{Hom}_{\operatorname{L} \cap K}(H^{S-p}(\mathcal{U} \cap \mathcal{K}, Z) \otimes D_1, \\
 &\quad S^m(\mathcal{U} \cap \mathcal{K}) \otimes W \otimes D).
 \end{aligned}$$

Here

$$D_1 = \Lambda^{\dim u \cap k} u \cap k.$$

If we define

$$(6.3.19) \quad D_2 = \Lambda^{\dim u \cap p} u \cap p,$$

then (6.3.18) can be written as

$$(6.3.20) \quad \bigoplus_{m=0}^n \operatorname{Hom}_{L \cap K} (H^P(u \cap k, Z), S^m(u \cap p) \otimes W \otimes D_2) \implies \operatorname{Ext}_{k, L \cap K}^{S-P}(Z, V).$$

Suppose then that π occurs in $\mathcal{R}^{S-i}(V)$, so that the limit of (6.3.20) is non-zero by (6.3.15). Then the corresponding E_2 terms are also non-zero; so there is an $L \cap K$ highest weight τ occurring in both

$$H^i(u \cap k, Z), S^m(u \cap p) \otimes W \otimes D_2.$$

By Proposition 3.2.12, there is an $L \cap K$ highest weight γ of W , and a weight Σn_β of $S^m(u \cap p)$, such that

$$\tau = \gamma + \Sigma n_\beta + 2\rho(u \cap p)$$

(the last term being the weight of D_2). By Corollary 3.2.16, there is an element w of W_K^1 of length i , and a highest weight δ of π , such that

$$\tau = w(\delta + \rho_C) - \rho_C.$$

Equating these two expressions for τ gives the first claim of the theorem. (The multiplicity estimate requires a corresponding elaboration of Proposition 3.2.12, which follows from the proof given in Section 3.2.) The second claim

follows from (6.3.15), and the Euler-Poincaré principle applied to (6.3.20). Q.E.D.

If $L \cap K$ is connected, or even if R^L is trivial (cf. Definition 5.1.2), then the right side of the multiplicity formula in Theorem 6.3.12 is easily written as an alternating sum over W_K (not W_K^1 , as one might guess) of terms involving the Kostant partition function for $\Delta(u \cap p)$. With a little more effort, one can get an uglier, but still very explicit, formula in the general case. Since we want the formula for theoretical purposes and not for actual calculation, we will not pursue these ideas.

Corollary 6.3.21 If $\mathfrak{g} = \mathfrak{k} + \mathfrak{u}$ is a θ -stable parabolic, W is an $(\mathfrak{u}, L \cap K)$ module, and $S = \dim u \cap k$,

then

$$\mathcal{R}^i W = 0 \quad (i > S).$$

For no element of W_K^1 can have length less than zero; so Theorem 6.3.12 shows that $\mathcal{R}^i W$ has no K -types for $i > S$.

Our next goal is to find conditions under which $\mathcal{R}^i W$ also vanishes for $i < S$. This is done by a series of tricks which are quite difficult to motivate. These tricks can be pushed a bit further, producing an improved vanishing theorem for which we would have no use, and which is therefore omitted. On the other hand, in Chapter 8 we will prove another vanishing theorem, which seems to be inaccessible by such tricks alone (although it is based on the vanishing theorem given here, and therefore relies on the tricks). This latter theorem, however, does not cover all the cases treated by

the theorem proved here. So there is room for a lot of work on vanishing theorems, although it is not clear how useful any further results might be.

Definition 6.3.22 Suppose $\sigma\tau = \ell + u$ is a θ -stable parabolic, $\Delta^+(\ell, \tau^C)$ is a positive system compatible with $\sigma \cap \ell$, and $\mu^L \in \hat{T}^C$ is dominant for $\Delta^+(\ell \cap \ell, \tau^C)$. Let $\lambda^L \in (\tau^C)^*$ be the weight associated to μ^L by Proposition 5.3.3. Define

$$\begin{aligned}\lambda^G &= \lambda^L + \rho(u) \in (\tau^C)^* \\ \mu^G &= \mu^L + 2\rho(u \cap \ell) \in \hat{T}^C.\end{aligned}$$

We say that μ^L is strictly $u \cap \ell$ -positive, if

$$\langle \alpha, \lambda^G \rangle > 0, \text{ all } \alpha \in \Delta(u \cap \ell, \tau^C).$$

Lemma 6.3.23 In the setting of Definition 6.3.22,
let μ^L be a strictly $u \cap \ell$ -positive, $\Delta^+(\ell \cap \ell, \tau^C)$ -
dominant weight; and put $\mu^G = \mu^L + 2\rho(u \cap \ell)$ as in
the definition. Then μ^G is dominant for $\Delta^+(\ell, \tau^C)$.

Proof. Fix a simple root α of τ^C in $\Delta^+(\ell, \tau^C)$; we want to show that

$$(*) \quad \langle \alpha, \lambda^G \rangle \geq 0.$$

If α lies in $\Delta^+(\ell \cap \ell, \tau^C)$, then

$$\langle \alpha, 2\rho(u \cap \ell) \rangle > 0$$

since $2\rho(u \cap \ell)$ is the weight of the one-dimensional representation $\Lambda^{\dim(u \cap \ell)}(u \cap \ell)$ of $L \cap K$. So

$$\langle \alpha, \mu^G \rangle = \langle \alpha, \mu^L \rangle \geq 0.$$

So we may assume α lies in $\Delta(u \cap \ell, \tau^C)$. Since μ^G is integral, $(*)$ amounts to

$$\langle \alpha, \mu^G \rangle > -1;$$

and by Lemma 3.2.4, this can be written as

$$(**) \quad \langle \alpha, \mu^G + \rho_C \rangle > 0.$$

Choose $\Delta^+(\ell)$ so that $\mu^L + 2\rho(\Delta^+(\ell \cap \ell))$ is dominant, and write

$$\lambda^L = \mu^L + 2\rho(\Delta^+(\ell \cap \ell)) - \rho(\Delta^+(\ell)) + \frac{1}{2} \sum c_i \beta_i$$

in accordance with Proposition 5.3.3. Put

$$\begin{aligned}\Delta^+(\sigma\tau) &= \Delta^+(\ell) \cup \Delta(u), \quad \rho = \rho(\Delta^+(\ell)) + \rho(u) \\ \rho_C &= \rho(\Delta^+(\ell, \tau^C)), \quad \rho_n = \rho(\Delta^+(\ell, \tau^C)).\end{aligned}$$

Then

$$\begin{aligned}\mu^G + \rho_C &= \mu^L + 2\rho(u \cap \ell) + \rho_C \\ &= \mu^L + \rho(\Delta^+(\ell \cap \ell)) - \rho(\Delta^+(\ell \cap \ell)) + \rho_n + \rho(u) \\ &= \lambda^L - \frac{1}{2} \sum c_i \beta_i + \rho_n + \rho(u) \\ &= \lambda^G + \rho_n - \frac{1}{2} \sum c_i \beta_i.\end{aligned}$$

So $(**)$ is equivalent to

$$(6.3.24) \quad \langle \alpha, \lambda^G \rangle + \langle \alpha, \rho_n - \frac{1}{2} \sum c_i \beta_i \rangle > 0.$$

The first term is positive by the definition of $u \cap \ell$ -positive; so it suffices to show that the second is non-negative. Now for any weight ξ ,

$$\xi - s_\alpha \xi = \langle \alpha, \xi \rangle \alpha;$$

so by Corollary 3.2.5 (since $\alpha \in \Delta(\mathcal{W})$), it is enough to show

$$(6.3.25) \quad \langle \rho(\mathcal{W}), [(\rho_n - \frac{1}{2} \sum c_i \beta_i) - s_\alpha (\rho_n - \frac{1}{2} \sum c_i \beta_i)] \rangle \geq 0.$$

By Lemma 5.4.5, the term in brackets is of the form

$$\sum_{\gamma \in \Delta^+} a_\gamma \gamma - \frac{1}{2} \sum c_i \beta_i,$$

with a_γ non-negative. So the inner product is non-negative by Corollary 3.2.5. This proves (6.3.25), and therefore the lemma.

Q.E.D.

Lemma 6.3.26 In the setting of (6.3.22), suppose $R^L = R^G$ (cf. (5.1.2)). Choose an element $y \in i\mathcal{L}_0^c$ satisfying (6.3.13). Suppose W is an (ℓ, Lok) module on which y acts by a scalar, and μ^L is a strictly $\mathcal{W} \cap \mathcal{K}$ -positive weight in $\hat{\mathfrak{t}}^c$ on which y acts by this same scalar. Suppose π is an irreducible representation of K occurring in $\mathcal{R}^{S-i} W$, and

$$\left[H^J(\mathcal{W} \cap \mathcal{K}, \pi) \otimes (\wedge^{J'}(\mathcal{W} \cap \mathcal{P}))^* \right] \mu^L \neq 0.$$

Then $i = 0$.

Proof. Choose $\Delta^+(\mathcal{W})$, $\{\beta_i\}$, $\{c_i\}$ as in the proof of the preceding lemma, and put $J'' = R - J'$. By the computation in the proof of Proposition 5.4.2, we can find

$$\{\alpha_1, \dots, \alpha_{J''}\} = A_1 \subseteq \Delta(\mathcal{W} \cap \mathcal{P}, \mathcal{L}^c) \\ \sigma \in W_K^1, \quad \ell(\sigma) = J',$$

and a highest weight δ of π , such that

$$\begin{aligned} \sigma(\delta + \rho_C) - \rho_C + 2\rho(A_1) &= \mu^G \\ &= \lambda^G + \rho - \frac{1}{2} \sum c_i \beta_i - 2\rho_C; \end{aligned}$$

the second equality is (6.3.25). Define

$$\rho_n = \rho(\Delta^+(\mathcal{P})) = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha;$$

then the preceding formula can be written

$$(6.3.27) \quad \sigma(\delta + \rho_C) = \lambda^G + [\rho_n - \frac{1}{2} \sum c_i \beta_i - 2\rho(A_1)].$$

We are going to apply σ^{-1} to both sides. The effect of this on the first term is described by the following lemma (which we will apply to $(\mathcal{K}, \mathcal{L}^c)$).

Lemma 6.3.28 Suppose $\mathcal{K} \subseteq \mathcal{W}$ is a Cartan subalgebra, $\Delta^+ \subseteq \Delta(\mathcal{W}, \mathcal{K})$ is a positive root system, and $\lambda \in \mathcal{K}^*$ satisfies

$$\langle \alpha, \lambda \rangle \geq 0, \quad (\alpha \in \Delta^+).$$

If $\sigma \in W(\mathcal{W}, \mathcal{K})$, then there are non-negative real numbers $\{n_\alpha \mid \alpha \in \Delta^+\}$ such that

$$\sigma \cdot \lambda = \lambda - \sum_{\alpha \in \Delta^+} n_\alpha \alpha.$$

Furthermore, all the n_α are zero if and only if σ is a product of reflections around simple roots orthogonal to λ .

(This is very well known; the last statement is Chevalley's theorem.)

Proof. We proceed by induction on the length of σ . If $\sigma = 1$, the result is obvious; so suppose $\sigma \neq 1$. Choose a simple root

$$\beta \in \Delta^+(\sigma) = \{\alpha \in \Delta^+ \mid \langle \alpha, \sigma\rho \rangle < 0\}.$$

Then

$$\begin{aligned} \Delta^+(s_\beta \sigma) &= \{\alpha \in \Delta^+ \mid \langle \alpha, s_\beta \sigma\rho \rangle < 0\} \\ &= \{\alpha \in \Delta^+ \mid \langle s_\beta(\alpha), \sigma\rho \rangle < 0\} \\ &= s_\beta(\Delta^+(\sigma) - \{\beta\}) \end{aligned}$$

since s_β preserves $\Delta^+ - \{\beta\}$. So

$$\ell(s_\beta \sigma) = \ell(\sigma) - 1,$$

and the lemma is known for $s_\beta \sigma$. So

$$s_\beta \sigma \lambda = \lambda - \sum_{\alpha \in \Delta^+} m_\alpha \alpha,$$

with $m_\alpha \geq 0$; equality holds for all α if and only if $s_\beta \sigma$ is a product of reflections around simple roots orthogonal to λ .

Now $\sigma \lambda$ lies in the same Weyl chamber as $\sigma\rho$ by hypothesis;

so since $\beta \in \Delta^+(\sigma)$,

$$\langle \beta, \sigma \lambda \rangle \leq 0.$$

So

$$\begin{aligned} s_\beta \sigma \lambda &= \sigma \lambda - \frac{2\langle \beta, \sigma \lambda \rangle}{\langle \beta, \beta \rangle} \cdot \beta \\ &= \sigma \lambda + c_\beta \beta, \end{aligned}$$

with $c_\beta \geq 0$. Define

$$n_\alpha = \begin{cases} m_\alpha & (\alpha \neq \beta) \\ m_\beta + c_\beta & (\alpha = \beta). \end{cases}$$

Then

$$\begin{aligned} \sigma \lambda &= s_\beta \sigma \lambda - c_\beta \beta \\ &= \lambda - \sum m_\alpha \alpha - c_\beta \beta \\ &= \lambda - \sum n_\alpha \alpha, \end{aligned}$$

proving the first claim. Suppose all the n_α are zero. Then c_β and all the m_α are zero. By inductive hypothesis, this means that $s_\beta \sigma$ is a product of reflections around roots orthogonal to λ ; so

$$s_\beta \sigma \lambda = \lambda.$$

Since $c_\beta = 0$, $\lambda = \sigma \lambda$. Conversely, suppose $\lambda = \sigma \lambda$. Since $\langle \beta, \sigma \lambda \rangle$ is non-positive by the choice of β , and $\langle \beta, \lambda \rangle$ is non-negative by hypothesis, both of these must be zero; so β is orthogonal to λ , and $s_\beta \sigma \lambda = \lambda$. By induction, $s_\beta \sigma$ is a product of reflections fixing λ ; so $\sigma = s_\beta(s_\beta \sigma)$ is also. Q.E.D.

By Lemmas 6.3.28 and 5.4.5, formula (6.3.27) gives

$$(6.3.29) \quad \delta + \rho_C = \lambda^G + \rho_n - \sum_{\alpha \in \Delta^+} m_\alpha \alpha - \sum_{\beta \in \Delta} c_\beta \beta,$$

with $m_\alpha \geq 0$ and $0 \leq c_\beta \leq 1$.

We now apply the hypothesis that π occurs in $\mathcal{R}^{S-i} w$.

By Theorem 6.3.12, we can find a highest weight δ' of π , an element $w \in W_K^1$ of length i , an $L_n K$ highest weight γ occurring in W , and non-negative integers

$$\{n_\beta \mid \beta \in \Delta(u \cap \mathfrak{p})\}$$

such that

$$(6.3.30) \quad (a) \quad w\delta' = \gamma + 2\rho(u \cap \mathfrak{p}) + \sum n_\beta \beta + \rho_C - w\rho_C.$$

By Proposition 5.1.9, there is an $r \in R^G$ such that

$$r\delta' = \delta.$$

By hypothesis, $r \in R^L$; so r normalizes w_K^L , $u_n \not\in \mathcal{P}$, and the set of highest weights occurring in W . So we can apply r to (6.3.30) (a), and deduce

$$(6.3.30) \text{ (b)} \quad w\delta = \gamma + 2\rho(u_n \not\in \mathcal{P}) + \sum n_\beta \beta + \rho_C - w\rho_C;$$

here we have changed the values of w , γ , and $\{n_\beta\}$, but not their properties given above. By Lemma 6.3.28 and (3.2.9),

$$(6.3.30) \text{ (c)} \quad \delta = \gamma + 2\rho(u_n \not\in \mathcal{P}) + \sum n_\beta \beta + \sum_{\alpha \in \Delta^+(\mathcal{K})} m'_\alpha \alpha;$$

here m'_α is non-negative, and $m'_\alpha \geq 1$ if α belongs to $\Delta_K^+(w)$. By (6.3.25),

$$\begin{aligned} \lambda^G &= \mu^G + 2\rho_C - \rho + \frac{1}{2} \sum c_i \beta_i \\ &= \mu^L + 2\rho(u_n \not\in \mathcal{P}) + \rho_C - \rho_n + \frac{1}{2} \sum c_i \beta_i. \end{aligned}$$

So (6.3.30) (c) can be written

$$\begin{aligned} \delta + \rho_C &= (\gamma - \mu^L) + [\mu^L + 2\rho(u_n \not\in \mathcal{P}) + \rho_C - \rho_n + \frac{1}{2} \sum c_i \beta_i] \\ (6.3.30) \text{ (d)} \quad &+ \rho_n - \frac{1}{2} \sum c_i \beta_i + \sum n_\beta \beta + \sum m'_\alpha \alpha \\ \delta + \rho_C &= (\gamma - \mu^L) + \lambda^G + \rho_n + \sum n_\beta \beta + \sum m'_\alpha \alpha - \frac{1}{2} \sum c_i \beta_i. \end{aligned}$$

Equating this with (6.3.29) and collecting terms, we get

$$\begin{aligned} (\gamma - \mu^L) + \sum_{\alpha \in \Delta^+(\ell_n \not\in \mathcal{K})} (m'_\alpha + m_\alpha) \alpha + \sum_{\beta \in \Delta^+(\ell_n \not\in \mathcal{P})} c_\beta \beta - \frac{1}{2} \sum c_i \beta_i \\ (6.3.31) \quad &= - \sum_{\alpha \in \Delta(u_n \not\in \mathcal{K})} (m'_\alpha + m_\alpha) \alpha - \sum_{\beta \in \Delta(u_n \not\in \mathcal{P})} (c_\beta + n_\beta) \beta. \end{aligned}$$

Since we assume that γ acts by the same scalar on W (and in particular on γ) as on μ^L , (6.3.13) shows that the left side of this equality is annihilated by γ ; so the right side is as well. Furthermore, every individual term on the right is non-positive on γ , so they must all be zero. Since $m'_\alpha \geq 1$ for α in $\Delta_K^+(w)$, it follows that $\Delta_K^+(w)$ does not meet $\Delta(u_n \not\in \mathcal{K})$. Since w belongs to W_K^1 , this implies that $w = 1$. Since $\ell(w) = i$, $i = 0$. Q.E.D.

We are going to use Frobenius reciprocity (Proposition 6.3.2) to study $\mathcal{Q}^i W$. Unfortunately, the nature of the spectral sequences requires that we control not only W and its submodules, but also any $(\ell, L \cap K)$ module having a non-zero Ext with W . The next definition states exactly what we need.

Definition 6.3.32 Let X be a (σ_f, K) module of finite length. A set $A \subseteq \hat{K}$ is called a strongly cogenerated set for X if whenever U is another (σ_f, K) module of finite length such that

$$\text{Ext}_{\sigma_f, K}^*(U, X) \neq 0,$$

then there is a $\pi \in A$ such that $U(\pi) \neq 0$.

The finite length assumptions are included only because we will need them to produce strongly cogenerated sets. Because they are included, we will need to use

Lemma 6.3.33 Suppose $\sigma_f = \ell + u$ is a θ -stable parabolic subalgebra of σ_f , X is a (σ_f, K) module of finite length, and W is an $(\ell, L \cap K)$ module of

finite length.

- a) $H^*(\mathcal{U}, X)$ is an $(\ell, L\cap K)$ module of finite length.
- b) $\mathcal{R}_W$ is a $(\mathcal{G}, K)$ module of finite length.

Proof. By Corollary 5.4.16, X is admissible and $Z(\mathcal{G})$ -finite; so by Corollary 5.2.4, $H^*(\mathcal{U}, X)$ is admissible and $Z(\ell)$ -finite. By Corollary 5.4.16, $H^*(\mathcal{U}, X)$ has finite length. Similarly, Theorem 6.3.12 guarantees that $\mathcal{R}^* W$ is admissible, and Proposition 6.3.11 guarantees that it is $Z(\mathcal{G})$ finite. So by Corollary 5.4.16, $\mathcal{R}^* W$ has finite length. Q.E.D.

Proposition 6.3.34 In the setting of (6.3.22),
choose $y \in i_{\ell_0}^C$ satisfying (6.3.13). Suppose
that y acts by a scalar on W , and that W admits
a strongly cogenerating set A (cf. (6.3.32)) on
which y acts by the same scalar. If A consists
of strictly $\mathcal{U}\cap\mathcal{K}$ -positive $L\cap K$ types (cf. (6.3.22)),
then

$$\mathcal{R}^i W = 0 \quad i \neq S = \dim \mathcal{U}\cap\mathcal{K}.$$

Proof. Set

$$K^1 = (L\cap K) \cdot K_0, \quad G^1 = K^1 \cdot G_0.$$

Then it is clear from Lemma 5.1.3 that

$$R^{G^1} = R^L$$

(notation 5.1.2). By Proposition 6.2.22, it is enough to prove the proposition with G replaced by G^1 ; so we may assume $R^G = R^L$.

Let i_0 be the least integer such that $\mathcal{R}^{i_0} W \neq 0$. By Corollary 6.3.21, it is enough to prove that $i = S$. Set

$$X = \mathcal{R}^{i_0} W.$$

then

$$\text{Hom}_{\mathcal{G}, K}(X, \mathcal{R}^{i_0} W) \neq 0;$$

and by the choice of i_0 ,

$$\text{Ext}_{\mathcal{G}, K}^p(X, \mathcal{R}^{q_0} W) = 0, \quad p+q < i_0.$$

So the I_2^{0, i_0} term in Proposition 6.3.2 persists to I_{∞}^{0, i_0} .

By the second spectral sequence of Proposition 6.3.2, we can find integers p and q , with $p+q = i_0$, such that

$$\text{Ext}_{\ell, L\cap K}^p(H^{\dim \mathcal{U}-q}(\mathcal{U}, X), W) \neq 0.$$

Now

$$H^{\dim \mathcal{U}-q}(\mathcal{U}, X)$$

has finite length by Lemma 6.3.33. By hypothesis, we can therefore find a strictly $\mathcal{U}\cap\mathcal{K}$ -positive weight μ^L on which y acts by the same scalars as in W , such that

$$H^{\dim \mathcal{U}-q}(\mathcal{U}, X) \mu^L \neq 0.$$

By Theorem 5.2.2, there is a K -type π occurring in X , such that

$$\left[H^*(\mathcal{U}\cap\mathcal{K}, \pi) \otimes (\Lambda^*(\mathcal{U}\cap\mathcal{K}))^* \right] \mu^L \neq 0.$$

Now $X = \mathcal{R}^{i_0} W$; so by Lemma 6.3.26, $i_0 = S$. Q.E.D.

§4. Principal series revisited.

Although most of the discussion in this section can be extended to induction from real parabolic subgroups without serious difficulty, such generality would only be an inconvenience for our purposes. So we assume throughout this section that

$$(6.4.1) \quad \begin{array}{l} G \text{ is quasisplit} \\ P = MAN \text{ is a Borel subgroup.} \end{array}$$

Our goal is to find strongly cogenerating sets of K -types for certain principal series representations (Definition 6.3.32). By Proposition 6.3.5(c), we have to study $\mathcal{N}$ cohomology of $(\mathcal{G}, K)$ modules. To do this, we need a structural result on $U(\mathcal{G})$, which has been noticed by several people independently.

Lemma 6.4.2 (cf. [49], Lemma 4.3.3). There is a finite dimensional subspace

$$\mathcal{H} \subseteq U(\mathcal{M} + \mathcal{O})$$

such that

$$U(\mathcal{G}) = U(\mathcal{N}) \cdot Z(\mathcal{G}) \cdot \mathcal{H} \cdot U(\mathcal{K});$$

that is, $U(\mathcal{G})$ is spanned by products of four terms as indicated.

Proof. Since G is quasisplit, $\mathcal{M} + \mathcal{O}$ is a Cartan subalgebra of $\mathcal{G}$. If $z \in Z(\mathcal{G})$, define

$$\gamma(z) \in U(\mathcal{M} + \mathcal{O}), \quad z - \gamma(z) \in \mathcal{N}U(\mathcal{G})$$

as usual. By Theorem 0.2.8, $U(\mathcal{M} + \mathcal{O})$ is integral over $\gamma(Z(\mathcal{G}))$; so we choose $\mathcal{H}_1$ to be a subspace of $U(\mathcal{M} + \mathcal{O})$ which generates $U(\mathcal{M} + \mathcal{O})$ as a module for $\gamma(Z(\mathcal{G}))$.

Filter $U(\mathcal{G})$ as usual by defining

$$U_m(\mathcal{G}) = \langle X_1 \cdot \dots \cdot X_i | X_i \in \mathcal{G}, i \leq m \rangle.$$

Then

$$(*) \quad \gamma(Z(\mathcal{G})) \cap U_m(\mathcal{G}) = \gamma(Z(\mathcal{G})) \cap U_m(\mathcal{M} + \mathcal{O}).$$

(To see this, pass to the associated graded algebra $S(\mathcal{G})$. Then $Z(\mathcal{G})$ is replaced by $S(\mathcal{G})^G$, the $\text{Ad}(G)$ -invariant polynomials; and γ is simply the restriction of functions from $\mathcal{G}^*$ to $(\mathcal{M} + \mathcal{O})^*$. Obviously this preserves degree. Then $(*)$ follows by induction on m .)

Introduce a new origin in the affine space $(\mathcal{M} + \mathcal{O})^*$, at the point $-\rho$. This gives a new notion of "homogeneous" in the polynomial functions $U(\mathcal{M} + \mathcal{O})$ on $(\mathcal{M} + \mathcal{O})^*$, which we call ρ -homogeneous. Write $U_{(m)}(\mathcal{M} + \mathcal{O})$ for the ρ -homogeneous polynomials of degree m . We have

$$U_m(\mathcal{M} + \mathcal{O}) = \bigoplus_{n \leq m} U_{(n)}(\mathcal{M} + \mathcal{O});$$

and by Theorem 0.2.8, $\gamma(Z(\mathcal{G}))$ is a ρ -homogeneous subring. We define $\mathcal{H}$ to be the ρ -homogeneous space generated by $\mathcal{H}_1$. We will show by induction on m that

$$U_m(\mathcal{G}) \subseteq U(\mathcal{N}) \cdot Z(\mathcal{G}) \cdot \mathcal{H} \cdot U(\mathcal{K}).$$

If $m = 0$ this is obvious (since $1 \in Z(\mathcal{G})$); so suppose $m > 0$, and the result is known for $m-1$. Fix $u \in U_m(\mathcal{G})$. By the

Iwasawa decomposition, we can write

$$u = \sum_{n=0}^m \sum_{t=1}^r x_t^n a_t^n y_t^n,$$

with $x_t^n \in U(\mathfrak{m})$, $y_t^n \in U(\mathfrak{k})$, and $a_t^n \in U(\mathfrak{a})$ ρ -homogeneous of degree n . If $n < m$, then by induction

$$a_t^n \in U(\mathfrak{m}) \cdot Z(\mathfrak{g}) \cdot \mathcal{H} \cdot U(\mathfrak{k}),$$

so clearly $x_t^n a_t^n y_t^n$ is also. So we need only show that a_t^m can be written in the desired form. Since $\mathcal{H}$ generates $U(\mathfrak{m} + \mathfrak{a})$ as a $\gamma(Z(\mathfrak{g}))$ module, we can write

$$a_t^m = \sum_i g_i h_i,$$

with $g_i \in \gamma(Z(\mathfrak{g}))$ and $h_i \in \mathcal{H}$, all ρ -homogeneous. By throwing out terms of the wrong ρ -degree, we may assume g_i has ρ -degree $n_i \leq m$, and h_i has ρ -degree $m - n_i$. Because of (*), we can find $z_i \in Z(\mathfrak{g}) \cap U_{n_i}(\mathfrak{g})$ such that

$$z_i = g_i + \sum_j x_{ij} u_{ij}; \quad x_{ij} \in \mathfrak{m}, \quad u_{ij} \in U_{n_i-1}(\mathfrak{g}).$$

Hence $u_{ij} h_i \in U_{m-1}(\mathfrak{g})$; so by induction

$$g_i h_i = z_i h_i - \sum_j x_{ij} (u_{ij} h_i) \in U(\mathfrak{m}) \cdot Z(\mathfrak{g}) \cdot \mathcal{H} \cdot U(\mathfrak{k}).$$

Adding these over i gives the desired result. Q.E.D.

The lemma obviously implies that we may take $\mathcal{H} \subseteq U(\mathfrak{a})$. Although we used the fact that G is quasipplit to simplify the proof a little, the result is true as stated (for a minimal parabolic) without this hypothesis.

Corollary 6.4.3 (Osborne). Suppose Y is a $(\mathfrak{g}, K)$ module of finite length. Then Y is finitely generated as a $U(\mathfrak{m})$ module.

Proof. Let $Y_0 \subseteq Y$ be a finite dimensional K -stable generating subspace for Y as a $\mathfrak{g}$ module. The annihilator I of Y in $Z(\mathfrak{g})$ has finite codimension; choose a linear complement $\mathcal{L}$ for I in $Z(\mathfrak{g})$. Then

$$\begin{aligned} [Z(\mathfrak{g}) \cdot \mathcal{H} \cdot U(\mathfrak{k})] Y_0 &= \mathcal{H} \cdot Z(\mathfrak{g}) \cdot Y_0 \\ &= \mathcal{H}(I + \mathcal{L}) \cdot Y_0 \\ &= (\mathcal{H} \cdot \mathcal{L}) Y_0 \end{aligned}$$

since Y_0 is $U(\mathfrak{k})$ -stable and is annihilated by I . So by Lemma 6.4.2, the finite dimensional space $(\mathcal{H} \cdot \mathcal{L}) Y_0$ generates Y as a $U(\mathfrak{m})$ module. Q.E.D.

There is a standard way in commutative algebra to localize the problem of computing cohomology. Since $\mathfrak{m}$ is nilpotent, $U(\mathfrak{m})$ is close enough to being commutative that this method still applies. Set $\mathcal{S} = \mathfrak{m} U(\mathfrak{m})$, the augmentation ideal in $U(\mathfrak{m})$. If $k \geq 1$, $\mathcal{S}^k$ is the ideal in $U(\mathfrak{m})$ spanned by products of at least k elements in $\mathcal{S}$. The next three lemmas are special cases of a result of Nouazé-Gabriel; the proof follows McConnell. For these lemmas only, we drop our special assumptions on $\mathfrak{m}$ (to allow inductive arguments).

Lemma 6.4.4 Let $\mathfrak{m}$ be any nilpotent complex Lie algebra, and $\mathcal{S} = \mathfrak{m} U(\mathfrak{m})$ the augmentation ideal.

Let M be a finitely generated $U(\mathfrak{m})$ module and $N \subseteq M$ a submodule. Suppose that if S is any submodule of M

such that $S \cap N \subseteq \mathcal{A}N$, then $S = 0$. Then there is an integer $k \geq 1$ such that $\mathcal{A}^k M = 0$.

Proof. We proceed by induction on $\dim \mathcal{N}$; if $\mathcal{N} = \{0\}$ the result is trivial, so suppose $\mathcal{N} \neq \{0\}$. Choose an element $x \neq 0$ in the center of $\mathcal{N}$. Notice first that $\mathcal{A}N = S$ satisfies the hypotheses of the assumption on M , so $\mathcal{A}N = 0$. Thus $\mathcal{N}$ acts trivially on N , and in particular $x \cdot N = 0$. For $i \geq 0$, define

$$N_{-i} = \{m \in M \mid x^i m \in N\}.$$

Then each N_{-i} is a submodule of M , and

$$N = N_0 \subseteq N_{-1} \subseteq \dots \subseteq M.$$

Since M is finitely generated and $U(\mathcal{N})$ is Noetherian, there is an $r \geq 1$ such that $N_{-r+1} = N_{-r}$. This means that if $m \in M$ and $x^r m \in N$, then $x^{r-1} m \in N$; $x^r m = x(x^{r-1} m) = 0$, since x annihilates N . Thus

$$N \cap x^r M = 0.$$

Setting $S = x^r M$, we see by the hypothesis that $x^r M = 0$. Define $r(M)$ to be the smallest integer such that $x^r M = 0$; we set $r(\{0\}) = 0$. We proceed by induction on $r(M)$. If $r(M) = 0$, then $M = 0$ and the result is trivial; so suppose $r(M) > 0$. Set

$$K = \{m \in M \mid xm = 0\}.$$

Then multiplication by x gives a short exact sequence of $U(\mathcal{N})$ modules

$$0 \rightarrow K \rightarrow M \rightarrow xM \rightarrow 0.$$

Clearly $r(xM) \leq r(M) - 1$; so by induction (applied to $xM \subseteq xM \cap N$) there is an integer $k_1 \geq 1$ so that $\mathcal{A}^{k_1}(xM) = 0$. Since x acts trivially on K , we can regard K as a module for the nilpotent Lie algebra $\mathcal{N}_1 = \mathcal{N}/\mathfrak{C} \cdot x$. Also $N \subseteq K$, and the pair $K \supseteq N$ satisfies the hypotheses of the lemma for $\mathcal{N}_1$. So (writing $\mathcal{A}_1 = \mathcal{N}_1 U(\mathcal{N}_1)$) there is an integer $k_2 \geq 1$ such that

$$\mathcal{A}^{k_2}_K = \mathcal{A}^{k_2}_1 = 0.$$

Now

$$x(\mathcal{A}^{k_1}_M) = \mathcal{A}^{k_1}_1(xM) = 0$$

since x is central; so $\mathcal{A}^{k_1}_1 M \subseteq K$. Hence $\mathcal{A}^{k_2}(\mathcal{A}^{k_1}_M) = 0$.

The lemma follows, with $k = k_1 + k_2$. Q.E.D.

Lemma 6.4.5 Let $\mathcal{N}$ be a nilpotent complex Lie algebra, $\mathcal{A} = \mathcal{N}U(\mathcal{N})$, M a finitely generated $U(\mathcal{N})$ module, and $N \subseteq M$ a submodule. Then there is an integer $k \geq 1$ such that

$$\mathcal{A}^k_M \cap N \subseteq \mathcal{A}N.$$

Proof. Let M_1 be a maximal submodule of M with respect to the condition $M_1 \cap N \subseteq \mathcal{A}N$; this exists since M is Noetherian. Then the pair $(M/M_1 \supseteq N/M_1 \cap N)$ satisfies the hypotheses of Lemma 6.4.4; so there is an integer $k \geq 1$ such that

$$\mathcal{A}^k_{(M/M_1)} = 0,$$

which means that

$$\mathcal{A}^k_M \subseteq M_1.$$

Hence

$$\mathcal{O}^k_M \cap N \subseteq M_1 \cap N \subseteq \mathcal{O}^j_N. \quad \text{Q.E.D.}$$

Lemma 6.4.5' Under the hypothesis of Lemma 6.4.5,

fix $j \geq 1$. Then there is an integer $k \geq 1$ such that

$$\mathcal{O}^k_M \cap N \subseteq \mathcal{O}^j_N.$$

Proof. We proceed by induction on j ; if $j = 1$, this is

Lemma 6.4.5. So suppose $j \geq 2$, and the result is known for

$j - 1$. Choose k_1 so that

$$\mathcal{O}^{k_1}_{1M} \cap N \subseteq \mathcal{O}^{j-1}_N.$$

Now apply Lemma 6.4.5 to $M \supseteq \mathcal{O}^{j-1}_N$; we find k_2 so that

$$\mathcal{O}^{k_2 M} \cap \mathcal{O}^{j-1}_N \subseteq \mathcal{O}^j_N.$$

Choose $k = \max(k_1, k_2)$; then

$$\mathcal{O}^k_M \cap N \subseteq \mathcal{O}^k_M \cap \mathcal{O}^{j-1}_N \subseteq \mathcal{O}^j_N. \quad \text{Q.E.D.}$$

Definition 6.4.6 If X is a finitely generated $U(\mathcal{N})$ module, define

$$\begin{aligned} \tilde{X} &= \{f \in X^* \mid f|_{\mathcal{O}^k_X} = 0 \text{ for some } k \geq 0\} \\ &= {}_k U(X/\mathcal{O}^k_X)^*, \end{aligned}$$

with the union taken using the obvious inclusions.

Notice that $X/\mathcal{O}^k_X$ is finite dimensional, since $\mathcal{O}^k$ has finite codimension in $U(\mathcal{N})$.

Proposition 6.4.7 (Casselman). The functor $X \mapsto \tilde{X}$ from finitely generated $U(\mathcal{N})$ modules to $U(\mathcal{N})$ modules is exact and contravariant. The inclusion

$$\tilde{X} \hookrightarrow X^*$$

induces isomorphisms

$$H^i(\mathcal{N}, \tilde{X}) \cong H^i(\mathcal{N}, X^*) \cong H^{\dim \mathcal{N} - i}(\mathcal{N}, X \otimes D)^*;$$

here we have written

$$D = \wedge^{\dim \mathcal{N}} \mathcal{N}.$$

If X is a $(\mathcal{O}_f, K)$ module, then $\tilde{X}$ is a $\mathcal{O}_f$ -stable, M -stable subspace of X^* , and the natural actions make $\tilde{X}$ a $(\mathcal{O}_f, M)$ module. The cohomology isomorphisms above are $(\mathcal{N} + \mathcal{O}, M)$ module maps.

Proof. (Compare [49], Section 4). That $X \mapsto \tilde{X}$ is contravariant is clear; and the exactness is a reformulation of Lemma 6.4.5'. The second cohomology isomorphism comes from the isomorphism of complexes

$$[\text{Hom}_{\mathbb{Q}}(\wedge^{\dim \mathcal{N} - i} \mathcal{N}, X \otimes D)]^* \cong \text{Hom}_{\mathbb{Q}}(\wedge^i \mathcal{N}, X^*),$$

which is purely formal and easily verified. For the first, we proceed by induction on i . If $i = 0$,

$$\begin{aligned} H^0(\mathcal{N}, X^*) &= \{f \in X^* \mid \mathcal{N} \cdot f = 0\} \\ &= \{f \in X^* \mid \mathcal{O}^j \cdot f = 0\} \\ &\subseteq \tilde{X}; \end{aligned}$$

so the inclusion

$$H^0(\mathcal{N}, \tilde{X}) \rightarrow H^0(\mathcal{N}, X^*)$$

is surjective. So suppose $i > 0$, and the result is known for $i - 1$. Choose a finite dimensional generating subspace X_0 of X , and define

$$P = U(\mathcal{N}) \otimes_{\mathbb{C}} X_0,$$

a free $U(\mathcal{N})$ module. Then there is a natural surjection

$$P \rightarrow X \rightarrow 0,$$

given by

$$\sum u_i \otimes x_i \rightarrow \sum u_i x_i \quad (u_i \in U(\mathcal{N}), x_i \in X_0).$$

Since $U(\mathcal{N})$ is Noetherian, the kernel of this map is finitely generated. We will be able to apply a dimension shifting argument as soon as we know that

$$H^i(\mathcal{N}, P^*) = H^i(\mathcal{N}, \tilde{P}) = 0.$$

Now

$$P^* = \text{Hom}_{\mathbb{C}}(U(\mathcal{N}), X_0^*)$$

is injective as a $U(\mathcal{N})$ module (by Corollary 6.1.24); so $H^i(\mathcal{N}, P^*) = 0$ since $i > 0$. For $\tilde{P}$, we make P an $\mathcal{N}$ module by letting $\mathcal{N}$ act trivially on X_0 and by the adjoint action on $U(\mathcal{N})$; then P is a semisimple $\mathcal{N}$ module, with finite multiplicities. So P^* is an infinite direct product of semisimple, inequivalent, finite dimensional $\mathcal{N}$ modules:

$$P^* = \prod_{\lambda \in \mathcal{N}^*} (P_{\lambda}^*), \text{ with } \dim P_{\lambda}^* < \infty.$$

It is more or less obvious that $\tilde{P}$ consists of the $\mathcal{N}$ -finite

part of this product:

$$\tilde{P} = \sum_{\lambda \in \mathcal{N}^*} P_{\lambda}^*.$$

Now consider the cochain complexes

$$\{C_j^*\} = \text{Hom}_{\mathbb{C}}(\wedge^j \mathcal{N}, P^*)$$

$$\{\tilde{C}_j\} = \text{Hom}_{\mathbb{C}}(\wedge^j \mathcal{N}, \tilde{P}).$$

Again each C_j^* is a direct product of $\mathcal{N}$ modules, and $\tilde{C}_j$ is the corresponding direct sum:

$$C_j^* = \prod_{\lambda \in \mathcal{N}^*} C_j^*(\lambda)$$

$$\tilde{C}_j = \sum_{\lambda \in \mathcal{N}^*} C_j^*(\lambda).$$

The differential commutes with the action of $\mathcal{N}$, so

$$\begin{aligned} H^i(\mathcal{N}, P^*) &= \prod_{\lambda \in \mathcal{N}^*} H^i(C^*(\lambda)) \\ H^i(\mathcal{N}, \tilde{P}) &= \sum_{\lambda \in \mathcal{N}^*} H^i(C^*(\lambda)). \end{aligned}$$

But the first group is zero; so the second is also, proving the claim. The desired cohomology isomorphism follows by dimension shifting just as in the proof of Theorem 3.1.5; we leave the details to the reader.

Suppose now that X is a $(\mathcal{G}, K)$ module. Since $\mathcal{N}$ acts nilpotently on $\mathcal{G}$, for any $j \geq 1$ we can find $k \geq 1$ so that

$$\mathcal{G}^k \mathcal{G} \subseteq U(\mathcal{G}) \mathcal{G}^j.$$

Suppose $f \in \tilde{X}$; say f is annihilated by $\mathcal{G}^j$. If $z \in \mathcal{G}$, then clearly $z \cdot f$ is annihilated by $\mathcal{G}^k$, so $z \cdot f$ belongs to $\tilde{X}$. If

$m \in M$, then m normalizes $\mathcal{N}$; so $m \cdot f$ is annihilated by $\mathcal{J}$.
So $\tilde{X}$ is M -stable; and in fact

$$(X/\mathcal{J}^j X)^* \subseteq \tilde{X}$$

is M -stable, so $\tilde{X}$ is M -finite. The compatibility condition required to make $\tilde{X}$ a $(\mathcal{G}, M)$ module follows formally from the $(\mathcal{G}, K)$ compatibility for X . The last assertion is obvious.

Q.E.D.

Suppose X is a $(\mathcal{G}, K)$ module of finite length. Since $m + \sigma$ normalizes $\mathcal{N}$, it preserves $(X/\mathcal{J}^k X)^* \subseteq \tilde{X}$, so $\tilde{X}$ is a direct sum of generalized weight spaces for $m + \sigma$:

$$(6.4.8) \quad \tilde{X} = \bigoplus_{\mu \in (m + \sigma)^*} \tilde{X}(\mu),$$

where

$$(Z - \mu(Z))^n X = 0 \quad \text{if } Z \in m + \sigma, \\ x \in \tilde{X}(\mu), \text{ and } n \text{ is large enough.}$$

We now describe the possible weights.

Lemma 6.4.9 Suppose X is a $(\mathcal{G}, K)$ module of finite length, and generalized infinitesimal character

$\gamma \in (m + \sigma)^*$ (with respect to the Harish-Chandra

map). Then every weight μ of $\tilde{X}$ is of the form

$$\mu = -w\gamma - \rho - \sum_{\beta \in \Delta(\mathcal{N}, m + \sigma)} n_\beta \beta \quad (n_\beta \in \mathbb{N})$$

for some $w \in W(\mathcal{G}, m + \sigma)$.

Proof. First of all, it is easy to check that X^* has generalized infinitesimal character $-\gamma$. Suppose μ is a weight of $(X/\mathcal{J}^k X)^*$; we proceed by induction on k . If $k = 0$, there is nothing to prove, so suppose $k > 0$. Choose

$f \in (X/\mathcal{J}^k X)^*$ of weight μ . Suppose first that there is a positive root α , such that if Z_α is the corresponding root vector, then $Z_\alpha \cdot f \neq 0$. Since $Z_\alpha \mathcal{J}^{k-1} \subseteq \mathcal{J}^k$, $Z_\alpha \cdot f \in (X/\mathcal{J}^{k-1} X)^*$; so by induction, $Z_\alpha \cdot f$ has weight

$$-w\gamma - \rho - \sum_{\beta \in \Delta^+} n_\beta \beta.$$

But this is also equal to $\mu + \alpha$; so μ has the desired form.

If no such α exists, then f is annihilated by $\mathcal{N}$, i.e.,

$$f \in H^0(\mathcal{N}, X^*).$$

By Corollary 3.1.6,

$$\mu = -w\gamma - \rho$$

for some $w \in W(\mathcal{G}, m + \sigma)$.

Q.E.D.

Lemma 6.4.10 Suppose $\delta \in \hat{M}$ is fine (Definition 4.3.8),

$\nu \in \hat{A}$ is negative (Definition 4.2.1), and for every

$\alpha \in \Delta(\mathcal{N}, m + \sigma)$ such that $\langle \alpha, \text{Re } \nu \rangle = 0$, we have

$$\text{Im} \langle \alpha, \nu \rangle \geq 0.$$

Write $\lambda \in m^*$ for the differential of δ , and

$\gamma = (\lambda, \nu) \in (m + \sigma)^*$. Then for $w \in W(\mathcal{G}, m + \sigma)$, we

have

$$w\gamma = \gamma + \sum_{\alpha \in \Delta(\mathcal{N}, \sigma + m)} c_\alpha^\alpha.$$

Here $c_\alpha \in \mathbb{C}$, $\operatorname{Re} c_\alpha \geq 0$; and if $\operatorname{Re} c_\alpha = 0$, then $\operatorname{Im} c_\alpha \geq 0$. In particular,

$$\operatorname{Re} \langle w\gamma, \rho \rangle \geq \operatorname{Re} \langle \gamma, \rho \rangle.$$

If equality holds,

$$\operatorname{Im} \langle w\gamma, \rho \rangle \geq \operatorname{Im} \langle \gamma, \rho \rangle;$$

and if equality holds here also, then $w\gamma = \gamma$.

Proof. Clearly we may assume G is semisimple; then by Definition 4.3.8, $\lambda = 0$; so $\gamma = \nu$. So γ satisfies the same dominance properties as ν . The formula for $w\gamma$ is proved in exactly the same way as Lemma 6.3.28; in fact it can even be deduced from that lemma without too much effort. We leave the argument to the reader. The statement about $\langle w\gamma, \rho \rangle$ follows trivially, since $\langle \alpha, \rho \rangle > 0$ whenever $\alpha \in \Delta(\mathfrak{g}, \mathfrak{m} + \sigma\mathfrak{g})$.
Q.E.D.

Corollary 6.4.11 (Schmid) Under the hypotheses of Lemma 6.4.10, suppose Y is a $(\mathfrak{g}, K)$ module of finite length, and that the $\mathfrak{m} + \sigma\mathfrak{g}$ weight $\gamma - \rho$ occurs in $H^i(\mathfrak{g}, Y)$. Then $i = \dim \mathfrak{g}$.

Proof. Since Y is a direct sum of its submodules of various generalized infinitesimal characters, we may as well assume that Y has some generalized infinitesimal character $\gamma' \in (\mathfrak{m} + \sigma\mathfrak{g})^*$. Since $\gamma - \rho$ occurs in $H^i(\mathfrak{g}, Y)$, Corollary 3.1.6 implies that $\gamma' \in W \cdot \gamma$; so we may as well assume $\gamma = \gamma'$. By Proposition 6.4.7, writing $t = \dim \mathfrak{g}$,

$$H^i(\mathfrak{g}, Y) \cong H^{t-i}(\mathfrak{g}, \tilde{Y} \otimes D^*)^*.$$

Since D has weight 2ρ , $-\gamma - \rho$ is a weight of $H^{t-i}(\mathfrak{g}, \tilde{Y})$. So it is also a weight of

$$\Lambda^{t-i}(\mathfrak{g})^* \otimes \tilde{Y}.$$

This means that we can write

$$-\gamma - \rho = -\sum \beta_j + \mu,$$

with $\{\beta_j\} \subseteq \Delta(\mathfrak{g}, \mathfrak{m} + \sigma\mathfrak{g})$ of cardinality $t - i$, and μ a weight of $\tilde{Y}$. By Lemma 6.4.9, it follows that

$$-\gamma = -w\gamma - \sum \beta_j - \sum_{\alpha \in \Delta^+} n_\alpha \alpha,$$

with $n_\alpha \geq 0$; so

$$\langle \gamma, \rho \rangle - \langle w\gamma, \rho \rangle \geq 0,$$

with equality only if $\{\beta_j\}$ is empty. The last two inequalities in Lemma 6.4.10 imply that this must be the case; so $\{\beta_j\}$ is empty, and

$$0 = |\{\beta_j\}| = t - i.$$

So i is equal to t , the dimension of $\mathfrak{g}$.
Q.E.D.

Corollary 6.4.12 Under the hypotheses of Lemma 6.4.10,

suppose Y is a $(\mathfrak{g}, K)$ module of finite length. Write

$$t = \dim \mathfrak{g}, \quad D = \Lambda^t \mathfrak{g}, \quad \tilde{\delta} = (\delta \otimes D^*) \Big|_{\mathfrak{M}}$$

$$H^t(\mathfrak{g}, Y)^{\tilde{\delta}}$$

for the $\tilde{\delta}$ -primary subspace of the cohomology (under the action of $\mathfrak{M}$). Write

$$H^t(\mathfrak{g}, Y)^{\tilde{\delta}, \nu - \rho}$$

for the $\nu - \rho$ generalized weight space of $\sigma\mathfrak{g}$. Then

a) The functor

$$Y \rightarrow H^t(\mathcal{N}, Y)^{\tilde{\delta}, \nu-\rho}$$

(from $(\mathcal{Q}, K)$ modules of finite length to finite dimensional $\mathcal{N}$ modules) is exact.

b) If $X(\delta \otimes \nu)$ denotes the principal series $(\mathcal{Q}, K)$ module, then there is a natural isomorphism

$$\text{Ext}_{\mathcal{Q}, K}^i(Y, X(\delta \otimes \nu)) \cong \text{Ext}_{\mathcal{N}}^i(H^t(\mathcal{N}, Y)^{\tilde{\delta}, \nu-\rho}, \mathbb{C}_{\nu-\rho})$$

Proof. Part (a) follows from Corollary 6.4.11 and the long exact sequence in $\mathcal{N}$ cohomology. (This uses the fact that D^* is trivial on $\mathcal{M}$.) For (b), Proposition 6.3.5 provides a spectral sequence

$$\text{Ext}_{\mathcal{M} + \sigma, M}^p(H^{t-q}(\mathcal{N}, Y), \tilde{\delta} \otimes \mathbb{C}_{\nu-\rho}) \Rightarrow \text{Ext}_{\mathcal{Q}, K}^{p+q}(Y, X(\delta \otimes \nu)).$$

Since any $(\mathcal{M} + \sigma, M)$ module is a direct sum of its M -primary subspaces, on which M acts semisimply, it is easy to check that

$$\text{Ext}_{\mathcal{M} + \sigma, M}^i(U, V) \cong \bigoplus_{Y \in \hat{M}} \text{Ext}_{\mathcal{N}}^i(U^Y, V^Y).$$

So the E_2^{pq} term is

$$\text{Ext}_{\mathcal{N}}^p(H^{t-q}(\mathcal{N}, Y)^{\tilde{\delta}}, \mathbb{C}_{\nu-\rho}).$$

We want to pass to the $\nu - \rho$ generalized weight space. This is more or less obviously allowed; for example, it is a very special case of Proposition 6.1.34. So we get

$$\text{Ext}_{\mathcal{N}}^p(H^{t-q}(\mathcal{N}, Y)^{\tilde{\delta}, \nu-\rho}, \mathbb{C}_{\nu-\rho}).$$

By Corollary 6.4.11, this vanishes for $q \neq 0$; so the spectral sequence collapses to the isomorphism we want. Q.E.D.

Corollary 6.4.13 Suppose $\delta \in \hat{M}$ is fine (Definition 4.3.8) and $\nu \in \hat{A}$ is negative (Definition 4.2.1). Let $A(\delta) \subseteq \hat{K}$ be the set of fine K -types containing δ (Definition 4.3.9); write

$$\{\bar{X}_i\} = \{\bar{X}(\delta \otimes \nu)(\mu) \mid \mu \in A(\delta)\}$$

for the set of Langlands submodules of $X(\delta \otimes \nu)$

(Definition 4.4.6 and (4.4.7)). Suppose Y is a

$(\mathcal{Q}, K)$ module of finite length. The following conditions on Y are equivalent:

- a) $\text{Hom}_{\mathcal{Q}, K}(Y, X(\delta \otimes \nu)) \neq 0$;
- b) One of the $\bar{X}_i$ is a subquotient of Y ;
- c) For some subquotient Z of Y ,

$$\text{Ext}_{\mathcal{Q}, K}^i(Z, X(\delta \otimes \nu)) \neq 0.$$

In particular, $A(\delta)$ is a strongly co-generating set for $X(\delta \otimes \nu)$.

Proof. If M^1 is defined as in (4.2.1), then modifying $\Delta^+(\mathcal{Q}, \mathcal{M} + \sigma)$ within $\mathcal{M}^1$ affects only the unitary part of the induction defining $X(\delta \otimes \nu)$, and therefore does not change $X(\delta \otimes \nu)$. We can make this modification so that $\text{Im } \nu$ becomes dominant for $\Delta^+(\mathcal{M}^1)$; then the hypotheses of Lemma 6.4.10 are satisfied. We will show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). Suppose (a) holds. Then Y has an irreducible subquotient Y_0 which is a submodule of $X(\delta \otimes \nu)$. By Theorem 4.2.12, Y_0 is one of the $\bar{X}_i$, so (b) holds. Assume (b). Taking $Z = \bar{X}_i$, we get

$$\text{Hom}_{\mathcal{Q}, K}(Z, X(\delta \otimes \nu)) \neq 0,$$

so (c) holds. Assume (c). By Corollary 6.4.12(b),

$$H^t(\mathcal{N}, Z)^{\tilde{\delta}, \nu-\rho} \neq 0$$

By Corollary 6.4.12(a),

$$H^t(\mathcal{N}, Y)^{\tilde{\delta}, \nu-\rho} \neq 0.$$

As a finite dimensional $\mathcal{O}$ module, this has an irreducible one dimensional quotient; so

$$\text{Hom}_{\mathcal{O}}(H^t(\mathcal{N}, Y)^{\tilde{\delta}, \nu-\rho}, \mathbb{C}_{\nu-\rho}) \neq 0.$$

By Corollary 6.4.12(b) again, this is equivalent to (a). So (a) - (c) are equivalent. For the last statement, suppose

$$\text{Ext}_{\mathcal{O}, K}^i(Y, X(\delta\mathcal{O}\nu)) \neq 0.$$

By (c) $\Rightarrow$ (b), it follows that

$$Y(\mu) \neq 0$$

for some $\mu \in A(\delta)$, as required.

Q.E.D.

Corollary 6.4.14 Suppose G is arbitrary (that is, not necessarily quasisplit), and $\mathcal{O} = \mathfrak{k} + \mathcal{U}$ is a θ -stable parabolic subalgebra of $\mathfrak{g}$, with L quasi-split. Suppose $H = TA$ is a maximally split Cartan subgroup of L , $\delta \in \hat{T}$ is fine for L , and $\nu \in \hat{A}$. Set

$$X_L(\delta\mathcal{O}\nu) = \text{Ind}_{TAN}^L(\delta \otimes \nu \otimes 1),$$

with N chosen to make ν negative (Definition 4.2.1).

Let i_0 be the least integer such that

$$\mathcal{R}og^{i_0}[X_L(\delta\mathcal{O}\nu)] \neq 0.$$

(If it is always zero, set $i_0 = +\infty$). Let Y be a $(\mathcal{O}, K)$ module of finite length.

a) $\overline{X}_L(\delta\mathcal{O}\nu)(\mu)$ (for $\mu \in A_L(\delta)$ -- cf. (4.4.6)) can occur in $H^i(\mathcal{U}, Y)$ only if $i \leq \dim \mathcal{U} - i_0$.

$$\text{b) } \quad \text{Hom}_{\mathcal{O}, L \cap K} \mathcal{Y}_K(Y, \mathcal{R}og^{i_0}[X_L(\delta\mathcal{O}\nu)]) \cong \text{Hom}_{\mathcal{O}, L \cap K} (H^{\dim \mathcal{U} - i_0}(\mathcal{U}, Y), X_L(\delta\mathcal{O}\nu)).$$

proof. Define q_0 to be the smallest integer such that for some $(\mathcal{O}, K)$ module Z of finite length, and some $p \geq 0$,

$$\text{Ext}_{\mathcal{O}, L \cap K}^p(H^{\dim \mathcal{U} - q_0}(\mathcal{U}, Z), X_L(\delta\mathcal{O}\nu)) \neq 0.$$

By Corollary 6.4.13,

$$\text{Hom}_{\mathcal{O}, L \cap K} (H^{\dim \mathcal{U} - q_0}(\mathcal{U}, Z), X_L(\delta\mathcal{O}\nu)) \neq 0.$$

So by the second spectral sequence of Proposition 6.3.2,

$$\begin{cases} \text{Ext}_{\mathcal{O}, L \cap K}^i(Y, \text{pro}(X_L(\delta\mathcal{O}\nu) \otimes \wedge \mathcal{U})) \cong \\ \text{Hom}_{\mathcal{O}, L \cap K} (H^{\dim \mathcal{U} - q_0}(\mathcal{U}, Y), X_L(\delta\mathcal{O}\nu)) & (i = q_0) \\ 0 & (i < q_0) \end{cases}$$

and the first of these may actually be non-zero.

By the first spectral sequence, on the other hand, this same Ext group is

$$\begin{cases} \text{Hom}_{\mathcal{O}, K} (Y, \mathcal{R}^{i_0}(X_L(\delta\mathcal{O}\nu))) & (i = i_0) \\ 0 & (i < i_0). \end{cases}$$

So $i_0 = q_0$, and the result follows.

Q.E.D.

§5. Construction of standard representations.

Definition 6.5.1 A set of θ -stable data for G is

a quadruple $(\mathfrak{g}, H, \delta, \nu)$, such that

a) $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra of $\mathfrak{g}$ (Definition 5.2.1). Let L be the normalizer of $\mathfrak{g}$ in G .

b) L is quasisplit, and $H \subseteq L$ is a maximally split θ -stable Cartan subgroup of L . Write $H = TA$ as in Definition 0.2.1.

c) $\delta \in \hat{T}$ is fine with respect to L (Definition 4.3.8).

d) $\nu \in \hat{A}$.

e) Write $\lambda^L \in \mathcal{T}^*$ for the differential of δ , and

$$\lambda^G = \lambda^L + \rho(\Delta(\mathfrak{u}, \mathfrak{t})) \in \mathcal{T}^* \subseteq \mathfrak{h}^* \quad (\text{notation (0.2.3)}).$$

Then λ^G is strictly dominant for $\Delta(\mathfrak{u}, \mathfrak{h})$:

$$\langle \alpha, \lambda^G \rangle > 0 \quad (\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})).$$

We may write simply $(\mathfrak{g}, \delta, \nu)$ or even $\delta\mathfrak{g}\nu$ if the rest is clear from context. A set of discrete θ -stable

data for G is a triple $(\mathfrak{g}, H, \delta)$ satisfying the conditions above (except for (d)).

The irreducible $(\mathfrak{g}, K)$ modules are going to be parametrized by $(K$ conjugacy classes of) sets of θ -stable data for G , up to some R -group questions. That is, we will prove a generalization of Theorem 4.4.8 which applies to all $(\mathfrak{g}, K)$ modules.

Definition 6.5.2 Suppose $(\mathfrak{g}, H, \delta, \nu)$ is a set of θ -stable data for G . Choose a minimal parabolic subgroup $P = TAN$ of L , in such a way that ν is

negative with respect to N (Definition 4.2.1). Let

$$X_L(P, \delta\mathfrak{g}\nu)$$

be the corresponding principal series $(\mathfrak{g}, K)$ module (Definition 4.1.3). The standard $(\mathfrak{g}, K)$ module with parameters $(\mathfrak{g}, H, \delta, \nu)$ is

$$X_G(\mathfrak{g}, H, \delta, \nu) = X(\mathfrak{g}, \delta\mathfrak{g}\nu) = \mathcal{R}_{\mathfrak{g}}^S(X_L(P, \delta\mathfrak{g}\nu))$$

(Definition 6.3.1); here $S = \dim \mathfrak{u} \cap \mathfrak{h}$.

Using P and $\mathfrak{g}$, one can construct a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. By induction by stages (Proposition 6.3.6),

$$(6.5.3) \quad X(\mathfrak{g}, \delta\mathfrak{g}\nu) = \mathcal{R}_{\mathfrak{b}}^S((\delta\mathfrak{g}\nu) \otimes (D_L^* \otimes \mathbb{C}_{\rho L}));$$

here the second factor is a twist defined in Proposition

6.3.5, which we could eliminate by changing the definitions

a bit. Philosophically this is a wonderful formula: it says that we can construct all the standard representations by (cohomological) parabolic induction from characters of Cartan subgroups, using a notion of induction which is not obviously patched together from separate considerations for the real

and complex parts. If this book were going to end with the classification, the definitions would have been arranged to lead up to (6.5.3) (without the twist) as the mountaintop of the subject. Unfortunately, we are going back down into the valleys to study the standard representations in more detail; so we prefer to think of them as defined in two steps which we can understand separately, as in Definition 6.5.2.

Lemma 6.5.4 Suppose $(\mathfrak{g}, H, \delta, \nu)$ is a set of θ -stable data for G . Fix a Cartan subalgebra $\mathcal{T}_0^C$ of $\mathfrak{g}_0 \cap \mathfrak{k}_0$ containing $\mathcal{T}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$, and a positive root system

$$\Delta^+(\mathfrak{k}, \mathcal{T}_0^C) \supseteq \Delta(\mathfrak{u} \cap \mathfrak{k}, \mathcal{T}_0^C).$$

Fix $\pi^L \in A_L(\delta)$ (Definition 4.3.15), and let $\mu^L \in \hat{\pi}^C$ be a highest weight of π^L . Put

$$\mu^G = \mu^L + 2\rho(\mathfrak{u} \cap \mathfrak{p}).$$

- a) μ^L is strictly $\mathfrak{u} \cap \mathfrak{k}$ -positive (Definition 6.3.22); so μ^G is $\Delta^+(\mathfrak{k}, \mathcal{T}_0^C)$ dominant (Lemma 6.3.23).
- b) Regard λ^L and λ^G (cf. (6.5.1)(e)) as weights in $(\mathcal{T}_0^C)^* \supseteq \mathcal{T}^*$. Then λ^L is the weight associated to μ^L for L by Proposition 5.3.3, and λ^G is the weight associated to μ^G for G .
- c) The parabolic associated to μ^G is $\mathfrak{g}_{\mathcal{T}}$ (Definition 5.3.22).
- d) $R_{\mu^L}^L = R_{\mu^G}^G$ (Definition 5.1.2).

Proof. First, let us show that the weight $(\lambda^L)^L$ associated to μ^L is λ^L . This is a statement only about the quasipoint group L , and can immediately be reduced to the case when L is semisimple. Since μ^L is fine, it is small (Corollary 5.4.14). This means that the parabolic defined by $(\lambda^L)^L$ is all of $\mathfrak{g}$ by definition; so since L is semisimple, $(\lambda^L)^L$ is zero. On the other hand, λ^L is the differential of δ , which is zero by Definition 4.3.8. This proves the first half of (b). Part (a) now follows from Definition 6.3.22 and Definition 6.5.1(e). For the second half of (b), choose a

positive root system $\Delta^+(\mathfrak{g}, \mathcal{T}^C)$ which makes

$$\mu^L + 2\rho(\Delta^+(\mathfrak{g} \cap \mathfrak{k}, \mathcal{T}^C))$$

dominant; and write

$$\lambda^L = \mu^L + 2\rho(\Delta^+(\mathfrak{g} \cap \mathfrak{k}, \mathcal{T}^C)) - \rho(\Delta^+(\mathfrak{g}, \mathcal{T}^C)) + \frac{1}{2} \sum \mathbf{c}_i \beta_i$$

in accordance with Proposition 5.3.3. Set

$$\Delta^+(\mathfrak{g}_{\mathcal{T}}, \mathcal{T}^C) = \Delta^+(\mathfrak{g}, \mathcal{T}^C) \cup \Delta(\mathfrak{u}, \mathcal{T}^C).$$

We claim that $\mu^G + 2\rho_G$ is dominant for $\Delta^+(\mathfrak{g}_{\mathcal{T}}, \mathcal{T}^C)$. In the proof of Lemma 6.3.23, an even stronger statement for the compact roots was deduced from the hypothesis that λ^G was strictly dominant on roots of $\Delta(\mathfrak{u} \cap \mathfrak{k})$. The present claim can be proved in the same way, using the stronger hypothesis (6.5.1)(e). We also showed (cf. (6.3.25)) that

$$\lambda^G = \mu^G + 2\rho_G - \rho + \frac{1}{2} \sum \mathbf{c}_i \beta_i.$$

It is trivial to verify the conditions in Proposition 5.3.3, using (6.4.1)(e) and the corresponding conditions for $\mathfrak{g}$; so λ^G is the weight attached to μ^G , proving (b). To prove (c), we only have to check that if $\alpha \in \Delta(\mathfrak{g}, \mathcal{T}^C)$, then

$$\langle \alpha, \lambda^G \rangle = 0.$$

But

$$\begin{aligned} \langle \alpha, \lambda^G \rangle &= \langle \alpha, \lambda^L + \rho(\mathfrak{u}) \rangle \\ &= \langle \alpha, \lambda^L \rangle \quad (\text{by Corollary 3.2.5}) \\ &= 0 \end{aligned}$$

since, as we saw at the beginning of the proof, λ^L is zero

on the roots of λ . The second equality in (d) is Proposition 5.4.2(b). For the first, it suffices to show that R^L fixes $2\rho(u_{\alpha})$. But since L normalizes σ , this is clear.

Q.E.D.

Definition 6.5.5 Suppose (σ, H, δ) is a set of discrete θ -stable data for G (Definition 6.5.1).

Consider $A_L(\delta) \subseteq (L\backslash K)^\wedge$ (Definition 4.3.15).

Define a subset

$$A_G(\sigma, H, \delta) = A(\sigma, \delta) \subseteq \hat{K}$$

in natural 1-1 correspondence with $A_L(\delta)$, as follows.

If $\pi^L \in A_L(\delta)$ has highest weight μ^L , we define $\pi^G \in \hat{K}$ by the conditions:

a) π^G has highest weight μ^G (Lemma 6.5.4).

b) $\pi^L \otimes (\wedge^{\dim u_{\alpha}} u_{\alpha})$ occurs in $\pi^G|_{L\backslash K}$.

(Such a π^G exists by Lemma 6.5.4(a), and is unique by Lemma 6.5.4(d) and Corollary 5.1.10.) Put

$$A(\sigma, \delta) = \{\pi^G|_{\pi^L} \in A_L(\delta)\}.$$

We also make $\hat{R}_\delta$ act on $A(\sigma, \delta)$ by this bijection (cf. Theorem 4.3.16 (c)).

There is no point in calling the set $A(\sigma, \delta)$ anything, since we will soon be able to describe it in terminology we already have.

Lemma 6.5.6 Suppose (σ, H, δ) is a set of discrete θ -stable data for G . Let π be an irreducible representation of G , and suppose

$$\text{Hom}_{L\backslash K}(H^j(u_{\alpha}, \pi), \text{Ind}_T^{L\backslash K}(\delta) \otimes S^m(u_{\alpha}) \otimes \wedge^{\dim u_{\alpha}} u_{\alpha}) \neq 0.$$

Then

$$||\pi||_{\text{lambda}} \geq \langle \lambda^G, \lambda^G \rangle$$

(notation (5.4.1)). Equality holds only if $m = 0$,

$j = 0$, and $\pi \in A(\sigma, \delta)$; and in that case the Hom

has dimension one.

proof. Arguing as in the proof of Theorem 6.3.12 (after

(6.3.20)), we can find a highest weight μ of π ; an element w of W_K^1 , having length j ; an $L\backslash K$ type π^0 occurring in $\text{Ind}_T^{L\backslash K}(\delta)$; a highest weight γ of π^0 ; and non-negative integers n_β , for $\beta \in \Delta(u_{\alpha})$, whose sum is m , such that

$$(6.5.7) \text{ (a) } w(\mu + \rho_C) - \rho_C = \gamma + 2\rho(u_{\alpha}) + \sum_{\beta \in \Delta(u_{\alpha})} n_\beta \beta.$$

If we apply Lemma 6.3.28 to (w, μ) and (3.2.9) to $w\rho_C - \rho_C$, we find non-negative integers n_α , for $\alpha \in \Delta^+(k, t)$, such that

$$w(\mu + \rho_C) - \rho_C = \mu - \sum_{\beta \in \Delta^+(k, t)} n_\alpha \alpha.$$

Furthermore (because w lies in W_K^1 , and thus $\Delta_K^+(w)$ is contained in $\Delta(u_{\alpha}, k)$),

$$\sum_{\alpha \in \Delta(u_{\alpha}, k)} n_\alpha \geq j.$$

So

$$(6.5.7) \text{ (b) } \mu = \gamma + 2\rho(u_{\alpha}) + \sum_{\alpha \in \Delta^+(k, t) \cup \Delta(u_{\alpha})} n_\alpha \alpha,$$

and

$$(6.5.7) \text{ (c)} \quad \sum_{\alpha \in \Delta(\mathcal{U})} n_{\alpha} \geq j + m.$$

Fix $\mu^G, \Delta^+(\mathcal{O}\mathcal{f}, \mathcal{L}^C), \{c_i, \beta_i\}$ as in Lemma 6.5.4 and its proof.
By (6.3.25),

$$\lambda^G = \mu^G + 2\rho_C - \rho + \frac{1}{2} \sum c_i \beta_i.$$

Choose $(\Delta^+)'(\mathcal{O}\mathcal{f}, \mathcal{L}^C)$ making $\mu + 2\rho_C$ dominant, and $\{c_j', \beta_j'\}$ as in Proposition 5.3.3 applied to μ ; then the weight λ associated to μ is

$$\begin{aligned} \lambda &= \mu + 2\rho_C - \rho' + \frac{1}{2} \sum c_j' \beta_j' \\ &= (\gamma + 2\rho(\mathcal{U} \cap \mathcal{P}) + \sum n_{\alpha} \alpha) + 2\rho_C - \rho' \\ &\quad + \frac{1}{2} \sum c_j' \beta_j' \\ &= \mu^L + 2\rho(\mathcal{U} \cap \mathcal{P}) + (\gamma - \mu^L) + 2\rho_C - \rho' \\ &\quad + \frac{1}{2} \sum c_j' \beta_j' + \sum n_{\alpha} \alpha \\ &= \mu^G + 2\rho_C - \rho + \frac{1}{2} \sum c_i \beta_i + \\ &\quad \left[\rho - \rho' + \frac{1}{2} \sum c_j' \beta_j' - \frac{1}{2} \sum c_i \beta_i + (\gamma - \mu^L) + \right. \\ &\quad \left. + \sum n_{\alpha} \alpha \right] \end{aligned}$$

$$(6.5.8) \text{ (a)} \quad \lambda = \lambda^G + [(\gamma - \mu^L) + \sum n_{\alpha} \alpha + \rho - \rho' + \frac{1}{2} \sum c_j' \beta_j' - \frac{1}{2} \sum c_i \beta_i].$$

Write Q for the term in square brackets. We claim that

$$(6.5.8) \text{ (b)} \quad Q = \sum_{\alpha \in \Delta(\mathcal{L}, \mathcal{L}^C)} a_{\alpha} \alpha + \sum_{\beta \in \Delta(\mathcal{U})} b_{\beta} \beta,$$

with $b_{\beta} \geq 0$, and

$$(6.5.8) \text{ (c)} \quad \sum b_{\beta} \geq j + m.$$

Now γ and μ^L are both highest weights in $\text{Ind}_T^{L \cap K}(\delta)$. Since

$\mathcal{L}$ contains the compact part of the center of $\mathcal{L}$, γ and μ^L must differ by a sum of roots in $\mathcal{L}$. All the β_i lie in $\Delta(\mathcal{L})$; so using (6.5.7) (c), (6.5.8) will follow if we can show

$$(*) \quad \rho - \rho' + \frac{1}{2} \sum c_j' \beta_j' = \sum_{\alpha \in \Delta^+} c_{\alpha} \alpha, \quad c_{\alpha} \geq 0.$$

By (3.2.9),

$$\rho - \rho' = \sum_{\substack{\alpha \in \Delta^+ \\ \alpha \notin (\Delta^+)'}} \alpha.$$

Suppose some β_j' does not belong to Δ^+ . Since it is in $(\Delta^+)', -\beta_j'$ occurs in the expression just given for $\rho - \rho'$. Since c_j' is between 0 and 1, (*) follows.

Now λ^G is strictly positive on $\Delta(\mathcal{U})$, and is orthogonal to $\Delta(\mathcal{L})$. So by (6.4.8),

$$\begin{aligned} ||\pi||_{\text{lambda}} &= \langle \lambda, \lambda \rangle \\ &= \langle \lambda^G, \lambda^G \rangle + 2\langle \lambda, Q \rangle + \langle Q, Q \rangle, \end{aligned}$$

and the last two terms are non-negative. This proves the inequality we want. If equality holds, then $Q = 0$; so by (6.5.8) (c), $j = m = 0$. Since w has length j , $w = 1$; so (6.5.7) (a) reads

$$\mu = \gamma + 2\rho(\mathcal{U} \cap \mathcal{P}).$$

Since $\lambda = \lambda^G$ (because $Q = 0$), the parabolic associated to μ is $\mathcal{O}\mathcal{f}$. By Corollary 5.3.25, our L $\cap$ K-type π^0 is fine. Since π^0 occurs in $\text{Ind}_T^{L \cap K}(\delta)$, δ occurs in $\pi^0|_T$; so π^0 belongs to $A_L(\delta)$. Since $m = j = 0$, our hypothesis reduces to

$$\text{Hom}_{L \cap K}(H^0(\mathcal{U} \cap \mathcal{L}, \pi), \pi^0 \otimes \wedge^{\dim \mathcal{U} \cap \mathcal{P}} \mathcal{U} \cap \mathcal{P}) \neq 0.$$

By Definition 6.5.5, this means that π belongs to $A(\mathcal{O}\mathcal{f}, \delta)$,

and is the K-type corresponding to π^0 . Since π^0 occurs exactly once in $\text{Ind}_T^{\text{Lok}}(\delta)$ (Theorem 4.3.16(b)), the Hom in the lemma is one dimensional. Q.E.D.

The next result is joint work with Zuckerman.

Theorem 6.5.9 Let (σ, H, δ, ν) be a set of θ -stable data for G, with $\sigma = \ell + \mathcal{U}$, $H = \text{TA}$ (Definition 6.5.1); and let $X(\sigma, \delta\mathfrak{v})$ be the corresponding standard representation (Definition 6.5.2).

a) If λ^G is as in (6.5.1)(e), then $X(\sigma, \delta\mathfrak{v})$ has infinitesimal character $(\lambda^G, \nu) \in \mathcal{H}^*$ (notation (0.2.9)).

b) The set of lambda-lowest K-types (Definition 5.4.1) of $X(\sigma, \delta\mathfrak{v})$ is $A(\sigma, \delta)$ (Definition 6.5.5); and these K-types occur with multiplicity one.

c) If $\nu \in \hat{A}$ is unitary, then every subquotient of $X(\sigma, \delta\mathfrak{v})$ contains some $\pi \in A(\sigma, \delta)$.

d) Choose $\mathcal{T}^C, {}^+_A(\mathcal{K}, \mathcal{T}^C)$ as in Lemma 6.5.4.

Then any K-type occurring in $X(\sigma, \delta\mathfrak{v})$ has a highest weight of the form

$$\gamma + 2\rho(\mathfrak{u}_{\cap}\mathfrak{p}) + \sum_{\alpha \in \Delta(\mathfrak{u}_{\cap}\mathfrak{p})} n_{\alpha}\alpha \quad (n_{\alpha} \in \mathbb{N}),$$

with γ an Lok -highest weight of an Lok -type whose restriction to T contains δ .

e) If $\pi \in \hat{K}$, the multiplicity of π in $X(\sigma, \delta\mathfrak{v})$ is

$$\sum_{j=0}^{\dim \mathfrak{u}_{\cap}\mathcal{K}} (-1)^j \dim \text{Hom}_{\text{Lok}}(H^j(\mathfrak{u}_{\cap}\mathcal{K}, \pi), \text{Ind}_T^{\text{Lok}}(\delta) \otimes S(\mathfrak{u}_{\cap}\mathfrak{p}) \otimes D_2);$$

here

$$D_2 = \wedge^{\dim \mathfrak{u}_{\cap}\mathfrak{p}}(\mathfrak{u}_{\cap}\mathfrak{p}).$$

In particular, this multiplicity is independent of ν .

f) Put $R = \dim \mathfrak{u}_{\cap}\mathfrak{p}$. If Y is any (σ, K) module, there is a spectral sequence

$$\text{Ext}_{\ell, \text{Lok}}^P(H^{R-q}(\mathcal{U}, Y), X_L(\delta\mathfrak{v})) \implies \text{Ext}_{\sigma, K}^{P+q}(Y, X(\sigma, \delta\mathfrak{v})).$$

If Y has finite length, this is a first quadrant spectral sequence (i.e., $E_2^{P,q} = 0$ for $q \leq 0$).

g) If Y is any (σ, K) module of finite length, then

$$\text{Hom}_{\sigma, K}(Y, X(\sigma, \delta\mathfrak{v})) \cong \text{Hom}_{\ell, \text{Lok}}(H^R(\mathcal{U}, Y), X_L(\delta\mathfrak{v})).$$

Proof. Statement (a) follows from Lemma 4.1.8, Proposition 6.3.11, and the definition of λ^G . For the rest, notice first that $A_L(\delta)$ is a strongly cogenerating set for $X_L(\delta\mathfrak{v})$ (Definition 6.3.32) by Corollary 6.4.13. By Lemma 6.5.4(a), $A_L(\delta)$ consists of strictly $\mathfrak{u}_{\cap}\mathcal{K}$ -positive Lok -types; so by

Proposition 6.3.34,

$$(6.5.10) \quad \text{Rg}_L^i(X_L(\delta\mathfrak{v})) = 0, \quad i \neq S = \dim \mathfrak{u}_{\cap}\mathcal{K}.$$

Now (d) and (e) follow from Theorem 6.3.12. For (b), notice first that the K-types in $A(\sigma, \delta)$ occur exactly once in $X(\sigma, \delta\mathfrak{v})$ by (e) and Lemma 6.5.6; so by Lemma 6.5.6 again, they are the lambda-lowest K-types of $X(\sigma, \delta\mathfrak{v})$. For (c), fix ν unitary, and suppose X' is an irreducible subquotient of $X(\sigma, \delta\mathfrak{v})$. Let π' be a lambda-lowest K-type of X' , and $\lambda' \in (\mathcal{T}^C)^*$ the associated weight. By (a), X' has infinitesimal character (λ^G, ν) ; since ν is imaginary, this has canonical real part λ^G (Definition 5.4.11). By Corollary 5.4.13,

$$\langle \lambda', \lambda' \rangle \leq \langle \lambda^G, \lambda^G \rangle.$$

By Lemma 6.5.6 and (e), this forces $\pi' \in A(\sigma, \delta)$, proving (c).

The first claim of (f) is Corollary 6.3.4 and (6.5.10). Fix Y of finite length, and choose q_0 as small as possible with

$$\text{Ext}_{\ell, L \cap K}^P(H^{R-q_0}(\mathcal{U}, Y), X_L(\delta \mathfrak{a} \mathfrak{v})) \neq 0.$$

By Corollary 6.4.13,

$$\text{Hom}_{\ell, L \cap K}(H^{R-q_0}(\mathcal{U}, Y), X_L(\delta \mathfrak{a} \mathfrak{v})) \neq 0;$$

and the other $E_2^{p,q}$ terms vanish for $q \leq q_0$. So this E_2 term persists to E^∞ , and we deduce that

$$\text{Ext}_{\mathcal{G}, K}^{q_0}(Y, X(\sigma, \delta \mathfrak{a} \mathfrak{v})) \neq 0.$$

So q_0 is non-negative, and the spectral sequence is first quadrant. Part (g) is just a corner isomorphism in the spectral sequence (compare Corollary 6.4.14). Q.E.D.

Theorem 6.5.10 Let (σ, H, δ, ν) be a set of θ -stable data for G . Let

$$X_L(\delta \mathfrak{a} \mathfrak{v}) = \bigoplus_{i=1}^r W_i$$

be the direct sum decomposition described by (4.2.7), coming from the unitary part of the induction. Define

$$\begin{aligned} X_i &= \mathcal{P}_{\sigma}^S(W_i). \\ \text{a) } X(\sigma, \delta \mathfrak{a} \mathfrak{v}) &= \bigoplus_{i=1}^r X_i. \end{aligned}$$

b) If Y is a (σ, K) module of finite length, then

$$\text{Hom}_{\mathcal{G}, K}(Y, X_i) \cong \text{Hom}_{\ell, L \cap K}(H^R(\mathcal{U}, Y), W_i).$$

- c) Let $R_0^1(\nu) \subseteq \hat{R}_0$ be defined by (4.4.9) (for the group L). If we make $\hat{R}_0$ act on $A(\sigma, \delta)$ as in Definition 6.5.5, then the set of lambda-lowest K -types of X_i is a single orbit of $R_0^1(\nu)$. These occur with multiplicity one in X_i , and not at all in X_j for $i \neq j$.
- d) Each X_i has a unique irreducible submodule, which we write as $\bar{X}_i$; its lowest K -types are those of X_i .
- e) Let $\bar{W}_i$ denote the Langlands subrepresentation of W_i (Definition 4.2.10). Then $\bar{W}_i$ is a sub-quotient of $H^R(\mathcal{U}, \bar{X}_i)$.

Proof. Obviously $\mathcal{R}^i$ commutes with finite direct sums, so (a) is obvious; and also

$$\mathcal{R}^j(W_i) = 0 \quad (j \neq i).$$

So the proof of Theorem 6.5.9 gives analogues of (6.5.9) (e) - (g), with X_i replacing $X_L(\delta \mathfrak{a} \mathfrak{v})$ everywhere. The analogue of (6.5.9) (g) is (b). The analogue of (6.5.9) (e), together with Theorem 4.4.10, gives (c) (if we keep in mind (6.5.9) (b)).

Now let M be any non-zero submodule of X_i . By (b),

$$\text{Hom}_{\ell, L \cap K}(H^R(\mathcal{U}, M), W_i) \neq 0.$$

By Theorem 4.2.12, $\bar{W}_i$ is a subquotient of $H^R(\mathcal{U}, M)$. So if $\pi^L \in A_L(\delta)$, then π^L occurs in $H^R(\mathcal{U}, M)$ (by Proposition 4.4.1). Let π^G be the corresponding K -type in $A(\sigma, \delta)$ (Definition 6.5.5). By Lemma 6.5.4, σ is a parabolic associated to π^G ; and by (c), no K -types π' with

$$||\pi'|||_{\text{lambda}} < ||\pi^G|||_{\text{lambda}}$$

(Definition 5.4.1) occur in M. By Corollary 5.4.7(c) (or rather its proof, since the statement excludes the possibility that π^G does not occur in M) it follows that π^G occurs in M. So M contains all the lambda-lowest K-types of X_i . Since M was arbitrary, it follows that the submodule generated by the lowest K-types of $\bar{X}_i$ is the unique irreducible submodule of X_i . We proved something stronger than (e) above. Q.E.D.

Definition 6.5.11 Let $(\mathcal{G}, H, \delta, \nu)$ be a set of θ -stable data for G. The Langlands subrepresentations of $X(\mathcal{G}, \delta\mathfrak{O}\nu)$ are the irreducible representations $\bar{X}_i$ defined by Theorem 6.5.10(d); that is, they are the submodules generated by the individual lambda-lowest K-types of $X(\mathcal{G}, \delta\mathfrak{O}\nu)$. Define

$$\bar{X}(\mathcal{G}, \delta\mathfrak{O}\nu) = \sum_{i=1}^r \bar{X}_i.$$

If $\pi \in A(\mathcal{G}, \delta)$, set

$$\begin{aligned} X(\mathcal{G}, \delta\mathfrak{O}\nu)(\pi) &= \text{unique } X_i \text{ containing } \pi \\ \bar{X}(\mathcal{G}, \delta\mathfrak{O}\nu)(\pi) &= \text{corresponding } \bar{X}_i. \end{aligned}$$

We can therefore write, for example,

$$X(\mathcal{G}, \delta\mathfrak{O}\nu) = \sum_{\pi \in A(\mathcal{G}, \delta)/R_G^1(\nu)} X(\mathcal{G}, \delta\mathfrak{O}\nu)(\pi).$$

Theorem 6.5.12 Suppose X is an irreducible $(\mathcal{G}, K)$ module, and $(\mathcal{G}, H, \delta, \nu)$ is a set of θ -stable data corresponding to X (cf. Definition 5.4.8 and Corollary 5.4.9); say $\bar{X}_i(\delta\mathfrak{O}\nu)(\pi^0)$ is

a canonical cohomology class of X, for some $\pi^0 \in A_L(\delta)$.

- a) $(\mathcal{G}, H, \delta, \nu)$ is a set of θ -stable data for G.
 b) Let $\pi \in A(\mathcal{G}, \delta)$ be the K-type corresponding to π^0 (Definition 6.5.5); this is just the lambda-lowest K-type of X used to construct $(\mathcal{G}, H, \delta, \nu)$. Then (notation 6.5.11)

$$X \cong \bar{X}(\mathcal{G}, \delta\mathfrak{O}\nu)(\pi).$$

- c) Suppose $(\mathcal{G}', H', \delta', \nu')$ is another set of θ -stable data for G, $\pi' \in A(\mathcal{G}', \delta')$, and

$$X \cong \bar{X}(\mathcal{G}', \delta'\mathfrak{O}\nu')(\pi').$$

Then $(\mathcal{G}', H', \delta', \nu')$ is conjugate to $(\mathcal{G}, H, \delta, \nu)$ under K, and π' lies in the $R_G^1(\nu)$ orbit of π in $A(\mathcal{G}, \delta)$.

Proof. For (a), all of the conditions of (6.5.1) except (e) are fulfilled by definition; and (6.5.1)(e) follows from Corollary 5.4.10(b). For (b), we know by definition that $\bar{X}_i(\delta\mathfrak{O}\nu)(\pi^0)$ occurs in $H^R(\mathcal{U}, X)$. By Corollary 6.4.13,

$$\text{Hom}_{\mathcal{L}, L \cap K}(H^R(\mathcal{U}, X), X_L(\delta\mathfrak{O}\nu)) \neq 0.$$

By Theorem 6.5.9 (g), X is a submodule of $X(\mathcal{G}, \delta\mathfrak{O}\nu)$. Now (b) follows from Theorem 6.5.10.

For (c), we know that π' is a lowest K-type of X, and therefore of both of the Langlands representations which are equal to X. By Lemma 6.5.4, both $(\mathcal{G}, \lambda^G)$ and $(\mathcal{G}', \lambda^G)$ are associated to π' ; since this association depends only on

choosing a Cartan subalgebra of $\mathfrak{k}_0$, a positive root system, and a highest weight, it follows that $(\mathfrak{g}, \lambda^G)$ and $(\mathfrak{g}', \lambda'^G)$ are conjugate under K . So we may as well assume $\mathfrak{g} = \mathfrak{g}'$, $\lambda^G = \lambda'^G$. H and H' are maximally split Cartan subgroups of L ; so they are conjugate by $L \cap K$, and we may as well assume $H = H'$. Both δ and δ' occur in the representation $(\pi')^L$ of $L \cap K$ generated by a certain K -highest weight space in π' [after twisting by $(\wedge^{\dim \mathfrak{p}} \mathfrak{p})^*$]; so by Theorem 4.3.16 (b), they are conjugate under the normalizer of A in $L \cap K$. So we may as well assume $\delta = \delta'$. The unique irreducible subquotient of $H^R(\mathcal{U}, X)$ containing the $L \cap K$ -type $(\pi')^L$ (which has multiplicity one by Corollary 5.4.7(c)) is both

$$X_L(\delta \otimes \nu) \left((\pi')^L \right), \quad X_L(\delta \otimes \nu') \left((\pi')^L \right)$$

by Theorem 6.5.10(e). By Theorem 4.4.8, ν and ν' are conjugate by the stabilizer of δ in the Weyl group of H in L , proving the conjugacy statement. The last assertion is Theorem 6.5.10(c) and (d). Q.E.D.

This theorem constitutes a complete classification of the irreducible $(\mathfrak{g}, K)$ modules. Since $\hat{R}_\delta$ acts simply transitively on $A(\mathfrak{g}, \delta)$ (Theorem 4.3.16(c) and Definition 6.5.5), the number of orbits of $R_\delta^1(\nu)$ is

$$|\hat{R}_\delta / R_\delta^1(\nu)| = |R_\delta(\nu)^\wedge| = |R_\delta(\nu)|.$$

The next two results are now fairly clear.

Corollary 6.5.13 There is a surjective map from $\hat{G}$ (the set of equivalence classes of irreducible

$(\mathfrak{g}, K)$ modules) to K conjugacy classes of sets of θ -stable data for G (Definition 6.5.1). The fiber over $(\mathfrak{g}, H, \delta, \nu)$ has cardinality $|R_\delta(\nu)|$ (cf. (4.4.9)); more precisely, there is a natural simply transitive action of $R_\delta(\nu)^\wedge$ on the fiber.

This is contained in Theorem 6.5.12.

Corollary 6.5.14 Suppose $(\mathfrak{g}, H, \delta, \nu)$ is a set of θ -stable data for G , and $\nu \in \hat{A}$ is unitary.

Then $X(\mathfrak{g}, \delta \otimes \nu)$ is a direct sum of $|R_\delta(\nu)|$ inequivalent irreducible components:

$$X(\mathfrak{g}, \delta \otimes \nu) = \bigoplus_{\pi \in A(\mathfrak{g}, \delta) / R_\delta^1(\nu)} \bar{X}(\mathfrak{g}, \delta \otimes \nu)(\pi).$$

This combines Theorem 6.5.9(c) and Theorem 6.5.10, using the notation (6.5.11).

Corollary 6.5.15 Suppose X is an irreducible $(\mathfrak{g}, K)$ module. Then the lambda-lowest K -types of X (cf. (5.4.1)) occur with multiplicity one in X . The pair $(\mathfrak{g}, Y)$, consisting of a parabolic and a canonical cohomology class in $H^*(\mathcal{U}, X)$ (Definition 5.4.8) is unique up to conjugacy by K ; and given $\mathfrak{g}$, Y is unique, and has multiplicity one in $H^*(\mathcal{U}, Y)$.

This essentially restates part of Theorem 6.5.12.

The most serious defect of this classification is that it fails to identify the unitary representations. It can be shown that Harish-Chandra's tempered representations

(temperedness is a strong version of unitarity) correspond exactly to unitary characters ν . If ν is unitary, $X_L(\delta\otimes\nu)$ is a unitary representation of L ; so this suggests

Conjecture 6.5.16 Suppose $(\mathfrak{g}_{\mathbb{H}}, \delta, \nu)$ is a set of θ -stable data for $\mathfrak{g}_{\mathbb{H}}$ and $\bar{X}_L(\delta\otimes\nu)(\pi^L)$ is unitary (some $\pi^L \in A_L(\delta)$). Let π^G be the corresponding representation of K (Definition 6.5.5). Then $\bar{X}(\mathfrak{g}_{\mathbb{H}}, \delta\otimes\nu)(\pi^G)$ is unitary.

(We call X unitary if it is the Harish-Chandra module of a unitary representation of G .) The converse is false (for example, for $\mathrm{Sp}(2, \mathbb{C})$). Another possibility along these lines is this:

Conjecture 6.5.17 Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{H}}$ and $\mathfrak{h} \subseteq \mathfrak{l}$ a Cartan subalgebra. Let W be an irreducible $(\mathfrak{l}, L\cap K)$ module of infinitesimal character $\xi \in \mathfrak{h}^*$. Assume that for all $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$,

$$\mathrm{Re}\langle \alpha, \xi + \rho(\mathfrak{u}) \rangle \geq 0.$$

Then (if $S = \dim \mathfrak{u} \cap \mathfrak{h}$), $\rho_{\mathfrak{g}}^S W$ is unitary whenever W is. If $\rho_{\mathfrak{g}}^S W \neq 0$, then the converse should also be true. (It can be shown that, under the hypothesis above, $\rho_{\mathfrak{g}}^{i,j} W$ is zero for $i \neq S$, and that $\rho_{\mathfrak{g}}^S W$ is irreducible or zero).

Neither direction of this conjecture is known in general; but it can be shown that (under the hypotheses of the conjecture),

if $\rho_{\mathfrak{g}}^S W$ is non-zero, then it is tempered if and only if W is. Even when W is one-dimensional, this conjecture would provide interesting new families of representations, as was pointed out by Zuckerman.

Another motivation for these conjectures is Langlands' functoriality conjecture (describing how cusp forms for different arithmetic groups should be related). From this point of view, one can convince oneself that there should be three ways to construct unitary representations from those of smaller groups: (real) parabolic induction; Conjecture 6.5.17; and a way to go from $\mathrm{SO}(2n)$ to $\mathrm{Sp}(n)$, etc. (Here "etcetera" means going from H to G , where the Dynkin diagram of coroots of H is obtained from the extended Dynkin diagram of coroots of G by dropping one vertex.) This elevates Conjecture 6.5.17 from trick to method; it is particularly encouraging that the unitary representation of $\mathrm{Sp}(2, \mathbb{C})$ not accounted for by (6.5.16) is a piece of the metaplectic representation, and hence presumably to be regarded as an example of the mysterious Third Way of constructing representations.

§6. Other parametrizations of $\hat{G}$, and the Langlands classification.

Definition 6.6.1 Let $H = TA$ be a θ -stable Cartan subgroup of G [with $T = H \cap K$, and $A = \exp(\mathfrak{h}_0 \cap \mathfrak{p}_0)$]. Let $MA = G^A$ be the Langlands decomposition of the centralizer of A in G . (Thus the Lie algebra $\mathfrak{m}_0$ of M is the orthogonal complement of $\mathfrak{a}_0$ in $\mathfrak{g}_0$, and

$$M = M_0 \cdot (G^A \cap K)$$

see [50], p. 81.) An M-regular pseudocharacter of H (or, briefly, a regular character) is an ordered pair $\gamma = (\Gamma, \bar{\gamma})$, with Γ an ordinary character of H , and $\bar{\gamma} \in \mathcal{H}^*$, satisfying

- a) if $\alpha \in \Delta(m, \mathcal{T})$, then $\langle \alpha, \bar{\gamma} \rangle$ is real and non-zero.

Define

$$\begin{aligned} \Delta^+(m, \mathcal{T}) &= \{\alpha \in \Delta(m, \mathcal{T}) \mid \langle \alpha, \bar{\gamma} \rangle > 0\} \\ \rho_m &= \rho(\Delta^+(m, \mathcal{T})) \in (\mathcal{T})^* \subseteq \mathcal{H}^* \\ \rho_{m \cap \mathcal{K}} &= \rho(\Delta^+(m \cap \mathcal{K}, \mathcal{T})). \end{aligned}$$

- b) The differential of Γ is

$$d\Gamma = \bar{\gamma} + \rho_m - 2\rho_{m \cap \mathcal{K}}$$

We write $\hat{H}'$ for the set of regular characters of H (which depends on G and not just on H). A set of character data for G is a pair (H, γ) with H a θ -stable Cartan subgroup of G , and $\gamma \in \hat{H}'$.

Proposition 6.6.2 There is a bijection between sets of θ -stable data for G (Definition 6.5.1)

and sets of character data for G , which preserves conjugacy under K . If $(\mathcal{G}, H, \delta, \nu)$ is a set of θ -stable data, then the corresponding set of character data is (H, γ) , where (using notation from (6.5.1)) if $\gamma = (\Gamma, \bar{\gamma})$, then

$$\begin{aligned} \text{a)} \quad \bar{\gamma} &= (\lambda^G, \nu) \in \mathcal{H}^* \\ \text{b)} \quad \Gamma|_A &= \nu \\ \text{c)} \quad \Gamma|_{\Gamma} &= \delta \otimes (\wedge^{\dim u \cap f(u \cap \mathcal{P})})|_{\Gamma}. \end{aligned}$$

Proof. Fix a set of θ -stable data $(\mathcal{G}, H, \delta, \nu)$, and define by (a) - (c). We want to check that $\gamma \in \hat{H}'$. To do this, it is convenient to choose a Cartan subalgebra $\mathcal{T}^C \subseteq \mathfrak{g} \cap \mathcal{K}$ and a positive system $\Delta^+(\mathcal{K}, \mathcal{T})$ as in Lemma 6.5.4, and to fix μ^G as in that lemma. Replacing H by a conjugate under K , we may assume that $\mathcal{K}$ is the Cartan subalgebra $\mathcal{H}^C$ constructed in the proof of Lemma 5.3.23; for we showed there that $\mathcal{H}_0^C$ is maximally split in $\mathfrak{g}_0$. Since the differential of δ is $\lambda^G - \rho(u)$ by (6.5.1), we compute (inside $\mathcal{T}^*$) that

$$\begin{aligned} d\Gamma &= d(\delta \otimes (\wedge^{\dim(u \cap \mathcal{P})} u \cap \mathcal{P})) \\ &= \lambda^G - \rho(u) + 2\rho(u \cap \mathcal{P}) \\ (*) \quad &= \lambda^G + (\rho - 2\rho_C) - (\rho(\mathcal{K}) - 2\rho(\mathfrak{g} \cap \mathcal{K})). \end{aligned}$$

By (5.3.23), the roots of $\mathcal{H} (= \mathcal{H}^C)$ in $\mathfrak{m} + \mathfrak{a}$ may be identified with the imaginary roots of $\mathcal{T}^C$ in $\mathfrak{g}$ orthogonal to all the β_i . If α is such a root, then by Proposition 5.3.3,

$$\langle \alpha, \lambda^G \rangle > 0 \iff \alpha \in \Delta^+.$$

So condition (a) of (6.6.1) is satisfied; and there is a natural bijection

$$\Delta^+(m, \mathcal{T}) \leftrightarrow \{\alpha \in \Delta^+ \mid \alpha \text{ is imaginary, and } \langle \alpha, \beta_i \rangle = 0 \text{ for all } i\},$$

given by restriction to $\mathcal{T}$. Applying Lemma 5.3.29 to G , we get

$$\rho - 2\rho_C|_{\mathcal{T}} = \rho_m - 2\rho_{m \cap \mathcal{K}}.$$

Since $\mathfrak{g}$ is quasisplit, $m \cap \mathfrak{g}_0 = \mathcal{T}$; so by Lemma 5.3.29 applied to L ,

$$\rho(\ell) - 2\rho(\ell \cap k)|_T = 0.$$

Now (*) can be written as

$$d\Gamma = \lambda^G + \rho_{\mathcal{M}} - 2\rho_{\mathcal{M} \cap k},$$

verifying 6.6.1(b).

So we have a well-defined map $(\sigma_{\mathcal{H}, \delta, \nu}) \rightarrow (H, \gamma)$. We claim that it is injective. Clearly it is enough to show how to recover $\sigma_{\mathcal{H}}$ from (H, γ) . Given H and γ , $\sigma_{\mathcal{H}}$ is that subalgebra containing k such that

$$\Delta(\sigma_{\mathcal{H}}, k) = \{\alpha \in \Delta(\sigma_{\mathcal{H}}, k) \mid \langle \alpha, \overline{\gamma} |_T \rangle > \geq 0\}.$$

So the correspondence is injective. To prove that it is surjective, we construct the inverse: given (H, γ) , define $\sigma_{\mathcal{H}}$ as above, and put

$$\begin{aligned} \nu &= \overline{\gamma} |_T \sigma_{\mathcal{H}} \\ \delta &= (\Gamma |_T) \otimes (\wedge^{\dim \mathcal{U} \cap \mathcal{P} \cap \mathcal{P}})^* |_T. \end{aligned}$$

The argument given for the first part of the proof can be reversed to show that $(\sigma_{\mathcal{H}, \delta, \nu})$ is a set of θ -stable data; we leave this to the reader. Obviously the map thus defined is an inverse for the one defined by 6.6.2(a) - (c), so this proves bijectivity. That the maps preserve conjugacy by K is obvious from the naturality of the definitions. Q.E.D.

Notational Convention 6.6.3 Any notation based on a set of θ -stable data will also be applied to character data, using the bijection (6.6.2). For example, suppose $(\sigma_{\mathcal{H}, \delta, \nu})$ is a set of θ -stable data, and (H, γ) is the corresponding set of character

data. Then the standard representation with parameters (H, γ) (cf. (6.5.2)) is

$$X_G(H, \gamma) = X(H, \gamma) = X(\gamma) = X_G(\sigma_{\mathcal{H}, \delta, \nu}).$$

Its set of lowest K -types $A(\sigma_{\mathcal{H}, \delta})$ (cf. (6.5.5)) is written $A(\gamma)$; and by definition

$$R(\gamma) = R_{\mathcal{O}}(\nu), \text{ the } R\text{-group for } \gamma \text{ (cf. (4.4.9))}$$

$$R^1(\gamma) = R_{\mathcal{O}}^1(\nu)$$

$$\overline{X}(\gamma)(\pi) = \overline{X}(\sigma_{\mathcal{H}, \delta \otimes \nu})(\pi) \quad (\pi \in A(\gamma) \text{ -- cf. (6.5.11)}).$$

and so forth. This will also apply to future notation.

Definition 6.6.4 Suppose (H, γ) is a set of character data for G . Write

$$\overline{\gamma} = (\lambda, \nu) \in \mathcal{X}^* + \sigma^*.$$

The lambda-norm of γ is

$$||\gamma||_{\text{lambda}} = \langle \lambda, \lambda \rangle.$$

The standard character with parameter γ , $\theta(\gamma)$ is the character of $X(\gamma)$. If $\pi \in A(\gamma)$, set

$$\theta(\gamma)(\pi) = \theta(X(\gamma)(\pi))$$

$$\overline{\theta}(\gamma)(\pi) = \theta(\overline{X}(\gamma)(\pi))$$

$$\overline{\theta}(\gamma) = \theta(\overline{X}(\gamma))$$

$$= \sum_{\pi \in A(\gamma)} \overline{\theta}(\gamma)(\pi) / R^1(\gamma)$$

(notation (6.5.11)).

Lemma 6.6.5 Suppose (H, γ) is a set of character data for G , and $\pi \in A(\gamma)$ (notation (6.6.3)). Then (notation (5.4.1), (6.6.4))

$$\begin{aligned} ||X(\gamma)||_{\text{lambda}} &= ||\bar{X}(\gamma)(\pi)||_{\text{lambda}} = \\ &= ||\pi||_{\text{lambda}} = ||\gamma||_{\text{lambda}}. \end{aligned}$$

If π^1 is any K -type of $X(\gamma)$, then

$$||\pi^1||_{\text{lambda}} \geq ||\pi||_{\text{lambda}}.$$

Equality holds if and only if $\pi^1 \in A(\gamma)$.

Proof. This is a restatement of Lemma 6.5.4(b) and Theorem 6.5.9(b). Q.E.D.

The next lemma reformulates the preceding one in terms of the composition series of $X(\gamma)$. It was noticed by several people independently; the first complete proof was given by B. Speh.

Lemma 6.6.6 Suppose (H, γ) and (H', γ') are sets of character data for G (notation (6.6.1)), and $\pi' \in A(\gamma')$ (notation (6.6.3)). Write

$$\begin{aligned} \bar{\gamma} &= (\lambda, \nu) \in \mathcal{X}^* + \sigma^* \\ \bar{\gamma}' &= (\lambda', \nu') \in (\mathcal{X}')^* + (\sigma')^*. \end{aligned}$$

Assume that $\bar{X}(\gamma')(\pi)$ is a subquotient of $X(\gamma)$.

- a) $X(\gamma)$ has infinitesimal character $\bar{\gamma}$.
b) $\langle \lambda, \lambda \rangle = ||\gamma||_{\text{lambda}} \leq ||\gamma'||_{\text{lambda}} = \langle \lambda', \lambda' \rangle$
 $\langle \text{Re } \nu, \text{Re } \nu \rangle \geq \langle \text{Re } \nu', \text{Re } \nu' \rangle.$

- c) Equality holds in one (or, equivalently, in both) cases in (b) if and only if (H, γ) is conjugate to (H', γ') ; that is, if and only if $\bar{X}(\gamma')(\pi')$ is a Langlands subrepresentation of $X(\gamma)$.

Proof. Part (a) is Theorem 6.5.9(a). Now $X(\gamma)$ and $X(\gamma')$ must have the same infinitesimal character; so by Corollary 5.4.13,

$$\langle \lambda, \lambda \rangle + \langle \text{Re } \nu, \text{Re } \nu \rangle = \langle \lambda', \lambda' \rangle + \langle \text{Re } \nu', \text{Re } \nu' \rangle.$$

So the two inequalities in (b) are equivalent. The first one is an immediate consequence of Lemma 6.6.5. Part (c) follows from Lemma 6.6.5 and Theorem 6.5.12(c). Q.E.D

This lemma provides us with a very nice basis for the Grothendieck group $\mathcal{V}(\mathfrak{g}, K)$ (Definition 0.3.16).

Proposition 6.6.7 Suppose (H, γ) is a set of character data for G . Then there are formulas:

$$\begin{aligned} \text{a) } \quad \theta(\gamma) &= \sum_{\pi \in A(\gamma)/R^1(\gamma)} \bar{\theta}(\gamma)(\pi) + \\ &\quad + \sum_{||\gamma_i||_{\text{lambda}} > ||\gamma||_{\text{lambda}}} n_i \bar{\theta}(\gamma_i)(\pi_i). \end{aligned}$$

Here n_i is a positive integer, and the various $\bar{\theta}(\gamma_i)(\pi_i)$ are distinct and have the same infinitesimal character as $\theta(\gamma)$.

$$\text{b) } \quad \theta(\gamma)(\pi) = \bar{\theta}(\gamma)(\pi) + \sum_{||\gamma_i||_{\text{lambda}} > ||\gamma||_{\text{lambda}}} n_i' \bar{\theta}(\gamma_i)(\pi_i)$$

Here the γ_i and π_i are the same as in (a), and n_i' is a non-negative integer.

$$\text{c) } \quad \bar{\theta}(\gamma)(\pi) = \theta(\gamma)(\pi) + \sum_{||\gamma_j||_{\text{lambda}} > ||\gamma||_{\text{lambda}}} m_j \theta(\gamma_j)(\pi_j)$$

Here $m_j \in \mathbb{Z}$, and the various $\Theta(\gamma_j)(\pi_j)$ are distinct and have the same infinitesimal character as $\Theta(\gamma)$.

d) The various $\Theta(\gamma')(\pi')$, as γ' runs over K conjugacy classes of regular pseudocharacters and π' runs over $R^1(\gamma')$ orbits in $A(\gamma)$, form a basis of $\mathcal{V}(\mathfrak{g}, K)$, the lattice of virtual characters of G (notation (0.3.16)).

(The last statement is essentially due to Zuckerman and, in an equivalent analytic formulation, to Harish-Chandra.)

Proof. Statement (a) is just a reformulation of Lemma 6.6.4. Then (b) follows trivially. For (c), we consider only representations with a fixed infinitesimal character. There are only finitely many distinct conjugacy classes of γ to consider (by Corollary 5.4.17); so it makes sense to argue by downward induction on $||\gamma||$. The trivial proof (which amounts to inverting an upper triangular matrix with ones on the diagonal) is left to the reader. To prove (d), we need only show that the $\Theta(\gamma')(\pi')$ are linearly independent when they are distinct (because of (c)). Consider the sublattice of characters with a fixed infinitesimal character. Its rank is the number of distinct irreducible characters of that infinitesimal character. By Theorem 6.5.12, this is equal to the number of distinct $\Theta(\gamma')(\pi')$ with that infinitesimal character; so they must be independent. Q.E.D.

The rest of this section is devoted to an account of the Langlands classification and its relationship to the algebraic classification given in Section 5. This has important

consequences in representation theory, but we will not use it in formulating our conjectures for the characters of irreducible representations. For that reason we will not give proofs of the results.

Definition 6.6.8 An irreducible $(\mathfrak{g}, K)$ module X is said to belong to the discrete series if there is an irreducible unitary representation $(\pi, \mathcal{H})$ of G such that

- the Harish-Chandra module of π is isomorphic to X ; and
- if $v, w \in \mathcal{H}$, then the matrix coefficient

$$g \mapsto (\pi(g)v, w)$$

is a square-integrable function on G .

Theorem 6.6.9 (Harish-Chandra, Schmid [37]). Suppose (H, γ) is a set of character data for G (Definition 6.6.1), and $\pi \in A(\gamma)$ (notation (6.6.3)). Then $\bar{X}(\gamma)(\pi)$ belongs to the discrete series if and only if H is compact; and in this case $X(\gamma)$ is irreducible.

(The last assertion follows from Corollary 6.5.14). This theorem can be proved in a more conceptual way using only the ideas of this book (including Chapter 7) and the following fact [which is not too hard to prove. One needs the theory of Section 6.4 for non-quasisplit G , but that is an easy generalization.]

Lemma 6.6.10 (Hecht-Schmid [16]). Suppose X is an irreducible $(\mathfrak{g}, K)$ module, and $\tilde{X}$ is defined by (6.4.6), with respect to an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} + \sigma + \mathfrak{n}.$$

List the simple restricted roots of σ in $\mathfrak{n}$ as

$$\{\alpha_i \mid i = 1, \dots, \ell\}.$$

Then X belongs to the discrete series if and only if the $\{\alpha_i\}$ span σ , and every σ -weight of $\tilde{X}$ is of the form

$$\sum_{i=1}^{\ell} c_i \alpha_i,$$

with c_i having strictly negative real parts.

We leave this program as a (very hard) exercise for the ambitious reader.

Definition 6.6.11 Suppose $H = TA$ is a θ -stable Cartan subgroup of G , and $MA = G^A$ is the Langlands decomposition of the centralizer of A in G . A set of cuspidal data for G is a triple (MA, δ, ν) , with δ a discrete series representation of M , and $\nu \in \hat{A}$. If $P = MAN$ is any parabolic subgroup with Levi factor MA , we call $\pi(P, \delta \otimes \nu)$ (or its Harish-Chandra module $X(P, \delta \otimes \nu)$) the generalized principal series representation with parameters $(P, \delta \otimes \nu)$ (notation (4.1.11)).

Lemma 6.6.12 There is a surjective map from sets of character data for G to sets of cuspidal data

for G , defined as follows: if $\gamma \in \hat{H}'$ (notation (6.6.1)), with $H = TA$, we set

$$MA = G^A, \quad \nu = \Gamma|_A, \quad \delta = X_M(\gamma|_T).$$

This map is bijective on the level of K conjugacy classes.

This is more or less obvious from Theorem 6.6.9. The following definition extends (4.2.1).

Definition 6.6.13 Suppose (MA, δ, ν) is a set of cuspidal data. We define a parabolic subgroup

$P^1 = M^1 A N^1$, the parabolic defined by $-\text{Re } \nu$, by

- a) $MA \subseteq M^1 A^1$
 b) $\Delta(\nu^1, \sigma) = \{\alpha \in \Delta(\mathfrak{g}, \sigma) \mid \text{Re } \langle \alpha, \nu \rangle < 0\}.$

Fix any parabolic subgroup

$$\tilde{P} = M(A \cap M^1) \tilde{N} \subseteq M^1$$

with Levi factor $M(A \cap M^1)$. The tempered representation of M^1 associated to (MA, δ, ν) is

$$\delta^1 = \text{Ind}_P^{M^1} (\delta \otimes (\nu|_{A \cap M^1}) \otimes 1).$$

Notice that δ^1 is a unitary representation of M^1 , so it decomposes as a direct sum of irreducibles.

Theorem 6.6.14 (Langlands [32]). Suppose (MA, δ, ν) is a set of cuspidal data for G (Definition 6.6.10), $P^1 = M^1 A N^1$ is the parabolic defined by $-\text{Re } \nu$, and δ^1 is the associated tempered representation of M^1 .

Write

described in Theorem 6.6.13. Then

$$X(\delta \otimes v) \cong X(\gamma)$$

(notation (6.6.3)); that is, the representations obtained by real parabolic induction and θ -stable cohomological parabolic induction coincide. More precisely, the decompositions

$$X(\delta \otimes v) = \bigoplus_{i=1}^r X(\delta_i^1 \otimes v^1)$$

$$X(\gamma) = \bigoplus_{\pi \in A(\gamma)/R^1(\gamma)} X(\gamma)(\pi)$$

of (6.5.11) and Theorem 6.6.14 are the same; so

$$\{\bar{X}(\gamma)(\pi)\} = \{\bar{X}(\delta_i^1 \otimes v^1)\}.$$

We will prove a very special case of this in Section 8.4. Although that argument resorts to tricks to avoid using Theorem 6.6.14, it illustrates some of the ideas involved in the general case. Because we need them later, we include here two of the main lemmas from the proof of Theorem 6.6.15.

Lemma 6.6.16 Suppose $\gamma = (\Gamma, \bar{\gamma}) \in \hat{H}'$ is a regular character of $H = TA$ (Definition 6.6.1).

a) Suppose $\pi \in A(\gamma)$ (notation (6.6.3)). Then

$\Gamma|_{\mathbb{T}}$ occurs exactly once in $\pi|_{\mathbb{T}^*}$.

b) Suppose $\pi \in \hat{K}$, and $\Gamma|_{\mathbb{T}}$ occurs in $\pi|_{\mathbb{T}^*}$. Then

$$||\pi||_{\text{lambda}} \geq \langle \bar{\gamma} | \chi, \bar{\gamma} | \chi \rangle$$

(notation (5.5.1)). Equality holds if and only if $\pi \in A(\gamma)$.

$$v^1 = v|_{A^1},$$

and let

$$\delta^1 = \delta_1^1 \oplus \dots \oplus \delta_r^1$$

be the decomposition of δ_1^1 into irreducibles. Put

$$\pi(\delta_i^1 \otimes v^1) = \text{Ind}_{P1}^G \delta_i^1 \otimes v^1 \otimes 1.$$

$$\pi(\delta \otimes v) = \text{Ind}_{P1}^G \delta^1 \otimes v^1 \otimes 1.$$

$$= \text{Ind}_P^G \delta \otimes v \otimes 1;$$

here $P = MA(\tilde{N}N^1)$, with $\tilde{N}$ as in (6.6.12).

a) $\pi(\delta_i^1 \otimes v^1)$ has a unique irreducible subrepresentation $\bar{\pi}(\delta_i^1 \otimes v^1)$, which we call a Langlands subrepresentation of $\pi(\delta \otimes v)$.

b) If $(M'A', \delta', v')$ is another set of cuspidal data for G , and $\bar{\pi}(\delta_i^1 \otimes v^1)$ is a Langlands subrepresentation of $\pi(\delta' \otimes v')$, then $(M'A', \delta', v')$ is conjugate under K to (MA, δ, v) .

(This formulation of (a) is due to Milićić.) The proof of this theorem given by Langlands is in the spirit of the special case we discussed in Section 4.2. Probably it can also be proved more or less algebraically, using the ideas of Section 6.4; the reader who has carried out the project suggested after Theorem 6.6.9 will have no difficulty with this.

Theorem 6.6.15 Let (H, γ) be a set of character data for G , and (MA, δ, v) the corresponding set of cuspidal data (Lemma 6.6.11). Write $X(\delta \otimes v)$, etc., for the Harish-Chandra modules of the various representations

Proof. Let $(\sigma_f, \delta_{\otimes v})$ be the θ -stable data associated to (H, γ) (Proposition 6.6.2). Choose a Cartan subgroup $T^C \supseteq T_0$ of $L \cap K$ (here $\sigma_f = \ell + u$) and a positive root system for $\mathcal{L}^C$ in $\ell \cap \mathcal{L}$. Put

$$\Delta^+(\mathcal{L}, \mathcal{L}^C) = \Delta^+(\ell \cap \mathcal{L}, \mathcal{L}^C) \cup \Delta(u \cap \mathcal{L}, \mathcal{L}^C).$$

Fix π as in (b), and choose a representation π^0 of $L \cap K$ such that

$$(6.6.17) \quad (a) \quad \pi^0 \subseteq \pi|_{L \cap K}$$

$$\Gamma|_T \subseteq \pi^0|_T.$$

Fix a highest weight $\mu^0 \in T^C$ of π^0 . By Proposition 0.4.3(g), we can find a highest weight μ of π , and non-negative integers n_α , so that

$$(6.6.17) \quad (b) \quad \mu = \mu^0 + \sum_{\alpha \in \Delta^+(\mathcal{L}, \mathcal{L}^C)} n_\alpha \alpha.$$

Define

$$\pi^1 = \pi^0 \otimes (\wedge^R u \cap \mathfrak{p})^*,$$

which has highest weight

$$\mu^1 = \mu^0 - 2\rho(u \cap \mathfrak{p}).$$

By (6.6.17) (a) and (6.6.2),

$$\pi^1|_T \supseteq \delta.$$

By (6.6.17) (b),

$$\mu = \mu^1 + 2\rho(u \cap \mathfrak{p}) + \sum_{\alpha \in \Delta^+(\mathcal{L}, \mathcal{L}^C)} n_\alpha \alpha.$$

In the proof of Lemma 6.5.6, we used exactly this kind of expression to deduce that

$$||\pi||_{\text{lambda}} \geq \langle \bar{\gamma} | \mathcal{L}, \bar{\gamma} | \mathcal{L} \rangle,$$

with equality if and only if

$$\sum n_\alpha = 0, \pi^1 \in A_L(\delta).$$

By Definition 6.5.5, therefore, these conditions for equality amount to $\pi \in A(\gamma)$. This proves (b). This calculation also reduces (a) to the corresponding statement for L , which is Theorem 4.3.16(b). Q.E.D.

Lemma 6.6.18 Suppose $\gamma \in \hat{H}^1$ (notation (6.6.1)); write $(\sigma_f, \delta_{\otimes v})$ for the associated set of θ -stable data -- cf. (6.6.2). Assume that γ has the following properties (a) and (b).

a) If $w \in W(\sigma_f, \mathcal{L})$, and

$$(w\bar{\gamma})|_{\mathcal{L}} = \bar{\gamma}|_{\mathcal{L}},$$

then $w \in W(L, H)$.

b) Let HN be a Borel subgroup of L , and $\Delta^+(\ell, \mathcal{L})$ the corresponding positive root system for $\mathcal{L}$ in ℓ .

For each simple real root $\alpha \in \Delta^+(\ell, \mathcal{L})$ define m_α

by (4.3.6). Then

$$\delta(m_\alpha) = 1.$$

Now let γ be any (σ_f, K) module of finite length having the same infinitesimal character as $X(\gamma)$, and having a lambda-lowest K -type in $A(\gamma)$

(notation (6.6.3)). Then

$$\text{Hom}_{\mathfrak{g},K}^{(\gamma, X(\gamma))} \neq 0.$$

In particular, Y contains a Langlands subrepresentation of $X(Y)$ as a composition factor.

Proof. Recall the root system $\Delta_\delta \subseteq \Delta(\mathfrak{k}, \mathfrak{h})$ of Definition 4.3.11. By hypothesis (b), Δ_δ contains every simple root of $\mathfrak{h}$ in $\mathfrak{l}$; so since it is a root system, $\Delta_\delta = \Delta(\mathfrak{k}, \mathfrak{h})$. By Lemma 4.3.12, $W(L, H)$ stabilizes δ in its action on $\hat{T}$.

Now let $\pi \in A(\gamma)$ be a lambda-lowest K -type of Y , and $\pi^L \in A_L(\delta)$ the corresponding $L \cap K$ -type (Definition 6.5.5). By Corollary 5.4.9, there is a $\nu' \in \hat{A}$ so that $\bar{X}_L(\delta \otimes \nu')(\pi^L)$ occurs in $H^R(\mathcal{U}, Y)$ as a composition factor. By Corollary 5.4.10, Y has infinitesimal character

$$(\lambda, \nu') \in \mathfrak{h}^*;$$

here

$$\lambda = \bar{\gamma} | \mathfrak{x}.$$

By hypothesis, this is the same as the infinitesimal character of $X(\gamma)$; so for some $w \in W(\mathfrak{g}, \mathfrak{h})$,

$$w(\lambda, \nu') = (\lambda, \nu).$$

By hypothesis (a), w belongs to $W(L, H)$. So by the first part of the proof, $w\delta = \delta$; and by Theorem 4.1.4:

$$\begin{aligned} \bar{X}_L(\delta \otimes \nu')(\pi^L) &\cong \bar{X}_L(w\delta \otimes w\nu')(\pi^L) \\ &\cong \bar{X}_L(\delta \otimes \nu)(\pi^L). \end{aligned}$$

By Corollary 6.4.13,

$$\text{Hom}_{\mathfrak{l}, L \cap K}^R(\mathcal{U}, Y), \quad X_L(\delta \otimes \nu) \neq 0.$$

By Theorem 6.5.9 (g), this is equivalent to the conclusion of the lemma. Q.E.D.

Theorem 6.6.15 can be interpreted as a surprisingly strong "independence of polarization" result; using Proposition 6.3.6, both $X(\gamma)$ and $X(\delta \otimes \nu)$ can be interpreted as cohomologically induced modules from two different Borel subalgebras over the same Cartan subalgebra. It would be very nice to have a more general theorem along these lines: presumably one should consider things like

$$\sum_i (-1)^i \mathcal{H}_i^1 W \in \mathcal{U}(\mathfrak{g}, K)$$

and show that, under some conditions, this depends only on the Levi factor of $\mathfrak{g}$ and on W . (Actually, W will be twisted slightly as $\mathfrak{g}$ varies.) One does need some hypotheses; for example, in the case of Borel subalgebras, one cannot change the positive imaginary roots arbitrarily (as is clear even for $SL(2, \mathbb{R})$). We will prove another independence of polarization result in Chapter 8 (Theorem 8.2.4).

§7. Proof of Theorem 2.2.4

Recall that Theorem 2.2.4 was a parametrization of the set of irreducible $(\mathfrak{g}, K)$ modules having the same infinitesimal character as the trivial representation. We know such a parametrization from Theorem 6.5.12; clearly we only need to match up the parameters.

Lemma 6.7.1 Let $H = \tau A$ be a θ -stable Cartan subgroup of G , and Δ^+ a positive root system for $\mathfrak{h}$ in $\mathfrak{g}$. Choose a set B of positive complex roots such that

- a) $B \cup \theta B = \{\alpha \in \Delta^+ \mid \alpha \text{ is complex, and } \theta\alpha \in \Delta^+\}$.
 b) $B \cap \theta B = \emptyset$.

Then

$$[\rho(\Delta^+) - 2\rho(B)]|_{\mathfrak{t}} = \rho(\Delta^+ \cap \Delta(m, \mathfrak{t}));$$

here $m + \mathfrak{a}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Furthermore the sum of roots

$$2\rho(B)|_{\mathfrak{T}} \in \hat{\Gamma}$$

is independent of the choice of B (subject to (a) and (b)).

Proof. Choose a set C of positive complex roots such that

$$\begin{aligned} C \cup (-\theta C) &= \{\alpha \in \Delta^+ \mid \alpha \text{ is complex, and } -\theta\alpha \in \Delta^+\} \\ C \cap (-\theta C) &= \emptyset. \end{aligned}$$

Then

$$\Delta^+ = \Delta^+(m, \mathfrak{t}) \cup (B \cup \theta B) \cup (C \cup (-\theta C)) \cup \{\text{positive real roots}\}.$$

So

$$\begin{aligned} \rho(\Delta^+)|_{\mathfrak{t}} &= \rho(\Delta^+(m, \mathfrak{t})) + \left[\frac{1}{2} \sum_{\alpha \in B} \alpha + \frac{1}{2} \sum_{\alpha \in B} \theta\alpha \right. \\ &\quad \left. + \frac{1}{2} \sum_{\alpha \in C} \alpha + \frac{1}{2} \sum_{\alpha \in C} -\theta\alpha \right] |_{\mathfrak{t}} \\ &= \rho(\Delta^+(m, \mathfrak{t})) + \left[\sum_{\alpha \in B} \alpha \right] |_{\mathfrak{t}} \\ &= \rho(\Delta^+(m, \mathfrak{t})) + 2\rho(B)|_{\mathfrak{t}}, \end{aligned}$$

since α and $\theta\alpha$ have the same restriction to $\mathfrak{t}$. This proves the first claim. For the second, notice that B is obtained by picking one root from each pair

$$\{\alpha, \theta\alpha\} \subseteq B \cup \theta B.$$

Since α and $\theta\alpha$ agree on $\mathfrak{T}$, the particular choice does not affect $2\rho(B)$. Q.E.D.

Proof of Theorem 2.2.4 Fix a set (H, γ) of character data for G (Definition 6.6.1) such that $\bar{X}(\gamma)$ has the same infinitesimal character as the trivial representation. We want to construct a corresponding set of $\mathbb{Z}/2\mathbb{Z}$ data (Definition 2.2.3); that is, a Cartan subgroup H' , a positive root system Δ^+ for $\mathfrak{h}'$ in $\mathfrak{g}'$, and a map

$$\chi: H' \rightarrow \{\pm 1\}.$$

Of course we take $H' = H$. To continue, recall that γ is an ordered pair $(\Gamma, \bar{\gamma})$, with $\Gamma \in \hat{H}$ and $\bar{\gamma} \in \mathfrak{h}^*$. Since $\bar{X}(\gamma)$ has the same infinitesimal character as the trivial representation, Lemma 6.6.4 guarantees that $\bar{\gamma}$ is half the sum of the positive roots in some positive system Δ^+ :

$$(6.7.2) \quad \bar{\gamma} = \rho(\Delta^+).$$

Choose $B \subseteq \Delta^+$ as in Lemma 6.7.1, and define $\chi \in \hat{H}$ by

$$(6.7.3) \quad \chi|_A = 1 \\ \chi|_{\mathfrak{T}} = \left[\Gamma|_{\mathfrak{T}} - 2\rho(\Delta^+(m \cap \mathfrak{t}, \mathfrak{t})) - 2\rho(B)|_{\mathfrak{T}} \right];$$

the $2\rho(\quad)$ terms are to be interpreted as sums of roots. By Lemma 6.7.1 this is well-defined; and its differential is

$$\begin{aligned}
\bar{\chi}|x &= \left[\bar{\gamma}|x + \rho(\Delta^+(m, x)) - 2\rho(\Delta^+(m \cap k, x)) \right] \\
&\quad - 2\rho(\Delta^+(m \cap p, x)) - 2\rho(B) \\
&= \rho(\Delta^+) \left[x - 2\rho(B) - \rho(\Delta^+(m, x)) \right] \\
&= 0
\end{aligned}$$

by (6.6.1) (b). Of course $\bar{\chi}|x$ is zero by (6.7.3). So

$$\chi: H/H_0 \rightarrow \Phi.$$

By Assumption (2.2.1), H/H_0 is a group of exponent 2; so actually

$$\chi: H \rightarrow \{\pm 1\},$$

as required. All of these steps can be reversed to recover (H, γ) from (H, Δ^+, χ) ; so this proves Theorem 2.2.4. Q.E.D.

Chapter 7. Coherent Continuation of Characters.

The rest of this book depends on an increasingly detailed analysis of tensor products of finite dimensional representations with $(\mathfrak{g}, K)$ modules of finite length, combined with the action of $Z(\mathfrak{g})$. These ideas seem to be intimately connected with the existence of differential operators on G . It is difficult to justify that claim easily; but the sort of phenomenon to be described is typified by the poles of the (real) gamma function. As one sees by integration by parts (in other words, by using a differential operator) these occur at the non-positive integers. The gamma functions of p -adic fields do not exhibit this periodicity, "because" there are no differential operators to cause it. Similarly, simple groups (real or p -adic) have only one obvious finite dimensional complex representation, namely, the trivial representation. For p -adic groups this is the only one, but for real groups there is a whole lattice (or rather a cone in a lattice) of others; many of them arise by the adjoint action of the group on differential operators. We will use periodicity as a technical tool in the form (7.3.3). The main ideas are discussed in more detail at the beginning of Section 3.

§1. Complex groups and ideals in $U(\mathfrak{g})$

In order to simplify the statements of various results, it is convenient to know that the strongly $Z(\mathfrak{g})$ -finite modules (Definition 4.5.1) are just the $Z(\mathfrak{g})$ -finite ones; as was pointed out after Definition 4.5.1, this is a result of Kostant. The simplest proof of it uses the fact that the

annihilator of a Verma module is generated as an ideal by its intersection with $Z(\mathfrak{g})$ ([7]). We will give here a different proof, using the machinery already at hand. This approach has the advantage of introducing the main idea in Duflo's beautiful paper [8].

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Since the map -1 on $\mathfrak{h}$ is obviously an automorphism of the root system, there is an automorphism σ of $\mathfrak{g}$ [unique up to an inner automorphism coming from $\exp(\mathfrak{h})$] such that

$$\sigma x = -x \quad (x \in \mathfrak{h}).$$

Define

$$(7.1.1) \quad (a) \quad \begin{cases} X = -X \\ t_X = -\sigma X \end{cases} \quad (X \in \mathfrak{g}).$$

(If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{h}$ consists of diagonal matrices, then t_X can be taken to be the ordinary transpose of X .) These maps, which are called "hat" and "transpose", are antiautomorphisms of $\mathfrak{g}$ of order 2. They extend uniquely to antiautomorphisms of $U(\mathfrak{g})$:

$$(7.1.1) \quad (b) \quad \begin{cases} (X_1 \dots X_n)^v = (-1)^n (X_n \dots X_1) \\ t(X_1 \dots X_n) = t_{X_n} \dots t_{X_1} \end{cases} \quad (X_i \in \mathfrak{g}).$$

Choose a connected compact group K^2 , with Lie algebra $\mathfrak{k}_0^2$, such that

$$(7.1.2) \quad (\mathfrak{k}_0^2)_{\mathbb{C}} = \mathfrak{k}^2 \cong \mathfrak{g}.$$

Now regard $\mathfrak{g}$ as a real Lie algebra, and write

$$(7.1.3) \quad \mathfrak{g}^2 = \left[\mathfrak{g} \middle| \mathbb{R} \right]_{\mathbb{C}}.$$

Write j for the action of $\sqrt{-1}$ on $\mathfrak{g}^2$ coming from the complexification, and i for the action coming from the original complex structure on $\mathfrak{g}$. Then $\mathfrak{g}^2$ is the direct sum of the two ideals

$$(7.1.4) \quad \begin{cases} \mathfrak{g}^L = \{ \frac{1}{2}(X + jX) \mid X \in \mathfrak{g} \} \\ \mathfrak{g}^R = \{ \frac{1}{2}(X - jX) \mid X \in \mathfrak{g} \}. \end{cases}$$

We will identify $\mathfrak{g}^L$ and $\mathfrak{g}^R$ with $\mathfrak{g}$, using the isomorphisms

$$(7.1.5) \quad \begin{cases} \phi^L: \mathfrak{g} \rightarrow \mathfrak{g}^L, \phi^L(X) = \frac{1}{2}(X - j(iX)) \\ \phi^R: \mathfrak{g} \rightarrow \mathfrak{g}^R, \phi^R(X) = \frac{1}{2}(-t_{\overline{X}} + j(i\overline{X})). \end{cases}$$

Here we have written bar for the complex conjugation on $\mathfrak{g}$ corresponding to the real form $\mathfrak{k}_0^2$. (The twist in ϕ^R is introduced in accordance with a common practice, to simplify subsequent notation.) Notice that ϕ^L and ϕ^R are both complex Lie algebra homomorphisms. So we can write

$$(7.1.6) \quad \begin{cases} \mathfrak{g}^2 = \mathfrak{g}^L \oplus \mathfrak{g}^R \cong \mathfrak{g} \oplus \mathfrak{g}. \\ U(\mathfrak{g}^2) \cong U(\mathfrak{g}^L) \otimes U(\mathfrak{g}^R) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g}). \end{cases}$$

We can identify the complexified Lie algebra $\mathfrak{k}^2$ of $\mathfrak{k}_0^2$ as follows. Define

$$(7.1.7) \quad \begin{aligned} \phi^D: \mathfrak{g} &\rightarrow \mathfrak{g}^2, \phi^D(X) = \phi^L(-t_X) + \phi^R(X) \\ &= -\frac{1}{2}(t_X + t_{\overline{X}}) - \frac{1}{2}j[(it_X + i\overline{t_X})] \end{aligned}$$

The reader can easily verify that ϕ^D is an isomorphism of $\mathfrak{g}$ onto $\mathfrak{k}^2$. Finally, define

$$(7.1.8) \quad \mathfrak{h}^2 = \phi^L(\mathfrak{h}) \oplus \phi^R(\mathfrak{h}).$$

It is easy to check that there is a complex reductive group G^2 , with maximal compact subgroup K^2 and complexified Lie algebra $\mathfrak{g}^2$.

Definition 7.1.9 Suppose M is a $U(\mathfrak{g})$ module.

Then

$$\text{Ann}(M) = \{u \in U(\mathfrak{g}) \mid \text{for all } m \in M, um = 0\},$$

the annihilator of M . If M is a $U(\mathfrak{g}^2)$ module, we can define

$$L \text{ Ann}(M) = \{u \in U(\mathfrak{g}) \mid \phi^L(u) \in \text{Ann}(M)\}$$

$$= \{u \in U(\mathfrak{g}) \mid u \otimes 1 \in \text{Ann}(M)\}$$

$$R \text{ Ann}(M) = \{u \in U(\mathfrak{g}) \mid 1 \otimes u \in \text{Ann}(M)\},$$

the left (respectively, right) annihilator of M ; here we make the identifications (7.1.6).

Proposition 7.1.10 Let I be a two-sided ideal in $U(\mathfrak{g})$. Make $U(\mathfrak{g})/I$ into a $U(\mathfrak{g}^2)$ module as follows: if

$a, b \in U(\mathfrak{g})$, $a \otimes b \in U(\mathfrak{g}^2)$ (cf. (7.1.6)), and $u \in U(\mathfrak{g})$, set

$$(a \otimes b) \cdot (u + I) = {}^{tv}_{\text{au}} b + I$$

(notation 7.1.1). Make it into a K^2 module by

$$k \cdot (u + I) = (\text{Ad}(k)u + I) \quad (k \in K^2).$$

a) $U(\mathfrak{g})/I$ is a $(\mathfrak{g}^2, K^2)$ module.

b) If $I \cap Z(\mathfrak{g})$ has finite codimension in $Z(\mathfrak{g})$, then $U(\mathfrak{g})/I$ has finite length.

c) If $I \cap Z(\mathfrak{g}) = \ker \xi_\lambda$ (notation (0.2.9)) then $U(\mathfrak{g})/I$ has infinitesimal character

$$(-\lambda, -\lambda) \in (-h^2)^*$$

$$\text{d) } L \text{ Ann } [U(\mathfrak{g})/I] = {}^t Y$$

$$R \text{ Ann } [U(\mathfrak{g})/I] = Y.$$

Proof. That the action of $U(\mathfrak{g}^2)$ is well-defined and makes $U(\mathfrak{g})/I$ a $U(\mathfrak{g}^2)$ module is very easy to check. If $X \in \mathfrak{k}_0^2$, then X corresponds to an element Y of $\mathfrak{g}$ such that

$$X = -\phi^L({}^t Y) + \phi^R(Y) = \phi^D(Y)$$

(cf. 7.1.7). So if $u \in U(\mathfrak{g})$,

$$X \cdot (u + I) = Yu - uY.$$

This is the differential of the action of K^2 , proving (a). For (b), notice that

$$\begin{aligned} Z(\mathfrak{g}^2) &= Z(\mathfrak{g}) \otimes Z(\mathfrak{g}) \\ (*) \text{ Ann}(U(\mathfrak{g})/I) \cap Z(\mathfrak{g}^2) &= Z(\mathfrak{g}) \otimes (I \cap Z(\mathfrak{g}))^\vee + \\ &\quad {}^t(I \cap Z(\mathfrak{g}))^\vee \otimes Z(\mathfrak{g}). \end{aligned}$$

So if $I \cap Z(\mathfrak{g})$ has finite codimension in $Z(\mathfrak{g})$, then

$$\text{Ann}(U(\mathfrak{g})/I) \cap Z(\mathfrak{g}^2)$$

has finite codimension in $Z(\mathfrak{g}^2)$. Since $U(\mathfrak{g})/I$ is always finitely generated (by $1 + I$), (b) follows from Corollary

5.4.16(b). For (c), (*) reduces us to showing that

$${}^t(\ker \xi_\lambda)^\vee = (\ker \xi_\lambda)^\vee = \ker \xi_{-\lambda}.$$

This is an immediate consequence of the definition of ξ_λ and its independence of the positive system used (Theorem 0.2.8); we leave the calculation to the reader. Finally, (d) is obvious. Q.E.D.

Lemma 7.1.11 Let

$$h: U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g})$$

be the unique algebra injection (the Hopf map) defined by

$$h(X) = X \otimes 1 + 1 \otimes X.$$

Suppose M_1 and M_2 are $U(\mathfrak{g})$ modules. Then h defines an injection of $(\mathfrak{g}^2, \kappa^2)$ modules (cf. Proposition 7.1.10)

$$U(\mathfrak{g})/\text{Ann}(M_1 \otimes M_2) \rightarrow [U(\mathfrak{g})/(\text{Ann } M_1)] \otimes [U(\mathfrak{g})/(\text{Ann } M_2)];$$

here the $\mathfrak{g}^2$ -module structure on the right is the tensor product of those on the two factors. In particular,

$$\text{Ann}(M_1 \otimes M_2) = h^{-1}(\text{Ann } M_1 \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \text{Ann } M_2).$$

Proof. Recall how $M_1 \otimes M_2$ is defined as a $U(\mathfrak{g})$ module: it is obviously a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ module, and we pull this structure back using h . The claims of the lemma are all straightforward verifications, which we leave to the reader. Q.E.D.

Lemma 7.1.12 Suppose X is a $(\mathfrak{g}, K)$ module of finite length, and F is a finite dimensional representation of G . Then $X \otimes F$ has finite length.

Proof. We can use any of the characterizations of finite length given in Corollary 5.4.16. For example, X is admissible and finitely generated, and we want to show that $X \otimes F$ is as well. Let Z be a finite dimensional representation of K ; we must show that

$$\text{Hom}_K(Z, X \otimes F)$$

is finite dimensional. But this space is naturally isomorphic to

$$\text{Hom}_K(Z \otimes F^*, X)$$

(Lemma 6.1.25, or compare the proof of Lemma 4.5.9). Since $Z \otimes F^*$ is finite dimensional and X is admissible, this latter space is finite dimensional. For the finite generation, let X_0 be a finite dimensional generating subspace for X ; then $X_0 \otimes F$ generates $X \otimes F$. Q.E.D.

Corollary 7.1.13 (Kostant [30]). Suppose M is a $Z(\mathfrak{g})$ -finite $\mathfrak{g}$ module. Then M is strongly $Z(\mathfrak{g})$ -finite: that is, if F is any finite dimensional $\mathfrak{g}$ -module, then $M \otimes F$ is $Z(\mathfrak{g})$ -finite. More precisely, if M has generalized infinitesimal character $\lambda \in \mathfrak{h}^*$, then every infinitesimal character occurring in h is of the form $\lambda + \gamma$, with γ a weight of h in F .

Proof. This is entirely a question about $\text{Ann}(M \otimes F)$; or, equivalently, (by Proposition 7.1.10(d)) about

$$R \text{Ann}(U(\mathfrak{g})/\text{Ann}(M \otimes F))^{\vee}.$$

So it is enough to show that the $\mathfrak{g}^2$ module

$$U(\mathfrak{g})/\text{Ann}(M \otimes F)$$

is $Z(\mathfrak{g}^2)$ finite, and that every infinitesimal character occurring is of the form

$$(-\lambda - \gamma_1, -\lambda - \gamma_2),$$

with γ_i weights of h in F . By Lemma 7.1.11, it is enough to prove the same thing about

$$[U(\mathfrak{g})/\text{Ann } M] \otimes [U(\mathfrak{g})/\text{Ann } F].$$

Clearly

$$U(\mathfrak{g})/\text{Ann } F \subseteq \text{End}_{\mathbb{Q}} F,$$

which is finite dimensional; and the weights of $\text{End } F$ as a $U(\mathfrak{g}^2)$ module are

$$(-\gamma_1, -\gamma_2),$$

with $\gamma_i \in \Delta(F)$; this follows from the isomorphism

$$\text{End } F \cong F \otimes F^*,$$

if one keeps track of all the twists. So we can apply Corollary 4.5.6 to the $(\mathfrak{g}^2, K^2)$ module $U(\mathfrak{g})/\text{Ann } M$; this is allowed by Proposition 7.1.10(b). Q.E.D.

§2. Translation functors revisited.

In Section 4.5, we used the simple behaviour of principal series under tensor products with finite dimensional representations to study more general representations. To refine that, we need to relate tensor products to the cohomological induction functors of Section 6.3. Since that involves several annoying technical problems, we will begin with the analogous but simpler problem of studying cohomology. The next lemma is really nothing but the proof of Proposition 3.2.12 given earlier.

Lemma 7.2.1 Suppose $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$, and X is a $Z(\mathfrak{g})$ -finite $\mathfrak{g}$ module. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{l}$, and a weight $\xi \in \mathfrak{h}^*$.

- a) $H^i(\mathfrak{u}, P_{\xi} X) = \bigoplus_{\sigma \in W(\mathfrak{g}, \mathfrak{h})} P_{\sigma\xi - \rho}(\mathfrak{u}) H^i(\mathfrak{u}, X)$ (notation (0.3.18));

the sum is over $W(\mathfrak{l}, \mathfrak{h})$ orbits of weights of the indicated form.

- b) Let F be a finite dimensional representation of G , and

$$0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = F$$

a $\mathfrak{g}$ -stable filtration, such that $\mathfrak{u}$ acts trivially on F_i/F_{i-1} . Then there is a natural spectral sequence

$$H^p(\mathfrak{u}, X) \otimes F_i/F_{i-1} \implies H^p(\mathfrak{u}, X \otimes F)$$

of $\mathfrak{l}$ modules. If we are in the setting of

Definition 6.3.1 and X is a $(\mathfrak{g}, L_0 K)$ module,

then there is a spectral sequence of $(\mathfrak{l}, L_0 K)$ modules.

Proof. Part (a) is Corollary 3.1.6. For (b), we compute the cohomology using the complex

$$\text{Hom}_{\mathbb{Q}}(\wedge^i \mathfrak{u}, X \otimes F)$$

of (3.1.1). Obviously the filtration of F induces a filtration of this complex, and the spectral sequence is the spectral sequence of the filtered complex. Q.E.D.

Lemma 7.2.2 Suppose $\mathfrak{h} \subseteq \mathfrak{g}$, $B \subseteq K$, and $(\mathfrak{h}, B)$ satisfies the hypotheses of Definition 6.1.1.

Let F be a finite dimensional $(\mathfrak{h}, B)$ module, and V an arbitrary $(\mathfrak{h}, B)$ module.

- a) If V is injective (Definition 6.1.4), so is $V \otimes F$. For the rest of the lemma, assume that F is a $(\mathfrak{g}, B)$ module.

b) $\text{pro } \mathcal{G}'_{\mathcal{K},B}(V) \otimes F \cong \text{pro } \mathcal{G}'_{\mathcal{K},B}(V \otimes F)$ in a natural way.

c) If X is any $(\mathcal{G}, B)$ module, and Γ^i is defined by Definition 6.2.11, then there is a natural isomorphism

$$\Gamma^i(X \otimes F) \cong (\Gamma^i X) \otimes F.$$

Proof. For (a), we must show that the functor

$$W \rightarrow \text{Hom}_{\mathcal{K},B}(W, V \otimes F)$$

is exact. By Lemma 6.1.25, this functor is equivalent to

$$W \rightarrow \text{Hom}_{\mathcal{K},B}(W \otimes F^*, V).$$

This is the composite of two exact functors (tensoring over $\mathbb{C}$ with F , and Hom into an injective), and is therefore exact.

For (b), let Y be any $(\mathcal{G}, B)$ module. Then

$$\text{Hom}_{\mathcal{G},B}(Y, \text{pro}(V) \otimes F) \cong \text{Hom}_{\mathcal{G},B}(Y \otimes F^*, \text{pro}(V))$$

by Lemma 6.1.25;

$$\cong \text{Hom}_{\mathcal{K},B}(\text{res}_{\mathcal{G},B}^{h,B}(Y \otimes F^*), V)$$

by (6.1.23);

$$\cong \text{Hom}_{\mathcal{K},B}((\text{res } Y) \otimes F^*, V)$$

$$\cong \text{Hom}_{\mathcal{K},B}(\text{res } Y, V \otimes F)$$

by Lemma 6.1.25;

$$\cong \text{Hom}_{\mathcal{G},B}(Y, \text{pro}(V \otimes F))$$

by (6.1.23). Since Y was arbitrary and the isomorphisms are all natural, a formal argument gives (b).

For (c), suppose first that $i = 0$. Then we can argue exactly as for (b), using Lemma 6.2.10 in place of (6.1.23).

In general, choose an injective resolution

$$X \hookrightarrow I_*$$

of X as a $(\mathcal{G}, B)$ module. By (a),

$$X \otimes F \hookrightarrow I_* \otimes F$$

is an injective resolution of $X \otimes F$. By definition,

$$\begin{aligned} \Gamma^i(X \otimes F) &= H^i(\Gamma(I_* \otimes F)) \\ &\cong H^i((\Gamma I_*) \otimes F) \quad (\text{by the case } i = 0) \\ &\cong H^i(\Gamma I_*) \otimes F \quad (\text{by the exactness of } \otimes F) \end{aligned}$$

$$\cong (\Gamma^i X) \otimes F$$

by definition again.

Q.E.D.

Lemma 7.2.3 Suppose $\mathcal{G} = \lambda + \mathcal{U}$ is a parabolic

subalgebra of $\mathcal{U}$ satisfying the hypotheses of

Definition 6.3.1 and W is a $Z(\lambda)$ -finite $(\lambda, L \cap K)$ module.

Fix a Cartan subgroup $\mathcal{H} \subseteq \lambda$, and a weight $\xi \in \mathcal{H}^*$.

$$\text{a) } P_{\xi}(\mathcal{R}_{\mathcal{G}}^i W) = \mathcal{R}_{\mathcal{G}}^i(\bigoplus_{\sigma \in W} (\mathcal{G}, \mathcal{H}) P_{\sigma\xi - \rho}(\mathcal{U}) W)$$

(notation (6.3.1), (0.3.18)); here the sum on the

right must be understood as a sum over $W(\lambda, \mathcal{H})$

orbits of weights of the indicated form.

b) Let F be a finite dimensional representation of G ,

and

$$0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = F$$

a $\mathcal{G}$ -stable filtration, such that $\mathcal{U}$ acts trivially

on F_i/F_{i-1} . Then there is a natural spectral sequence

$$\mathcal{R}_{\mathcal{G}}^P(W \otimes F_i/F_{i-1}) \implies \mathcal{R}_{\mathcal{G}}^P(W \otimes F) \cong (\mathcal{R}_{\mathcal{G}}^P W) \otimes F$$

of $(\mathcal{G}, K)$ modules.

Proof. Part (a) is an immediate consequence of Proposition 6.3.11. For (b), the last isomorphism combines Lemma 7.2.2(b) and (c). To get the spectral sequence, choose an injective resolution

$$0 \rightarrow W \rightarrow I_*$$

of W as a $(\mathcal{O}, \text{L}\mathcal{O}K)$ module. By Lemma 7.2.2(a), all of the modules

$$I_* \otimes F_i/F_j \quad (j < i)$$

are injective; and therefore

$$0 \rightarrow W \otimes F_i/F_j \rightarrow I_* \otimes F_i/F_j$$

is an injective resolution of $W \otimes F_i/F_j$. By (6.1.23),

$$0 \rightarrow \text{pro}(W \otimes F_i/F_j) \rightarrow \text{pro}(I_* \otimes F_i/F_j)$$

is an injective resolution in $\mathcal{M}(\mathcal{O}, \text{L}\mathcal{O}K)$. By Definition 6.3.1,

$$\mathcal{R}\mathcal{O}\mathcal{F}^p(W \otimes F) = H^p(\Gamma \circ \text{pro}(I_* \otimes F)).$$

We filter this complex by

$$\Gamma \circ \text{pro}(I_* \otimes F_i).$$

Now Γ is exact on injective $(\mathcal{O}, \text{L}\mathcal{O}K)$ modules (Lemma 6.2.12); so the subquotients of the filtration are

$$\Gamma \circ \text{pro}(I_* \otimes F_i/F_{i-1}).$$

The cohomology of the associated graded complex is therefore

$$\mathcal{R}\mathcal{O}\mathcal{F}^p(W \otimes F_i/F_{i-1});$$

so what we want is the spectral sequence of this filtered complex.

Q.E.D.

To define the formal framework for the kind of periodicity we want to study, one more definition is helpful.

Definition 7.2.4 Suppose X and Y are $(\mathcal{O}, K)$ modules of finite length, and F is a finite dimensional representation of G . Define $\mathcal{O}F$ in $\mathcal{V}(\mathcal{O}, K)$ by

$$(X \dashv Y) \otimes F = X \otimes F - Y \otimes F.$$

(Tensoring with F is exact and preserves finite length (Lemma 7.1.12), so this preserves the equivalence relations defining $\mathcal{V}(\mathcal{O}, K)$ - cf. Definition 0.3.16).

We can now define coherent continuation; this formulation of the definition is due to Duflo [9].

Definition 7.2.5 Fix a Cartan subgroup H of G , and let $\Lambda \subseteq \hat{H}$ denote the lattice of weights of finite dimensional representations (Definition 0.4.11). Fix a weight $\xi \in \mathfrak{h}^*$, and write

$$\xi + \Lambda = \{\xi + \lambda \mid \lambda \in \Lambda\}$$

(which is a set of formal symbols); $\xi + \Lambda$ is called the translate of Λ by ξ . If $\lambda \in \Lambda$, recall that $\bar{\lambda}$ is the differential of λ ; thus $\xi + \bar{\lambda} \in \mathfrak{h}^*$. A coherent family of virtual $(\mathcal{O}, K)$ modules (based on H and $\xi + \Lambda$) is a map

$$\phi: \xi + \Lambda \rightarrow \mathcal{V}(\mathcal{O}, K)$$

(notation (0.3.16)) satisfying

- a) $\phi(\xi + \lambda)$ has infinitesimal character $\xi + \bar{\lambda} \in \mathfrak{h}^*$; and
 b) if F is any finite dimensional representation of G ,
 then

$$\phi(\xi + \lambda) \otimes F = \sum_{\mu \in \Delta(F)} \phi(\xi + (\lambda + \mu))$$

(notation (0.2.3), (7.2.4)).

A number of examples are given in 7.2.11-13.

It will be convenient to consider coherent families based on arbitrary Cartan subgroups of G , but we would also like to relate them to a fixed Cartan. The necessary technicalities have already been treated in Section 0.4.

Lemma 7.2.6 Let H be an arbitrary Cartan subgroup of G , and H^S a maximally split one (Definition 0.4.1); write Λ and Λ^S for the lattices of weights of finite dimensional representations of G . Fix systems Δ^+ , $(\Delta^+)^S$ of positive roots.

- a) There is a unique $\text{Ad}(\mathfrak{g})$ -inner automorphism

$$\psi : (\mathfrak{h}^S)^* \rightarrow \mathfrak{h}^*$$

which takes $(\Delta^+)^S$ to Δ^+ .

- b) There is a unique group homomorphism

$$\psi : \Lambda^S \rightarrow \Lambda$$

such that if $\lambda \in \Lambda^S$, and F is any finite dimensional representation of G , then (writing bar for differentials)

$$\begin{aligned} \bar{\psi}(\bar{\lambda}) &= \psi(\bar{\lambda}) \\ \Delta(F, H) &= \psi(\Delta(F, H^S)). \end{aligned}$$

ψ is surjective, and has finite kernel.

- c) Suppose ϕ is a coherent family of virtual $(\mathfrak{g}, K)$ modules based on H and $\xi + \Lambda$ (Definition 7.1.1).

Set $\xi^S = \psi^{-1}(\xi)$, and define

$$\phi^S(\xi^S + \lambda^S) = \phi(\xi + \psi(\lambda^S)) \quad (\lambda^S \in \Lambda^S)$$

Then ϕ^S is a coherent family based on H^S and

$\xi^S + \Lambda^S$, and ϕ^S determines ϕ .

Proof. Part (a) is obvious. For (b), fix $\lambda \in \Lambda^S$, and let F_λ be the unique finite dimensional irreducible representation of G of extremal weight λ (Proposition 0.4.12). By Proposition 0.4.7, F_λ has a one-dimensional $\mathfrak{h}^S$ weight space of weight $\bar{\lambda}$; so it has a one-dimensional $\mathfrak{h}$ weight space of weight $\psi(\bar{\lambda})$. We define $\psi(\lambda)$ to be the weight of H on this space. The first property claimed for ψ then holds by definition, and the second follows from Proposition 0.4.9. That ψ is surjective is obvious from this second property, and that it has finite kernel follows from the first (since H^S/H_0^S is finite). Part (c) is immediate from (b) and Definition 7.2.5. Q.E.D.

Theorem 7.2.7 (Schmid, Zuckerman - cf. [39], Section 5)

Fix a maximally split Cartan subgroup H^S of G (Definition 0.4.1), and a virtual $(\mathfrak{g}, K)$ module Θ having infinitesimal character $\xi^S \in (\mathfrak{h}^S)^*$. Then there is a coherent family ϕ based on H^S and $\xi^S + \Lambda^S$ (Definition 7.2.5), such that

$$\phi(\xi^S + 0) = \Theta.$$

There were several stages in the evolution of this theorem which are blurred in the proof we will give. Schmid first formulated the notion of coherent families in his work on discrete series (e.g., [36], [15]), and proved Theorem 7.1.4 when θ is a discrete series character. (The proof was more or less trivial, using Harish-Chandra's regularity theorem for characters; the main point was to make the definition.) Although the result was not stated, Schmid's proof applies whenever ξ^S is nonsingular (Definition 0.2.11). When ξ^S is singular, the theorem is substantially harder; as far as I know it was probably first proved by Zuckerman in that case (also unpublished). A detailed proof (still using Harish-Chandra's regularity theorem) can be found in [39]. The proof given here, which avoids the regularity theorem, is based in part on conversations with Zuckerman.

Definition 7.2.8 Suppose $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra as in Definition 6.3.1. Assume that

- (A) whenever W is an $(\mathfrak{l}, \mathfrak{L}(\mathfrak{K}))$ module of finite length, then $\mathcal{B}_{\mathfrak{g}}^{\mathfrak{l}, W}$ is a $(\mathfrak{g}, \mathfrak{K})$ module of finite length.

(We have proven this for $\mathfrak{g}$ θ -stable (Lemma 6.3.33) or $\mathfrak{g}$ real (Proposition 6.3.5, Proposition 4.1.12, and Corollary 5.4.16).) If W is an $(\mathfrak{l}, \mathfrak{L}(\mathfrak{K}))$ module of finite length, set

$$\chi_{\mathfrak{g}}^W(\mathfrak{w}) = \sum_i (-1)^i \mathcal{B}_{\mathfrak{g}}^{\mathfrak{l}, W} \in \mathcal{V}(\mathfrak{g}, \mathfrak{K}).$$

Lemma 7.2.9 In the setting of Definition 7.2.8, suppose W is an $(\mathfrak{l}, \mathfrak{L}(\mathfrak{K}))$ module of finite length, and F is a finite dimensional representation of G .

- a) The functor $W \mapsto \chi_{\mathfrak{g}}(W)$ is exact, and therefore extends to a homomorphism of Grothendieck groups

$$\chi_{\mathfrak{g}}: \mathcal{V}(\mathfrak{l}, \mathfrak{L}(\mathfrak{K})) \rightarrow \mathcal{V}(\mathfrak{g}, \mathfrak{K}).$$

- b) If $\theta \in \mathcal{V}(\mathfrak{l}, \mathfrak{L}(\mathfrak{K}))$, then

$$\chi_{\mathfrak{g}}(\theta) \otimes F = \chi_{\mathfrak{g}}(\theta \otimes F|_{\mathfrak{l}, \mathfrak{L}(\mathfrak{K})}).$$

Proof. For (a), suppose

$$0 \rightarrow W_1 \rightarrow W_2 \rightarrow W_3 \rightarrow 0$$

is a short exact sequence of $(\mathfrak{g}, \mathfrak{L}(\mathfrak{K}))$ modules. Since ρ is exact, we get a short exact sequence of $(\mathfrak{g}, \mathfrak{L}(\mathfrak{K}))$ modules, and thus (by Lemma 6.2.12) a long exact sequence

$$\dots \rightarrow \mathcal{R}^{\mathfrak{l}, W_1} \rightarrow \mathcal{R}^{\mathfrak{l}, W_2} \rightarrow \mathcal{R}^{\mathfrak{l}, W_3} \rightarrow \mathcal{R}^{\mathfrak{l}+1, W_1} \rightarrow \dots$$

By the Euler-Poincaré principle,

$$\sum_i (-1)^i \mathcal{R}^{\mathfrak{l}, W_2} = \sum_i (-1)^i \mathcal{R}^{\mathfrak{l}, W_1} + \sum_i (-1)^i \mathcal{R}^{\mathfrak{l}, W_3},$$

or

$$\chi_{\mathfrak{g}}^{W_2} = \chi_{\mathfrak{g}}^{W_1} + \chi_{\mathfrak{g}}^{W_3}.$$

This is (b). Part (c) is the Euler-Poincaré principle applied to the spectral sequence (7.2.3) (b). Q.E.D.

Corollary 7.2.10 In the setting of Definition 7.2.8, suppose $H \subseteq L$ is a Cartan subgroup, $\xi_L \in \mathfrak{h}_L^*$, and Λ_L (respectively, Λ) is the lattice of weights of finite

dimensional representations of L (respectively, G).
Let ϕ_L be a coherent family of virtual $(\ell, L \cap K)$ modules
based on H and $\xi_L + \Lambda_L$. Define

$$\begin{aligned} \xi &= \xi_L + \rho(u) \in \mathcal{h}^* \\ \phi(\xi + \lambda) &= \chi_{\mathcal{O}\mathcal{F}}(\phi_L(\xi_L + \lambda)) \quad (\lambda \in \Lambda) \end{aligned}$$

(notation (7.2.8)). Then ϕ is a coherent family of
virtual $(\mathcal{O}\mathcal{F}, K)$ modules based on H and $\xi + \Lambda$.

Proof. This is an immediate consequence of Proposition 6.3.11,
 Lemma 7.2.9(b), and the definitions. Q.E.D.

Proof of Theorem 7.2.7 Case I. G satisfies (1.5.1) (that is,
 G is essentially a product of copies of $SL(2, \mathbb{R})$), and θ is
 the character of one of the representations $X(\delta \otimes 1)(\mu)$ defined
 in Corollary 1.5.12. Fix ε in $S(\mu)$, and let $\mathcal{O}\mathcal{F} = \mathcal{T} + u$ be
 the Borel subalgebra of $\mathcal{O}\mathcal{F}$ containing $\mathcal{T}$, defined in Proposi-
 tion 1.5.26. Then the conclusion of that proposition, in the
 notation of Definition 6.3.1, says that

$$X(\delta \otimes 1)(\mu) \cong \mathcal{P}\mathcal{O}\mathcal{F}^0(\mathbb{C}_{\mu_\varepsilon} \otimes (\wedge^{\dim u} u)^*).$$

Since $u \cap \mathcal{h}$ is zero, Corollary 6.3.21 guarantees that the
 higher $\mathcal{P}\mathcal{O}\mathcal{F}^i$ are zero; so this can be written as

$$\theta = \chi_{\mathcal{O}\mathcal{F}}(\mathbb{C}_{\lambda_0} \otimes (\wedge^{\dim u \cap \mathcal{h}} u \cap \mathcal{h})^*).$$

Obviously any weight of T occurs in the coherent family $\hat{\tau}$
 of $(\mathcal{T}, T)$ modules; so by Corollary 7.2.10, we deduce that θ
 belongs to a coherent family of $(\mathcal{O}\mathcal{F}, K)$ modules based on T .
 By Lemma 7.2.6, Theorem 7.2.7 follows in this case.

Case II. G is quasisplit, $\delta \in \hat{T}^s$ is fine, $\pi \in A(\delta)$,
 $\cdot \in \hat{A}^s$, and

$$\theta = \theta(\delta \otimes v)(\pi)$$

(notation 6.6.5)). By the proof of Theorem 4.4.8 (especially
 Lemma 4.4.17), we can find a parabolic subgroup

$$P^1 = M^1 A_N^1$$

of G , and an $M^1 \cap K$ -type $\pi^1 \in A_{M^1}^1(\delta)$, such that

- a) M^1 satisfies (1.5.1)
- b) $v|_{A \cap M^1}$ is trivial
- c) δ satisfies the hypotheses of Proposition 1.5.9
for M^1
- d) $X(\delta \otimes v)(\pi) = \text{Ind}_{P^1}^G [X_{M^1}^1(\delta \otimes (v|_{A \cap M^1}(\pi^1))$
 $\otimes (v|_A^1) \otimes 1$.

By Case I, the inducing representation occurs in a coherent
 family for $M^1 A^1$; so by Corollary 7.2.10, the induced repre-
 sentation occurs in a coherent family for G .

Case III. (H, γ) is a set of character data for G ,
 $\pi \in A(\gamma)$, and

$$\theta = \theta(\gamma)(\pi)$$

(notation (6.6.5)). Let $(\mathcal{O}\mathcal{F}, H, \delta, v)$ be the corresponding set
 of θ -stable data for G (Proposition 6.6.2), and let $\pi^L \in A_L(\delta)$
 be the $L \cap K$ -type corresponding to π (Definition 6.5.5). Then

$$\mathcal{R}^i(X_L(\delta \otimes v)(\pi^L)) = \begin{cases} X(\gamma)(\pi) & i = S \\ 0 & i \neq S \end{cases};$$

the first formula is the definition and the second is noted
 at the beginning of the proof of Theorem 6.5.10. Accordingly,

$$\Theta(\gamma)(\pi) = \pm \chi_{\mathcal{G}}(\Theta_L(\delta \otimes \nu)(\pi^L)).$$

By Case II, $\Theta_L(\delta \otimes \nu)(\pi^L)$ occurs in a coherent family for L ; so by Corollary 7.2.10, $\Theta(\gamma)(\pi)$ occurs in a coherent family for G . We can transfer this family to H^S by Lemma 7.2.6.

Case IV. General case. By Proposition 6.6.6 (d), Θ is an integral combination of the characters treated in Case III, with the same infinitesimal character as Θ . Adding the corresponding coherent families, we get the conclusion of the theorem.

Q.E.D.

Before continuing, we will describe some examples of coherent families.

Example 7.2.11 Suppose G is quasisplit, $H^S = T^S_A$ is a maximally split Cartan subgroup, $\delta \in \hat{T}^S$, and $\nu \in \hat{A}^S$.

Let $\xi \in (\mathcal{H}^S)^*$ be the differential of $\delta \otimes \nu$. If λ lies in the weight lattice Λ^S in $\hat{H}^S$, set

$$\phi(\xi + \lambda) = \Theta((\delta \otimes \nu) + \lambda),$$

the character of the principal series representation parametrized by

$$\delta \otimes \nu + \lambda \in \hat{H}^S.$$

This is a coherent family by Corollary 7.2.10.

Example 7.2.12 Let H^S be a maximally split Cartan subgroup of G , $(\Delta^+)^S$ a positive system for $\mathcal{H}^S$ in $\mathcal{G}$, and

$$\rho^S = \rho((\Delta^+)^S).$$

If $\xi \in \Lambda^S$, define $\phi(\xi + \rho^S)$ as follows: if $\xi + \rho^S$ is singular (Definition 0.2.11), set

$$\phi(\xi + \rho^S) = 0.$$

If $\xi + \rho^S$ is nonsingular, choose $w \in W(\mathcal{G}, \mathcal{H}^S)$ so that $w(\xi + \rho^S)$ is dominant. Since

$$\rho^S - w^{-1}\rho^S$$

is a sum of roots (by (3.2.9)), we can regard it canonically as a character of H^S . Let F be the finite dimensional irreducible representation of G of extremal weight

$$\xi + (\rho^S - w^{-1}\rho^S) \in \hat{H}^S,$$

and define

$$\phi(\xi + \rho^S) = (-1)^{\ell(w)} \Theta(F).$$

That this is a coherent family is substantially equivalent to Exercise 9, p. 142, of [22] (which is attributed to Klimyk).

Example 7.2.13 Suppose $G = SL(2, \mathbb{R})$. Let $\mathcal{G} = \mathcal{X} + \mathcal{U}$ be the Borel subalgebra of $\mathcal{G}$ spanned by H and X (notation (1.2.1)). Recall that $\hat{T}$ was identified with $\mathbb{Z}$; this coincides with the weight lattice for T . Define

$$\begin{aligned} Y(n) &= \mathcal{R}\mathcal{G}^{0}(\mathbb{C}_{n-1}) \\ &= \text{pro}_{\mathcal{G}, T}^{g, T}(\mathbb{C}_{n+1}^{T}) \\ &= \text{Hom}_{\mathcal{G}^{(U(\mathcal{G}), \mathbb{C}_{n+1})_{T\text{-finite}}}} \end{aligned}$$

As a trivial special case of Theorem 6.3.12, or directly, one sees that the K-types of $Y(n)$ are

$$\{n+1, n+3, n+5, \dots\}.$$

Then

$$\phi(0 + n) = \phi(Y(n))$$

defines a coherent family based on T and $0 + \Lambda$, by

Corollary 7.2.10. By the results of Section 1.2, if $n > 0$

$$\begin{aligned} \phi(n) &= \text{character of } X_d(n), \text{ a discrete series} \\ &\quad \text{(cf. (1.2.13))} \\ \phi(0) &= \text{character of } \bar{X}_C(0)(+1), \text{ a piece of a} \\ &\quad \text{reducible unitary principal series} \\ &\quad \text{(cf. (1.2.13))} \end{aligned}$$

$$\phi(-n) = \phi(n) + (\text{character of } n \text{ dimensional irreducible representation}).$$

Because of Proposition 1.3.3, one might suspect that this coherent family is closely connected with the reducibility of principal series for $SL(2, \mathbb{R})$. One of our goals will be to generalize this.

We want to relate coherent continuation to the Jantzen-Zuckerman translation functors.

Lemma 7.2.14 The Jantzen-Zuckerman translation functors (Definition 4.5.7) preserve $\mathcal{Z}(\mathfrak{g}, K)$, the category of $(\mathfrak{g}, K)$ modules of finite length. Accordingly, they define homomorphisms from $\mathcal{V}(\mathfrak{g}, K)$ to itself.

Proof. The first claim follows from the definition and Lemma 7.1.12. The second is a consequence of the exactness of the translation functors (Proposition 4.5.8). Q.E.D.

Lemma 7.2.15 Suppose $H^S \subseteq G$ is a maximally split Cartan subgroup of G , Λ^S is its weight lattice, $\xi \in (\mathcal{H}^S)^*$, and ϕ is a coherent family based on H^S and ξ (Definition 7.2.5). If λ_1 and λ_2 belong to Λ^S , then

$$\psi_{\xi+\lambda_1}^{\xi+\lambda_2}(\phi(\xi + \lambda_1)) = \sum_{\substack{\lambda \in \Delta(F_{\lambda_1-\lambda_2}) \\ \xi+(\bar{\lambda}+\bar{\lambda}_1) \in W \cdot (\xi+\lambda_2)}} \phi(\xi + (\lambda+\lambda_1))$$

Here W denotes the Weyl group of $\mathcal{H}^S$ in $\mathfrak{g}$, and other notation is as in Definition 4.5.7.

Proof. By definition (cf. (4.5.7)) ,

$$\psi_{\xi+\lambda_1}^{\xi+\lambda_2}(\phi(\xi + \lambda_1)) = P_{\xi+\bar{\lambda}_2}(P_{\xi+\bar{\lambda}_1}(\phi(\xi + \lambda_1)) \otimes F_{\lambda_2^{-\lambda_1}}).$$

By (7.2.5)(a), the first P has no effect; so by (7.2.5)(b),

$$= P_{\xi+\bar{\lambda}_2} \left(\sum_{\lambda \in \Delta(F_{\lambda_2^{-\lambda_1}})} \phi(\xi + (\lambda + \lambda_1)) \right).$$

By (7.2.5)(a) and Corollary 0.2.10, the terms in this sum with infinitesimal character $\xi + \bar{\lambda}_2$ are precisely those with

$$\xi + (\bar{\lambda} + \bar{\lambda}_1) \in W \cdot (\xi + \bar{\lambda}_2). \quad \text{Q.E.D.}$$

We want to compute the set

$$\{\lambda \in \Delta(F_{\lambda_2^{-\lambda_1}}) \mid \xi + \bar{\lambda} + \bar{\lambda}_1 \in W \cdot (\xi + \bar{\lambda}_2)\}$$

under some hypotheses on λ_1 and λ_2 . The following notation conflicts slightly with notation introduced earlier for the R-group, but we won't need them simultaneously.

Definition 7.2.16 Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, and $\gamma \in \mathfrak{h}^*$ a weight. Put

$$R(\gamma) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \langle \alpha, \gamma \rangle \in \mathbb{Z}\} \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$$

(notation 0.2.5), the set of roots integral with respect to γ . Put

$$W(\gamma) = W(R(\gamma)) \subseteq W(\mathfrak{g}, \mathfrak{h}),$$

the integral Weyl group for γ . If γ is nonsingular, define

$$R^+(\gamma) = \{\alpha \in R(\gamma) \mid \langle \alpha, \gamma \rangle > 0\},$$

the system of positive integral roots defined by γ .

The set $R(\gamma)$ is an abstract root system (since by definition $R(\gamma)^\vee$ is obviously closed under addition and -1 in $\Delta^\vee$); but it does not in general correspond to a subalgebra of $\mathfrak{g}$. A typical example is $\mathfrak{g} = \mathfrak{so}(5)$, and γ equal to half a short root. Then $R(\gamma)$ consists of all four short roots, and so is of type $A_1 \times A_1$. The only subalgebra of $\mathfrak{g}$ of type $A_1 \times A_1$ is (up to automorphisms) $\mathfrak{so}(4)$, which corresponds to the four long roots.

Lemma 7.2.17 ([4]). In the setting of Definition 7.2.16, suppose $w \in W(\mathfrak{g}, \mathfrak{h})$. Then w belongs to $W(\gamma)$ if and only if $\gamma - w\gamma$ is a sum of roots.

The "only if" part is very easy. The "if" part is Chevalley's theorem for the affine Weyl group; we will not digress to prove it here.

Lemma 7.2.18 Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, and $\gamma \in \mathfrak{h}^*$ is non-singular; define $R^+(\gamma)$ as in (7.2.16). Let $\mu \in \mathfrak{h}^*$ be a weight of a finite dimensional representation of G . Let $W_0 \subseteq W(\mathfrak{g}, \mathfrak{h})$ denote the stabilizer of $\gamma + \mu$.

$$a) \quad R(\gamma + \mu) = R(\gamma)$$

$$W_0 \subseteq W(\gamma).$$

For the rest of the lemma, assume that $\gamma + \mu$ is dominant (possibly singular) for $R^+(\gamma)$. Let F be the finite dimensional representation of $\mathfrak{g}$ of extremal weight μ .

b) The equation

$$\gamma + \lambda = w(\gamma + \mu),$$

for $\lambda \in \Delta(F)$, $w \in W(\mathfrak{g}, \mathfrak{h})$, holds if and only if

$$w \in W_0, \text{ and } \lambda = \mu.$$

c) The equation

$$(\gamma + \mu) - \lambda = w\gamma$$

for $\lambda \in \Delta(F)$, $w \in W(\mathfrak{g}, \mathfrak{h})$, holds if and only if

$$w \in W_0, \text{ and } \lambda = w\mu.$$

Proof. Since μ is necessarily integral for all of $\Delta(\mathfrak{g}, \mathfrak{h})$, we have for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$

$$\langle \alpha, \gamma \rangle \equiv \langle \alpha, \gamma + \mu \rangle \pmod{\mathbb{Z}}.$$

This proves the first claim in (a). For the second, Chevalley's

theorem says that W_0 is generated by

$$\{s_\alpha \mid \langle \alpha, \mu + \gamma \rangle = 0\}.$$

Obviously these reflections lie in $W(\gamma + \mu)$, which is equal to $W(\gamma)$ by the first part. Consider (b). The "if" part is trivial, so suppose the equality holds. It can be written

$$(\gamma + \mu) - w(\gamma + \mu) = \mu - \lambda.$$

Since λ and μ are both weights of F , the right side is a sum of roots; so by Lemma 7.2.17, w belongs to $W(\gamma + \mu) = W(\gamma)$.

Write the equation as

$$\lambda = \gamma - w(\gamma + \mu).$$

Taking the length of both sides and using the W -invariance of

$<, >$, we get

$$(7.2.19) \text{ (a) } \langle \lambda, \lambda \rangle = \langle \gamma + \mu, \gamma + \mu \rangle - 2\langle w(\gamma + \mu), \gamma \rangle + \langle \gamma, \gamma \rangle.$$

Now apply Lemma 6.3.28 to the root system $R(\gamma)$ and the weight $\gamma + \mu$. We find non-negative real numbers n_α ($\alpha \in R^+(\gamma)$) such that

$$w(\gamma + \mu) = \gamma + \mu - \sum_{\alpha \in R^+(\gamma)} n_\alpha \alpha.$$

Since γ is dominant and regular for $R^+(\gamma)$, this means that

$$(7.2.19) \text{ (b) } w(\gamma + \mu), \gamma \rangle \leq \langle \gamma + \mu, \gamma \rangle;$$

with equality if and only if all the n_α are zero; that is, if and only if

$$(7.2.20) \quad w(\gamma + \mu) = \gamma + \mu.$$

Now (7.2.19) gives

$$\begin{aligned} \langle \lambda, \lambda \rangle &\geq \langle \gamma + \mu, \gamma + \mu \rangle - 2\langle \gamma + \mu, \gamma \rangle + \langle \gamma, \gamma \rangle \\ &= \langle (\gamma + \mu) - \gamma, (\gamma + \mu) - \gamma \rangle \\ &= \langle \mu, \mu \rangle, \end{aligned}$$

with equality if and only if (7.2.20) holds. On the other hand, since λ is a weight of F which has extremal weight μ , Proposition 0.4.3(b) gives

$$\langle \lambda, \lambda \rangle \leq \langle \mu, \mu \rangle.$$

So the equality holds above, forcing (7.2.20). By definition, this means that $w \in W_0$; and it also allows us to write the assumption as

$$\gamma + \lambda = \gamma + \mu,$$

so that $\lambda = \mu$. This proves (b). The hypothesis in (c) can be written

$$\gamma + w^{-1}\lambda = w^{-1}(\lambda + \mu).$$

Since $\Delta(F)$ is stable under W (by Proposition 0.4.3(a)), $w^{-1}\lambda$ is also a weight of F ; so (c) follows from (b). Q.E.D.

To state the next result, we need one more definition.

Definition 7.2.21 Let $H \subseteq G$ be a Cartan subgroup, and Λ the weight lattice. Fix $\xi \in \mathcal{H}^*$, $\lambda \in \Lambda$, and $w \in W(\xi + \bar{\lambda})$ (notation 7.2.16). By Lemma 7.2.17,

$$w(\xi + \bar{\lambda}) = \xi + \bar{\lambda} + \bar{Q},$$

where $\bar{Q}$ is a sum of roots. By Lemma 0.4.5, $\bar{Q}$ is the differential of a unique sum of roots Q in $\hat{H}$; and we define

$$w(\xi + \lambda) = \xi + (\lambda + Q).$$

Thus $w(\xi + \bar{\lambda}) = w(\xi)$ (cf. Lemma 7.2.18(a)) acts on $\xi + \Lambda$. (In particular, $w(q, h)$ acts on Λ , as we already saw in Definition 3.2.15.)

Proposition 7.2.22 (Zuckerman [54]) Suppose $H^S \subseteq G$ is a maximally split Cartan subgroup, $\xi \in (h^S)^*$, and ϕ is a coherent family based on H^S and $\xi + \Lambda^S$ (notation 7.2.5)). Fix $\lambda_1, \lambda_2 \in \Lambda^S$, and assume that

$$\begin{aligned} \xi + \bar{\lambda}_1 & \text{ is nonsingular} \\ \xi + \bar{\lambda}_2 & \text{ is dominant (possibly singular) for} \\ & \quad R^+(\xi + \bar{\lambda}_1) \end{aligned}$$

(notation 7.2.16)). Let W_0 denote the stabilizer of $\xi + \bar{\lambda}_2$ in W . Then (notation (4.5.7)),

$$a) \quad \psi_{\xi+\lambda_1}^{\xi+\lambda_1}(\phi(\xi + \lambda_1)) = \phi(\xi + \lambda_2)$$

In particular, if $\phi(\xi + \lambda_1)$ is the character of a representation (and not merely a difference of such characters) then $\phi(\xi + \lambda_2)$ is as well.

$$b) \quad \psi_{\xi+\lambda_2}^{\xi+\lambda_1}(\phi(\xi + \lambda_2)) = \sum_{w \in W_0} \phi(w(\xi + \lambda_1)) \text{ (notation (7.2.21)).}$$

Proof. Consider for example (b). By Lemma 7.2.15,

$$\psi_{\xi+\lambda_2}^{\xi+\lambda_1}(\phi(\xi + \lambda_2)) = \sum_{\lambda \in \Delta(F_{\lambda_1-\lambda_2})} \phi(\xi + (\lambda + \lambda_2)) \cdot \xi + \bar{\lambda}_2 \in W \cdot (\xi + \bar{\lambda}_1)$$

Apply Lemma 7.2.18(c), with $\gamma = \xi + \bar{\lambda}_1$, $\mu = \bar{\lambda}_1 - \bar{\lambda}_2$. We get

$$= \sum_{w \in W_0} \phi(\xi + [w(\lambda_1 - \lambda_2) + \lambda_2]).$$

$$= \sum_{w \in W_0} \phi(w(\xi + [\lambda_2 + \lambda_1 - \lambda_2]))$$

since w fixes $\xi + \lambda_2$

$$= \sum_{w \in W_0} \phi(w(\xi + \lambda_1)),$$

as claimed. Part (a) is similar and easier. Q.E.D.

Proposition 7.2.23 (Schmid) Let ϕ be a coherent family based on H and $\xi + \Lambda$ (notation (7.2.5)). Fix $\lambda \in \Lambda$ such that $\xi + \bar{\lambda}$ is nonsingular, and suppose

$$\phi(\xi + \lambda) = 0.$$

Then ϕ is identically zero.

Proof. By Lemma 7.2.6(c), we may as well assume $H = H^S$ is maximally split. We will first show that

$$(7.2.24) \quad \phi(w(\xi + \lambda)) = 0, \quad w \in W(\xi + \bar{\lambda})$$

(notation 7.2.16) and (7.2.21). To prove this, choose a positive root system $\Delta^+(q, h^S)$ containing $R^+(\xi + \bar{\lambda})$, and set

$$\rho = \rho(\Delta^+(q, h^S)).$$

Now list the elements of $W(\xi + \bar{\lambda})$ as $\{w_1, \dots, w_n\}$ in such a way that if $i \leq j$, then

$$(7.2.25) \quad \langle w_i(\xi + \bar{\lambda}) - w_j(\xi + \bar{\lambda}), \rho \rangle \geq 0.$$

We will prove (7.2.24) by induction on i . So fix i , and suppose it is known for $j < i$. Choose a $\Delta^+(q, h^S)$ -dominant weight $\gamma \in \Lambda^S$ in such a way that

$$w_i(\xi + \bar{\lambda}) + \bar{\gamma}$$

is dominant for $R^+(\xi + \bar{\lambda})$; for example, we could take for γ a large multiple of 2ρ . Proposition 7.2.22(a) then gives

$$\begin{aligned}\phi(w_{\bar{1}}(\xi + \lambda) + \gamma) &= \psi_{\xi + \lambda}^{w_{\bar{1}}(\xi + \lambda) + \gamma} \phi(\xi + \lambda) \\ &= 0.\end{aligned}$$

We now compute, using Lemma 7.2.15,

$$\begin{aligned}(7.2.26) \quad 0 &= \psi_{w_{\bar{1}}(\xi + \lambda) + \gamma}^{w_{\bar{1}}(\xi + \lambda)} \phi(w_{\bar{1}}(\xi + \lambda) + \gamma) \\ 0 &= \sum_{\delta \in \Delta(F_{-\gamma})} \phi[w_{\bar{1}}(\xi + \lambda) + (\gamma + \delta)]. \\ &\quad w_{\bar{1}}(\xi + \bar{\lambda}) + \bar{\delta} + \bar{\gamma} \in W \bullet (\xi + \bar{\lambda})\end{aligned}$$

Fix a weight δ occurring in this sum. Since γ is dominant, $F_{-\gamma}$ has lowest weight $-\gamma$; so by Proposition 0.4.9(b),

$$\delta = -\gamma + \sum_{\alpha \in \Delta^+} n_{\alpha} \alpha,$$

with n_{α} non-negative. The other condition on δ reads

$$w_{\bar{1}}(\xi + \bar{\lambda}) + \sum n_{\alpha} \alpha = w(\xi + \bar{\lambda}),$$

some $w \in W(\mathfrak{g}, \mathfrak{h})$. By Lemma 7.2.17, w lies in $W(\xi + \bar{\lambda})$; say $w = w_j$. We have

$$\begin{aligned}\langle w_{\bar{1}}(\xi + \bar{\lambda}) - w_j(\xi + \bar{\lambda}), \rho \rangle &= \langle \sum n_{\alpha} \alpha, \rho \rangle \\ &\geq 0,\end{aligned}$$

with equality exactly when all the n_{α} are zero; that is, when $\delta = -\gamma$. Suppose this is not the case. By (7.2.25), it follows that $j < i$. On the other hand,

$$\begin{aligned}\phi(w_{\bar{1}}(\xi + \lambda) + (\gamma + \delta)) &= \phi(w_j(\xi + \lambda)) \\ &= 0\end{aligned}$$

by Definition 7.2.21, the calculations just given, and inductive

hypothesis. So the only term in (7.2.26) which can be non-zero is for $\delta = -\gamma$; and so

$$0 = \phi(w_{\bar{1}}(\xi + \lambda)),$$

as we wanted to show. This proves (7.2.24).

To prove the lemma, fix $\gamma \in \Lambda$, and choose $w \in W(\xi + \bar{\lambda})$ so that

$$\xi + \bar{\gamma}, w(\xi + \bar{\lambda})$$

lie in the same $W(\xi + \bar{\lambda})$ Weyl chamber; this is certainly possible.

By Proposition 7.2.22(a) again,

$$\phi(\xi + \gamma) = \psi_{w(\xi + \lambda)}^{\xi + \gamma} \phi(w(\xi + \lambda)),$$

and the right side is zero by (7.2.24). Q.E.D.

Corollary 7.2.27 In the setting of Theorem 7.2.7,

suppose ξ^S is nonsingular. Then the coherent family ϕ based on H^S and $\xi^S + \Lambda^S$, satisfying

$$\phi(\xi^S + 0) = 0,$$

is unique.

If ξ^S is singular, the coherent family is never unique, although there are at least two ways to find an essentially canonical one; one of these is described in Corollary 7.3.23. Our proof of Theorem 7.2.7 may not have appeared to involve any choices, even for singular ξ^S , but the choices were there (in the applications of Lemma 7.2.6).

Definition 7.2.28 (Zuckerman) Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, $\xi \in \mathfrak{h}^*$ is a nonsingular weight,

and $\emptyset$ is a virtual $(\mathfrak{g}, K)$ module of infinitesimal character ξ . If $w \in W(\xi)$ (notation (7.2.16)), we define a new virtual $(\mathfrak{g}, K)$ module $w \cdot \emptyset$, also having infinitesimal character ξ , as follows. Choose a maximally split Cartan subgroup H^S of G , and a weight $\xi^S \in (\mathfrak{h}^S)^*$ defining the same infinitesimal character as ξ ; and choose a coherent family ϕ such that

$$\phi(\xi^S + 0) = 0$$

(Theorem 7.2.7). Let $\psi: \mathfrak{h} \rightarrow \mathfrak{h}^S$ be the unique $\text{Ad}(\mathfrak{g})$ inner automorphism taking ξ to ξ^S , and let w^S be the element of $W(\xi^S)$ corresponding to w under this automorphism. Define

$$w \cdot \emptyset = \phi(w^{-1}(\xi^S + 0))$$

(notation (7.2.21)).

Lemma 7.2.29 In the setting of Definition 7.2.28,

$w \cdot \emptyset$ is well-defined.

a) If $w_1, w_2 \in W(\xi)$, then

$$w_1(w_2 \emptyset) = (w_1 w_2) \emptyset$$

b) If ϕ is a coherent family based on H and $\xi + \Lambda$,

and $w \in W(\xi)$, then if $\xi + \bar{\lambda}$ is nonsingular,

$$w \cdot \phi(\xi + \lambda) = \phi(w^{-1}(\xi + \lambda)).$$

Proof. The first claim follows from Corollary 7.2.27. For (a), in the definition of w_1 applied to $w_2 \emptyset$, we can use for the coherent family in Definition 7.2.28

$$\phi'(\xi^S + \lambda) = \phi(w_2^{-1}(\xi^S + \lambda));$$

this has the required property that

$$\phi'(\xi^S) = w_2 \cdot \emptyset.$$

Then

$$\begin{aligned} w_1 \cdot (w_2 \emptyset) &= \phi'(w_1^{-1} \xi^S) \\ &= \phi(w_2^{-1} w_1^{-1} \xi^S) \\ &= \phi((w_1 w_2)^{-1} \xi^S) \\ &= (w_1 w_2) \emptyset. \end{aligned}$$

Part (b) requires a similar formal argument, using Lemma 7.2.6. We leave it to the reader. Q.E.D.

§3. The translation principle, and an introduction to wall crossing.

Fix a nonsingular infinitesimal character ξ , and let $\mathcal{L}_\xi$ denote the lattice of virtual $(\mathfrak{g}, K)$ modules having infinitesimal character ξ . Lemma 7.2.29 defines a representation of the Weyl group $W(\xi)$ on $\mathcal{L}_\xi$. We know two bases for $\mathcal{L}_\xi$: the standard characters of Definition 6.6.5, and the irreducible characters. As explained in Chapter 2, our goal is to express the second basis in terms of the first. We will approach this problem in the following way.

- 1) Compute the action of $W(\xi)$ explicitly on the standard characters (Propositions 8.2.7, 8.4.3 and 8.4.9, and Corollary 8.4.6).
- 2) Say something (or, modulo a technical conjecture, everything) about the action of $W(\xi)$ on the irreducible characters.

(See Corollary 7.3.19 and Section 9.6,

respectively.)

- 3) Combine these computations to deduce something (or, using the conjecture, everything) about how the irreducible characters must be expressed in terms of the standard ones (Theorem 8.6.6 and Section 9.6).

(The technical conjecture is (7.3.25).) This philosophy is due to Jantzen ([23]). For the rest of this chapter, we will be concentrating on part (2) of this program, and the formulation of the technical conjecture mentioned in it. Parts (1) and (3) present no essential difficulties (but many inessential ones); they are treated in Chapters 8 and 9, respectively.

We begin with a useful formal observation.

Lemma 7.3.1 In the setting of Definition 4.5.7, fix $w \in W(\xi)$ (notation (7.2.16)). Then

$$\frac{w(\xi+\lambda_1)}{w(\xi+\lambda_2)} = \frac{\xi+\lambda_1}{\xi+\lambda_2} ;$$

here we use also Definition 7.2.21.

Proof. The definition of the translation functor uses the projection operators $P_{\xi+\bar{\lambda}_i}$ and the finite dimensional representation F of extremal weight $\lambda_1 - \lambda_2$. By Corollary 0.2.10,

$$P_{\xi+\bar{\lambda}_i} = P_{w(\xi+\bar{\lambda}_i)} .$$

If we write

$$\psi_{\xi+0}^{\xi+\lambda} \theta = \phi(\xi + \lambda) .$$

$$w(\xi + \lambda_i) = \xi + \gamma_i ,$$

then γ_i differs from λ_i by a sum of roots; so $\gamma_1 - \gamma_2$ differs from $\lambda_1 - \lambda_2$ by a sum of roots. Also

$$\begin{aligned} \overline{\gamma_1 - \gamma_2} &= w(\xi + \bar{\lambda}_1) - w(\xi + \bar{\lambda}_2) \\ &= w(\bar{\lambda}_1 - \bar{\lambda}_2) ; \end{aligned}$$

so $\gamma_1 - \gamma_2$ is an extremal weight of F . So the two translation functors are defined by exactly the same formulas. Q.E.D.

Proposition 7.3.2 (Zuckerman [54]) Fix a maximally split Cartan subgroup $H^S \subseteq G$, and a nonsingular weight $\xi \in \mathcal{K}^S$. Suppose $\lambda \in \Lambda^S$ is such that $\xi + \bar{\lambda}$ is dominant (possibly singular) for $R^+(\xi)$ (notation 7.2.16). Let W_0 be the stabilizer of $\xi + \bar{\lambda}$ in $W(\mathcal{K}, \mathcal{K}^S)$.

a) If θ is a virtual $(\mathcal{K}, K)$ module of infinitesimal

character ξ , then

$$\psi_{\xi+\lambda}^{\xi+0} \psi_{\xi+0}^{\xi+\lambda}(\theta) = \sum_{w \in W_0} w \cdot \theta$$

(notation (7.2.28)).

b) If θ is a virtual $(\mathcal{K}, K)$ module of infinitesimal character $\xi + \bar{\lambda}$, then

$$\psi_{\xi+0}^{\xi+\lambda} \psi_{\xi+\lambda}^{\xi+0}(\theta) = |W_0| \theta .$$

Proof. Consider (a). By Theorem 7.2.7, we can choose a coherent family ϕ on H^S , $\xi + \Lambda^S$, so that

$$\phi(\xi + 0) = \theta .$$

By Proposition 7.2.22(a),

By Proposition 7.2.22(b) and Lemma 7.2.29(b),

$$\begin{aligned}\psi_{\xi+\lambda}^{\xi+0} \phi(\xi + \lambda) &= \sum_{w \in W_0} \phi(w^{-1}(\xi + 0)) \\ &= \sum_{w \in W_0} w \cdot 0,\end{aligned}$$

as required. The proof of (b) is similar (although it uses also Lemma 7.3.1); it is left to the reader. Q.E.D.

The Jantzen-Zuckerman translation principle is

Proposition 7.3.3 ([23], [54]) Let $H^S \subseteq G$ be a maximally split Cartan subgroup, $\xi \in (h^S)^*$, and $\lambda \in \Lambda^S$. Assume that

a) ξ is nonsingular

b) $\xi + \bar{\lambda}$ is nonsingular and dominant for $R^+(\xi)$ (notation (7.1.16)). Write $\mathcal{K}(\mathfrak{g}, K)_\gamma$ for the category of $(\mathfrak{g}, K)$ modules of finite length and generalized infinitesimal character
 $\gamma \in (h^S)^*$. Then

$$\psi_{\xi+0}^{\xi+\lambda}: \mathcal{K}(\mathfrak{g}, K)_\xi \rightarrow \mathcal{K}(\mathfrak{g}, K)_{\xi+\bar{\lambda}}$$

is an equivalence of categories, with natural inverse $\psi_{\xi+\lambda}^{\xi+0}$. In particular, if X is an irreducible $(\mathfrak{g}, K)$ module of infinitesimal character ξ , then $\psi_{\xi+0}^{\xi+\lambda} X$ is irreducible.

Proof. By Proposition 7.3.2, if 0 is any virtual character of infinitesimal character ξ , then

$$(7.3.4) \quad \psi_{\xi+\lambda}^{\xi+0} \psi_{\xi+0}^{\xi+\lambda} 0 = 0.$$

So if Y is in $\mathcal{K}(\mathfrak{g}, K)_\xi$ and $Y \neq 0$,

$$\psi_{\xi+0}^{\xi+\lambda}(Y) \neq 0.$$

It follows that $\psi_{\xi+0}^{\xi+\lambda}(Y)$ has at least as many irreducible composition factors as Y . By symmetry and (7.3.4), they have exactly the same number of irreducible composition factors. This proves the last claim. By Proposition 4.5.8, there is a natural isomorphism

$$\text{Hom}_{\mathfrak{g}, K}^{(\psi_{\xi+\lambda}^{\xi+0} \psi_{\xi+0}^{\xi+\lambda})} Y \cong \text{Hom}_{\mathfrak{g}, K}^{(\psi_{\xi+0}^{\xi+\lambda} \psi_{\xi+\lambda}^{\xi+0})} Y, \quad \psi_{\xi+0}^{\xi+\lambda} Y.$$

Define

$$\delta_Y: Y \rightarrow \psi_{\xi+\lambda}^{\xi+0} \psi_{\xi+\lambda}^{\xi+0} Y$$

to be the map corresponding to the identity on the right in (7.3.5). We claim that δ_Y is an isomorphism; by (7.3.4), it suffices to show that it is injective. Let $Z \subseteq Y$ be the kernel. By naturality,

$$\delta_Z = \delta_Y|_Z = 0.$$

By definition, this means that the identity map on $\psi_{\xi+0}^{\xi+\lambda} Z$ is zero; so $\psi_{\xi+0}^{\xi+\lambda} Z$ is zero. As we observed before, it follows that $Z = 0$, proving that δ_Y is an isomorphism. The rest of the proposition (which we don't need, in any case) follows by a formal argument, which is left to the reader. Q.E.D.

It is possible (but not trivial) to prove the corresponding result in $\mathcal{M}_f(\mathfrak{g})$ (Definition 4.5.1).

Although this proposition will be used constantly for technical reasons, the main ideas we want to explore involve translation to a singular infinitesimal character. So one thing we want is to have a lot of singular weights of the

form $\xi + \bar{\lambda}$ (in our usual notation). To get them, we may have to pass to a covering group of G .

Lemma 7.3.5 There is a finite covering group $\tilde{G}$ of G (that is, there is a surjective map

$$\pi: \tilde{G} \rightarrow G$$

with finite kernel) which satisfies our hypotheses (0.1.2), and has in addition the following property:
suppose $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ is a Cartan subalgebra, and $\bar{\lambda} \in \mathfrak{h}^*$ is an integral weight (that is, $R(\bar{\lambda}) = \Delta(\mathfrak{g}, \mathfrak{h})$ -- cf. (7.2.16)). Then there is a weight $\gamma \in \hat{H}$ of a finite dimensional representation of $\tilde{G}$, such that

$$\langle \alpha, \bar{\gamma} - \bar{\lambda} \rangle = 0, \text{ all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}).$$

Proof. Let $\tilde{G}_{\mathbb{C}}^d$ denote the simply connected complex semisimple Lie group with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$; we can regard $\tilde{G}_{\mathbb{C}}^d$ as acting on all of $\mathfrak{g}$ in a natural way, by an action we call Ad . Set

$$\tilde{G} = \{(x, y) \in \tilde{G}_{\mathbb{C}}^d \times G \mid \text{Ad}(x) = \text{Ad}(y)\}.$$

The map

$$\pi: \tilde{G} \rightarrow G$$

by projection on the second factor is surjective by (0.1.2) (b). Its kernel is clearly isomorphic to the center of $\tilde{G}_{\mathbb{C}}^d$, which is finite by the theory of complex semisimple groups. The finite dimensional representations of $\tilde{G}$ required by the lemma can be obtained from those of $\tilde{G}_{\mathbb{C}}^d$, by projection on the first

factor. We leave the straightforward verification of this to the reader. Q.E.D.

Lemma 7.3.6 Suppose $H \subseteq G$ is a Cartan subgroup, and $\xi \in \mathfrak{h}^*$ is nonsingular. Fix a simple root α of $R^+(\xi)$ (notation (7.2.16)).

- a) There is a positive root system $\Delta^+(\mathfrak{g}, \mathfrak{h}) \supseteq R^+(\xi)$, such that α is simple in $\Delta^+(\mathfrak{g}, \mathfrak{h})$.
- b) Possibly after passing to a finite cover of G in accordance with Lemma 7.3.5, we can find a weight $\lambda \in \Lambda$ (of a finite dimensional representation) such that

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0$$

$$\langle \beta, \xi + \bar{\lambda} \rangle > 0, \quad \beta \in R^+(\xi) - \{\alpha\}.$$

We can also arrange

$$\text{Re} \langle \gamma, \xi + \bar{\lambda} \rangle > 0, \quad \gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h}) - \{\alpha\}.$$

Proof. Choose a weight $\phi_1 \in \mathfrak{h}^*$ such that

$$\langle \alpha, \phi_1 \rangle = 0$$

$$\langle \beta, \phi_1 \rangle > 0, \quad \beta \in R^+(\xi) - \{\alpha\}.$$

(For example, Corollary 3.2.4 guarantees that

$$\sum_{\beta \in R^+(\xi) - \{\alpha\}} \beta$$

has this property.) Next, choose $\phi_2 \in \mathfrak{h}^*$ so that

$$\langle \alpha, \phi_2 \rangle = 0$$

$$\text{Re} \langle \beta, \phi_2 \rangle > 0, \quad \beta \in R^+(\xi) - \{\alpha\}$$

$$\text{Re} \langle \gamma, \phi_2 \rangle \neq 0, \quad \gamma \in \Delta(\mathfrak{g}, \mathfrak{h}) - \{\alpha\}.$$

(Almost every element orthogonal to α and near ϕ_1 will have this property.) Define

$$\Delta^+(\alpha, \hbar) = \{\alpha\} \cup \{\gamma \in \Delta(\alpha, \hbar) \mid \operatorname{Re}\langle \gamma, \phi_2 \rangle > 0\}.$$

Obviously this satisfies (a). Now choose a weight $\lambda_1 \in \Lambda$ so that for every root γ of $\Delta^+(\alpha, \hbar)$,

$$(*) \quad \operatorname{Re}\langle \gamma, \xi + \bar{\lambda}_1 \rangle > 0.$$

(By Lemma 3.2.4, we could take for λ_1 a large multiple of the sum of all the positive roots.) Since α is integral for ξ and $\bar{\lambda}_1$,

$$\langle \alpha, \xi + \bar{\lambda}_1 \rangle = n,$$

a positive integer. By Lemma 7.3.5, there is a weight $\lambda_2 \in \Lambda$ (possibly after passing to a covering group) so that

$$\begin{aligned} \langle \alpha, \bar{\lambda}_2 \rangle &= n \\ \langle \gamma, \bar{\lambda}_2 \rangle &= 0, \quad \gamma \in \Delta^+(\alpha, \hbar) - \{\alpha\} \\ &\text{simple.} \end{aligned}$$

We define

$$\lambda = \lambda_1 - \lambda_2.$$

Then

$$\begin{aligned} \langle \alpha, \xi + \bar{\lambda} \rangle &= \langle \alpha, \xi + \bar{\lambda}_1 \rangle - \langle \alpha, \bar{\lambda}_2 \rangle \\ &= n - n \\ &= 0. \end{aligned}$$

If γ is a simple root of $\Delta^+(\alpha, \hbar)$ other than α , then

$$\begin{aligned} \operatorname{Re}\langle \gamma, \xi + \bar{\lambda} \rangle &= \operatorname{Re}\langle \gamma, \xi + \bar{\lambda}_1 \rangle \\ &> 0. \end{aligned}$$

So this holds for all roots in $\Delta^+(\alpha, \hbar)$ other than α (since they are sums of such roots and α). This proves (b). Q.E.D.

This lemma cannot be improved to produce a Δ^+ making all the simple roots of R^+ simple for Δ^+ at once; this is impossible in the example given after Definition 7.2.16.

We can now begin to study the representation of $W(\xi)$ discussed at the beginning of this section.

Proposition 7.3.7 ([39], Theorem 5.20). Suppose

$H^S \subseteq G$ is a maximally split Cartan subgroup,

$\xi \in (\mathcal{H}^S)^*$ is nonsingular, and $\alpha \in R^+(\xi)$ is simple

(notation (7.2.16)). Let X be an irreducible

(α, K) module with infinitesimal character ξ , and

let ϕ be the unique coherent family based on H^S

and $\xi + \Lambda^S$ satisfying

$$\phi(\xi + 0) = \phi(X)$$

(Corollary 7.2.27). The following conditions are

equivalent. (For (e) and (f), we may have to pass

to a covering group of G in accordance with Lemma 7.2.5.)

a) There exists a $\lambda \in \Lambda^S$ such that $\xi + \bar{\lambda}$ is dominant and regular for $R^+(\xi)$, and

$$s_\alpha \cdot \phi(\xi + \lambda) = \phi(s_\alpha(\xi + \lambda)) = -\phi(\xi + \lambda)$$

(notation (7.2.28)).

b) For every $\lambda \in \Lambda^S$ such that $\xi + \bar{\lambda}$ is dominant and regular for $R^+(\xi)$,

$$s_\alpha \cdot \phi(\xi + \lambda) = \phi(s_\alpha(\xi + \lambda)) = -\phi(\xi + \lambda).$$

c) For every $\lambda \in \Lambda^S$ such that $\xi + \bar{\lambda}$ is dominant and regular for $R^+(\xi)$, $s_\alpha \cdot \phi(\xi + \lambda)$ is not the character of a representation.

d) There exists a $\lambda \in \Lambda^S$ such that $\xi + \bar{\lambda}$ is dominant and regular for $R^+(\xi)$, and $s_\alpha \cdot \phi(\xi + \lambda)$ is not the character of a representation.

e) For every $\lambda \in \Lambda^S$ such that

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0$$

$$\langle \beta, \xi + \bar{\lambda} \rangle > 0 \quad (\beta \in R^+(\xi) - \{\alpha\}),$$

$\phi(\xi + \bar{\lambda})$ is zero.

f) There exists a λ as in (e) such that $\phi(\lambda) = 0$.

Proof. We will show that (a) $\implies$ (b) $\implies \dots \implies$ (f) $\implies$ (a).

(a) $\implies$ (b): Fix λ_0 as in (a), and let $\lambda \in \Lambda^S$ be any other weight with $\xi + \bar{\lambda}$ dominant and regular for $R^+(\xi)$. Then

$$\begin{aligned} s_\alpha \cdot \phi(\xi + \lambda) &= \phi(s_\alpha(\xi + \lambda)) && \text{by (7.2.29) (b)} \\ &= \psi_{s_\alpha(\xi + \lambda_0)}^{s_\alpha(\xi + \lambda)} \phi(s_\alpha(\xi + \lambda_0)) && \text{by (7.3.2)} \\ &= \psi_{\xi + \lambda}^{\xi + \lambda} \phi(s_\alpha(\xi + \lambda_0)) && \text{by (7.3.1)} \\ &= \psi_{\xi + \lambda_0}^{\xi + \lambda} (-\phi(\xi + \lambda_0)) && \text{by hypothesis} \\ &= -\phi(\xi + \lambda) && \text{by (7.3.2)}. \end{aligned}$$

This is (b). The implications (b) $\implies$ (c) $\implies$ (d) are obvious.

(d) $\implies$ (e): Suppose (e) fails for some weight $\lambda_0 \in \Lambda$;

we will show that (d) fails as well. So fix λ as in (d). The stabilizer of $\xi + \lambda_0$ in $W(\mathfrak{g}, \mathfrak{h})$ is $\{1, s_\alpha\}$ by Chevalley's theorem. By Proposition 7.2.22(a), $\phi(\xi + \lambda)$ is the character

of a representation Y ; and Y is irreducible by Proposition 7.3.3. Define

$$Z = \psi_{\xi + \lambda}^{\xi + \lambda_0} Y.$$

By Proposition 7.2.22,

$$\Theta(Z) = \phi(\xi + \lambda_0),$$

which is non-zero by hypothesis; so Z is non-zero. By Proposition 4.5.8,

$$\begin{aligned} 0 &\neq \text{Hom}_{\mathfrak{g}, K}(Z, Z) \\ &= \text{Hom}_{\mathfrak{g}, K}(\psi_{\xi + \lambda}^{\xi + \lambda_0} Y, \psi_{\xi + \lambda}^{\xi + \lambda_0} Y) \\ &\cong \text{Hom}_{\mathfrak{g}, K}(Y, \psi_{\xi + \lambda_0}^{\xi + \lambda} \psi_{\xi + \lambda}^{\xi + \lambda_0} Y). \end{aligned}$$

Set

$$W = \psi_{\xi + \lambda_0}^{\xi + \lambda} \psi_{\xi + \lambda}^{\xi + \lambda_0} Y;$$

then since Y is irreducible, we have just shown that Y may be regarded as a submodule of W . So

$$\Theta(W) - \Theta(Y) = \Theta(W/Y)$$

is the character of a representation. On the other hand, by Proposition 7.2.22(b),

$$\begin{aligned} \Theta(W) &= \phi(\xi + \lambda) + s_\alpha \cdot \phi(\xi + \lambda) \\ &= \Theta(Y) + s_\alpha \cdot \phi(\xi + \lambda); \end{aligned}$$

so $s_\alpha \cdot \phi(\xi + \lambda)$ is the character of a representation, and (d) fails. To show that (e) $\implies$ (f), we only have to find such a λ ; and its existence (on a covering group) is guaranteed by Lemma 7.3.5.

f $\Rightarrow$ a: Fix λ_0 as in (f), and λ as in (a). By Proposition 7.2.22(b),

$$\begin{aligned} s_\alpha \cdot \phi(\xi + \lambda) + \phi(\xi + \lambda) &= \psi_{\xi+\lambda_0}^{\xi+\lambda} \phi(\xi + \lambda_0) \\ &= \psi_{\xi+\lambda_0}^{\xi+\lambda} (0) \\ &= 0, \end{aligned}$$

which is (a).

Q.E.D.

Definition 7.3.8 ([8], [23]). In the setting of

Proposition 7.3.7, let Π denote the set of simple roots of $R^+(\xi)$. The Borho-Jantzen-Duflo τ -invariant of X , $\tau(X)$, is the subset of Π consisting of those simple roots satisfying any of the equivalent conditions

$$7.3.7(a) - (f).$$

To be more precise, what we have defined is what Borho-

Jantzen and Duflo call the τ -invariant of the annihilator of X in $U(\mathfrak{g})$. The τ -invariant turns out to be a (very crude) measure of the size of X , with small representations having a large τ invariant. We will not dwell on this idea here; the interested reader is directed to the papers of Borho, Jantzen, and Joseph.

Corollary 7.3.9 In the setting of Definition 7.3.8,

- a) $\alpha \in \tau(X)$ if and only if $s_\alpha \cdot \theta(X) = -\theta(X)$.
- b) $\alpha \notin \tau(X)$ if and only if $s_\alpha \cdot \theta(X)$ is the character of a representation.

We will often use the more cumbersome formulation of Proposition 7.3.7, but this corollary is the best way to think about

the τ -invariant. The corollary provides decisive information about the action of s_α when $\alpha \in \tau(X)$; so we turn now to the investigation of what happens when $\alpha \notin \tau(X)$.

Proposition 7.3.10 Suppose $H^S \subseteq G$ is a maximally split Cartan subgroup, and X and X' are irreducible $(\mathfrak{g}, K)$ modules of nonsingular infinitesimal character $\xi \in (h^S)^*$. Suppose $\lambda \in \Lambda^S$, and $\xi + \bar{\lambda}$ is dominant (but possibly singular) for $R^+(\xi)$ (notation (7.2.16)). Let W_0 be the stabilizer of $\xi + \bar{\lambda}$ in $W(\mathfrak{g}, h)$.

a) There is an irreducible $(\mathfrak{g}, K)$ module Z and a non-negative integer m such that

$$\psi_{\xi+0}^{\xi+\lambda} \theta(X) = m \cdot \theta(Z)$$

b) If

$$\psi_{\xi+0}^{\xi+\lambda} \theta(X') = m' \cdot \theta(Z),$$

and $m' \neq 0$, then X' is isomorphic to a quotient of

$$\psi_{\xi+\lambda}^{\xi+0}(Z). \quad \text{If also } m \neq 0, \text{ then } \theta(X') \text{ occurs in } w \cdot \theta(X) \text{ for some } w \in W_0.$$

c) If there is a simple root $\alpha \in \tau(X)$ (Definition 7.3.8) such that

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0,$$

then $\psi_{\xi+0}^{\xi+\lambda} X$ is zero.

Proof. For (a), we may obviously assume that

$$\psi_{\xi+0}^{\xi+\lambda} X \neq 0.$$

Choose an irreducible $(\mathfrak{g}, K)$ submodule Z of this module. By

Proposition 4.5.8,

$$\operatorname{Hom}_{\mathcal{G}, K}(\psi_{\xi+\lambda}^{\xi+0} Z, X) \cong \operatorname{Hom}_{\mathcal{G}, K}(Z, \psi_{\xi+0}^{\xi+\lambda} X) \neq 0$$

by the choice of Z . So X is a quotient of $\psi_{\xi+\lambda}^{\xi+0} Z$. By the exactness of the translation functor, $\psi_{\xi+0}^{\xi+\lambda}(X)$ is a quotient of

$$\psi_{\xi+0}^{\xi+\lambda} \psi_{\xi+\lambda}^{\xi+0} Z.$$

By Proposition 7.3.2(b), the character of this module is

$$|W_0| \cdot \Theta(Z);$$

so the character of $\psi_{\xi+0}^{\xi+\lambda}(X)$ is also a multiple of $\Theta(Z)$.

The first claim of (b) came out in the argument just given for (a). If $m \neq 0$, then by Proposition 7.3.2(a),

$$\begin{aligned} \Theta(\psi_{\xi+\lambda}^{\xi+0}(Z)) &= \frac{1}{m} [\Theta(\psi_{\xi+\lambda}^{\xi+0} \psi_{\xi+0}^{\xi+\lambda} X)] \\ &= \frac{1}{m} \sum_{w \in W_0} w \cdot \Theta(X); \end{aligned}$$

so the quotient X' of $\psi_{\xi+\lambda}^{\xi+0}(Z)$ must occur in some $w \cdot \Theta(X)$, as claimed.

For (c), let Φ be the coherent family with

$$\Phi(\xi + 0) = \Theta(X).$$

Since s_α fixes $\xi + \lambda$, we compute

$$\begin{aligned} \psi_{\xi+0}^{\xi+\lambda}(\Theta(X)) &= \psi_{\xi+0}^{\xi+\lambda}(\Phi(\xi + 0)) \\ &= \Phi(\xi + \lambda) \text{ by (7.2.22) (a)} \\ &= \psi_{s_\alpha(\xi+0)}^{s_\alpha(\xi+\lambda)} \Phi(s_\alpha(\xi + 0)) \end{aligned}$$

$$\begin{aligned} &= \psi_{\xi+0}^{\xi+\lambda} \Phi(s_\alpha(\xi+0)) \text{ by (7.3.1)} \\ &= -\psi_{\xi+0}^{\xi+\lambda} \Phi(\xi+0) \text{ since } \alpha \in \tau(X) \\ &= -\psi_{\xi+0}^{\xi+\lambda}(\Theta(X)). \end{aligned}$$

So both sides are zero, proving (c). Q.E.D.

To refine this, we must study a special case more carefully.

Proposition 7.3.11 ([39], Theorems 5.16 and 5.20).

Suppose $H^S \subseteq G$ is a maximally split Cartan subgroup,

and X is an irreducible $(\mathcal{G}, K)$ module having non-singular infinitesimal character $\xi \in (\mathcal{H}^S)^*$. Fix

a simple root α in $R^+(\xi)$, and a weight $\lambda \in \Lambda^S$

(possibly for a covering group of G) satisfying

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0$$

$$\langle \beta, \xi + \bar{\lambda} \rangle > 0, \quad \beta \in R^+(\xi) - \{\alpha\}$$

(cf. Lemma 7.3.6). Suppose $\alpha \notin \tau(X)$. Define

$$W = \psi_{\xi+\lambda}^{\xi+0} \psi_{\xi+0}^{\xi+\lambda} X.$$

a) There are natural non-zero maps $X \rightarrow W, W \rightarrow X$; and

$$s_\alpha \cdot \Theta(X) = \Theta(W/X).$$

b) There is a unique irreducible $(\mathcal{G}, K)$ module Y such that

$$\text{i) } \alpha \notin \tau(Y)$$

$$\text{(ii) } \Theta(Y) \text{ occurs in } s_\alpha \cdot \Theta(X).$$

$$\text{c) } \Theta(Y) \text{ occurs exactly once in } s_\alpha \cdot \Theta(X).$$

Proof. That $X \subseteq W$ appeared in the proof of (d) $\implies$ (e) in Proposition 7.2.7, along with the formula for $s_\alpha \cdot \theta(X)$. The proof that X is a quotient of W is identical. Write

$$(7.3.12) \quad s_\alpha \cdot \theta(X) = \theta_1 + \dots + \theta_r + \theta_{r+1} + \dots + \theta_{r+s},$$

with each θ_i irreducible, and $\alpha \in \tau(\theta_i)$ if and only if $i > r$. Write ϕ for the coherent family with

$$\phi(\xi + 0) = \theta(X).$$

If μ is as in (d), then s_α fixes $\xi + \mu$; so

$$\begin{aligned} \psi_{\xi+0}^{\xi+\mu}(\theta(X)) &= \psi_{\xi+0}^{\xi+\mu} \phi(\xi + 0) \\ &= \phi(\xi + \mu) \text{ by (7.2.22) (a)} \\ &= \phi(s_\alpha(\xi + \mu)) \\ &= \psi_{s_\alpha(\xi+0)}^{s_\alpha(\xi+\mu)} \phi(s_\alpha(\xi + 0)) \text{ (by (7.2.22) (a))} \\ (7.3.13) \text{ (a)} \quad \psi_{\xi+0}^{\xi+\mu} \theta(X) &= \psi_{\xi+0}^{\xi+\mu} [s_\alpha \cdot \theta(X)] \text{ (by 7.3.1)} \end{aligned}$$

By Proposition 7.2.10(c), $\psi_{\xi+0}^{\xi+\mu}$ annihilates all θ_i with $i > r$; so

$$(7.3.13) \text{ (b)} \quad \psi_{\xi+0}^{\xi+\mu} \left[\sum_{i=1}^r \sum_{j=1}^{r+s} \theta_i \right] = \psi_{\xi+0}^{\xi+\mu} \theta(X).$$

If $\mu = \lambda$, the right side is non-zero by the definition of $\tau(X)$; so the left side is as well, and $r \geq 1$. Now apply s_α to (7.3.12). We get

$$\theta(X) = \sum_{i=1}^r \left[\sum_{j=1}^{r+s} \theta_{ij} \right] - \sum_{i=r+1}^s \theta_i;$$

here $r_i \geq 1$, and $\alpha \in \tau(\theta_{ij})$ if and only if $j > r_i$. So the right side has exactly

$$\sum_{i=1}^r r_i$$

irreducible constituents not having α in their τ invariants. Since X is irreducible, it follows that $r = 1$. This proves (b) and (c). Q.E.D.

Proposition 7.3.14 Suppose $H^S \subseteq G$ is a maximally split Cartan subgroup, and X is a $\mathfrak{g}$ module of nonsingular infinitesimal character $\xi \in (H^S)^*$. Fix a simple root $\alpha \in R^+(\xi)$, and a weight $\lambda \in \Lambda^S$ (possibly for a covering group of G) such that

$$\begin{aligned} \langle \alpha, \xi + \bar{\lambda} \rangle &= 0 \\ \langle \beta, \xi + \bar{\lambda} \rangle &> 0, \quad \beta \in R^+(\xi) - \{\alpha\} \end{aligned}$$

(cf. Lemma 7.3.6). Let

$$\psi_\alpha = \psi_{\xi+0}^{\xi+\lambda} \quad \phi_\alpha = \psi_{\xi+\lambda}^{\xi+0};$$

and let

$$X \xrightarrow{i} \phi_\alpha \psi_\alpha X, \quad \phi_\alpha \psi_\alpha X \xrightarrow{\pi} X$$

be the natural maps corresponding to the identity in the equivalences (Proposition 4.5.8)

$$\begin{aligned} \text{Hom}_{\mathfrak{g}}(X, \phi_\alpha \psi_\alpha X) &\cong \text{Hom}_{\mathfrak{g}}(\psi_\alpha X, \psi_\alpha X) \\ &\cong \text{Hom}_{\mathfrak{g}}(\phi_\alpha \psi_\alpha X, X). \end{aligned}$$

Then the composite map $i \circ \pi$ is zero.

The proof is based on the following useful trick.

Lemma 7.3.15 Let $I \subseteq U(\mathfrak{g})$ be any (two-sided) ideal, and let $M_I(\mathfrak{g})$ denote the category of $\mathfrak{g}$ modules M with

$$\text{Ann}(\mathcal{M}) \supseteq I.$$

Let ϕ be any natural construction which associates to each object X in $\mathcal{M}_I(\mathcal{G})$ a $\mathcal{G}$ -module map

$$\phi_X: X \rightarrow X.$$

Then there is a unique element

$$\bar{z} \in Z(\mathcal{G})/I \cap Z(\mathcal{G})$$

such that for all X in $\mathcal{M}_I(\mathcal{G})$,

$$\phi_X(x) = z \cdot x \quad (x \in X);$$

here z is a representative of $\bar{z}$ in $Z(\mathcal{G})$. In particular, if $I \cap Z(\mathcal{G})$ is maximal in $Z(\mathcal{G})$, then there is a constant $c \in \mathbb{C}$ such that

$$\phi_X(x) = cx.$$

Proof. The module

$$F = U(\mathcal{G})/I$$

is in $\mathcal{M}_I(\mathcal{G})$. Write bar for the map from $U(\mathcal{G})$ to $U(\mathcal{G})/I$.

Set

$$\bar{u}_0 = \phi_F(1) \in U(\mathcal{G})/I,$$

and fix a representative u_0 of $\bar{u}_0$ in $U(\mathcal{G})$. If x is in $\mathcal{M}_I(\mathcal{G})$ and $x \in X$, there is a unique map

$$\delta_x: F \rightarrow X$$

$$\delta_x(\bar{u}) = ux \quad (u \in U(\mathcal{G}));$$

this is well defined since I annihilates x . By the naturality of ϕ ,

$$\begin{aligned} \phi_X(x) &= \phi_X(\delta_x(1)) \\ &= \delta_x(\phi_F(1)) \\ &= \delta_x(\bar{u}_0) \end{aligned}$$

$$(*) \quad \phi_X(x) = u_0 \cdot x.$$

In particular, for $X = F$, we find

$$\phi_F(\bar{u}) = \bar{u}u.$$

Since this is a $\mathcal{G}$ -module map,

$$\overline{gu} = \overline{u_0gu} \quad (g \in \mathcal{G}, u \in U(\mathcal{G}))$$

and in particular

$$\text{ad}(g) u_0 \in I$$

for all $g \in \mathcal{G}$. Since $\mathcal{G}$ acts reductively on $U(\mathcal{G})$, we can choose

$$z \in \bar{u}_0 = u_0 + I$$

such that

$$\text{ad}(g)z = 0$$

for all $g \in G$. This means that $z \in Z(\mathcal{G})$. If we replace u_0 by z , then $(*)$ gives the first assertion of the proposition. The uniqueness and the other statements are obvious. Q.E.D.

Proof of Proposition 7.3.14 Because of Lemma 7.3.15, there is a constant $c \in \mathbb{C}$ such that

$$i \circ \pi = c$$

for all X having infinitesimal character ξ . So we need only find a single non-zero X with $i \circ \pi = 0$. Obviously it is enough to arrange $\psi_\alpha X = 0$, by the construction of i and π . To do this, choose $\Delta^+(g, h^s)$ so that

$$\Delta^+(g, h^s) \supseteq R^+(\xi)$$

α is simple in $\Delta^+(g, h^s)$;

this is possible by Lemma 7.3.6. Let

$$g = \ell + u$$

be the parabolic subalgebra of g with

$$\ell \supseteq h^s, \Delta(u) = \Delta^+(g, h^s) - \{\alpha\};$$

thus ℓ is $\mathcal{sl}(2)$ plus center. Let V be the irreducible finite dimensional representation of ℓ of highest weight $\xi - \rho$ (which exists since α is integral). Then V has infinitesimal character $\xi - \rho(u)$. Set

$$X = \text{pr}_{g^s}^g(V);$$

X has infinitesimal character ξ by the proof of Proposition 6.3.11. To calculate $\psi_\alpha X$, it is convenient simply to refer to the proof of Proposition 7.4.1 (although we could repeat the argument here). The conclusion is

$$\psi_\alpha X \cong \text{pr}_{g^s}^g [\psi_{\xi-\rho(u)+\lambda}^{\xi-\rho(u)+\lambda} V].$$

The module in brackets is finite dimensional (since V is),

but has infinitesimal character $\xi - \rho(u) + \lambda$. Now

$$\langle \alpha, \xi - \rho(u) + \lambda \rangle = 0$$

by hypothesis; and there is no finite dimensional representation of $\mathcal{sl}(2)$ of infinitesimal character 0. So the term in brackets is zero, and $\psi_\alpha X = 0$. Q.E.D.

Theorem 7.3.16 ([39], Corollary 6.17). Suppose

$H^S \subseteq G$ is a maximally split Cartan subgroup, and

X is an irreducible (g, K) module with nonsingular

infinitesimal character $\xi \in (h^S)^*$. Fix a simple

root $\alpha \in R^+(\xi)$ (notation (7.2.16)), and suppose

$\alpha \notin \tau(X)$ (Definition 7.3.8). Suppose that $\lambda \in \Lambda^S$

satisfies

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0$$

$$\langle \beta, \xi + \bar{\lambda} \rangle > 0,$$

$$\beta \in R^+(\xi) - \{\alpha\}.$$

Set

$$W = \begin{matrix} \xi+0 & \xi+\lambda \\ \xi+\lambda & \psi_{\xi+0} \end{matrix} X.$$

a) There is a chain complex with non-zero maps

$$X \rightarrow W \rightarrow X$$

b) X occurs in $s_\alpha(\Theta(X))$, as the unique irreducible constituent not having α in its τ invariant.

Proof. The maps were given in Proposition 7.3.11(a); that they form a chain complex is Proposition 7.3.14. This is (a).

In particular, X occurs at least twice in W . Now (b) is Proposition 7.3.11(b) and (c). Q.E.D.

Definition 7.3.17 In the setting of Theorem 7.3.16, define

$$\begin{aligned} K_\alpha &= K_\alpha(X) = \ker(W \rightarrow X) \\ Q_\alpha &= Q_\alpha(X) = W/X \\ U_\alpha &= U_\alpha(X) = K_\alpha/X = \ker(Q_\alpha \rightarrow X), \end{aligned}$$

the cohomology of the complex (7.3.16) (a).

Corollary 7.3.18 In the setting of Definition 7.3.17,

$$\begin{aligned} \Theta(K_\alpha) &= \Theta(Q_\alpha) = s_\alpha \Theta(X) \\ \Theta(U_\alpha) &= s_\alpha \Theta(X) - \Theta(X) \end{aligned}$$

Every irreducible constituent of U_α has α in its τ invariant.

Proof. This is contained in Proposition 7.3.11(a) and Theorem 7.3.16. Q.E.D.

Corollary 7.3.19 Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, and Θ is an irreducible character with nonsingular infinitesimal character $\xi \in \mathfrak{h}^*$. Fix a simple root $\alpha \in R^+(\xi)$. Then

- a) $\alpha \in \tau(\Theta)$ if and only if $s_\alpha \Theta = -\Theta$.
- b) $\alpha \notin \tau(\Theta)$ if and only if

$$s_\alpha \Theta = \Theta + \Theta_0.$$

Here Θ_0 is the character of a representation, and every irreducible constituent of Θ_0 has α in its τ

invariant. (If $\Theta = \Theta(X)$, $\Theta_0 = \Theta(U_\alpha(X))$.)

Proof. Part (a) is Corollary 7.3.9(a). Part (b) follows from Corollary 7.3.18. Q.E.D.

Corollary 7.3.20 In the setting of Corollary 7.3.19, suppose ϕ is a virtual character of infinitesimal character ξ . In the notation (0.3.12),

- a) if $\alpha \in \tau(\Theta)$, then

$$m(\Theta, s_\alpha \phi) = \sum_{\Theta'} m(\Theta', \phi) m(\Theta, U_\alpha(\Theta')) - m(\Theta, \phi) \alpha \notin \tau(\Theta')$$

- b) if $\alpha \notin \tau(\Theta)$, then

$$m(\Theta, s_\alpha \phi) = m(\Theta, \phi).$$

This is a formal consequence of Corollary 7.3.19.

Corollary 7.3.21 In the setting of Corollary 7.3.19, suppose $(\alpha_1 \dots \alpha_m)$ is a sequence of simple roots in $R^+(\xi)$ (possibly with repetitions), and

$$w = s_{\alpha_1} \dots s_{\alpha_m} \in W(\xi).$$

Then

$$w \cdot \Theta = \Theta + \Theta_0;$$

and every irreducible constituent of the virtual character Θ_0 has some α_i in its τ invariant.

This follows from Corollary 7.3.19 by induction on i . Corollary 7.3.22 ([39], Theorem 6.18). In the setting of Proposition 7.3.10, $\psi_{\xi+0}^{\xi+\lambda}$ X is non-zero

if and only if there is no simple root $\alpha \in \tau(X)$ such that

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0.$$

In that case, $\theta(X)$ is the unique irreducible character such that

$$\psi_{\xi+\theta}^{\xi+\lambda} \theta(X) = m \theta(Z),$$

with Z as in (7.3.10)(a) and m non-zero.

Proof. The "only if" part of the first claim is (7.3.10)(c); so suppose $\tau(X)$ has no such roots. The stabilizer W_0 of $\xi + \bar{\lambda}$ in $W(\xi)$ is generated by simple reflections around those roots; so if $w \in W_0$, Corollary 7.3.21 says (with $\theta = \theta(X)$)

$$w \cdot \theta = \theta + \theta_w;$$

and if θ_0 occurs in θ_w , then $\tau(\theta_0)$ contains a root orthogonal to $\xi + \bar{\lambda}$. In particular,

$$m(\theta, w \cdot \theta) = 1.$$

By Proposition 7.3.2(a),

$$\begin{aligned} m(\theta, \psi_{\xi+\lambda}^{\xi+\theta} \psi_{\xi+\theta}^{\xi+\lambda} \theta) &= \sum_{w \in W_0} m(\theta, w \cdot \theta) \\ &= |W_0| \neq 0. \end{aligned}$$

In particular, $\psi_{\xi+\theta}^{\xi+\lambda} \theta$ is non-zero. If θ' is another character satisfying the second condition in the statement, then Proposition 7.3.10 says that θ' must occur in some $w \cdot \theta$, and have no α in its τ invariant such that

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0.$$

As we have just seen, this forces $\theta = \theta'$. Q.E.D.

Corollary 7.3.23 Fix a maximally split Cartan subgroup H^S of G , and an irreducible (σ, K) module X of (possibly singular) infinitesimal character $\xi \in (h^S)^*$. Choose a positive root system R^+ for $R(\xi)$ (notation (7.2.16)) in such a way that ξ is dominant. Then there is a unique coherent family ϕ based on H^S and $\xi + \Lambda^S$ (Definition 7.1.1), characterized by the following two properties:

- a) $\phi(\xi + 0) = \theta(X)$
- b) Whenever $\lambda \in \Lambda^S$ and $\xi + \bar{\lambda}$ is R^+ -dominant, $\phi(\xi + \lambda)$ is irreducible or zero.

Write $\Pi \subseteq R^+$ for the set of simple roots. Fix

$\lambda_0 \in \Lambda^S$ so that $\xi + \bar{\lambda}_0$ is R^+ -dominant and nonsingular, and set

$$\tau(X) = \tau(\phi(\xi + \lambda_0)) \subseteq \Pi$$

(Definition 7.3.8). Then ϕ has in addition the

following property:

- c) Suppose $\lambda \in \Lambda^S$, and $\xi + \bar{\lambda}$ is R^+ -dominant. Then

$\phi(\xi + \lambda)$ is zero if and only if there is a root α in $\tau(X)$ such that

$$\langle \alpha, \xi + \bar{\lambda} \rangle = 0.$$

proof. By Theorem 7.2.7, we can choose ϕ_0 to satisfy (a).

Choose $\lambda_0 \in \Lambda^S$ so that $\xi + \bar{\lambda}_0$ is R^+ -dominant and nonsingular. By Proposition 7.2.22(a),

$$(*) \quad \psi_{\xi+\lambda_0}^{\xi+0} (\phi_0(\xi + \lambda_0)) = \phi_0(\xi + 0) = \phi(X).$$

So we can find an irreducible constituent γ of $\phi_0(\xi + \lambda_0)$

such that X occurs in

$$\psi_{\xi+\lambda_0}^{\xi+0} Y.$$

put

$$r = m(Y, \phi_0(\xi + \lambda_0))$$

and define m as in Proposition 7.3.10(a). By Corollary 7.3.22 and (*)

$$l = m(X, \phi_0(\xi + 0)) = mr;$$

so $m = r = l$. Now define ϕ to be the coherent family with

$$\phi(\xi + \lambda_0) = \phi(Y);$$

then (a) holds by the calculation just given, and (c) by Corollary 7.3.22. For (b), we know by Corollary 7.3.10 that $(\xi + \lambda)$ is a multiple of an irreducible character $\phi(Z)$; and we apply the first part of the argument to Z instead of Y , find that the multiple must be zero or one. Finally, suppose ϕ' has the same properties. By Corollary 7.3.22,

$$\phi'(\xi + \lambda_0) = \phi(Y) = \phi(\xi + \lambda_0),$$

$\phi = \phi'$ by Corollary 7.1.27.

Q.E.D.

Corollary 7.3.24 In the setting of Theorem 7.3.16, X is the unique submodule and the unique quotient of W .

Proof. By Corollary 7.3.23(b),

$$\psi_{\xi+0}^{\xi+\lambda} X = Y$$

is irreducible, and has X as its unique irreducible preimage under $\psi_{\xi+0}^{\xi+\lambda}$. So if Z is any irreducible $(\mathfrak{g}, K)$ module,

$$\begin{aligned} \text{Hom}_{\mathfrak{g}, K}(Z, W) &\cong \text{Hom}_{\mathfrak{g}, K}(\psi_{\xi+0}^{\xi+\lambda} Z, Y) \\ &\cong \begin{cases} \mathbb{C} & X \cong Z \\ 0 & X \not\cong Z. \end{cases} \end{aligned}$$

The quotient statement is similar.

Q.E.D.

We can now formulate the technical conjecture referred to at the beginning of Section 2.

Conjecture 7.3.25 ([42], Conjecture 3.15) In the setting of Theorem 7.3.16, the cohomology group $U_\alpha(Z)$ of the complex

$$0 \rightarrow X \rightarrow W \rightarrow X \rightarrow 0$$

is a completely reducible $(\mathfrak{g}, K)$ module.

Using this conjecture, we will construct (in Chapter 9) an algorithm for computing the composition series of the standard representations (or, equivalently, for expressing the irreducibles as integer combinations of the standard representations). This algorithm is what we call the Kazhdan-Lusztig conjecture.

As it stands, this conjecture looks a little ridiculous.

If one had a serious understanding of why it ought to be true, one could probably prove it; this is not a minor technical problem, but rather one which may require some fundamental new insight into the structure of $(\mathfrak{g}, K)$ modules. We can give some evidence for the conjecture, however. It is true

- a) when G has split rank one;
- b) when every simple factor of the root system $R(\xi)$

has rank at most 2 -- in particular, when

$\mathcal{G}$ has rank at most 2;

c) for $\mathrm{Sp}(3, \mathbb{R})$ and $\mathrm{SL}(4, \mathbb{R})$;

d) for complex classical groups of rank at most 4.

Many of the techniques needed to check these examples will be developed in Chapters 8 and 9. On the scale of what was known until recently about composition series, a conjecture which holds in all these cases should be on very solid ground. Nevertheless, the conjecture is really made only because it is not clear how to manage without it, and so it should still be viewed with a little skepticism.

§4. Translation functors and parabolic subalgebras.

As part of the program outlined at the beginning of Section 3, we need to study how the translation functors affect (cohomologically) induced representations.

Proposition 7.4.1 Suppose $\mathcal{G} = \ell + \mathfrak{u}$ is a parabolic subalgebra of $\mathcal{G}$ satisfying the hypotheses of Definition 6.3.1, and $H \subseteq L$ is a maximally split Cartan subgroup. Suppose W is an $(\ell, L \cap K)$ module of generalized infinitesimal character $\xi - \rho(\mathfrak{u}) \in \mathfrak{h}^*$, and assume that ξ is nonsingular. Let $\Lambda \subseteq \hat{H}$ be the lattice of weights of finite dimensional representations of G . Fix $\lambda \in \Lambda$, and suppose $\xi + \bar{\lambda}$ is dominant (possibly singular) for $R^+(\xi)$ (notation (7.2.16)). Choose a maximally split Cartan subgroup H^S of G , and an $\mathrm{Ad}(\mathcal{G})$ -inner automorphism

$$\psi: (\mathfrak{h}^S)^* \rightarrow \mathfrak{h}^*$$

as in Lemma 7.2.6. Write

$$\xi^S = \psi^{-1}(\xi) \in (\mathfrak{h}^S)^*$$

and fix a preimage

$$\lambda^S \in \Lambda^S$$

of λ under the map ψ of Lemma 7.2.6. Then

$$\psi_{\xi^S + \lambda^S}^S(\mathcal{R}_{\mathcal{G}}^{\mathcal{P}} W) \cong \mathcal{R}_{\mathcal{G}}^{\mathcal{P}}(\psi(\xi - \rho(\mathfrak{u})) + \lambda^S W).$$

Here the translation functor on the right is for the

group L .

Proof. Let F denote the finite dimensional representation of G of extremal weight λ^S . By Lemma 7.2.6, F has extremal H -weight λ . By Proposition 6.3.11, $\mathcal{R}_{\mathcal{G}}^{\mathcal{P}} W$ has generalized infinitesimal character ξ ; so by Definition 4.5.7,

$$(7.4.2) \text{ (a)} \quad \psi_{\xi^S + \lambda^S}^S(\mathcal{R}_{\mathcal{G}}^{\mathcal{P}} W) = P_{\xi + \bar{\lambda}}[(\mathcal{R}_{\mathcal{G}}^{\mathcal{P}} W) \otimes F].$$

Filter F as in Lemma 7.2.3; we may assume in addition that each F_i/F_{i-1} is irreducible as an L module. By Lemma 7.2.3, there is a spectral sequence converging to (7.4.2) (a), with E_2 terms

$$(7.4.2) \text{ (b)} \quad \bigoplus_{\sigma \in W(\mathcal{G}, \mathfrak{h})} \mathcal{R}_{\mathcal{G}}^{\mathcal{P}}(\sigma(\xi + \bar{\lambda}) - \rho(\mathfrak{u})[W \otimes F_i/F_{i-1}]).$$

Suppose that such a term is non-zero. By Corollary 7.1.13, there is a weight γ of F_i/F_{i-1} such that

$$[\xi - \rho(\mathfrak{u})] + \bar{\gamma} = \sigma' \sigma(\xi + \bar{\lambda}) - \rho(\mathfrak{u}),$$

for some $\sigma' \in W(\ell, \mathfrak{h})$. Since γ is also a weight of F ,

Lemma 7.2.18 implies that $\gamma = \lambda$, and $\sigma' \sigma = 1$. So F_i/F_{i-1} is

the irreducible representation E of L of extremal weight λ , and the E_2 term above is

$$\begin{aligned} & \mathcal{P}_{\mathcal{F}}^p(P_{\xi+\bar{\lambda}-\rho}(u) [W \otimes E]) \\ &= \mathcal{P}_{\mathcal{F}}^p(\psi_{\xi-\rho}(u) + \lambda \quad W) \end{aligned}$$

So the spectral sequence of (7.4.2)(b) collapses to the isomorphism we want. Q.E.D.

Translation away from a wall is much messier, but we will need to study it in a special case.

Proposition 7.4.3 Under the same general hypotheses as in Proposition 7.4.1, suppose that there is a simple root $\alpha \in R^+(\xi)$ such that

$$\begin{aligned} \langle \alpha, \xi + \bar{\lambda} \rangle &= 0 \\ \langle \beta, \xi + \bar{\lambda} \rangle &> 0, \quad \beta \in R^+(\xi) - \{\alpha\}. \end{aligned}$$

Suppose W_0 is an $(\ell, L \cap K)$ module of generalized infinitesimal character $\xi + \bar{\lambda} - \rho(u) \in \mathcal{K}^*$.

a) Suppose $\alpha \in \Delta(u, \mathcal{K})$. Set

$$\begin{aligned} W_1 &= \psi_{(\xi-\rho(u)+\lambda)}^{(\xi-\rho(u))+0} (W_0) \\ W_2 &= \psi_{S_{\alpha}\xi-\rho(u)+S_{\alpha}\lambda}^{S_{\alpha}\xi-\rho(u)+0} W. \end{aligned}$$

Then there is a long exact sequence

$$\dots \mathcal{P}_{\mathcal{F}}^{p-1} W_2 \rightarrow \mathcal{P}_{\mathcal{F}}^p W_1 \rightarrow \psi_{\xi+\lambda}^{\xi S+0} (\mathcal{P}_{\mathcal{F}}^p W_0) \rightarrow \mathcal{P}_{\mathcal{F}}^p (W_2) \rightarrow \mathcal{P}_{\mathcal{F}}^{p+1} W_1 \rightarrow \dots$$

b) Suppose $\alpha \in \Delta(\ell, \mathcal{K})$. Then

$$\psi_{\xi+\lambda}^{\xi S+0} (\mathcal{P}_{\mathcal{F}}^p W_0) \cong \mathcal{P}_{\mathcal{F}}^p (\psi_{(\xi-\rho(u))+\lambda}^{(\xi-\rho(u))+0} W_0)$$

c) Suppose $-\alpha \in \Delta(u)$. Then the long exact sequence of (a) exists, with W_1 and W_2 interchanged.

Proof. Let F denote the finite dimensional representation of G of extremal weight $-\lambda^S$. By Lemma 7.2.6, F has extremal H -weight $-\lambda$. Filter F by $\mathcal{O}_{\mathcal{F}}$ -invariant subspaces F_i so that F_1/F_{i-1} is irreducible. Since $\alpha \in R^+(\xi)$ and α kills $\xi + \bar{\lambda}$,

$$\langle \alpha, -\lambda \rangle = m,$$

a positive integer. So if X_{α} is a root vector for α ,

$$(7.4.4) \quad X_{\alpha}^m \cdot [F(-s_{\alpha}\lambda)] = F(-\lambda);$$

here $F(\gamma)$ is the γ weight space. Just as in the proof of Proposition 7.4.1, we get a spectral sequence converging to

$$\psi_{\xi+\lambda}^{\xi S+0} (\mathcal{P}_{\mathcal{F}}^p W_0)$$

with E_2 terms

$$(7.4.5) \quad \bigoplus_{\sigma \in W(\mathcal{O}_{\mathcal{F}}, \mathcal{K})} \mathcal{P}_{\mathcal{F}}^{p(P_{\sigma\xi-\rho}(u) [W_0 \otimes F_1/F_{i-1}])}.$$

Suppose such a term is non-zero. By Corollary 7.1.13, there is a weight $-\gamma$ of F_1/F_{i-1} , such that

$$\sigma^* \sigma \xi - \rho(u) = [\xi + \bar{\lambda} - \rho(u)] - \bar{\gamma};$$

that is,

$$\sigma' \sigma \xi = \xi + \bar{\lambda} - \bar{\gamma}.$$

By Lemma 7.2.18, $\sigma' \sigma$ is 1 or $s_{\alpha'}$, and γ is λ or $s_{\alpha} \lambda$, accordingly. So F_i / F_{i-1} has extremal weight $-\lambda$ or $-s_{\alpha} \lambda$, and the operator $P_{\sigma \xi - \rho(\mathcal{U})}$ is $P_{\xi - \rho(\mathcal{U})}$ or $P_{s_{\alpha} \xi - \rho(\mathcal{U})}$, accordingly. In case (b), these projection operators coincide, and the L-module of extremal weight $-\lambda$ also has extremal weight $-s_{\alpha} \lambda$; so the only non-zero term in (7.4.5) is

$$P_{\mathcal{U}}^P \left(\psi_{\left(\begin{smallmatrix} \xi - \rho(\mathcal{U}) + 0 \\ \xi - \rho(\mathcal{U}) + \lambda \end{smallmatrix} \right)} w_0 \right),$$

which is what we want. In case (a), let

$$F_{i_1} / F_{i_1 - 1} = E_1$$

be the L module of extremal weight $-\lambda$, and

$$F_{i_2} / F_{i_2 - 1} = E_2$$

the one of extremal weight $-s_{\alpha} \lambda$. By (7.4.4), $i_1 < i_2$. The non-zero terms in (7.4.5) are those for i_1 and i_2 , and they are clearly

$$P_{\mathcal{U}}^P w_1, P_{\mathcal{U}}^P w_2.$$

So the spectral sequence collapses to the long exact sequence we want. Part (c) is identical. Q.E.D.

Proposition 6.3.2 suggests that there should be "dual" results for cohomology. This is the case, and the proofs are identical (using Lemma 7.2.1 instead of Lemma 7.2.3). So we will simply state the results, and leave the proofs to the reader.

Proposition 7.4.6 Suppose $\mathcal{U} = \ell + \mathcal{U}$ is a parabolic subalgebra of $\mathfrak{g}$, and X is a $Z(\mathfrak{g})$ -finite $\mathfrak{g}$ module. With the formal hypotheses and notation of Proposition 7.4.1, there is a natural isomorphism of ℓ modules

$$P_{\xi - \rho(\mathcal{U})} H^i(\mathcal{U}, \psi_{\xi + \lambda}^{\xi + 0} X) \cong \psi_{\xi - \rho(\mathcal{U}) + \lambda}^{\xi - \rho(\mathcal{U}) + 0} H^i(\mathcal{U}, X).$$

Proposition 7.4.7 In the formal setting of Proposition 7.4.3, suppose X is a $Z(\mathfrak{g})$ -finite $\mathfrak{g}$ module.

Set

$$X_0 = \psi_{\xi + 0}^{\xi + \lambda} X$$

$$w_1^i = \psi_{\xi - \rho(\mathcal{U}) + 0}^{\xi - \rho(\mathcal{U}) + \lambda} H^i(\mathcal{U}, X)$$

$$w_2^i = \psi_{s_{\alpha} \xi - \rho(\mathcal{U}) + 0}^{s_{\alpha} \xi - \rho(\mathcal{U}) + s_{\alpha} \lambda} H^i(\mathcal{U}, X).$$

a) If $\alpha \in \Delta(\mathcal{U}, \mathcal{h})$, there is a long exact sequence

$$\dots w_1^{i-1} \rightarrow w_2^i \rightarrow P_{\xi - \rho(\mathcal{U}) + \bar{\lambda}} H^i(\mathcal{U}, X_0) \rightarrow w_1^i \rightarrow w_2^{i+1} \rightarrow \dots$$

b) If $\alpha \in \Delta(\ell, \mathcal{h})$, then

$$P_{\xi - \rho(\mathcal{U}) + \bar{\lambda}} H^i(\mathcal{U}, X_0) \cong w_1^i$$

c) If $-\alpha \in \Delta(\mathcal{U}, \mathcal{h})$, we have the exact sequence of (a) with w_1^* and w_2^* interchanged.

If X is a $(\mathfrak{g}, L \cap K)$ module, we get $(\ell, L \cap K)$ module maps in both cases.

To understand this proposition, think of the case when $\ell = \mathcal{h}$ is a Cartan subalgebra. "Translation" for an abelian group simply means adding the specified weight. So (7.4.7)(a)

says roughly that the $\xi - \rho + \bar{\lambda}$ weight space of $H^i(\mathcal{U}, X_0)$ is built out of the $\xi - \rho$ and $s_\alpha \xi - \rho$ weight spaces of $H^i(\mathcal{U}, X)$. As an example, suppose X is a finite dimensional representation of infinitesimal character ξ . Then X_0 is a finite dimensional representation having singular infinitesimal character, so it must be zero. So (7.4.7) (a) says that

$$w_2^i \cong w_1^{i-1};$$

that is, the $s_\alpha \xi - \rho$ weight space of $H^i(\mathcal{U}, X)$ is isomorphic to the $\xi - \rho$ weight space of $H^{i-1}(\mathcal{U}, X)$. It is a good exercise to check that this agrees with Kostant's theorem (3.2.3). A slightly harder exercise is to deduce Theorem 3.2.3 from Proposition 7.4.7, using a knowledge of H^0 to get the induction going.

Chapter 8. Reducibility of the Standard Representations.

In the preceding chapter, we proved a strong result about tensor products of finite dimensional and infinite dimensional irreducible representations (Theorem 7.3.16). This can be used to find reducibility of the standard representations: whenever a representation fails to satisfy Theorem 7.3.16, it must be reducible. Perhaps surprisingly, this technique detects all the reducibility of the standard representations with nonsingular infinitesimal character (Theorem 8.6.6). For singular infinitesimal character, there is a weaker result (Proposition 8.7.6). To carry out this program, we must study tensor products of finite dimensional and standard representations. This would be a little easier if we had realized the standard representations as ordinary induced representations, in accordance with Theorem 6.6.15 (which we did not prove). The interested reader can find such an approach in [39]. However, most of the technical problems connected with using cohomological induction must be treated eventually to formulate the conjecture of Section 9.6 on composition series of standard representations; so we may as well treat them now. The key intermediate result is Theorem 8.1.6, which describes some more cohomology of an irreducible representation (extending the construction of Section 5.4).

§1. Cohomology of irreducible representations.

Lemma 8.1.1 ([46], Definition 6.8). Suppose $H = TA$

is a θ -stable Cartan subgroup of G , and $\mathfrak{g} = \mathfrak{h} + \mathfrak{u}$ is a θ -stable parabolic subalgebra of $\mathfrak{g}$, such that $\mathfrak{h} \subseteq \mathfrak{h}$. Let MA be the Langlands decomposition of the centralizer of A in G , and fix a positive root system

$$\Delta^+(m, t)$$

for t in m . Let $\mathfrak{g}^1 = \mathfrak{h}^1 + \mathfrak{u}^1$ be any other

θ -stable parabolic subalgebra of $\mathfrak{g}$, satisfying

$$a) \quad \mathfrak{h} \supseteq \mathfrak{h}^1 \supseteq \mathfrak{h};$$

$$b) \quad \mathfrak{h}_0^1 \text{ is quasisplit, with } \mathfrak{h}_0 \text{ as a maximally split Cartan;}$$

$$c) \quad \Delta^+(m, t) \subseteq \Delta(\mathfrak{u}^1, t).$$

$$\text{Set } t_1 = \dim(\mathfrak{u}^1 \cap \mathfrak{p} / \mathfrak{u}^1 \cap \mathfrak{h} \cap \mathfrak{p}), \quad t_2 = \dim(\mathfrak{u} / \mathfrak{u} \cap \mathfrak{u}^1).$$

Then the one dimensional representation

$$V = [\Lambda^1(\mathfrak{u}^1 \cap \mathfrak{p} / \mathfrak{u}^1 \cap \mathfrak{h} \cap \mathfrak{p}) \otimes \Lambda^2(\mathfrak{u} / \mathfrak{u} \cap \mathfrak{u}^1)]$$

of T is independent of $\mathfrak{g}^1$ (subject to the conditions

$$(a), (b), \text{ and } (c)). \text{ Furthermore, if } t = \dim(\mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{p} + \mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{h}), \text{ then}$$

$$V \cong \Lambda^t(\mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{p} + \mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{h}).$$

Proof. Observe first that if V_1 is any T -invariant subspace of $\mathfrak{g}$, then $\bar{V}_1 \cong V_1^*$ as T modules. For if $t \in T$ acts on $v \in V_1$ by the scalar z , then it acts on $\bar{v}$ by $\bar{z}$. Since T is compact, $|z| = 1$, so $\bar{z} = z^{-1}$. This proves the claim.

Because of the triangular decompositions

$$\mathfrak{g} = \mathfrak{u} + \mathfrak{h} + \mathfrak{u} = \mathfrak{u}^1 + \mathfrak{h}^1 + \mathfrak{u}^1,$$

there are T module isomorphisms

$$\begin{aligned} \mathfrak{u}^1 \cap \mathfrak{p} / \mathfrak{u}^1 \cap \mathfrak{h} \cap \mathfrak{p} &\cong \mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{p} + \mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{h} \\ \mathfrak{u} / \mathfrak{u} \cap \mathfrak{u}^1 &\cong \mathfrak{u} \cap \mathfrak{u}^1 + \mathfrak{u} \cap \mathfrak{h}^1 \\ &\cong \mathfrak{u} \cap \mathfrak{u}^1 \\ &\cong \mathfrak{u} \cap \mathfrak{u}^1 \cap \mathfrak{p} + \mathfrak{u} \cap \mathfrak{u}^1 \cap \mathfrak{h}; \end{aligned}$$

The third isomorphism uses (a), which implies that $\mathfrak{u} \cap \mathfrak{h}^1 = 0$. By the observation just made, the top exterior powers of $\mathfrak{u} \cap \mathfrak{u}^1 \cap \mathfrak{p}$ and $\mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{p}$ are dual T modules. So if

$$V_0 = \mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{p} + \mathfrak{u}^1 \cap \mathfrak{u} \cap \mathfrak{h}$$

has dimension t , then

$$V \cong \Lambda^t(V_0).$$

This proves the last claim. We want to identify this with some T module which is independent of $\mathfrak{u}^1$. To do that, fix a parabolic subgroup

$$P = M A N$$

with Levi factor MA . List the positive roots of $\mathfrak{h}$ in m as

$$\{\alpha_1, \dots, \alpha_s\};$$

by (c), these are also roots of $\mathfrak{h}$ in $\mathfrak{u}^1$. The remaining roots of $\mathfrak{h}$ in $\mathfrak{u}^1$ are complex roots; they occur in pairs $\{\beta, \theta\beta\}$. Exactly one root in such a pair is a root of $\mathfrak{h}$ in $\mathfrak{u}$ (the Lie algebra of the nil radical of P); so we can list the complex roots of $\mathfrak{h}$ in $\mathfrak{u}^1$ as

$$\{\beta_1, \dots, \beta_r, \theta\beta_1, \dots, \theta\beta_r\},$$

with

$$\{\beta_j\} = \Delta(n \cap u^1, h).$$

It follows that a basis for V_0 can be given as

$$\begin{aligned} & \{X_{\alpha_i} |_{\alpha_i} \mid \alpha_i \text{ is noncompact, and } \alpha_i \in \Delta(u, h)\} \\ & \cup \{\bar{X}_{\alpha_i} |_{\alpha_i} \mid \alpha_i \text{ is compact, and } -\alpha_i \in \Delta(u, h)\} \\ & \cup \{X_{\beta_j} - \theta X_{\beta_j} \mid \beta_j \in \Delta(u, h)\} \\ & \cup \{\bar{X}_{\beta_j} + \theta \bar{X}_{\beta_j} \mid -\beta_j \in \Delta(u, h)\}; \end{aligned}$$

here X_δ denotes a root vector for the root δ . Since T commutes with θ , the action of T on (for example) $X_{\beta_j} - \theta X_{\beta_j}$ is the same as on X_{β_j} . So the action of T on V_0 is the same as on

$$\left(\begin{array}{c} \Delta \\ \alpha_i \in \Delta(u \cap p, h) \end{array} X_{\alpha_i} \right) \left(\begin{array}{c} \Delta \\ -\alpha_i \in \Delta(u \cap h, h) \end{array} X_{-\alpha_i} \right) \left(\begin{array}{c} \Delta \\ \beta_j \in \Delta(u, h) \end{array} X_{\beta_j} \right) \\ \left(\begin{array}{c} \Delta \\ -\beta_j \in \Delta(u, h) \end{array} \bar{X}_{\beta_j} \right).$$

Now the first two factors are independent of u^1 by (c). For the last two, notice that since $\overline{\beta_j} = -\theta\beta_j$,

$$\begin{aligned} & \{\beta_j, \beta_j \in \Delta(u, h)\} \cup \{\overline{\beta_j} \mid -\beta_j \in \Delta(u, h)\} \\ & = \Delta(u \cap u^1 \cap n, h) \cup \Delta(u \cap \bar{u}_1 \cap \bar{n}, h) \\ & = \Delta(u \cap n, h). \end{aligned}$$

So the last two factors are

$$\begin{array}{c} \Delta \\ \beta \in \Delta(u \cap n, h) \end{array} X_\beta,$$

which is independent of u^1 .

Q.E.D.

Lemma 8.1.1.2 Suppose $H = TA$ is a θ -stable Cartan

subgroup of G , and $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable

parabolic subalgebra such that $\mathfrak{h} \subseteq \mathfrak{l}$. Fix

$\gamma \in (\hat{H}')_G$, a regular character of H with respect

to G (Definition 6.6.1). Let MA be the Langlands

decomposition of the centralizer of A in G ; and

if $\gamma = (\Gamma, \bar{\gamma})$, put

$$\Delta^+(m, \mathfrak{x}) = \{\alpha \in \Delta(m, \mathfrak{x}) \mid \langle \alpha, \bar{\gamma} \rangle > 0\}.$$

Choose a θ -stable Cartan subalgebra $\mathfrak{g}^1 = \mathfrak{l}^1 + \mathfrak{u}^1$

satisfying (a) - (c) of Lemma 8.1.1; and with V as

in that lemma define

$$\begin{aligned} \bar{\gamma} \mathfrak{g}^1 &= \bar{\gamma} - \rho(u) \\ \Gamma \mathfrak{g}^1|_A &= \Gamma|_A \\ \Gamma \mathfrak{g}^1|_T &= \left(\Gamma|_T \right) \otimes V^*. \end{aligned}$$

Then $\gamma \mathfrak{g}^1 = (\Gamma \mathfrak{g}^1, \bar{\gamma} \mathfrak{g}^1) \in (\hat{H}')_L$ is a regular character of H with respect to L .

We may occasionally write $\gamma \mathfrak{g}^1$ instead of $\gamma \mathfrak{g}$ for precision.

Proof. The centralizer of A in L is $(M \cap L)A$. Every root of $\mathfrak{h}$ in $\mathfrak{l}$ is orthogonal to $\rho(u)$; so if α is a root of $\mathfrak{x}$ in $m \cap \mathfrak{l}$, 4.5.1(a) implies that

$$\langle \alpha, \bar{\gamma} \mathfrak{g}^1 \rangle = \langle \alpha, \bar{\gamma} - \rho(u) \rangle = \langle \alpha, \bar{\gamma} \rangle \neq 0.$$

So $\bar{\gamma} \mathfrak{g}^1$ satisfies condition (a) of Definition 4.5.1, and (in the notation there)

$$\Delta^+(m_{n\ell}, t) = \Delta^+(m, t) \cap \Delta(\ell, k).$$

We must verify condition (b) of Definition 4.5.1, which is

$$(*) \quad d\Gamma_{\mathcal{G}} = \bar{\gamma}_{\mathcal{G}} + \rho_{m_{n\ell}} - 2\rho_{m_{n\ell}k}.$$

We know that the analogous condition for Γ holds. By the proof of Proposition 6.6.2, we have an identity of weights of $\mathcal{L}$

$$\begin{aligned} \rho_{m_{n\ell}} - 2\rho_{m_{n\ell}k} &= \rho(u^1_{n\ell}) - 2\rho(u^1_{n\ell}k) \\ &= 2\rho(u^1_{n\ell}k) - \rho(u^1_{n\ell}); \end{aligned}$$

the second equation is just a rearrangement of the first.

The corresponding fact for $\mathcal{G}$ is

$$\rho_m - 2\rho_{mk} = 2\rho(u^1_{n\ell}k) - \rho(u^1_{n\ell}).$$

Taking into account the definition of $\bar{\gamma}_{\mathcal{G}}$, we can therefore rewrite (*) as

$$(**) \quad d\Gamma_{\mathcal{G}} = \bar{\gamma} - \rho(u) + 2\rho(u^1_{n\ell}k) - \rho(u^1_{n\ell}).$$

Write $\mu \in (t)^*$ for the weight of V . Then

$$\begin{aligned} d\Gamma_{\mathcal{G}} &= d\Gamma - \mu \\ &= \bar{\gamma} - \rho_m - 2\rho_{mk} - \mu \\ &= \bar{\gamma} + 2\rho(u^1_{n\ell}k) - \rho(u^1_{n\ell}) - \mu. \end{aligned}$$

So the desired identity (**) is just

$$(***) \quad \mu = (2\rho(u^1_{n\ell}k) - 2\rho(u^1_{n\ell}k)) + (\rho(u) - \rho(u^1_{n\ell}) + \rho(u^1_{n\ell}k)).$$

Obviously the first bracket is just

$$2\rho(u^1_{n\ell}k) - \rho(u^1_{n\ell}k),$$

which is the weight of the first factor of V (Lemma 8.1.1).

By the triangular decompositions $\mathcal{G} = u + \ell + \bar{u}$, $\mathcal{G} = u^1 + \ell^1 + \bar{u}^1$, the second term is

$$\begin{aligned} &(\rho(u) - \rho(u^1_{n\ell}k) - (\rho(u^1_{n\ell}k))) \\ &= (\rho(u_{n\ell}k) - (\rho(u_{n\ell}k))) - (\rho(u_{n\ell}k)) \\ &= 2\rho(u_{n\ell}k) = 2\rho(u_{n\ell}k). \end{aligned}$$

Here we have used the fact that a weight of $\mathcal{L}$ in $\mathcal{G}$ is purely imaginary, and hence equal to the negative of its complex conjugate; and the fact that $\ell^1 \subseteq \ell$ (hypothesis (a) of Lemma 8.1.1). This is the weight of the second factor of V ; so (**) holds, and the lemma is proved. Q.E.D.

Corollary 8.1.3 Suppose $H = TA$ is a θ -stable

Cartan subgroup of G , and $\mathcal{G} = \ell + u$ is a

θ -stable parabolic subalgebra, such that $\mathcal{L} \subseteq \ell$.

Fix $\gamma \in (\hat{H}')_{\Gamma}$, a regular character of H with

respect to L . Let MA be the Langlands decomposition-

tion of the centralizer of A in G . Call γ G -

admissible if $\bar{\gamma} + \rho(u)$ is not orthogonal to any

root of $\mathcal{L}$ in m . If γ is G -admissible, then

there is a unique regular character

$$\gamma_{\mathcal{G}} = \gamma_{\mathcal{G}}^{\theta} \in (\hat{H}')_G,$$

with the property that

$$(\gamma_g)_{\gamma_g} = \gamma.$$

This correspondence defines a bijection between G-admissible regular characters of H for L, and regular characters of H for G.

Proof. Define

$$\Delta^+(m, \mathcal{L}) = \{\alpha \in \Delta(m, \mathcal{L}) \mid \langle \alpha, \bar{\gamma} \rangle + \rho(u) > 0\},$$

and choose $\gamma_g^1 = \ell^1 + u^1$ as in Lemma 8.1.1. Then γ_g must be defined by

$$\begin{aligned} \bar{\gamma}_g &= \bar{\gamma} + \rho(u) \\ \gamma_g|_A &= \gamma|_A \\ \gamma_g|_T &= \left(\gamma|_T \right) \otimes V, \end{aligned}$$

with V as in Lemma 8.1.1. The proof of Lemma 8.1.2 can be read backwards to show that $\gamma_g \in (\hat{H}')_G$; the rest of the corollary is trivial.

Q.E.D.

Definition 8.1.4 Let c_0 be one half the dimension of the split part of a fundamental (that is, maximally compact) Cartan subalgebra of G. Fix a θ -stable Cartan subgroup $H = TA$ of G, and $\gamma \in \hat{H}'$. Suppose $\bar{\gamma} \in \mathcal{H}^*$ is regular. Define the integral length of γ , $\ell^I(\gamma)$, by

$$\ell^I(\gamma) = \frac{1}{2} \left| \{ \alpha \in R^+(\bar{\gamma}) \mid \theta \alpha \notin R^+(\bar{\gamma}) \} \right| + \frac{1}{2} \dim A - c_0;$$

here $R^+(\bar{\gamma})$ is the set of positive integral roots for $\bar{\gamma}$ (notation (7.1.16)).

It is not difficult to verify that $\ell^I(\gamma)$ is an integer; since we will not use that fact, we leave its verification to the reader. This notion of length will play the same role in inductive arguments that the length function on the Weyl group does in the theory of Verma modules; and in fact for complex groups the two kinds of length can be identified in a natural way. If γ is associated to a set of $\mathbb{Z}/2\mathbb{Z}$ data $\tilde{\gamma}$ (cf. (6.7.2), (2.2.3)), then clearly $\ell^I(\gamma) = \ell(\tilde{\gamma})$ (notation (2.2.8)).

Definition 8.1.5 In the setting of Definition

8.1.4, suppose $\gamma_g = \ell + u$ is a θ -stable parabolic, with $\mathcal{H} \subseteq \ell$. Define the γ_g -length of γ , $\ell_{\gamma_g}(\gamma)$, by

$$\begin{aligned} \ell_{\gamma_g}(\gamma) &= |\{ \alpha \in \Delta(u, \mathcal{H}) \mid \alpha \text{ is compact imaginary,} \\ &\quad \text{and } \langle \alpha, \bar{\gamma} \rangle < 0 \}| \\ &\quad + |\{ \alpha \in \Delta(u, \mathcal{H}) \mid \alpha \text{ is noncompact} \\ &\quad \text{imaginary, and } \langle \alpha, \bar{\gamma} \rangle > 0 \}| \\ &\quad + \frac{1}{2} |\{ \alpha \in \Delta(u, \mathcal{H}) \mid \theta \alpha \neq \alpha \}|. \end{aligned}$$

Some of these definitions are tied together by

Theorem 8.1.6 ([46], Theorem 6.10). Suppose

$\gamma_g = \ell + u$ is a θ -stable parabolic subalgebra of $\mathfrak{g}$, and $H = TA$ is a θ -stable Cartan subgroup of L. Fix regular characters $\gamma, \gamma' \in \hat{H}'$ (notation (6.6.1)),

and assume that

- a) $\bar{\gamma} \in \mathcal{h}^*$ (cf. (6.6.1)) is nonsingular;
- b) There is an element $w \in W(\mathfrak{g}, \mathcal{h})$ commuting with θ , such that $w\bar{\gamma} = \bar{\gamma}'$.

Then the irreducible module $\bar{X}_L(\gamma'_\mathfrak{g})$ (notation (6.5.11), (6.6.3), and (8.1.2)) occurs in

$$H^*(\mathcal{U}, \bar{X}_G(\gamma))$$

if and only if γ and γ' are conjugate under K .

In that case it occurs exactly once, in degree $\ell_{\mathfrak{g}}(\gamma)$.

The proof of this theorem requires some preparation. We will use the translation principle of Proposition 7.2.3 to reduce to the case when γ is in "general position" in $\mathcal{h}^*$, in a sense to be made precise later. To do that, we need to know that the translation functors respect the parameters of the Langlands classification. The reason that this is not obvious is that if $\gamma^1 \in \hat{H}'$ is a second regular character, even with $\bar{\gamma}$ and $\bar{\gamma}^1$ in the same Weyl chamber, and $(\mathfrak{g}, \delta_{\mathfrak{v}})$, $(\mathfrak{g}^1, \delta_{\mathfrak{v}}^1)$ are the corresponding sets of θ -stable data (Proposition 6.6.2), then we may have $\mathfrak{g} \neq \mathfrak{g}^1$; so the standard representations $X(\gamma)$ and $X(\gamma^1)$ are constructed in apparently unrelated ways. Exactly the same problem appears if one views the Langlands classification in terms of induced representations (Theorems 6.6.14 and 6.6.15). So what we want is a sort of "independence of polarization" theorem. Our best result in that direction will be a consequence of Theorem 8.1.6 (Theorem 8.2.4), but we can prove a special case by hand (Lemma 8.1.13). First, however,

we do the calculation which is the formal part of Theorem 8.1.6.

Lemma 8.1.7 Let $\mathfrak{g} = \mathfrak{l} + \mathcal{U}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$. Fix a Cartan subgroup T^C of $L\cap K$, and a positive root system $\Delta^+(\mathfrak{l}_\cap \mathcal{h}, \mathcal{L}^C)$.

Set

$$\Delta^+(\mathcal{h}, \mathcal{L}^C) = \Delta^+(\mathfrak{l}_\cap \mathcal{h}, \mathcal{L}^C) \cup \Delta(\mathcal{U} \cap \mathcal{h}, \mathcal{L}^C).$$

Fix an $L\cap K$ -type $\pi_{\mathfrak{g}}$, of highest weight $\mu_{\mathfrak{g}} \in \hat{T}^C$; and let $\lambda_{\mathfrak{g}} \in (\mathcal{L}^C)^*$ be the weight associated to $\mu_{\mathfrak{g}}$ by Proposition 5.3.3. Let π be a representation of K , of highest weight μ , and let $\lambda \in (\mathcal{L}^C)^*$ be the associated weight.

- a) Suppose that $\pi_{\mathfrak{g}}$ occurs in

$$H^*(\mathcal{U} \cap \mathcal{h}, \pi) \otimes (\Lambda^* \mathcal{U} \cap \mathfrak{p})^*.$$

Then (cf. Definition 5.4.1)

$$||\pi||_{\text{lambda}} = \langle \lambda, \lambda \rangle \leq \langle \lambda_{\mathfrak{g}} + \rho(\mathcal{U}), \lambda_{\mathfrak{g}} + \rho(\mathcal{U}) \rangle.$$

- b) More precisely, suppose that for some $\sigma \in W_{K^0}^1$ (defined by 3.2.2) and subset A of $\Delta(\mathcal{U} \cap \mathfrak{p}, \mathcal{L}^C)$, that

$$\text{i) } \mu_{\mathfrak{g}} = \sigma(\mu + \rho_C) - \rho_C - 2\rho(A).$$

(Here $\rho_C = \rho(\Delta^+(\mathcal{h}, \mathcal{L}^C))$. If also

$$\text{ii) } \langle \alpha, \lambda_{\mathfrak{g}} + \rho(\mathcal{U}) \rangle \neq 0, \quad \alpha \in \Delta(\mathcal{U}, \mathcal{L}^C),$$

and equality holds in (a), then

$$\sigma\lambda = \lambda_{\mathfrak{g}} + \rho(\mathcal{U})$$

$$\Delta^+(\sigma) = \{\alpha \in \Delta(u \cap k, t^C) \mid \langle \alpha, \lambda_{\mathcal{G}} + \rho(u) \rangle < 0\}$$

$$A = \{\alpha \in \Delta(u \cap p, t^C) \mid \langle \alpha, \lambda_{\mathcal{G}} + \rho(u) \rangle > 0\}.$$

Proof. As we saw in the beginning of the proof of Proposition 5.4.2, the hypothesis of (a) guarantees hypothesis (i) of (b), at least for some choice of the highest weights. So we will use (b) (i) to deduce the inequality in (a), and then apply (b) (ii) to get the more precise claims which follow. Choose positive root systems

$$(\Delta^+)'(\mathcal{G}, t^C), \Delta^+(l, t^C)$$

making $\mu + 2\rho_C$ and $\mu_{\mathcal{G}} + 2\rho(\Delta^+(l \cap k, t^C))$, respectively, dominant.

Set

$$\begin{aligned} \Delta^+(\mathcal{G}, t^C) &= \Delta^+(l, t^C) \cup \Delta(u, t^C) \\ \rho_n &= \rho(\Delta^+(p)), \quad \rho_n' = \rho((\Delta^+)'(p)). \end{aligned}$$

Choose $\{\beta_i\} \subseteq \Delta^+(l, t^C)$, $\{\beta_j'\} \subseteq (\Delta^+)'(\mathcal{G}, t^C)$, and $\{c_i\}$, $\{c_j'\}$ in accordance with Proposition 5.3.3, so that

$$\begin{aligned} \lambda_{\mathcal{G}} &= \mu_{\mathcal{G}} + \rho(\Delta^+(l \cap k)) - \rho(\Delta^+(l \cap p)) + \frac{1}{2} \sum c_i \beta_i \\ \lambda &= \mu + \rho_C - \rho_n' + \frac{1}{2} \sum c_j' \beta_j'. \end{aligned}$$

We calculate from (b) (i) that

$$\begin{aligned} \lambda_{\mathcal{G}} + \rho(u) &= \mu_{\mathcal{G}} + \rho(\Delta^+(l \cap k)) - \rho(\Delta^+(l \cap p)) + \frac{1}{2} \sum c_i \beta_i + \rho(u) \\ &= (\sigma(\mu + \rho_C) - \rho_C - 2\rho(A)) + \rho_C + \rho_n - \\ &\quad 2\rho(\Delta^+(l \cap p)) + \frac{1}{2} \sum c_i \beta_i \\ &= \sigma(\mu + \rho_C) + \left[\rho_n - 2\rho(A) - (2\rho[\Delta^+(l \cap p)] + \frac{1}{2} \sum c_i \beta_i) \right] \end{aligned}$$

By Lemma 5.5.6, the right side is of the form

$$= \sigma(\mu + \rho_C) + \sigma(-\rho_n' + \sum_{\gamma \in (\Delta^+)'(p)} a_{\gamma} \gamma),$$

with a_{γ} between 0 and 1; so

$$\begin{aligned} &= \sigma(\mu + \rho_C - \rho_n' + \frac{1}{2} \sum c_j' \beta_j' + \sum a_{\gamma} \gamma - \frac{1}{2} \sum c_j' \beta_j') \\ (8.1.8) \quad &\lambda_{\mathcal{G}} + \rho(u) = \sigma(\lambda + \sum a_{\gamma} \gamma - \frac{1}{2} \sum c_j' \beta_j'). \end{aligned}$$

Now λ is orthogonal to all the β_j' , and dominant for $(\Delta^+)'$; so

$$\langle \lambda, \sum a_{\gamma} \gamma - \frac{1}{2} \sum c_j' \beta_j' \rangle \geq 0.$$

Taking the lengths of both sides of (8.1.8) and expanding the right side, we get (a). Suppose equality holds. Then

$$(8.1.9) \quad \sum a_{\gamma} \gamma - \frac{1}{2} \sum c_j' \beta_j' = 0$$

by the proof of the inequality, so (8.1.8) reads

$$\lambda_{\mathcal{G}} + \rho(u) = \sigma \lambda.$$

This is the first claim of (b). Now assume (b) (ii). Since λ is dominant, we have

$$\begin{aligned} \Delta^+(\sigma) &= \{\alpha \in \Delta(u \cap k, t^C) \mid \langle \sigma^{-1} \alpha, \rho_C \rangle > 0\} \\ &= \{\alpha \in \Delta(u \cap k, t^C) \mid \langle \sigma^{-1} \alpha, \lambda \rangle > 0, \text{ or} \\ &\quad \langle \sigma^{-1} \alpha, \lambda \rangle = 0 \text{ and } \langle \sigma^{-1} \alpha, \rho_C \rangle > 0\} \\ &= \{\alpha \in \Delta(u \cap k, t^C) \mid \langle \alpha, \sigma \lambda \rangle > 0, \text{ or } \dots\} \\ &= \{\alpha \in \Delta(u \cap k, t^C) \mid \langle \alpha, \lambda_{\mathcal{G}} + \rho(u) \rangle > 0\}; \end{aligned}$$

the second alternative does not arise because of (b)(ii). This is the second claim of (b). Finally, if we recall where $\{a_\gamma\}$ came from, (8.1.9) reads

$$(8.1.10) \quad \rho_n - 2\rho(A) - 2\rho(\Delta^+(\mathfrak{L}_n \mathfrak{p})) + \frac{1}{2} \sum c_{1i} \beta_i = \sigma(-\rho_n' + \frac{1}{2} \sum c_j' \beta_j')$$

Define

$$B = \{\alpha \in \Delta(\mathcal{U} \cap \mathfrak{p}, \mathcal{U}^C) \mid \langle \alpha, \lambda_{\mathcal{G}} + \rho(\mathcal{U}) \rangle > 0\}.$$

By applying (3.2.9) to $\mathcal{G}$ and $\mathfrak{k}$ separately, we compute

$$\rho_n - 2\rho(B) - 2\rho(\Delta^+(\mathfrak{L}_n \mathfrak{p})) = -\sigma\rho_n' + Q,$$

with

$$\langle Q, \lambda_{\mathcal{G}} + \rho(\mathcal{U}) \rangle = 0.$$

Since β_i is orthogonal to $\lambda_{\mathcal{G}} + \rho(\mathcal{U})$, and β_j' to λ , (8.1.10) can now be written

$$-\sigma\rho_n' - 2\rho(A - B \cap A) + 2\rho(B - B \cap A) = -\sigma\rho_n' + Q',$$

$$\langle Q', \lambda_{\mathcal{G}} + \rho(\mathcal{U}) \rangle = 0.$$

If we take the inner product of both sides with $\lambda_{\mathcal{G}} + \rho(\mathcal{U})$, we get

$$\langle -2\rho(A - B \cap A), \lambda_{\mathcal{G}} + \rho(\mathcal{U}) \rangle + \langle 2\rho(B - B \cap A), \lambda_{\mathcal{G}} + \rho(\mathcal{U}) \rangle$$

$$= 0.$$

By the definition of B and hypothesis (b)(ii), each term on the left is positive. So there must be no terms; that is,

$$A - B \cap A = B - B \cap A = \emptyset.$$

This is the last assertion of (b).

Q.E.D.

Corollary 8.1.11 Let $\mathcal{G} = \mathfrak{l} + \mathcal{U}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$, and $H \subseteq G$ and $H^1 \subseteq L$ θ -stable Cartan subgroups. Fix $\gamma \in \hat{H}'$, $\gamma^1 \in (\hat{H}^1)'$, and suppose $\bar{X}_L(\gamma_{\mathcal{G}}^1)$ (notation (8.1.2)) occurs in

$$H^*(\mathcal{U}, \bar{X}(\gamma)).$$

Then

$$||\gamma||_{\text{lambda}} \leq ||\gamma^1||_{\text{lambda}}$$

(notation (6.6.5)). More generally, if Y is any $(\mathcal{G}, K)$ module, and $\bar{X}_L(\gamma_{\mathcal{G}}^1)$ occurs in the cohomology of Y , then

$$||Y||_{\text{lambda}} \leq ||\gamma^1||_{\text{lambda}}.$$

Proof. Let $\pi_{\mathcal{G}}^1$ be a lambda-lowest $L \cap K$ -type of $\bar{X}_L(\gamma_{\mathcal{G}}^1)$, and $\lambda_{\mathcal{G}}^1$ the associated weight. By Lemma 6.5.4(b) (applied to $G = L$),

$$\lambda_{\mathcal{G}}^1 = \gamma_{\mathcal{G}}^1 \Big|_{\mathcal{U}^1}.$$

By Lemma 8.1.2, it follows that

$$\lambda_{\mathcal{G}}^1 + \rho(\mathcal{U}) = \gamma^1 \Big|_{\mathcal{U}^1};$$

and so (by Definition 6.6.5)

$$\langle \lambda_{\mathcal{G}}^1 + \rho(\mathcal{U}), \lambda_{\mathcal{G}}^1 + \rho(\mathcal{U}) \rangle = ||\gamma^1||_{\text{lambda}}.$$

By Theorem 5.2.2, there is a K -type π^1 of $\bar{X}(\gamma)$ such that $\pi_{\mathcal{G}}^1$ occurs in

$$H^*(\mathcal{U} \cap \mathfrak{k}, \pi^1) \otimes [\Lambda^*(\mathcal{U} \cap \mathfrak{p})]^*.$$

By Lemma 8.1.7, and Definition 5.4.1,

$$||\pi^1||_{\text{lambda}} \leq ||\gamma^1||_{\text{lambda}}.$$

Let π be a lambda-lowest K-type of $\bar{X}(\gamma)$. By Lemmas 6.6.5 and 6.6.6

$$||\gamma||_{\text{lambda}} = ||\pi||_{\text{lambda}} \leq ||\pi^1||_{\text{lambda}}$$

Together with the preceding inequality, this proves the first assertion. The second is identical. Q.E.D.

We will use Lemma 6.6.18 and the translation principle to identify constituents of cohomology. To do that, we need to know that, after an appropriate translation, the hypotheses of (6.6.18) are often satisfied.

Lemma 8.1.12 Suppose $\gamma \in \hat{H}'$ is a regular character of the θ -stable Cartan subgroup $H = TA$ of G . Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra, with L quasisplit, and $H \subseteq L$ a maximally split Cartan subgroup. Write

$$\Gamma_{\mathfrak{g}} = \delta \oplus \nu \in \hat{T} \times \hat{A}$$

(notation 8.1.2). Let $\Lambda \subseteq \hat{H}$ be the lattice of weights of finite dimensional representations of G .

a) There is a finite set $\{V_i\}$ of proper subspaces of $\mathfrak{h}^*$ such that if $\mu \in \Lambda$, and $\bar{\gamma} + \bar{\mu}$ does not belong to any of the V_i , then property (a) of Lemma 6.6.18 holds for $\bar{\gamma} + \bar{\mu}$. That is, if $w \in W(\mathfrak{g}, \mathfrak{h})$, and

$$(w(\bar{\gamma} + \bar{\mu}))|_{\mathfrak{t}} = (\bar{\gamma} + \bar{\mu})|_{\mathfrak{t}},$$

then $w \in W(L, H)$.

b) There is a subgroup $\Lambda_0 \subseteq \Lambda$, of finite index, and a coset $\mu_0 + \Lambda_0$ of Λ_0 in Λ , such that whenever $\mu \in \mu_0 + \Lambda_0$, then property (b) of Lemma 6.6.18 holds for $\bar{\gamma} + \bar{\mu}$. More precisely, if $\mu \in \mu_0 + \Lambda_0$, and α is any real root of $\mathfrak{h}$ in $\mathfrak{l}$, then

$$(\delta + \mu|_{\mathfrak{T}})(m_{\alpha}) = 1.$$

Proof. For (a), fix $w \in W(\mathfrak{g}, \mathfrak{h})$ and consider the equation in $\mathfrak{h}^*$

$$(*) \quad w(\lambda|_{\mathfrak{t}}) = \lambda|_{\mathfrak{t}}.$$

This is obviously linear, so it holds on a subspace of $\mathfrak{h}^*$. What (a) says is that if this subspace is not proper, then $w \in W(L, H)$. So suppose the subspace is not proper; that is, that (*) holds for all λ . For $\lambda \in \mathfrak{t}^*$, the equation reads

$$w\lambda = \lambda + (w\lambda)|_{\mathfrak{u}}.$$

Taking lengths of both sides, we conclude that $(w\lambda)|_{\mathfrak{u}} = 0$. By Chevalley's theorem, w is a product of reflections fixing λ . For generic λ , this means that w is a product of reflections in real roots; so in particular w belongs to $W(L, H)$.

For (b), let

$$\Lambda_0 = \{\mu \in \Lambda \mid (\mu|_{\mathfrak{T}})(m_{\alpha}) = 1, \text{ all real } \alpha \in \Delta(\mathfrak{l}, \mathfrak{h})\}.$$

Obviously Λ_0 has finite index in Λ . To prove (b), we only have to find one weight μ_0 with the desired property. Since

G is linear and T is compact, we can find a weight $\mu_0 \in \Lambda$ such that

$$\mu_0|_T = \delta^{-1}.$$

Obviously this satisfies (b).

Q.E.D.

Lemma 8.1.13 Suppose G is quasisplit, and $H^S = T^S A^S$ is a maximally split θ -stable Cartan subgroup of G .

Fix a fine representation $\delta \in \hat{T}^S$ (Definition 4.3.8)

and a character $\nu \in \hat{A}^S$; and regard $\gamma = \delta \otimes \nu$ as a

regular character of H^S (with $\Gamma = \delta \otimes \nu$ and $\bar{\gamma}$ its differential). Suppose $\bar{\gamma}$ is nonsingular. Let

$X(\gamma)$ denote the standard representation of G with

parameter γ (cf. 6.6.3 and 6.5.2). (Thus $X(\gamma)$ is

by definition an ordinary principal series representation of G .) Then $X(\gamma)$ can also be realized

in the following way. Choose a θ -stable parabolic

subalgebra $\mathfrak{a}\mathfrak{f} = \mathfrak{l} + \mathfrak{u}$ with $\mathfrak{h} \subseteq \mathfrak{l}$ and $\mathfrak{l}_0$ split;

$\mathfrak{a}\mathfrak{f}$ may be taken to be the parabolic defined by

a generic element of $i(\mathcal{T}_0^S)^*$ (cf. Definition 5.2.1).

Put $S = \dim(\mathfrak{u} \cap \mathfrak{k})$. Then (notation (6.3.1))

$$\mathcal{P}_{\mathfrak{a}\mathfrak{f}}^i(X_L(\gamma_{\mathfrak{a}\mathfrak{f}})) \cong \begin{cases} 0, & i \neq S \\ X(\gamma), & i = S. \end{cases}$$

Proof. Choose a Borel subgroup $B = T^S A^S N$ of G , in such a way that $\text{Re}(\nu)$ is negative:

$$\text{Re}\langle \alpha, \nu \rangle \leq 0, \text{ all } \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^S).$$

Write $\Delta^+(\mathfrak{a}\mathfrak{f}, \mathfrak{h}^S)$ for the set of roots of $\mathfrak{h}^S$ not in $\mathfrak{u}$.

Since δ is fine, its differential is orthogonal to all roots

Definition 4.3.8); so

$$\text{Re}\langle \alpha, \bar{\gamma} \rangle \geq 0, \text{ all } \alpha \in \Delta^+(\mathfrak{a}\mathfrak{f}, \mathfrak{h}).$$

We want to choose a weight $\mu \in \hat{H}^S$ of a finite dimensional representation of G , to use the translation principle. We require μ to have several special properties. (Recall that car denotes the differential of a weight.)

a) $\text{Re}\langle \alpha, \bar{\gamma} + \bar{\mu} \rangle > 0$, all $\alpha \in \Delta^+(\mathfrak{a}\mathfrak{f}, \mathfrak{h}^S)$;

b) $\langle \alpha, (\bar{\gamma} + \bar{\mu})|_{\mathfrak{t}^S} \rangle > 0$, all $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^S)$;

(here $\mathfrak{u}$ is as in the statement of the lemma)

c) if $w \in W(\mathfrak{a}\mathfrak{f}, \mathfrak{h}^S)$, and

$$(w(\bar{\gamma} + \bar{\mu})|_{\mathfrak{t}^S}) = (\bar{\gamma} + \bar{\mu})|_{\mathfrak{t}^S},$$

then $w \in W(\mathfrak{l}, \mathfrak{h}^S)$;

d) if α is a simple root of $\mathfrak{h}^S$ in Δ^+ , and α is real,

then (with m_α defined by (4.3.6))

$$(\gamma_{\mathfrak{a}\mathfrak{f}} + \mu)(m_\alpha) = 1.$$

First we will show why such a μ must exist. Let μ_0 be a very large multiple of 2ρ , the sum of the weights of H^S in $\mathfrak{u}$. Then $\bar{\mu}_0|_{\mathfrak{t}^S} = 0$, since $\theta\mathfrak{u}$ is opposite to $\mathfrak{u}$; so for every root α of $\mathfrak{h}^S$ in $\mathfrak{u}$

$$\text{Re}\langle \alpha, \bar{\gamma} + \bar{\mu}_0 \rangle \text{ is very large}$$

$$\langle \alpha, (\bar{\gamma} + \bar{\mu}_0)|_{\mathfrak{t}^S} \rangle = 0.$$

Next, let μ_1 be a large (but not very large) multiple of $2\rho(\mathfrak{u})$, the sum of the weights of H^S in $\mathfrak{u}$. Then $\bar{\mu}_1|_{\mathfrak{u}} = 0$, since $\theta\mathfrak{u} = \mathfrak{u}$; so for every root α of $\mathfrak{h}$ in $\mathfrak{u}$,

$$\langle \alpha, \bar{\mu}_1 \rangle = \langle \alpha, \bar{\mu}_1 | \chi_s \rangle \text{ is large (but not very large).}$$

Put $\mu_2 = \mu_0 + \mu_1$; then

$$\operatorname{Re} \langle \alpha, \bar{\gamma} + \bar{\mu}_2 \rangle \text{ is very large for all } \alpha \in \Delta^+$$

$$\langle \alpha, (\bar{\gamma} + \bar{\mu}_2) | \chi_s \rangle \text{ is large for all } \alpha \in \Delta(\mathcal{U}).$$

So (a) and (b) hold for any μ which is not too far from μ_2 . By Lemma 8.1.12, we can arrange (c) and (d) by picking μ so that $\mu \in \mu_0 + \Lambda_0$, and $\bar{\gamma} + \bar{\mu}$ is not in any V_i ; this is obviously possible for some μ near μ_2 .

Write $\gamma + \mu \in \hat{H}^S$ for the regular character $(\Gamma_{\emptyset\mu}, \bar{\gamma} + \bar{\mu})$, and consider the standard representation $X(\gamma + \mu)$. Because of condition (b) on μ , the $\emptyset$ -stable data corresponding to $\gamma + \mu$ are (cf. Proposition 6.6.2)

$$\left[\sigma_f, (\gamma + \mu) \right]_{TS} \otimes (\wedge^R(\mathcal{U} \cap \mathcal{P})^*) \Big|_{TS}, (\gamma + \mu) \Big|_{AS} = (\sigma_f, \delta^1, \nu^1).$$

In the definition of γ_{σ_f} , we can take $\sigma_f^1 = \sigma_f$; it follows that

$$\gamma_{\sigma_f} \Big|_{TS} = \gamma \Big|_{TS} \otimes (\wedge^R(\mathcal{U} \cap \mathcal{P}))^*$$

(with $R = \dim \mathcal{U} \cap \mathcal{P}$). So

$$(\sigma_f, \delta^1 \otimes \nu^1) = (\sigma_f, \gamma_{\sigma_f} + \mu).$$

By definition of the standard representations, therefore

$$(8.1.14) \text{ (a)} \quad \mathcal{R}_{\sigma_f}^i(X_L(\gamma_{\sigma_f} + \mu)) = \begin{cases} X(\gamma + \mu) & (i = S) \\ 0 & (i \neq S) \end{cases}.$$

We propose to show that there is a non-zero map

$$(8.1.14) \text{ (b)} \quad \operatorname{Ind}_{\mathcal{B}}^G(\gamma + \mu) \rightarrow X(\gamma + \mu).$$

Assume this for a moment, and apply the translation functor $\bar{\gamma} + \mu$ to everything. By Proposition 7.4.1

$$\begin{aligned} \psi_{\bar{\gamma} + \mu}^{\bar{\gamma} + 0}(\operatorname{Ind}_{\mathcal{B}}^G(\gamma + \mu)) &= \operatorname{Ind}_{\mathcal{B}}^G(\gamma) = X(\gamma) \\ \psi_{\bar{\gamma} + \mu}^{\bar{\gamma} + 0}(X(\gamma + \mu)) &= \mathcal{R}_{\sigma_f}^S \left(\psi_{(\bar{\gamma} - \rho(\mathcal{U})) + \mu}^{(\bar{\gamma} - \rho(\mathcal{U})) + 0} X_L(\gamma_{\sigma_f} + \mu) \right) \\ &= \mathcal{R}_{\sigma_f}^S(X_L(\gamma_{\sigma_f})). \end{aligned}$$

[The same argument proves the vanishing of $\mathcal{R}_{\sigma_f}^i(X_L(\gamma_{\sigma_f}))$ for $i \neq S$; so we only have to worry about $\mathcal{R}_{\sigma_f}^S$.] Because of Proposition 7.3.3, (8.1.14) now gives a non-zero map

$$\phi: X(\gamma) \rightarrow \mathcal{R}_{\sigma_f}^S(X_L(\gamma_{\sigma_f})).$$

By Corollary 6.4.14(b),

$$\operatorname{Hom}_{\mathcal{G}, K} \left[Y, \mathcal{R}^S(X_L(\gamma_{\sigma_f})) \right] \cong \operatorname{Hom}_{\ell, L \cap K} (H^R(\mathcal{U}, Y), X_L(\gamma_{\sigma_f}))$$

whenever Y has finite length. Apply this to $Y = \operatorname{image} \phi$.

By Corollary 6.4.13, $H^R(\mathcal{U}, Y)$ contains $\bar{X}_L(\gamma_{\sigma_f})$. By Corollary 8.1.11,

$$||Y||_{\text{lambda}} \leq ||\gamma||_{\text{lambda}}.$$

By Lemma 6.6.6, Y contains $\bar{X}(\gamma)$. By Theorem 6.5.10, ϕ is injective. In particular, $\bar{X}(\gamma)$ is a submodule of $\mathcal{R}^S(X_L(\gamma_{\sigma_f}))$. Now $X(\gamma + \mu)$ has a unique irreducible submodule, which occurs once in the composition series (by Theorem 6.5.10); so by the translation principle, this holds for $\mathcal{R}^S(X_L(\gamma_{\sigma_f}))$ as well. So this submodule must be $\bar{X}(\gamma)$. By Corollary 6.4.13, there

is a non-zero map

$$\psi: \mathcal{Q}^S(X_L(\gamma_g)) \rightarrow X(\gamma).$$

The image of ψ contains the unique irreducible submodule $\bar{X}(\gamma)$ of $X(\gamma)$; and since that is the unique irreducible submodule of the left side, ψ must be injective. Since we already have an injective map in the other direction, ψ must be an isomorphism. This proves the lemma; it remains to verify (8.1.14) (b).

By Lemma 4.1.5,

$$\left(\text{Ind}_B^G(\gamma + \mu) \right) \Big|_K = \text{Ind}_{TS}^K \left((\gamma + \mu) \Big|_{TS} \right).$$

By Lemma 6.6.16, it follows that the set of lambda-lowest K-types of the left side is exactly $A(\gamma + \mu)$. By our hypotheses (c) and (d) on μ , $\gamma + \mu$ satisfies conditions (a) and (b) of Lemma 6.6.18; and the conclusion of that lemma, applied to $\gamma = \text{Ind}_B^G(\gamma + \mu)$, is exactly (8.1.14) (b). Q.E.D.

As usual, once a result is proved for quasisplit groups, the machinery of Chapter 6 begins to grind; and we get

Corollary 8.1.15 Suppose $(\gamma^1, H, \delta, \nu)$ is a set of θ -stable data for G (Definition 6.5.1); here $\gamma^1 = \ell^1 + \mathcal{W}^1, \ell_0^1$ is quasisplit, $H = TA$ is a maximally split Cartan subgroup of L . Let $\gamma \in \hat{H}'$ be the corresponding regular character (Proposition 6.6.2), and suppose $\bar{\gamma} \in \mathcal{L}^*$ is nonsingular. Then the standard representation $X(\gamma)$ (notation (6.5.2), (6.6.3)) can also be realized as follows. Choose a θ -stable parabolic subalgebra $\mathfrak{g}^0 = \mathfrak{g} + \mathcal{W}^0$ of ℓ^1 , in such a way

that ℓ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$; thus L is split, with split Cartan subgroup H . Put

$$\mathcal{U} = \mathcal{U}^0 + \mathcal{U}^1, \quad S = \dim(\mathcal{U} \cap \mathcal{L}).$$

Then

$$\mathcal{R}_{\mathfrak{g}^1}^i(X_L(\gamma_{\mathfrak{g}})) = \begin{cases} X(\gamma) & (i = S) \\ 0 & (i \neq S). \end{cases}$$

Proof. Set

$$S^0 = \dim \mathcal{U}^0 \cap \mathcal{L}, \quad S^1 = \dim \mathcal{U}^1 \cap \mathcal{L}.$$

By definition and (6.5.10),

$$\mathcal{R}_{\mathfrak{g}^1}^i(X_{L^1}(\gamma_{\mathfrak{g}^1})) = \begin{cases} X(\gamma) & i = S^1 \\ 0 & i \neq S^1 \end{cases}$$

By Lemma 8.1.12,

$$\mathcal{R}_{\mathfrak{g}^0}^i \left[X_L((\gamma_{\mathfrak{g}^1})_{\mathfrak{g}^0}) \right] = \begin{cases} X_{L^1}(\gamma_{\mathfrak{g}^1}) & i = S^0 \\ 0 & i \neq S^0 \end{cases}.$$

So by induction by stages (Proposition 6.3.6)

$$\mathcal{R}^i \left[X_L((\gamma_{\mathfrak{g}^1})_{\mathfrak{g}^0}) \right] = \begin{cases} X(\gamma) & i = S = S^0 + S^1 \\ 0 & i \neq S \end{cases}.$$

So the only thing to check is that

$$(\gamma_{\mathfrak{g}^1})_{\mathfrak{g}^0} = \gamma_{\mathfrak{g}}.$$

This is a trivial consequence of the definition, and we leave it to the reader. Q.E.D.

Proof of Theorem 8.1.6 Just as in the proof of Lemma 8.1.13, we will use the translation principle to reduce to the case when γ satisfies some special hypotheses. Define

$$\Delta_Y^+ = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \operatorname{Re}\langle \alpha, \bar{\gamma} \rangle > 0; \text{ or } \operatorname{Re}\langle \alpha, \bar{\gamma} \rangle = 0, \text{ and} \\ (8.1.16) \quad \operatorname{Re}\langle \theta\alpha, \bar{\gamma} \rangle > 0; \text{ or } \operatorname{Re}\langle \alpha, \bar{\gamma} \rangle = \operatorname{Re}\langle \theta\alpha, \bar{\gamma} \rangle = 0, \text{ and} \\ \operatorname{Im}\langle \alpha, \bar{\gamma} \rangle > 0\}.$$

[This does not quite agree with the notation at (0.2.11) (a). Let $\mathfrak{g}^1 = \mathfrak{l}^1 + \mathfrak{u}^1$ be the parabolic subalgebra corresponding to γ (Proposition 4.5.2); and choose $\mathfrak{g}^0 = \mathfrak{l}^2 + \mathfrak{u}^0 \subseteq \mathfrak{l}^1$ so that $\mathfrak{l}^2$ is the centralizer of $\mathfrak{x}$ in $\mathfrak{g}$. Set

$$\mathfrak{u}^2 = \mathfrak{u}^0 + \mathfrak{u}^1, \quad \mathfrak{g}^2 = \mathfrak{l}^2 + \mathfrak{u}^2.$$

Choose a weight $\mu \in \hat{\mathfrak{H}}$ of a finite dimensional representation of G , with the following properties.

$$(8.1.17) \quad \left\{ \begin{array}{l} \text{(a) } \operatorname{Re}\langle \alpha, \bar{\gamma} + \bar{\mu} \rangle > 0, \quad \alpha \in \Delta_Y^+; \\ \text{(b) } \langle \alpha, (\bar{\gamma} + \bar{\mu}) \mid \mathfrak{x} \rangle > 0, \quad \alpha \in \Delta(\mathfrak{u}^2, \mathfrak{h}); \\ \text{(c) If } w \in W(\mathfrak{g}, \mathfrak{h}), \text{ and} \\ \qquad w(\bar{\gamma} + \bar{\mu}) \mid \mathfrak{x} = \bar{\gamma} + \bar{\mu} \mid \mathfrak{x}, \\ \text{then } w \in W(\mathfrak{l}^2, \mathfrak{h}); \\ \text{(d) If } \alpha \text{ is a simple positive root for } \Delta(\mathfrak{l}^2, \mathfrak{h}), \\ \text{and } m_\alpha \text{ is defined by (4.3.6), then} \\ \qquad (\gamma_2 + \mu)(m_\alpha) = 1. \end{array} \right.$$

The first problem is to prove that such μ exist. Let γ_0 denote the projection of $\bar{\gamma}$ on the real span of roots. Then

$$\langle \alpha, \gamma_0 \mid \mathfrak{x} \rangle > 0 \quad (\alpha \in \Delta(\mathfrak{u}^1, \mathfrak{h})) \\ \langle \alpha, \gamma_0 \rangle \geq 0 \quad (\alpha \in \Delta_Y^+).$$

Fix a large number N , and define

$$\gamma_1 = N\gamma_0 + \rho(\mathfrak{u}^1).$$

Obviously $\gamma_1 \mid \mathfrak{x}$ still defines the parabolic $\mathfrak{g}^1$. If $\alpha \in \Delta_Y^+$, we claim that

$$\langle \alpha, \gamma_1 \rangle \geq 0,$$

with equality if and only if

$$\operatorname{Re}\langle \alpha, \bar{\gamma} \rangle = \operatorname{Re}\langle \theta\alpha, \bar{\gamma} \rangle = 0.$$

If $\langle \alpha, \gamma_0 \rangle \neq 0$, then $\langle \alpha, N\gamma_0 \rangle$ is large, so $\langle \alpha, \gamma_1 \rangle$ is positive; and

$$0 \neq \langle \alpha, \gamma_0 \rangle = \operatorname{Re}\langle \alpha, \bar{\gamma} \rangle.$$

So the claim is true in this case. So suppose $\langle \alpha, \gamma_0 \rangle = 0$, but $\langle \theta\alpha, \gamma_0 \rangle \neq 0$. By (8.1.16), $\theta\alpha$ is positive, so

$$\langle \theta\alpha, \gamma_0 \rangle > 0.$$

So $\theta\alpha \in \Delta(\mathfrak{u}^1, \mathfrak{h})$; and since this is θ -stable, $\alpha \in \Delta(\mathfrak{u}^1, \mathfrak{h})$.

So

$$\langle \alpha, \gamma_1 \rangle = \langle \alpha, \rho(\mathfrak{u}^1) \rangle > 0.$$

Finally, suppose $\langle \alpha, \gamma_0 \rangle = \langle \theta\alpha, \gamma_0 \rangle = 0$. Then

$$\langle \alpha, \bar{\gamma} \mid \mathfrak{x} \rangle = \frac{1}{2} \langle \alpha + \theta\alpha, \gamma_0 \rangle = 0,$$

so $\alpha \in \Delta(\ell^1, h)$. This implies that

$$\langle \alpha, \gamma_1 \rangle = \langle \alpha, \rho(\mathcal{U}^1) \rangle = 0,$$

proving the claim.

Put

$$\gamma_2 = N\gamma_1 + \sum_{\substack{\alpha \in \Delta_{\bar{\gamma}} \\ \langle \alpha, \gamma_0 \rangle = \langle \theta\alpha, \gamma_0 \rangle = 0}} \alpha.$$

If α is as in the summation, then $\langle \alpha, \bar{\gamma} | \mathcal{U} \rangle = 0$, so

$$\langle \alpha, \bar{\gamma} \rangle = \langle \alpha, \bar{\gamma} | \sigma \rangle = \langle -\theta\alpha, \bar{\gamma} | \sigma \rangle = \langle -\theta\alpha, \bar{\gamma} \rangle.$$

By the definition of $\Delta_{\bar{\gamma}}^+$, $-\theta\alpha \in \Delta_{\bar{\gamma}}^+$; so the sum lies in σ^* .

So $\gamma_2 | \mathcal{U}$ still defines $\sigma_{\bar{\gamma}}^1$; and the reader can check that γ_2 is dominant and regular for $\Delta_{\bar{\gamma}}^+$. So

$$\gamma_3 = N\gamma_2 + \rho(\mathcal{U}^2)$$

is still dominant and regular for $\Delta_{\bar{\gamma}}^+$; but $\gamma_3 | \mathcal{U}$ defines the parabolic $\sigma_{\bar{\gamma}}^2 = \ell^2 + \mathcal{U}^2$. More precisely,

$$\begin{aligned} \langle \alpha, \gamma_3 \rangle &> 0 & (\alpha \in \Delta_{\bar{\gamma}}^+) \\ \langle \alpha, \gamma_3 | \mathcal{U} \rangle &= 0 & (\alpha \in \Delta(\ell^2)) \\ \langle \alpha, \gamma_3 | \mathcal{U} \rangle &> 0 & (\alpha \in \Delta(\mathcal{U}^2)) \end{aligned}$$

(Of course the second of these is trivial, since ℓ^2 is the centralizer of $\mathcal{U}$ in $\sigma_{\bar{\gamma}}$.) If μ is any weight of a finite dimensional representation which is not too far from $N\gamma_3$, then (8.1.17)(a) and (b) will be satisfied. By Lemma 8.1.12, such μ exist satisfying (c) and (d).

Suppose for a moment that Theorem 8.1.6 is known with γ replaced by $\gamma + \mu$; we will deduce it for γ . With w as in (8.1.6)(b), set $\mu' = w\mu$. Fix a maximally split Cartan subgroup H^S of G , and choose $\bar{\gamma}^S, \mu^S$ in $(h^S)^*$ and Λ^S as in Proposition 7.4.1; here $\bar{\gamma}$ and μ are playing the roles of ξ and λ there. By (8.1.17)(b), the θ -stable parabolic defined by $\bar{\gamma} + \bar{\mu}$ is $\sigma_{\bar{\gamma}}^2 = \ell^2 + \mathcal{U}^2$. So by Definition 6.5.2,

$$X(\gamma + \mu) = \mathcal{B}_{\sigma_{\bar{\gamma}}^2}^{S^2}(X_{L^2}(\gamma_{\sigma_{\bar{\gamma}}^2} + \mu)),$$

and $\bar{X}(\gamma + \mu)$ is its unique irreducible submodule. So

$$\bar{\gamma}_{\bar{\gamma}^S + \mu^S}^{S+0}(X(\gamma + \mu)) = \mathcal{B}_{\sigma_{\bar{\gamma}}^2}^{S^2} \left(\bar{\gamma}_{\bar{\gamma}^S - \rho(\mathcal{U}^2) + 0}^{S-\rho} \psi_{\bar{\gamma}^S - \rho(\mathcal{U}^2) + \mu^S} X_{L^2}(\gamma_{\sigma_{\bar{\gamma}}^2} + \mu) \right)$$

(by Proposition 7.4.1)

$$= \mathcal{B}_{\sigma_{\bar{\gamma}}^2}^{S^2}(X_{L^2}(\gamma_{\sigma_{\bar{\gamma}}^2}))$$

(by Proposition 7.4.1 applied to L^2 and the real Borel subgroup from which $X_{L^2}(\gamma_{\sigma_{\bar{\gamma}}^2} + \mu)$ is induced). By Corollary 8.1.15, we deduce

$$(8.1.18) \text{ (a)} \quad \bar{\gamma}_{\bar{\gamma}^S + \mu^S}^{S+0}(X(\gamma + \mu)) = X(\gamma).$$

So

$$(8.1.18) \text{ (b)} \quad \bar{\gamma}_{\bar{\gamma}^S + \mu^S}^{S+0}(\bar{X}(\gamma + \mu)) = \text{unique irreducible submodule of } \bar{X}(\gamma) = \bar{X}(\gamma).$$

For similar reasons,

$$\psi \frac{\bar{\gamma}-\rho(u)+0}{\bar{\gamma}-\rho(u)+\mu} \bar{X}_L(\gamma'_{\mathfrak{g}} + \mu') = \bar{X}_L(\gamma'_{\mathfrak{g}});$$

more precisely, we must transfer everything to a maximally split Cartan in L to make sense of this. Now Proposition 7.4.6 guarantees that the statements of Theorem 8.1.6 for γ and $\gamma+\mu$ are equivalent.

So we may as well assume that γ has the properties of $\gamma+\mu$ specified in (8.1.17). In particular, ℓ^1 is the centralizer of $\mathcal{X}$ in $\mathfrak{g}$, so $\ell^1 \subseteq \ell$.

Suppose $\bar{X}_L(\gamma'_{\mathfrak{g}})$ occurs in $H^1(u, \bar{X}(\gamma))$. Choose a Cartan subgroup T^C of $L \cap K$, and a positive root system $\Delta^+(\ell^1 \cap \mathfrak{k}, \mathcal{X}^C)$. Fix a lowest $L \cap K$ -type $\pi'_{\mathfrak{g}}$ of $\bar{X}_L(\gamma'_{\mathfrak{g}})$. By Lemma 6.5.4, the associated weight in $(\mathcal{X}^C)^*$ (Proposition 5.3.3) is

$$\bar{\gamma}'_{\mathfrak{g}}|_{\mathcal{X}} = \bar{\gamma}'|_{\mathcal{X}} - \rho(u)$$

(Lemma 8.1.2). By Theorem 5.2.2, there is a K -type π of $\bar{X}(\gamma)$ such that $\pi'_{\mathfrak{g}}$ occurs in

$$H^*(u \cap \mathfrak{k}, \pi) \otimes [\Lambda^* u \cap \mathfrak{p}]^*.$$

By Lemma 8.1.7(a),

$$||\pi||_{\text{lambda}} \leq \langle \bar{\gamma}'|_{\mathcal{X}}, \bar{\gamma}'|_{\mathcal{X}} \rangle = \langle \bar{\gamma}'|_{\mathcal{X}}, \bar{\gamma}|_{\mathcal{X}} \rangle = ||\gamma||_{\text{lambda} \mathfrak{d} \mathfrak{a}};$$

the second equality is hypothesis (b) of the theorem. By Lemma 6.6.5, π is necessarily a lowest K -type of $\bar{X}(\gamma)$. We now apply Lemma 8.1.7(b); hypothesis (ii) there is a

consequence of (8.1.17)(b). The first conclusion is that for

some $\sigma \in W(K, T^C)$,

$$\sigma \cdot \bar{\gamma}|_{\mathcal{X}} = \bar{\gamma}'|_{\mathcal{X}};$$

here we must regard both sides as elements of $\mathcal{X}^C$ by extending them to be zero on the orthogonal complement of $\mathcal{X}$, as usual. Choose an element k_0 in the normalizer of $\mathcal{X}^C$ in K realizing this Weyl group element. If α is a root of $\mathcal{X}^C$ in ℓ^1 , then

$$\begin{aligned} \langle \sigma \alpha, \bar{\gamma}'|_{\mathcal{X}} \rangle &= \langle \alpha, \sigma^{-1} \left(\bar{\gamma}'|_{\mathcal{X}} \right) \rangle \\ &= \langle \alpha, \bar{\gamma}|_{\mathcal{X}} \rangle \\ &= 0. \end{aligned}$$

So $\sigma \alpha$ must also be a root of $\mathcal{X}^C$ in ℓ^1 . So k_0 normalizes L^1 . After modifying k_0 by an element of $L^1 \cap K_0$, we get $k_1 \in K$ which normalizes H , and maps $\bar{\gamma}|_{\mathcal{X}}$ to $\bar{\gamma}'|_{\mathcal{X}}$. After replacing γ by $k_1 \cdot \gamma$, we may as well assume that

$$\bar{\gamma}|_{\mathcal{X}} = \bar{\gamma}'|_{\mathcal{X}}.$$

So far we have been vague about our choice of $\Delta^+(\mathfrak{k}, \mathcal{X}^C)$. We now specify it as

$$\Delta^+(\mathfrak{k}, \mathcal{X}^C) = \Delta^+(\ell^1 \cap \mathfrak{k}, \mathcal{X}^C) \cup \Delta(u^1 \cap \ell \cap \mathfrak{k}, \mathcal{X}^C) \cup \Delta(u \cap \mathfrak{k}, \mathcal{X}^C).$$

Let w be the unique element of $W(\mathfrak{k}, \mathcal{X}^C)$ specified by

$$\begin{aligned} w(\Delta^+(\mathfrak{k}, \mathcal{X}^C)) &= \Delta^+(\ell^1 \cap \mathfrak{k}, \mathcal{X}^C) \cup \Delta(u^1 \cap \mathfrak{k}, \mathcal{X}^C) \\ 8.1.19(a) \quad \Delta^+(w) &= \{\alpha \in \Delta^+(\mathfrak{k}, \mathcal{X}^C) \mid \alpha \notin w(\Delta^+(\mathfrak{k}, \mathcal{X}^C))\} \\ &= \Delta(\bar{u}^1 \cap u \cap \mathfrak{k}). \end{aligned}$$

Since π is a lowest K-type of $\bar{X}(\gamma)$, the weight associated to π (by Proposition 5.3.3) in the $w(\Delta^+(\mathcal{L}, \mathcal{L}^C))$ positive system is $\bar{\gamma}|_{\mathcal{L}}$; so the weight associated to π in the $\Delta^+(\mathcal{L}, \mathcal{L}^C)$ positive system is

$$\lambda = w^{-1}(\bar{\gamma}|_{\mathcal{L}}).$$

So w is the element σ specified in Lemma 8.1.7(b). Since we have seen that what is there called $\lambda_{\mathcal{G}} + \rho(\mathcal{U})$ is just $\bar{\gamma}|_{\mathcal{L}}$, condition (8.1.17)(b) guarantees that the set A of the lemma is

$$(8.1.19) \text{ (b)} \quad A = (\mathcal{U}^1 \cap \mathcal{U} \cap \mathcal{P}).$$

If we put all this together, and use the explicit description of $\mathcal{U} \cap \mathcal{L}$ cohomology given by Corollary 3.2.16, this is what we find. Let $\phi'_{\mathcal{G}}$ be a highest weight of $\pi'_{\mathcal{G}}$, such that the associated weight in $(\mathcal{L}^C)^*$ is $(\bar{\gamma}_{\mathcal{G}}|_{\mathcal{L}})$. The only way to make this highest weight appear in

$$(8.1.20) \text{ (a)} \quad H^*(\mathcal{U} \cap \mathcal{L}, \bar{X}(\gamma)) \otimes (\Lambda^* \mathcal{U} \cap \mathcal{P})^*$$

is as follows. Take a lowest K-type π of $\bar{X}(\gamma)$, and a highest weight vector whose associated weight (Proposition 5.3.3) is $w^{-1}(\bar{\gamma}|_{\mathcal{L}})$. Apply w to this weight, to get an extremal weight ϕ of π (a highest weight for the $\Delta^+(\mathcal{L}^1 \cap \mathcal{L}) \cup \Delta(\mathcal{U}^1 \cap \mathcal{L})$ positive system). Let $V_0 \subseteq \pi$ denote the span of these weight spaces. Put

$$\begin{aligned} t_1 &= \dim \mathcal{U} \cap \mathcal{U}^1 \cap \mathcal{L} \\ t_2 &= \dim \mathcal{U} \cap \mathcal{U}^1 \cap \mathcal{P}. \end{aligned}$$

Then there is a class in $H^{t_1}(\mathcal{U} \cap \mathcal{L}, \pi)$ with representatives in

$$V_0 \otimes \Lambda^{t_1} \mathcal{U} \cap \mathcal{U}^1 \cap \mathcal{L})^*.$$

In this way we can regard

$$(8.1.20) \text{ (b)} \quad V_0 \otimes \Lambda^{t_1+t_2}[\mathcal{U} \cap \mathcal{U}^1 \cap \mathcal{L} + \mathcal{U} \cap \mathcal{U}^1 \cap \mathcal{P}]^* = V_1$$

as a subspace of (8.1.20) (a); and it is only in V_1 that the highest weights of $\pi'_{\mathcal{G}}$ can be found. By Lemma 6.6.16,

$$m\left[\pi'_{\mathcal{G}}, H^*(\mathcal{U} \cap \mathcal{L}, \bar{X}(\gamma)) \otimes (\Lambda^*(\mathcal{U} \cap \mathcal{P}))^*\right] \leq m\left[\Gamma'_{\mathcal{G}}|_{\mathcal{T}}, V_1\right].$$

By the last statement of Lemma 8.1.1, and Lemma 8.1.2, this is

$$= m\left[\Gamma'|_{\mathcal{T}'}, V_0\right].$$

Since we are assuming that the multiplicity on the left is non-zero, Lemma 6.6.16 implies that V_0 contains a lowest K-type of $\bar{X}(\gamma')$. So $\bar{X}(\gamma)$ and $\bar{X}(\gamma')$ have a common lowest K-type. By hypotheses (8.1.17)(c) and (d), and Lemma 6.6.18, γ is conjugate to γ' , as we wished to show.

If $\gamma = \gamma'$, the computation above shows that a lowest

$L \cap K$ -type $\pi'_{\mathcal{G}}$ of $\bar{X}_L(\gamma'_{\mathcal{G}})$ occurs in

$$H^*(\mathcal{U} \cap \mathcal{L}, \bar{X}(\gamma)) \otimes (\Lambda^*(\mathcal{U} \cap \mathcal{P}))^*$$

exactly once, in total degree

$$i = \dim(\mathcal{U} \cap \mathcal{U}^1 \cap \mathcal{L} + \mathcal{U} \cap \mathcal{U}^1 \cap \mathcal{P}).$$

By Theorem 5.2.2, $\pi'_{\mathcal{G}}$ occurs exactly once

$$H^*(\mathcal{U}, \bar{X}(\gamma)),$$

in degree i. Now put

$$\mathfrak{C} = (\text{center of } \mathfrak{L}) \cap \mathfrak{h} \subseteq \mathfrak{X}$$

$\mathfrak{Y}^i =$ subspace of $H^i(\mathfrak{u}, \bar{X}(\gamma))$ on which $\mathfrak{C}$ acts by $\bar{\gamma}_{\mathfrak{G}}|_{\mathfrak{C}}$.

Clearly $\pi_{\mathfrak{G}}$ occurs in $\mathfrak{Y}^i$. We claim that it is a lambda-lowest $L\cap K$ -type of $\mathfrak{Y}^i$ (Definition 5.4.1). This follows easily from Lemma 8.1.7; the only point which requires comment is the fact that if λ and λ' are in $(\mathfrak{X}^C)^*$, and

$$\lambda|_{\mathfrak{C}} = \lambda'|_{\mathfrak{C}},$$

then

$$\langle \lambda, \lambda \rangle = \langle \lambda', \lambda' \rangle = \langle \lambda + \rho(\mathfrak{u}), \lambda + \rho(\mathfrak{u}) \rangle = \langle \lambda' + \rho(\mathfrak{u}), \lambda' + \rho(\mathfrak{u}) \rangle.$$

So let Z be the unique irreducible constituent of $\mathfrak{Y}^i$ containing $\pi_{\mathfrak{G}}$. By Corollary 5.4.10, Z has infinitesimal character

$$\left[\bar{\gamma}_{\mathfrak{G}}|_{\mathfrak{X}, \mathfrak{V}'} \right]$$

for some $\mathfrak{V}' \in \sigma^*$. By Lemma 6.6.6(a), $\bar{X}(\gamma)$ has infinitesimal character $\bar{\gamma}$. By Corollary 3.1.6, there is a $w \in W(\mathfrak{G}, \mathfrak{h})$ such that

$$w\bar{\gamma} = \left\{ \bar{\gamma}_{\mathfrak{G}}|_{\mathfrak{X}, \mathfrak{V}'} \right\} + \rho(\mathfrak{u}) \\ = \left\{ \bar{\gamma}|_{\mathfrak{X}, \mathfrak{V}'} \right\};$$

the second equality is Lemma 8.1.2. By condition 8.1.17(c), $w \in W(\mathfrak{L}^1, \mathfrak{h})$; so in particular, Z has the same infinitesimal character as $\bar{X}_L(\gamma_{\mathfrak{G}})$. By Lemma 6.6.18, $Z = \bar{X}_L(\gamma_{\mathfrak{G}})$. To complete the proof, we only have to check that the integer i

defined above is $\ell_{\mathfrak{G}}(\gamma)$. This can be proved using some of the ideas in the proof of Lemma 8.1.1, and we leave it to the reader. Q.E.D.

The proof used only the translation principle (in the form 8.1.18) and knowledge of the lambda-lowest K -types of $\bar{X}(\gamma)$. Both of these were proved for $X(\gamma)$ as well; so we have:

Corollary 8.1.21 Let γ be any $(\mathfrak{G}, K)$ module of finite length, all of whose composition factors occur in $X(\gamma)$. Then we get essentially the same conclusion as in Theorem 8.1.6; except that the multiplicity of $\bar{X}_L(\gamma_{\mathfrak{G}})$ in the cohomology is now equal to the multiplicity of $\bar{X}(\gamma)$ in γ .

§2. Coherent continuation of the standard representations.

Theorem 8.2.1 ([39], Corollary 5.17) Suppose $\gamma \in \hat{H}$ is a regular character, and $\bar{\gamma} \in \mathfrak{h}^*$ is nonsingular (Definition 6.6.1). Let $\mu \in \hat{H}$ be a weight of a finite dimensional representation of G , and suppose that

$$\langle \alpha, \bar{\gamma} + \bar{\mu} \rangle > 0 \quad (\alpha \in R^+(\bar{\gamma}))$$

(notation (7.2.16)). Choose $H^S, \bar{\gamma}^S, \mu^S \in \Lambda^S$ as in Proposition 7.4.1. Then

$$\psi_{\bar{\gamma}^S + \mu^S}^{-S} [X(\gamma)] = X(\gamma + \mu)$$

$$\psi_{\bar{Y}S+0}^{-S+\mu} [\bar{X}(\gamma)] = \bar{X}(\gamma+\mu) .$$

(notation (4.5.7), (6.6.3)).

Proof. Since the translation functors used here take irreducibles to irreducibles (Proposition 7.3.3), and $\bar{X}(\gamma)$ is the unique irreducible submodule of $X(\gamma)$, it is enough to prove the first statement. We proceed by induction on $\dim G$. Choose a θ -stable parabolic subalgebra $\mathfrak{g}$ of $\mathfrak{g}$ for $\gamma+\mu$ as in Corollary 8.1.15; in particular, ℓ is the centralizer of $\mathcal{I}$ in $\mathfrak{g}$ (with $\mathcal{I} = \mathcal{I} + \mathfrak{a}$). Set $S = \dim \mathfrak{u} \cap \mathfrak{k}$. By Corollary 8.1.15,

$$\mathcal{O}_{\mathfrak{g}}^i \left(X_L((\gamma+\mu))_{\mathfrak{g}} \right) = \begin{cases} X(\gamma+\mu) & i = S \\ 0 & i \neq S \end{cases}$$

By Corollary 6.4.14,

$$(8.2.2) \quad \text{Hom}_{\mathfrak{g},K}(Y, X(\gamma+\mu)) \cong \text{Hom}_{\ell, \text{In}K} \left(H^R(\mathfrak{u}, Y), X_L((\gamma+\mu)_{\mathfrak{g}}) \right);$$

here $R = \dim \mathfrak{u} \cap \mathfrak{p}$, and Y is any $(\mathfrak{g}, K)$ module of finite length. We will apply this to

$$Y = \psi_{\bar{Y}S+0}^{-S+\mu} [X(\gamma)]$$

$$\bar{Y} = \psi_{\bar{Y}S+0}^{-S+\mu} [\bar{X}(\gamma)] .$$

By Proposition 7.5.8,

$$P_{\bar{Y}+\bar{\mu}-\rho} H^i(\mathfrak{u}, Y) = \bar{Y}_{\bar{Y}-\rho(\mathfrak{u})+0}^{-\rho(\mathfrak{u})+\mu} H^i(\mathfrak{u}, X(\gamma)) ,$$

and similarly for $Y/\bar{Y}$. By Corollary 8.1.21, therefore,

$$(8.2.3) \quad \bar{Z} = \psi_{\bar{Y}-\rho(\mathfrak{u})+0}^{-\rho(\mathfrak{u})+\mu} \bar{X}_L(\gamma)_{\mathfrak{g}})$$

occurs exactly once in $H^*(\mathfrak{u}, Y)$, in degree R ; it does not occur at all in $H^*(\mathfrak{u}, Y/\bar{Y})$.

Case I. $L \neq G$. By inductive hypothesis, we may apply the theorem to L , and conclude that

$$\bar{Z} = \bar{X}_L((\gamma + \mu)_{\mathfrak{g}}) .$$

By (8.2.2) and Corollary 6.4.13,

$$\begin{aligned} \text{Hom}_{\mathfrak{g},K}(Y, X(\gamma)) &\neq 0 \\ \text{Hom}_{\mathfrak{g},K}(Y/\bar{Y}, X(\gamma)) &= 0. \end{aligned}$$

Since $\bar{Y}$ is the unique irreducible submodule of Y , any non-zero map from Y to $X(\gamma)$ must be injective. So we have proved

$$(*) \quad \psi_{\bar{Y}S+0}^{-S+\mu} (X(\gamma)) \subseteq X(\gamma + \mu) .$$

Replacing μ by $-\mu$, we get

$$\psi_{\bar{Y}S+\bar{\mu}S}^{-S+0} (X(\gamma + \mu)) \subseteq X(\gamma) .$$

Now translate both sides. By Proposition 7.3.3, this gives

$$(**) \quad X(\gamma + \mu) \subseteq \psi_{\bar{Y}S+0}^{-S+\mu} X(\gamma) .$$

Together (*) and (**) prove the theorem.

Case II. $L = G$. By Case I, the translation functors under consideration take representations attached to non-split Cartan subgroups to other representations attached to the same Cartan subgroup. So $\bar{Z}$ [cf. (6.2.3)] is attached to the split Cartan subgroup H of G ; that is, $\bar{Z}$ is the Langlands subrepresentation of an ordinary principal series. By

Proposition 4.4.1, $\bar{Z}$ contains a fine K-type. Choose a Borel subgroup $B = HN$ of G , negative with respect to γ (Definition 4.2.1); then

$$X(\gamma) = \text{Ind}_B^G(\gamma).$$

By Proposition 7.5.2,

$$\begin{aligned} Z &= \psi_{\gamma+0}^{\bar{\gamma}+\mu} X(\gamma) \\ &= \text{Ind}_B^G(\gamma + \mu), \end{aligned}$$

and $\bar{Z}$ is the unique irreducible submodule of Z . We are trying to prove that Z is isomorphic to $X(\gamma + \mu)$. Choose a Borel subgroup B' of G , negative with respect to $\gamma + \mu$. Then

$$X(\gamma + \mu) = \text{Ind}_{B'}^G(\gamma + \mu);$$

so $X(\gamma + \mu)$ has the same composition series as Z . So $\bar{Z}$ is a subquotient of $X(\gamma + \mu)$ containing a fine K-type; so

$$\bar{Z} = \bar{X}(\gamma + \mu).$$

By Corollary 6.4.13, we can find a non-zero map

$$\phi: Z \rightarrow X(\gamma + \mu).$$

The image of ϕ must contain $\bar{X}(\gamma + \mu)$; since this is the unique irreducible submodule of Z , ϕ is injective. We already know Z and $X(\gamma + \mu)$ have the same composition series, so ϕ is an isomorphism.

Q.E.D.

Since the two Borel subgroups B and B' considered at the end of the last proof differ only in non-integral roots, the

isomorphism

$$\text{Ind}_B^G(\gamma + \mu) \cong \text{Ind}_{B'}^G(\gamma + \mu)$$

can also be produced by standard intertwining operator considerations. If standard representations are regarded in the Langlands picture (as induced from discrete series), the whole theorem can be proved by much less standard intertwining operator considerations; one needs the irreducibility theorem for fundamental series proved by Enright in [11]. This is the approach adopted in [39].

We turn now to independence of polarization. As was remarked after the statement of Theorem 8.1.6, this is intimately connected with the fact that the classification of irreducibles behaves well under translation. Accordingly, it should be possible to deduce the next theorem directly from Theorem 8.2.1, using only induction by stages. The proof below is along those lines, but is more roundabout than should be necessary.

Theorem 8.2.4 (Independence of polarization). Suppose
 $\gamma \in \mathfrak{h}'$ is a regular character such that $\bar{\gamma} \in \mathfrak{h}^*$ is
nonsingular; and $\eta\gamma = \ell + \mathfrak{u}$ is a θ -stable parabolic
subalgebra, with $\mathfrak{h} \subseteq \ell$. Suppose that for every root

$$\alpha \in \Delta(\mathfrak{u}) \cap R(\bar{\gamma})$$

(Definition 7.2.16), either

$$\langle \alpha, \bar{\gamma} \rangle > 0$$

or

$$\langle \theta\alpha, \bar{\gamma} \rangle > 0.$$

Put $S = \dim u \cap k$. Then

$$\rho_{\mathfrak{g}}^i(x_L(\gamma_{\mathfrak{g}})) = \begin{cases} 0 & i \neq S \\ x_G(\gamma) & i = S \end{cases}.$$

Proof. We proceed by induction on $\dim(G)$. If $L = G$, there is nothing to prove; so we assume $L \neq G$. Choose a parabolic subalgebra $\mathfrak{g}^1 = \mathfrak{l}^1 + u^1$ as in Corollary 8.1.15; then H is split in L^1 , and the roots of k in $\mathfrak{l}^1$ are just the real roots of k in $\mathfrak{g}$. If α is a real root, then $\theta\alpha = -\alpha$, so α cannot be a root of k in u . It follows that $\mathfrak{l}^1 \subseteq \mathfrak{l}$. We consider two cases separately. The most difficult case is $\mathfrak{l}^1 = \mathfrak{l}$, so we treat it last. But suppose that case is known; we will prove the theorem in general. Define

$$\begin{aligned} u^2 &= (u^1 \cap \mathfrak{l}), \quad \mathfrak{g}^2 = \mathfrak{l}^1 + u^2 \subseteq \mathfrak{l} \\ u^3 &= u^2 + u, \quad \mathfrak{g}^3 = \mathfrak{l}^1 + u^3 \subseteq \mathfrak{g}. \end{aligned}$$

The parabolic $\mathfrak{g}^2$ satisfies the hypotheses of the theorem for L and the regular character

$$\gamma_{\mathfrak{g}} \in (\hat{H}^1)_L;$$

for if α is a root of k in u^2 , then (if $k = x + \alpha$)

$$\begin{aligned} \langle \alpha + \theta\alpha, \bar{\gamma}_{\mathfrak{g}} \rangle &= \langle \alpha + \theta\alpha, \bar{\gamma} - \rho(u) \rangle \\ &= \langle \alpha + \theta\alpha, \bar{\gamma} \rangle \\ &= 2\langle \alpha, \bar{\gamma} \rangle|_x \\ &\geq 0 \end{aligned}$$

by the definition of u^2 and the choice of u^1 . So either α or $\theta\alpha$ is positive on $\bar{\gamma}_{\mathfrak{g}}$. By inductive hypothesis, if $t_2 = \dim u^2$, $s^2 = \dim u^2 \cap k$, then

$$\rho_{\mathfrak{g}^2}^i(x_L((\gamma)_{\mathfrak{g}^2})) = \begin{cases} 0 & i \neq s^2 \\ x_L(\gamma), & i = s^2. \end{cases}$$

Similarly, one can check that $\mathfrak{g}^3$ satisfies the hypotheses of the theorem; and by the "first case" $\mathfrak{l} = \mathfrak{l}^1$ (still to be treated), one can analyze parabolic induction via $\mathfrak{g}^3$ from L^1 to G . Comparing the result with Proposition 6.3.6 (induction by stages) gives the equality we want; details are left to the reader.

So we may assume $\mathfrak{l} = \mathfrak{l}^1$. We proceed by downward induction on $\dim(u \cap u^1)$. If u and u^1 coincide, there is nothing to prove, so we may suppose they do not. Fix a positive root system for k in $\mathfrak{l}$, and let

$$\Delta^+(\mathfrak{g}, k) = \Delta^+(\mathfrak{l}, k) \cup \Delta(u, k).$$

Since $u \neq u^1$, we can find a simple root α of k in Δ^+ , so that $\alpha \notin \Delta(u^1, k)$. Define subspaces $\mathfrak{l}^2, u^2$, and u^3 of $\mathfrak{g}$, stable under k , by

$$\Delta(\mathfrak{l}^2, k) = \{\beta \in \Delta(\mathfrak{g}, k) \mid \beta|_x \text{ is proportional to } \alpha|_x\}$$

$$\Delta(u^2, k) = \{\beta \in \Delta(u, k) \mid \beta \notin \Delta(\mathfrak{l}^2, k)\}$$

$$\Delta(u^3, k) = \Delta(u^2, k) \cup [-\Delta(u, k) \cap \Delta(\mathfrak{l}^2, k)].$$

We claim that

$$\mathfrak{g}^2 = \mathfrak{l}^2 + u^2, \quad \mathfrak{g}^3 = \mathfrak{l} + u^3$$

are parabolic subalgebras. To prove this, it is enough to show that $\Delta(\mathfrak{L}^2, \mathfrak{h})$ is spanned by simple roots. Let w_0 be the longest element of $W(\mathfrak{L}, \mathfrak{h})$. Then θ commutes with w_0 , and θw_0 preserves Δ^+ . So $\alpha' = \theta w_0(\alpha)$ is simple. We leave to the reader the verification that $\Delta(\mathfrak{L}^2, \mathfrak{h})$ is spanned by α, α' , and the simple real roots.

Next, we claim that $\mathfrak{g}^3$ also satisfies the hypotheses of the theorem. For suppose

$$\beta \in \Delta(\mathfrak{u}^3, \mathfrak{h}) \cap R(\bar{\gamma}).$$

If $\beta \in \Delta(\mathfrak{u}^2)$, then $\beta \in \Delta(\mathfrak{u})$; so

$$\langle \beta, \bar{\gamma} \rangle > 0 \text{ or } \langle \theta\beta, \bar{\gamma} \rangle > 0$$

by hypothesis. Otherwise, by definition of $\mathfrak{u}^3$,

$$\beta|_{\mathfrak{t}} = -c\alpha|_{\mathfrak{t}}, \quad c > 0.$$

So

$$\begin{aligned} \langle \beta + \theta\beta, \bar{\gamma} \rangle &= 2\langle \beta, \bar{\gamma} \rangle|_{\mathfrak{t}} > \\ &= -2c\langle \alpha, \bar{\gamma} \rangle|_{\mathfrak{t}} > \\ &\geq 0, \end{aligned}$$

since $\alpha \notin \Delta(\mathfrak{u}^1, \mathfrak{h})$, and $\mathfrak{u}^1$ was chosen positive for $\bar{\gamma}|_{\mathfrak{t}}$. So either β or $\theta\beta$ is positive on $\bar{\gamma}$, as we wished to show.

By the choice of α ,

$$\dim(\mathfrak{u}^3 \cap \mathfrak{u}^1) > \dim(\mathfrak{u} \cap \mathfrak{u}^1);$$

so by induction, the theorem is known when $\mathfrak{g}$ is replaced by $\mathfrak{g}^3$. Since $\mathfrak{g}^2 \supseteq \mathfrak{g}^3$, the first part of the proof shows that

it is also known for $\mathfrak{g}^2$. We are interested in going from $\mathfrak{g}$ to $\mathfrak{g}$; we can do this (using Proposition 6.3.6) by first going from $\mathfrak{g}$ to $\mathfrak{g}^2$, then from $\mathfrak{g}^2$ to $\mathfrak{g}$. By what we have just said, the second step is understood, so we need only consider the first. This means that we may assume $\mathfrak{g} = \mathfrak{L}^2$. (Details of the application of Proposition 6.3.6 are again left to the reader.) In this case, if β is any non-real root of $\mathfrak{h}$ in $\mathfrak{g}$, then

$$\beta|_{\mathfrak{t}} = c\left[\alpha|_{\mathfrak{t}}\right], \quad c \neq 0;$$

so $\mathfrak{g}$ and $\mathfrak{g}^3$ are opposite parabolics. Furthermore, they are the only parabolic subalgebras with Levi factor $\mathfrak{L}$, so $\mathfrak{g}^3 = \mathfrak{g}^1$. Both satisfy the hypotheses of the theorem; since they are opposite, it follows that

(*) if β is a non-real integral root, then $\langle \beta, \gamma \rangle$ and $\langle \theta\beta, \gamma \rangle$ have opposite sign.

In particular, there are no imaginary roots; so G is quasi-split, with H a maximally split Cartan subgroup. Fix a Borel subgroup $B = HN$ of G , positive for $\gamma|_A$, and let Δ_1^+ be the corresponding positive root system. It follows from (*) that $\bar{\gamma}$ is dominant for $R(\bar{\gamma}) \cap \Delta_1^+$. Set

$$\rho_1 = \rho(\Delta_1^+).$$

Choose positive integers m_1, m_2 so that

$$\begin{aligned} &\langle \alpha, \bar{\gamma} + m_1 \alpha|_{\mathfrak{t}} \rangle > \geq 0 \\ &m_1 \alpha + 2m_2 \rho_1 \text{ is } R(\bar{\gamma}) \cap \Delta_1^+ \text{-dominant;} \end{aligned} \quad (**)$$

clearly this is possible. Put

$$\mu = m_1 \alpha + 2m_2 \rho_1;$$

then $\bar{\gamma} + \bar{\mu}$ and $\bar{\gamma}$ lie in the same $R(\bar{\gamma})$ Weyl chamber. By Theorem 8.2.1 and Proposition 7.4.1, it is enough to prove the theorem with γ replaced by $\gamma + \mu$. But $\rho_1|_{\mathcal{L}} = 0$; so the first condition in (**) means that the θ -stable parabolic subalgebra defined by $\bar{\gamma} + \bar{\mu}|_{\mathcal{L}}$ is $\mathfrak{g}$. So the desired formula (for $\gamma + \mu$) is precisely the definition of $X_G(\gamma + \mu)$. Q.E.D.

Our next task will be to study the translation functors across a wall on the standard representations. One case of that is rather easy, so we will do it in this section to illustrate the kind of result to be obtained; what we want is Corollary 8.2.9. It is convenient to have a technical lemma.

Lemma 8.2.5 Suppose $\gamma \in \hat{H}'$ is a regular character (Definition 6.6.1). Then $R(\bar{\gamma})$ is θ -stable (notation (7.2.16)).

Proof. Let $(\mathfrak{g}, \delta \otimes \nu)$ be the θ -stable data corresponding to (H, γ) (Proposition 6.6.2); then

$$\begin{aligned} \bar{\delta} &= \text{differential of } \delta \\ &= (\bar{\gamma} + \rho(\mathfrak{u})|_{\mathcal{L}}) \Big|_{\mathcal{L}}. \end{aligned} \quad (a)$$

Since L is quasisplit, we can choose a positive root system for $\mathfrak{h}$ in L (corresponding to a Borel subgroup of L) so that

$$\rho(\Delta^+(\mathfrak{h})|_{\mathcal{L}}) \Big|_{\mathcal{L}} = 0.$$

Then (a) becomes

$$b) \quad \bar{\gamma}^S = \psi^{-1}(\bar{\gamma}) \in (\mathfrak{h}^S)^*.$$

$$b) \quad \bar{\delta} = \bar{\gamma} + \rho|_{\mathcal{L}}.$$

As a character of the compact torus T_0 inside G , δ is the restriction of some weight $\mu \in \hat{H}$ of a finite dimensional representation of G . By (b),

$$c) \quad \bar{\mu}|_{\mathcal{L}} = (\bar{\gamma} + \rho)|_{\mathcal{L}}.$$

Since $\rho - \mu$ is integral,

$$R(\bar{\gamma} + \rho - \bar{\mu}) = R(\bar{\gamma}).$$

But by (c)

$$\bar{\gamma} + \rho - \bar{\mu}|_{\mathcal{L}} = 0,$$

so $R(\bar{\gamma} + \rho - \bar{\mu})$ is obviously θ -stable. Q.E.D.

The following situation will occur often enough that we don't want to repeat the notation every time.

Situation 8.2.6 Let H be a θ -stable Cartan subgroup of G , and $\gamma \in \hat{H}'$ a regular character (notation 6.6.1). Assume that $\bar{\gamma}$ is nonsingular. Fix a simple root $\alpha \in R^+(\bar{\gamma})$ (notation (7.2.16)). Choose a maximally split Cartan subgroup H^S of G , and an $\text{Ad}(\mathfrak{g})$ -inner automorphism

$$a) \quad \psi: (\mathfrak{h}^S)^* \rightarrow \mathfrak{h}^*$$

as in Lemma 7.2.6. Write

Possibly after passing to a covering group of G , choose a weight

$$\lambda^S \in \Lambda^S \subseteq \hat{H}^S$$

(the lattice of weights of finite dimensional representations of G) satisfying

$$\begin{aligned} \text{c)} \quad & \langle \alpha^S, \bar{\gamma}^S + \bar{\lambda}^S \rangle = 0 \\ & \langle \beta^S, \bar{\gamma}^S + \bar{\lambda}^S \rangle > 0 \quad \beta^S \in R^+(\bar{\gamma}^S) - \{\alpha^S\}; \end{aligned}$$

here $\alpha^S = \psi^{-1}(\alpha)$. (This is possible by Lemma 7.3.6).

Write

$$\text{d)} \quad \lambda = \psi(\lambda^S),$$

with ψ as in Lemma 7.2.6. Write

$$\begin{aligned} \text{e)} \quad \psi_\alpha &= \psi_{\bar{\gamma}^S + \lambda^S}^S \\ \phi_\alpha &= \psi_{\bar{\gamma}^S + \lambda^S}^S \end{aligned}$$

Finally, define

$$\text{f)} \quad n = \langle \alpha, \bar{\gamma} \rangle,$$

a non-negative integer.

We may invoke this as "the situation (8.2.6) for (β, γ') instead of (α, γ) ".

Proposition 8.2.7 In the situation (8.2.6),

suppose α is a complex root, and $\theta\alpha \in R^+(\bar{\gamma})$.

Then there is a short exact sequence

$$0 \rightarrow X(\gamma) \rightarrow \phi_\alpha \psi_\alpha [X(\gamma)] \rightarrow X(\gamma - n\alpha) \rightarrow 0.$$

If $\theta\alpha \notin R^+(\bar{\gamma})$, we get the same sequence with the arrows reversed. Finally,

$$\psi_\alpha [X(\gamma)] \cong \psi_\alpha [X(\gamma - n\alpha)].$$

Proof. Since α is complex, $\theta\alpha \neq \pm\alpha$; so since α is simple in $R^+(\bar{\gamma})$, and $\theta\alpha \in R^+(\bar{\gamma})$,

$$s_\alpha(\theta\alpha) \in R^+(\bar{\gamma}).$$

So

$$\langle \theta\alpha, \bar{\gamma} - n\alpha \rangle = \langle s_\alpha(\theta\alpha), \bar{\gamma} \rangle > 0.$$

So γ and $\gamma - n\alpha$ play dual roles: α and $\theta\alpha$ have the same sign on γ , and opposite signs on $\gamma - n\alpha$. So the second claim follows from the first and the last. To prove those, choose

$$\sigma_f = \ell + u$$

for γ as in Corollary 8.1.15; then σ_f satisfies the hypotheses of Theorem 8.2.4, as is easy to check. We claim that it also satisfies those hypotheses for $\gamma - n\alpha$. Since

$$R^+(\bar{\gamma} - n\alpha) = (R^+(\bar{\gamma}) - \{\alpha\}) \cup \{-\alpha\},$$

we only have to check them for the root α . Since α and $\theta\alpha$ have opposite signs on $\bar{\gamma} - n\alpha$, they are satisfied. So

$$\begin{aligned} \text{(8.2.8)} \quad \mathcal{R}^i(x_L(\gamma\sigma_f)) &= \begin{cases} X(\gamma) & i = S \\ 0 & i \neq S \end{cases} \\ \mathcal{R}^i(x_L((\gamma - n\alpha)\sigma_f)) &= \begin{cases} X(\gamma - n\alpha) & i = S \\ 0 & i \neq S \end{cases}. \end{aligned}$$

By Proposition 7.4.1,

$$\begin{aligned}\psi_\alpha[X(\gamma)] &= \frac{\overline{\gamma} + \lambda}{\overline{\gamma} + 0} \left[\mathcal{R}^S(x_L(\gamma_{\overline{\gamma}})) \right] \\ &= \mathcal{R}^S \left(\psi_{\overline{\gamma} - \rho(\overline{u}) + 0}^{\overline{\gamma} - \rho(\overline{u}) + \lambda} x_L(\gamma_{\overline{\gamma}}) \right) \\ &= \mathcal{R}^S \left(x_L((\gamma + \lambda)_{\overline{\gamma}}) \right) \quad (\text{by Theorem 8.2.1.})\end{aligned}$$

Similarly, we calculate

$$\psi_\alpha(X(\gamma - n\alpha)) = \mathcal{R}^S(x_L(\gamma + \lambda)_{\overline{\gamma}}),$$

proving the last assertion. (One can show easily that

$$\mathcal{R}^S(x_L(\gamma + \lambda)_{\overline{\gamma}}) = X(\gamma + \lambda);$$

because of (8.2.8), this follows from Propositions 8.2.12 or 8.2.15.) The long exact sequence in Proposition 7.4.3(a) degenerates to the short exact sequence in the first assertion. Q.E.D.

Corollary 8.2.9 In the setting (8.2.6), suppose α is a complex root. Then $\alpha \notin \tau(\overline{X}(\gamma))$ (notation 7.3.8) if and only if $\theta\alpha$ also belongs to $R^+(\overline{\gamma})$.

In that case

a) there is a short exact sequence

$$0 \rightarrow Y_1 \rightarrow U_\alpha(\overline{X}(\gamma)) \rightarrow Y_2 \rightarrow 0$$

(notation (7.4.15)), with

$$\begin{aligned}Y_1 &\subseteq X(\gamma)/\overline{X}(\gamma) \\ 0 &\neq Y_2 \subseteq X(\gamma - n\alpha).\end{aligned}$$

- b) $\overline{X}(\gamma - n\alpha)$ occurs exactly once in $U_\alpha(\overline{X}(\gamma))$.
 c) $X(\gamma - n\alpha)$ is reducible, and contains $\overline{X}(\gamma)$ exactly once as a subquotient.

Proof. Suppose $\theta\alpha \in R^+(\overline{\gamma})$. We will show that $\alpha \notin \tau(\overline{X}(\gamma))$, and prove (a) - (c). By Corollary 7.3.18, it will follow from (b) that

$$\alpha \in \tau(\overline{X}(\gamma - n\alpha)).$$

The "only if" part of the first statement follows by interchanging the roles of $\gamma - n\alpha$ and γ .

By Proposition 8.2.7, $X(\gamma)$ is a submodule of $\phi_\alpha \psi_\alpha X(\gamma)$.

So

$$\begin{aligned}0 &\neq \text{Hom}_{\mathcal{G}, K}(\overline{X}(\gamma), \phi_\alpha \psi_\alpha X(\gamma)) \\ &= \text{Hom}_{\mathcal{G}, K}(\psi_\alpha \overline{X}(\gamma), \psi_\alpha \overline{X}(\gamma))\end{aligned}$$

by Proposition 4.5.8. So $\psi_\alpha \overline{X}(\gamma) \neq 0$. By Definition 7.3.8, this means that $\alpha \notin \tau(\overline{X}(\gamma))$. By the exactness of the translation functors,

$$\phi_\alpha \psi_\alpha \overline{X}(\gamma) \subseteq \phi_\alpha \psi_\alpha X(\gamma).$$

Now Proposition 8.2.7 provides

$$(8.2.10) \text{ (a)} \quad Z_1 \subseteq X(\gamma), \quad Z_2 \subseteq X(\gamma - n\alpha)$$

and a short exact sequence

$$(8.2.10) \text{ (b)} \quad 0 \rightarrow Z_1 \rightarrow \phi_\alpha \psi_\alpha \overline{X}(\gamma) \rightarrow Z_2 \rightarrow 0.$$

Since $\overline{X}(\gamma)$ occurs as a submodule and a quotient in the middle term (Theorem 7.3.16) and only once in $X(\gamma) \supseteq Z_1$, Z_2 must be

non-zero, and must contain $\bar{X}(\gamma)$ as a quotient. Since $\bar{X}(\gamma)$ is a submodule of the middle term, but not of $Z_2 \subseteq X(\gamma - n\alpha)$, Z_1 must be non-zero. Set

$$Y_1 = Z_1 / \bar{X}(\gamma), \quad Y_2 = \ker(Z_2 \rightarrow \bar{X}(\gamma)).$$

By Definition 7.3.17, we get a short exact sequence

$$0 \rightarrow Y_1 \rightarrow U_\alpha(\bar{X}(\gamma)) \rightarrow Y_2 \rightarrow 0.$$

Since Z_2 is a non-zero submodule of $X(\gamma - n\alpha)$, it contains $\bar{X}(\gamma - n\alpha)$ exactly once. This proves (a). By Proposition 7.3.2 and Proposition 8.2.7,

$$\begin{aligned} s_\alpha \cdot \theta(\gamma) &= \theta[\phi_\alpha \psi_\alpha X(\gamma)] - \theta(\gamma) \\ &= \theta(\gamma - n\alpha). \end{aligned} \quad (8.2.11)$$

By Corollary 7.3.20(b), since $\alpha \notin \tau(\bar{X}(\gamma))$,

$$\begin{aligned} m(\bar{X}(\gamma), \theta(\gamma - n\alpha)) &= m(\bar{X}(\gamma), s_\alpha \cdot \theta(\gamma - n\alpha)) \\ &= m(\bar{X}(\gamma), \theta(\gamma)) \\ &= 1. \end{aligned}$$

This proves (c). By Lemma 6.6.4, it follows that $\bar{X}(\gamma - n\alpha)$ cannot also occur in $X(\gamma)$; so $\bar{X}(\gamma - n\alpha)$ does not occur in Y_1 , and (b) follows from (a). Q.E.D.

We will occasionally use analogues of Theorems 8.2.1 and 8.2.4 for singular infinitesimal characters. (In the proofs which follow, we have been slightly careless about changing real parabolic subgroups. The arguments can be fixed just as in Case II of the proof of Theorem 8.2.1; we leave this to the reader.)

Proposition 8.2.12 Suppose $\gamma \in \hat{H}'$ is a regular character, (with $\bar{\gamma}$ possibly singular). Let $\mu \in \hat{H}$ be a weight of a finite dimensional representation of G , and suppose that

- a) $\bar{\gamma} + \bar{\mu}$ is nonsingular;
- b) If $\alpha \in R(\bar{\gamma})$, and $\langle \alpha, \bar{\gamma} \rangle > 0$, then

$$\langle \alpha, \bar{\gamma} + \bar{\mu} \rangle > 0.$$

Choose $H^S, \bar{\gamma}^S, \mu^S \in \Lambda^S$ as in Proposition 7.4.1.

Then

$$\psi_{\bar{\gamma}^S + \mu^S}^{\bar{\gamma}^S + 0}[X(\gamma + \mu)] = X(\gamma)$$

(notation (4.5.7), (6.6.3)).

[Notice that the analogue of the last statement of Theorem 8.2.1 has been omitted; it is often false for $\bar{\gamma}$ singular, since the translation functor may take $\bar{X}(\gamma + \mu)$ to zero (cf. Corollary 7.3.22).]

Proof. Let $\sigma_f = \ell + u$ be the θ -stable parabolic associated to γ (Proposition 6.6.2). We claim that σ_f satisfies the hypotheses of Theorem 8.2.4 for $\gamma + \mu$. To see this, suppose $\alpha \in \Delta(u)$. By Proposition 6.6.2,

$$\begin{aligned} \langle \alpha + \theta\alpha, \bar{\gamma} \rangle &= 2\langle \alpha, \bar{\gamma} \rangle \\ &> 0; \end{aligned}$$

so either $\langle \alpha, \bar{\gamma} \rangle > 0$ or $\langle \theta\alpha, \bar{\gamma} \rangle > 0$. By hypothesis (b), either $\langle \alpha, \bar{\gamma} + \bar{\mu} \rangle > 0$ or $\langle \theta\alpha, \bar{\gamma} + \bar{\mu} \rangle > 0$,

which is the hypothesis of Theorem 8.2.4. Accordingly,

$$\mathcal{P}og \left\{ X_L((\gamma + \mu)og) \right\} = \begin{cases} X(\gamma + \mu) & i = S \\ 0 & i \neq S \end{cases}.$$

By Proposition 7.4.1,

$$\psi_{\bar{\gamma}S+\mu S}^{\bar{\gamma}S+0} X(\gamma + \mu) = \mathcal{P}og^S \left(\psi_{(\bar{\gamma}-\rho(u))+\mu}^{\bar{\gamma}-\rho(u)+0} X_L((\gamma + \mu)og) \right).$$

Because of the definition of $X(\gamma)$, this formula reduces the proposition to the case $L = G$, which we therefore assume.

Thus G is quasisplit, H is maximally split in G , and $\gamma|_T$ is fine with respect to G (Definition 4.3.8). So

$$\langle \bar{\gamma} | \mathcal{t}, \alpha \rangle = 0 \quad (\alpha \in \Delta(og, h)).$$

Choose a Borel subgroup

$$B = TAN$$

of G making $\text{Re}(\bar{\gamma} | \alpha)$ negative (Definition 4.2.1). Set

$$\begin{aligned} \Delta^+(og, h) &= -\Delta(\tau, h) \\ \mu^2 &= 2\rho(\Delta^+(og, h)). \end{aligned}$$

Then

$$R^+(\bar{\gamma} + \bar{\mu}^2) \subseteq \Delta^+(og, h),$$

so

$$(8.2.13) \quad \alpha \in R^+(\bar{\gamma} + \bar{\mu}^2) \implies \theta\alpha \notin R^+(\bar{\gamma} + \bar{\mu}^2).$$

The parabolic attached to $\gamma + \mu^2$ is G itself, and $(\bar{\gamma} + \bar{\mu}^2) | \sigma$ is negative for B [since $2\rho(\Delta^+(og, h))$ is zero on $\mathcal{t}$]; so

by definition

$$X(\gamma + \mu^2) = \text{Ind}_B^G(\gamma + \mu^2).$$

By Proposition 7.4.1,

$$(8.2.14) \quad \begin{aligned} \psi_{\bar{\gamma}+\mu^2}^{\bar{\gamma}+0} (X(\gamma + \mu^2)) &= \text{Ind}_B^G(\gamma) \\ &= X(\gamma). \end{aligned}$$

Now choose $og^1 = \mathcal{t}^1 + \mathcal{u}^1$ for $\gamma + \mu$ in accordance with Corollary 8.1.15; thus $\mathcal{t}^1$ is split, and

$$X(\gamma + \mu) = \mathcal{P}og^1 \left\{ X_L((\gamma + \mu)og^1) \right\}.$$

By (8.2.13), og^1 satisfies the hypotheses of Theorem 8.2.4 for $\gamma + \mu^2$; so

$$X(\gamma + \mu^2) = \mathcal{P}og^1 \left\{ X_L(\gamma + \mu^2)og^1 \right\}.$$

By Proposition 7.4.1 [applied first to og and og^1 , then to $\mathcal{t}^1$ and the real parabolic induction defining $X_L(*)$], we get

$$\begin{aligned} \psi_{\bar{\gamma}+\mu}^{\bar{\gamma}+0} [X(\gamma + \mu)] &= \mathcal{P}og^1 \left(X_L(\gamma_{og}) \right) \\ \psi_{\bar{\gamma}+\mu^2}^{\bar{\gamma}+0} [X(\gamma + \mu^2)] &= \mathcal{P}og^1 \left(X_L(\gamma_{og^1}) \right). \end{aligned}$$

These two formulas and (8.2.14) prove the proposition. Q.E.D.

Proposition 8.2.15 Suppose $\gamma \in \hat{H}'$ is a regular character, with $\bar{\gamma}$ possibly singular. Let

$og = \mathcal{t} + \mathcal{u}$ be a θ -stable parabolic subalgebra, with $\mathcal{h} \subseteq \mathcal{t}$. Suppose that for every root

$$\alpha \in \Delta(\mathcal{u}) \cap R(\bar{\gamma})$$

(Definition 7.2.16), either

$$\langle \alpha, \bar{\gamma} \rangle \geq 0$$

or

$$\langle \theta\alpha, \bar{\gamma} \rangle \geq 0.$$

Put $S = \dim u \cap k$. Then

$$\rho_{\mathfrak{g}}^i(x_L(\gamma_{\mathfrak{g}})) = \begin{cases} 0 & i \neq S \\ X_G(\gamma) & i = S \end{cases}$$

Proof. Choose a positive root system

$$R^+ \subseteq R(\bar{\gamma})$$

in such a way that if $\alpha \in R(\bar{\gamma})$,

$$\langle \alpha, \bar{\gamma} \rangle > 0 \implies \alpha \in R^+$$

$$\langle \alpha, \bar{\gamma} \rangle = 0, \alpha \in \Delta(u, k) \implies \alpha \in R^+;$$

this is certainly possible. Choose a weight $\mu \in \hat{H}$ of a finite dimensional representation of G so that $\bar{\gamma} + \bar{\mu}$ is dominant and nonsingular for R^+ . We claim that $\mathfrak{g}$ satisfies the hypotheses of Theorem 8.2.4 for $\gamma + \mu$. To see this, suppose $\alpha \in \Delta(u, k)$, and $\alpha \notin R^+$. By the conditions above on R^+ , this means that

$$\langle \alpha, \bar{\gamma} \rangle < 0.$$

By the hypotheses of the proposition, it follows that

$$\langle \theta\alpha, \bar{\gamma} \rangle \geq 0;$$

so by the conditions on R^+ , $\theta\alpha \in R^+$. By Theorem 8.2.4,

$$\rho_{\mathfrak{g}}^i(x_L(\gamma_{\mathfrak{g}} + \mu)) = \begin{cases} 0 & i \neq S \\ X(\gamma + \mu) & i = S \end{cases}.$$

By Proposition 7.4.1, and Proposition 8.2.12 for L ,

$$\psi_{\bar{\gamma}+\mu}^{\bar{\gamma}+0} \rho_{\mathfrak{g}}^i(x_L(\gamma_{\mathfrak{g}} + \mu)) = \rho_{\mathfrak{g}}^i(x_L(\gamma_{\mathfrak{g}})).$$

On the other hand, by Proposition 8.2.12 applied to G ,

$$\psi_{\bar{\gamma}+\mu}^{\bar{\gamma}+0} X(\gamma + \mu) = X(\gamma).$$

Q.E.D.

§3. Cayley transforms of regular characters.

In this section we collect a number of structural results which will be the formal framework for extending Corollary 8.2.9 to non-complex roots.

Definition 8.3.1 Suppose $\gamma = (\Gamma, \bar{\gamma}) \in (\hat{H})'$ is a regular character (Definition 6.6.1), with $\bar{\gamma}$ non-singular, and $w \in W(\bar{\gamma})$ [the integral Weyl group -- see (7.2.16)]. By Lemma 7.2.17, write

$$w\bar{\gamma} = \bar{\gamma} + \sum_{\alpha \in \Delta(\mathfrak{g}, k)} n_{\alpha} \alpha \quad (n_{\alpha} \in \mathbb{Z})$$

In the general notation of Definition 6.6.1, set

$$\Delta_w^+(m, t) = \{\alpha \in \Delta(m, t) \mid \langle \alpha, w\bar{\gamma} \rangle > 0\}$$

$$\rho_m^w = \rho(\Delta_w^+(m, t))$$

$$\rho_{m \cap k}^w = \rho(\Delta_w^+(m \cap k, t))$$

By (3.2.9), write

$$\rho_m^w - 2\rho_m^w k = \rho_m - 2\rho_m k + \sum_{\alpha \in \Delta(m, \mathcal{L})} m_\alpha.$$

By Lemma 0.4.5, $\Sigma(n_\alpha + m_\alpha)\alpha$ is a well-defined character of H ; so we can define a new regular character w cross γ , by

$$w \times \gamma = (\Gamma + \Sigma(n_\alpha + m_\alpha)\alpha, w\bar{\gamma}).$$

That $w \times \gamma$ is a regular character is clear from the definitions. There are some subtle points here. First of all, it is important to keep in mind that if w actually belongs to $W(G, H)$, it need not happen that $w \times \gamma$ is equal to the conjugate $w\bar{\gamma}$. We will study with some care exactly how this fails.

Lemma 8.3.2 Suppose $\gamma \in \hat{H}'$, $\bar{\gamma}$ is nonsingular, and $\alpha \in R^+(\bar{\gamma})$ is simple (notation (7.2.16)). In the notation of (8.3.1), set $n = \langle \alpha, \bar{\gamma} \rangle$.

- a) If α is not imaginary, then
$$s_\alpha \times \gamma = \gamma - n\alpha = (\Gamma - n\alpha, \bar{\gamma} - n\alpha).$$
- b) If α is noncompact imaginary, then
$$s_\alpha \times \gamma = (\Gamma - (n+1)\alpha, \bar{\gamma} - n\alpha).$$
- c) If α is compact imaginary, then
$$s_\alpha \times \gamma = (\Gamma - (n-1)\alpha, \bar{\gamma} - n\alpha).$$

We leave the straightforward proof to the reader.

Lemma 8.3.3 In the setting (8.3.1), suppose w acts trivially on $\mathcal{O}_V$, and $w \in W(G, H)$. Then

$$w \times \gamma = w \cdot \gamma.$$

proof. On the level of differentials, this is obvious; we only have to check that (in the notation of (8.3.1))

$$(*) \quad (w \cdot \Gamma) \Big|_T = \Gamma + \Sigma(n_\alpha + m_\alpha)\alpha \Big|_T.$$

By Chevalley's theorem, w is a product of reflections around imaginary roots; so we may assume that α is an imaginary root whenever $n_\alpha \neq 0$. So the question is entirely about the group M . Since T is a compact Cartan subgroup of M , and M is linear, there is a finite dimensional representation F of M , of extremal weight $\Gamma \Big|_T$. Obviously $(w \cdot \Gamma) \Big|_T$ is another extremal weight of F ; so by Proposition 0.4.9,

$$(**) \quad (w \cdot \Gamma) \Big|_T = \Gamma \Big|_T + \Sigma p_\alpha \alpha.$$

Since $(*)$ is true on the level of differentials,

$$\Sigma p_\alpha \alpha = \Sigma (n_\alpha + m_\alpha) \alpha$$

on $\mathcal{L}$. By Lemma 0.4.5, this is true on T as well, so $(*)$ follows from $(**)$. Q.E.D.

Definition 8.3.4 Suppose H is a θ -stable Cartan subgroup of G , and $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a noncompact imaginary root. We say that α is type I if the reflection $s_\alpha \in W(\mathfrak{g}, \mathfrak{h})$ does not belong to $W(G, H)$, and type II if it does. Define a new Cartan subgroup H^α (the Cayley transform of H through α) as in Proposition 5.3.1: that is, choose a root vector X_α for α , and set

Definition 8.3.6 In the setting (8.3.4), fix a real root $\tilde{\alpha}$ of $\mathcal{h}^\alpha$ in $\mathcal{g}$ supported on Z_α . (This root is determined up to sign.) Suppose $\gamma \in \hat{H}$ is a regular character, and α is simple in the corresponding positive system $\Delta^+(m, \mathcal{E})$ (notation 6.6.1).

a) If α is type I, define $\gamma^\alpha = (\Gamma^\alpha, \bar{\gamma}^\alpha) \in (\hat{H}^\alpha)$, by

$$\begin{aligned} \bar{\gamma}^\alpha \Big|_{\bar{\gamma}^\alpha} h_\alpha h^\alpha &= \bar{\gamma} \Big|_{\bar{\gamma}} h_\alpha h^\alpha \\ \langle \bar{\gamma}^\alpha, \tilde{\alpha} \rangle &= \langle \bar{\gamma}, \alpha \rangle \\ \Gamma^\alpha \Big|_{\Gamma^\alpha} &= \Gamma \Big|_{\Gamma} \\ \Gamma^\alpha \Big|_{A^\alpha} &= \exp \left(\bar{\gamma}^\alpha \Big|_{\sigma_0^\alpha} \right) \end{aligned}$$

b) If α is type II, define an (unordered) pair $\{\gamma_\pm^\alpha\} = \{(\Gamma_\pm^\alpha, \gamma_\pm^\alpha)\} \subseteq (\hat{H}^\alpha)$, by

$$\begin{aligned} \bar{\gamma}_\pm^\alpha \Big|_{\bar{\gamma}_\pm^\alpha} h_\alpha h^\alpha &= \bar{\gamma} \Big|_{\bar{\gamma}} h_\alpha h^\alpha \\ \langle \bar{\gamma}_\pm^\alpha, \tilde{\alpha} \rangle &= \langle \bar{\gamma}, \alpha \rangle \\ (\Gamma_+^\alpha \oplus \Gamma_-^\alpha) \Big|_{\Gamma^\alpha} &= \text{Ind}_{\Gamma_1^\alpha}^{\Gamma^\alpha} \left(\Gamma \Big|_{\Gamma_1^\alpha} \right) \\ \Gamma_\pm^\alpha \Big|_{A^\alpha} &= \exp \left(\bar{\gamma}_\pm^\alpha \Big|_{\sigma_0^\alpha} \right) \end{aligned}$$

Write

$$c^\alpha(\gamma) = \{\gamma^\alpha\} \text{ or } \{\gamma_+^\alpha, \gamma_-^\alpha\},$$

respectively, the Cayley transform of γ by α .

Lemma 8.3.7 In the setting (8.3.6), all of the constructions described give regular characters, whose conjugacy class is independent of the

$$\begin{aligned} Z_\alpha &= X_\alpha + \bar{X}_\alpha \in \mathfrak{p}_0 \\ \mathcal{T}_0^\alpha &= \{x \in \mathcal{T}_0 \mid \alpha(x) = 0\} \\ \sigma_0^\alpha &= \sigma_0 + \mathbb{R} \cdot Z_\alpha \\ \mathcal{h}_0^\alpha &= \mathcal{T}_0^\alpha + \sigma_0^\alpha, \quad H^\alpha = T^{A^\alpha} \\ T_1^\alpha &= T^\alpha \cap T \\ &= \{x \in T \mid \alpha(x) = 1\}. \end{aligned}$$

We often write $\tilde{\alpha}$ for a choice of a real root of $\mathcal{h}^\alpha$ in $\mathcal{g}$ supported on Z_α .

Lemma 8.3.5 In the setting (8.3.4):

- a) If α is type I, then $T_1^\alpha = T^\alpha$;
b) If α is type II, then s_α has a representative in T_1^α .
This representative does not lie in T_1^α , and $|T^\alpha/T_1^\alpha| = 2$.

Proof. Suppose $t \in T^\alpha$. Since t commutes with $\mathcal{h}^\alpha$, it normalizes the centralizer $\mathcal{L}_0^\alpha$ of $\mathcal{h}_0 \cap \mathcal{h}_0^\alpha$ in $\mathcal{g}_0$. Clearly,

$$\mathcal{L}_0^\alpha = \mathcal{h} + \mathbb{Q}X_\alpha + \mathbb{Q}X_{-\alpha},$$

so $\mathcal{L}_0^\alpha$ is $\mathcal{sl}(2, \mathbb{R})$ plus center. Since t is in K , it normalizes:

$$\mathcal{L}_0^\alpha \cap \mathcal{h}_0 = \mathcal{T}_0.$$

So t normalizes $\mathcal{h}_0$; and since we have seen that it acts trivially on the kernel $\mathcal{h}_0 \cap \mathcal{h}_0^\alpha$ of α , it acts either by the identity or by s_α . The lemma follows easily. Q.E.D.

Thus type I and type II correspond to $SL(2, \mathbb{R})$ and $SL^\pm(2, \mathbb{R})$, respectively (cf. Chapter 1).

choice of $\tilde{\alpha}$. If α is type I, then (notation (8.3.1))

$$a) \quad s_{\tilde{\alpha}} \cdot \gamma^{\alpha} = s_{\tilde{\alpha}} \times \gamma^{\alpha} = (s_{\alpha} \times \gamma)^{\alpha}.$$

If α is type II, then

$$b) \quad s_{\tilde{\alpha}} \cdot \gamma_{\pm}^{\alpha} = s_{\tilde{\alpha}} \times \gamma_{\mp}^{\alpha}.$$

In particular, $s_{\tilde{\alpha}} \times \gamma_{\pm}^{\alpha}$ is not conjugate to $\gamma_{\pm}^{\alpha}$ in this case.

Proof. That the definitions make sense is a simple consequence of Lemma 5.3.4, which we leave to the reader. Assertions (a) and (b) are similar, so we will prove only (b) (which is a little harder). On the level of differentials, it is trivial; so we only have to show that for $t \in T^{\alpha}$, (with $n = \langle \tilde{\alpha}, \bar{\gamma} \rangle$)

$$(*) \quad \Gamma_{+}^{\alpha}(s_{\tilde{\alpha}}t) = \Gamma_{-}^{\alpha}(t)\tilde{\alpha}(t)^{-n}.$$

Define $\sigma_{\tilde{\alpha}}, m_{\tilde{\alpha}}$ as in (1.5.5)(o),(q) for the $SL(2, \mathbb{R})$ through α ; then $\sigma_{\tilde{\alpha}}$ lies in T and is a representative for $s_{\tilde{\alpha}}$. If $t \in T_1^{\alpha}$ (notation 8.3.4), then t is in T , so t commutes with $\sigma_{\tilde{\alpha}}$. Furthermore, $\tilde{\alpha}(t)$ is one; so (*) reduces to

$$\Gamma_{+}^{\alpha}(t) = \Gamma_{-}^{\alpha}(t),$$

which is true since $t \in T_1^{\alpha}$. If t is not in T_1^{α} , then

$$\tilde{\alpha}(t) = -1$$

$$s_{\tilde{\alpha}}t = \sigma_{\tilde{\alpha}}t\sigma_{\tilde{\alpha}}^{-1} = tm_{\alpha},$$

as was computed at (1.5.23). Since t is not in T_1^{α} ,

$$\Gamma_{+}^{\alpha}(t) = -\Gamma_{-}^{\alpha}(t);$$

so (*) reduces to

$$\Gamma_{+}^{\alpha}(m_{\alpha}) = (-1)^{n+1}.$$

Since m_{α} lies in T_0 , this can be written as

$$\begin{aligned} (-1)^{n+1} &= \Gamma(m_{\alpha}) \\ &= (-1)^{\langle \tilde{\alpha}, \bar{\Gamma} \rangle}, \end{aligned}$$

or

$$n + 1 \equiv \langle \tilde{\alpha}, \bar{\Gamma} \rangle \pmod{2}.$$

By (6.6.1)(b), this amounts to

$$(**) \quad \langle \tilde{\alpha}, \rho_m \rangle - 2\langle \tilde{\alpha}, \rho, m_n k \rangle \equiv 1 \pmod{2}.$$

Since α is assumed to be simple, the first term on the left is one by Lemma 3.2.4. For the second, recall that s_{α} is assumed to be in $W(M, T)$. Necessarily $W(M, T)$ preserves the compact and noncompact roots separately; and s_{α} preserves

$$\Delta^{+}(m, t) - \{\alpha\}$$

since α is simple. So s_{α} preserves $\Delta^{+}(m_n k, t)$; so

$$\begin{aligned} \rho m_n k &= s_{\alpha}(\rho m_n k) \\ &= \rho m_n k - \sum_{\alpha} \rho m_n k \langle \alpha, \rho m_n k \rangle^{\alpha}. \end{aligned}$$

So the second term on the left in (**) is zero. This proves (**), and so the lemma. Q.E.D.

We turn next to real roots.

Definition 8.3.8 Suppose H is a θ -stable Cartan subgroup of G , and $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is a real root. Choose

$$\begin{aligned}\phi_\alpha: \mathfrak{sl}(2, \mathbb{R}) &\rightarrow \mathfrak{g}_0 \\ \Phi_\alpha: \mathfrak{SL}(2, \mathbb{R}) &\rightarrow G \\ z_\alpha &= \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{h}_0 \\ \sigma_\alpha &= \Phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m_\alpha = \sigma_\alpha^2\end{aligned}$$

exactly as in (4.3.6). We say that α is type I if

$$\alpha: T \rightarrow \{\pm 1\}$$

is not onto, and type II if it is. Define a new Cartan subgroup H_α , the Cayley transform of H through α , by

$$\begin{aligned}(t_0)_\alpha &= \mathbb{R}Z_\alpha \oplus \mathfrak{t}_0 \\ (\sigma_0)_\alpha &= \{x \in \sigma_0 \mid \alpha(x) = 0\} \\ (h_0)_\alpha &= (t_0)_\alpha \oplus (\sigma_0)_\alpha, \quad H_\alpha = T_\alpha A_\alpha.\end{aligned}$$

Thus σ_α is a representative of s_α in $W(G, H)$, and

$$\begin{aligned}\sigma_\alpha &\in T_\alpha \\ m_\alpha &\in T_\alpha \cap T.\end{aligned}$$

We will often write $\tilde{\alpha}$ for a choice of a noncompact imaginary root of $\mathfrak{h}_\alpha$ in $\mathfrak{g}$ supported on Z_α .

We want to identify the regular characters of H which arise by the construction (8.3.6) from regular characters of H_α .

As can be seen already in $\mathfrak{SL}(2, \mathbb{R})$, what is critical is the sign of $\gamma(m_\alpha)$. To describe the condition exactly, a technical lemma is helpful.

Lemma 8.3.9 In the setting (8.3.8), let $M_\alpha A_\alpha$ be the Langlands decomposition of the centralizer of A_α in G . Fix a noncompact imaginary root $\tilde{\alpha}$ of $\mathfrak{t}_\alpha$ in $\mathfrak{m}_\alpha$ supported on Z_α , and a positive root system for $\mathfrak{t}_\alpha$ in $\mathfrak{m}_\alpha \cap \mathfrak{k}$, and set

$$\rho \mathfrak{m}_\alpha \cap \mathfrak{k} = \frac{1}{2} \sum_{\beta \in \Delta^+} (\mathfrak{m}_\alpha \cap \mathfrak{k}, \mathfrak{t}_\alpha)_\beta.$$

Fix a θ -stable parabolic subalgebra $\mathfrak{g}$ of $\mathfrak{g}$ such that $\mathfrak{k}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$. $(\mathfrak{h} = \mathfrak{t} + \sigma)$. Set

$$\begin{aligned}d_1 &= \frac{1}{2} \dim (-1 \text{ eigenspace of } \text{Ad}(m_\alpha) \text{ on } \mathfrak{m}_\alpha \cap \mathfrak{k}) \\ d_2 &= \langle \tilde{\alpha}, 2\rho \mathfrak{m}_\alpha \cap \mathfrak{k} \rangle \\ d_3 &= \frac{1}{2} \dim (-1 \text{ eigenspace of } \text{Ad}(m_\alpha) \text{ on } \mathfrak{k}/\mathfrak{k} \cap \mathfrak{k}) \\ &= \dim (-1 \text{ eigenspace of } \text{Ad}(m_\alpha) \text{ on } \mathfrak{u} \cap \mathfrak{k}) \\ d_4 &= \langle \tilde{\alpha}, 2\rho(\mathfrak{u} \cap \mathfrak{k}) \rangle \\ d_5 &= \dim (-1 \text{ eigenspace of } \text{Ad}(m_\alpha) \text{ on } \mathfrak{u} \cap \mathfrak{p}) \\ d_6 &= \langle \tilde{\alpha}, 2\rho(\mathfrak{u} \cap \mathfrak{p}) \rangle\end{aligned}$$

Then all the d_i are integers of the same parity.

Proof. If T_α acts on a one-dimensional space with weight

$$\begin{aligned}\lambda \in \mathfrak{t}_\alpha^* \text{, then } m_\alpha \text{ acts by} \\ (-1)^{\langle \tilde{\alpha}, \lambda \rangle}\end{aligned}$$

[compare Lemma 4.3.19(e); this is just a fact about $SL(2, \mathbb{R})$].

It follows that

$$d_1 \equiv d_2, d_3 \equiv d_4, d_5 \equiv d_6 \pmod{2}.$$

The group L acts by scalars on $\Lambda^{\dim \mathcal{U}} \mathcal{U}$, so the commutator subgroup acts trivially. Since m_α lies in this subgroup [it is already a product of commutators in $\Phi_\alpha(SL(2, \mathbb{R})) \subseteq L$], we conclude that

$$\begin{aligned} (-1)^{d_3+d_5} &= \text{Ad}(m_\alpha) \text{ on } \Lambda^{\dim \mathcal{U}} \mathcal{U} \\ &= 1. \end{aligned}$$

So

$$d_3 \equiv d_5 \pmod{2}.$$

To complete the proof, we need only show that d_2 and d_4 have the same parity. If $\Delta^+(m_\alpha \mathcal{K}, \mathcal{T}_\alpha)$ is changed, then $\rho_{m_\alpha \cap \mathcal{K}}$ changes by a sum of roots; so $2\rho_{m_\alpha \cap \mathcal{K}}$ changes by twice an integral weight, and d_2 does not change parity. So we may assume

$$(8.3.10) \quad \Delta(m_\alpha \cap \mathcal{U} \cap \mathcal{K}, \mathcal{T}_\alpha) \subseteq \Delta^+(m_\alpha \cap \mathcal{K}, \mathcal{T}_\alpha).$$

The group $M_\alpha \cap L$ is isomorphic to $SL(2, \mathbb{R})$ [up to center and covering], so $\mathcal{T}_\alpha$ has no roots in $m_\alpha \cap \mathcal{L} \cap \mathcal{K}$; so equality holds in (8.3.10). So

$$d_4 - d_2 = \sum_{\beta \in \Delta(\mathcal{U} \cap \mathcal{K}/m_\alpha \cap \mathcal{U} \cap \mathcal{K}, \mathcal{T}_\alpha)} \langle \tilde{\alpha}, \beta \rangle.$$

Set

$$S = \{ \beta \in \Delta(\mathcal{U} \cap \mathcal{K}/m_\alpha \cap \mathcal{U} \cap \mathcal{K}, \mathcal{T}_\alpha) \mid \langle \tilde{\alpha}, \beta \rangle \text{ is odd} \}.$$

Then

$$d_4 - d_2 \equiv |S| \pmod{2}.$$

We will partition S into subsets of order 2; this will prove the result. Let γ be a root of $\mathcal{K}_\alpha$ in $\mathcal{U}/m_\alpha \cap \mathcal{U}$, and X_γ a root vector. A root vector β in S is the restriction to $\mathcal{T}_\alpha$ of such a γ , satisfying

$$(*) \quad \langle \tilde{\alpha}, \gamma \rangle \text{ is odd.}$$

Since γ is not a root in $\mathcal{L}$, it is non-zero on $\mathcal{T}$; since it is not a root in m_α , it is non-zero on $\mathcal{U}_\alpha$; and by (*)

$$\gamma \neq s_{\tilde{\alpha}} \gamma.$$

So we get four distinct roots in $\mathcal{U}/m_\alpha \cap \mathcal{U}$ satisfying (*),

$$\gamma, s_{\tilde{\alpha}} \gamma, \theta \gamma, \theta s_{\tilde{\alpha}} \gamma.$$

These define two elements of S , corresponding to the $\mathcal{T}_\alpha$ root vectors

$$X_\gamma + \theta X_\gamma, X_{s_{\tilde{\alpha}} \gamma} + \theta X_{s_{\tilde{\alpha}} \gamma}.$$

It is easy to see that this gives a partition of S into two element subsets. Q.E.D.

Definition 8.3.11 In the setting (8.3.8) define

$$\varepsilon_\alpha = \varepsilon_\alpha^G = (-1)^{d_i+1},$$

with d_i any of the integers of Lemma 8.3.9. (This is well-defined by that lemma, and independent of all choices by the definition of d_1 .) Suppose $\gamma \in \hat{H}'$. Define $R(\gamma) = R(\tilde{\gamma})$ (notation (7.2.16)), and

$$R^{\text{IR}}(\gamma) = \{\alpha \in R(\gamma) \mid \alpha \text{ is real}\}$$

$$R_0^{\text{IR}}(\gamma) = \{\alpha \in R^{\text{IR}}(\gamma) \mid \Gamma(m_\alpha) = -\varepsilon_\alpha(-1)^{\check{\alpha}(\bar{\gamma})}\},$$

the roots which do not satisfy the parity condition.

Similarly,

$$\begin{aligned} R_1^{\text{IR}}(\gamma) &= \{\alpha \in R^{\text{IR}}(\gamma) \mid \Gamma(m_\alpha) = \varepsilon_\alpha(-1)^{\check{\alpha}(\bar{\gamma})}\} \\ &= R^{\text{IR}}(\gamma) - R_0^{\text{IR}}(\gamma), \end{aligned}$$

the roots which satisfy the parity condition.

Lemma 8.3.12 In the setting of Definition 8.3.11,

let $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra,

with $H \subseteq L$.

a) If $\alpha \in R^{\text{IR}}(\gamma)$, then α satisfies the parity

condition for γ and G if and only if α satisfies

the parity condition for $\gamma_{\mathfrak{g}}$ and L .

b) If H is split in L (that is, every root of

$\mathfrak{h}$ in $\mathfrak{l}$ is real; or $\mathfrak{l}$ is the centralizer of $\mathfrak{x}$ in

$\mathfrak{g}^1$,

$$R_0^{\text{IR}}(\gamma) = R_0^{\text{IR}}(\gamma_{\mathfrak{g}}) = \{\alpha \in R^{\text{IR}}(\gamma) \mid \Gamma_{\mathfrak{g}}(m_\alpha) = (-1)^{\check{\alpha}(\bar{\gamma})}\}$$

$$R_1^{\text{IR}}(\gamma) = \{\alpha \in R^{\text{IR}}(\gamma) \mid \Gamma_{\mathfrak{g}}(m_\alpha) = -(-1)^{\check{\alpha}(\bar{\gamma})}\}.$$

proof. For (a), we have to recall the definition of $\gamma_{\mathfrak{g}}$. So choose $\mathfrak{g}^1$ as in Lemma 8.1.1, with $\mathfrak{l}^1$ split. What we have to show [after sorting through the definitions in (8.3.11) and (8.1.2)] is that

$$\varepsilon_\alpha^L[\det(m_\alpha) \text{ on } (u^1 \cap \mathfrak{p}) / (u^1 \cap \mathfrak{l} \cap \mathfrak{p})][\det(m_\alpha) \text{ on } u/u \cap u^1] = \varepsilon_\alpha^G.$$

Exactly as in the proof that $d_3 \equiv d_5$ in Lemma 8.3.9, one checks that the second factor on the left is one. So we want to show

$$\varepsilon_\alpha^L[\det(m_\alpha) \text{ on } (u^1 \cap \mathfrak{p})] = \varepsilon_\alpha^G \cdot [\det(m_\alpha) \text{ on } (u^1 \cap \mathfrak{l} \cap \mathfrak{p})].$$

If we use d_5 to define ε_α in (8.3.11), this reads

$$-\varepsilon_\alpha^L \varepsilon_\alpha^G = -\varepsilon_\alpha^G \varepsilon_\alpha^L.$$

This proves (a). For (b), we must show that $\varepsilon_\alpha^L = -1$. Since $M_\alpha \cap L$ is essentially $SL(2, \mathbb{R})$ (in the notation of Lemma 8.3.9), the integer d_1 (defined for α and L) is zero; so the result is clear from the definition. Q.E.D.

Lemma 8.3.13 In the setting (8.3.8), set

$$B_\alpha = \exp(\text{IRZ}_\alpha) = \phi_\alpha(\text{SO}(2)).$$

Let T^0 be the kernel of the root α in T (so that

$T^0 = T$ if α is type I, and T/T^0 has order 2 if α

is type II). Then

$$T_\alpha = T^0 \cdot B_\alpha$$

$$T \cap B_\alpha = T^0 \cap B_\alpha = \{1, m_\alpha\}.$$

proof. Let $\check{\alpha}$ be the root of $\mathfrak{h}_\alpha$ in $\phi_\alpha(\mathfrak{sl}(2, \mathbb{R}))_{\mathfrak{q}}$ such that $\check{\alpha}(Z_\alpha) = 2i$. Then there are root vectors $X_{\check{\alpha}}, X_{-\check{\alpha}}$ such that

$$Y_\alpha = X_{\check{\alpha}} + X_{-\check{\alpha}}$$

lies in $\mathfrak{u}$; and Y_α and $\mathfrak{u}_\alpha$ span $\mathfrak{u}$.

Suppose $t \in T_\alpha$. Then

$$\begin{aligned} \text{Ad}(t)X_{\check{\alpha}} &= zX_{\check{\alpha}} \\ \text{Ad}(t)X_{-\check{\alpha}} &= z^{-1}X_{-\check{\alpha}}, \end{aligned}$$

for some complex number z of absolute value one. Choose $r \in \mathbb{R}$ so that

$$z = e^{2ir},$$

and set

$$b = \exp(r Z_\alpha) \in B_\alpha.$$

Then

$$\begin{aligned} \text{Ad}(b) \cdot X_{\tilde{\alpha}} &= \exp(r \tilde{\alpha}(Z_\alpha)) X_{\tilde{\alpha}} \\ &= e^{2ir} X_{\tilde{\alpha}} \\ &= z X_{\tilde{\alpha}}. \end{aligned}$$

So tb^{-1} centralizes $X_{\tilde{\alpha}}$, and similarly $X_{-\tilde{\alpha}}$; so it centralizes Y_α , and $\alpha(tb^{-1}) = 1$. Since Y_α and σ_α span $\mathfrak{a}$, tb^{-1} centralizes σ_α ; and it also centralizes $\mathcal{X} \subseteq \mathcal{X}_\alpha$. So

$$tb^{-1} \in T^0,$$

which proves the first claim. For the second, we know that

$$\{1, m_\alpha\} \subseteq T^0 \cap B_\alpha;$$

so suppose

$$b = \exp(r Z_\alpha) \in T.$$

By the calculation above

$$\text{Ad}(b)(Y_\alpha) = e^{2ir} X_{\tilde{\alpha}} + e^{-2ir} X_{-\tilde{\alpha}}.$$

But T centralizes $\mathfrak{a}$, and $Y_\alpha \in \mathfrak{a}$; so r must be a multiple of π , say $r = n\pi$. By definition of m_α , this means that b is $(m_\alpha)^n$. Since m_α has order 2 (or 1), the lemma follows. Q.E.D.

Definition 8.3.14 In the setting (8.2.6), suppose

α is a real root which satisfies the parity condition

(Definition 8.3.11). We use the notation (8.3.8). Let $\tilde{\alpha}$ be the (noncompact imaginary) root of $\mathcal{H}_\alpha$ in $\phi_\alpha(\mathcal{A}(2, \mathbb{R}))_{\mathbb{C}}$ satisfying $\tilde{\alpha}(Z_\alpha) = 2i$. Define

$$\bar{Y}_\alpha \in (\mathcal{H}_\alpha)^*, \quad \Gamma_\alpha \in \hat{H}_\alpha$$

by

$$\bar{Y}_\alpha | \mathcal{H}_\alpha \cap \mathcal{H}_\alpha = \bar{Y} | \mathcal{H}_\alpha \cap \mathcal{H}_\alpha$$

$$\langle \tilde{\alpha}, \bar{Y}_\alpha \rangle = \langle \alpha, \bar{Y} \rangle$$

$$\Gamma_\alpha |_{\text{TA}_\alpha} = \Gamma |_{\text{TA}_\alpha}$$

$$\Gamma_\alpha(\exp(r Z_\alpha)) = e^{(n+d)ir}.$$

Here d is defined as follows: let $M_\alpha A_\alpha$ be the Langlands decomposition of the centralizer of A_α in G . Choose a positive root system $\Delta^+(\mathfrak{m}_\alpha, \mathcal{X}_\alpha)$ containing $R^+(\bar{Y}_\alpha) \cap \Delta(\mathfrak{m}_\alpha)$, and making α simple (this is possible by (7.2.6)). Define $\rho \mathfrak{m}_\alpha'$, $\rho \mathfrak{m}_{\alpha \cap \mathcal{K}}$ as in Definition 6.6.1, and set

$$d = \langle \tilde{\alpha}, \rho \mathfrak{m}_\alpha' - 2\rho \mathfrak{m}_{\alpha \cap \mathcal{K}} \rangle.$$

Lemma 8.3.15 In the setting of Definition 8.3.14,

$Y_\alpha = (\bar{Y}_\alpha, \Gamma_\alpha)$ is a well-defined regular character of H_α (Definition 6.6.1). Furthermore,

$$\gamma \in c^{\tilde{\alpha}}(\gamma_\alpha)$$

(notation (8.3.6)).

Proof. Since $\tilde{\alpha}$ is simple in $\Delta^+(\mathfrak{m}_\alpha, \mathcal{X}_\alpha)$, we have

$$\langle \check{\alpha}, \rho m_{\alpha} \rangle = 1.$$

Set

$$d_2 = \langle \check{\alpha}, 2\rho m_{\alpha \cap k} \rangle;$$

then it follows that

$$(-1)^d = -(-1)^{d_2} = \varepsilon_{\alpha}$$

(cf. Definition 8.3.11). We want to show that Γ_{α} is well-defined. By Lemma 8.4.9, this amounts to showing that

$$\begin{aligned} \Gamma(m_{\alpha}) &= (-1)^{n+d} \\ &= (-1)^d (-1)^{\langle \check{\alpha}, \bar{\gamma} \rangle} \\ &= \varepsilon_{\alpha} (-1)^{\langle \check{\alpha}, \bar{\gamma} \rangle}. \end{aligned}$$

By Definition 8.3.11, this is true since α satisfies the parity condition. So Γ_{α} is well-defined. Next, we check the compatibility condition (6.6.1)(b). On σ_{α} it is obvious.

On $\mathcal{L}$,

$$\begin{aligned} d\Gamma_{\alpha} \Big|_{\mathcal{L}} &= d\Gamma \Big|_{\mathcal{L}} \\ &= (\bar{\gamma} + \rho m - 2\rho m_{\alpha \cap k}) \Big|_{\mathcal{L}} \end{aligned}$$

(by the compatibility for γ)

$$= (\bar{\gamma}_{\alpha} + \rho m_{\alpha} - 2\rho m_{\alpha \cap k}) \Big|_{\mathcal{L}}$$

(by Lemma 5.3.5 and the definition of $\bar{\gamma}_{\alpha}$). In the direction of Z_{α} ,

$$(d\Gamma_{\alpha})(Z_{\alpha}) = i(n+d);$$

by definition of Γ_{α} . On the other hand, calculation in $SL(2, \mathbb{C})$ shows that for any $\lambda \in \mathcal{L}_{\alpha}^*$,

$$\lambda(Z_{\alpha}) = i \langle \check{\alpha}, \lambda \rangle.$$

It is enough to check this for $\lambda = \check{\alpha}$; there it is obvious.) So by the definition of d ,

$$(\bar{\gamma}_{\alpha} + \rho m_{\alpha} - 2\rho m_{\alpha \cap k})(Z_{\alpha}) = i(n+d),$$

as desired. This proves (6.6.1)(b). It also follows that

$\bar{\gamma}_{\alpha} \Big|_{\mathcal{L}_{\alpha}}$ is integral for m_{α} , since it differs by ρm_{α} from the differential of a character of T_{α} . So

$$R(\bar{\gamma}_{\alpha}) \supseteq \Delta(m_{\alpha}, \mathcal{L}_{\alpha}).$$

It follows that our choice of $\Delta^+(m_{\alpha}, \mathcal{L}_{\alpha})$ was unique, and consistent with (6.6.1)(a). So γ_{α} is a regular character. The last assertion follows easily by checking the definitions.

Q.E.D.

The regular character γ_{α} constructed in Definition 8.3.14 depends on the choice of the map ϕ_{α} of $\mathcal{SL}(2, \mathbb{R})$ into $\mathcal{O}_0$; replacing ϕ_{α} by its conjugate by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ replaces $\tilde{\alpha}$ by $-\tilde{\alpha}$, and γ_{α} by $s_{\tilde{\alpha}} \times \gamma_{\alpha}$ (notation (8.3.1)). Of course, this is conjugate to γ_{α} if $\tilde{\alpha}$ is type II (by Lemma 8.3.3 and Definition 8.3.4). But $\tilde{\alpha}$ is type II if and only if α is (Lemma 8.3.5(b) and Definition 8.3.8). So we make the following definition.

Definition 8.3.16 In the setting (8.3.14), define

$$\begin{aligned} c_{\alpha}(\gamma) &= \{\gamma_{\alpha}\} && \text{(if } \alpha \text{ is type II)} \\ c_{\alpha}(\gamma) &= \{\gamma_{\alpha}, s_{\tilde{\alpha}} \times \gamma_{\alpha}\} = \{\gamma_{\alpha}^{\pm}\} && \text{(if } \alpha \text{ is type I),} \end{aligned}$$

the Cayley transform of γ by α . (Here $\tilde{\alpha}$ is the noncompact imaginary root of H_{α} corresponding to α -- see (8.3.8); and $s_{\tilde{\alpha}} \times \gamma_{\alpha}$ is defined by (8.3.1).

Lemma 8.3.17 In the setting (8.3.11), suppose $\alpha \in R_0^{\mathbb{R}}(\gamma)$ [that is, α does not satisfy the parity condition for γ]. Then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

Proof. Write $n = \langle \alpha, \bar{\gamma} \rangle$. On the level of differentials, the claim is clear; we only have to check it on T . So fix $t \in T$; then

$$\begin{aligned} (s_\alpha \cdot \Gamma)(t) &= \Gamma(\sigma_\alpha t \sigma_\alpha^{-1}) \\ &= \Gamma(t) \Gamma(m_\alpha)^{1/2(1 - \alpha(t))} \end{aligned}$$

by (4.3.19) (d). On the other hand,

$$\begin{aligned} (s_\alpha \times \Gamma)(t) &= (\Gamma - n\alpha)(t) \\ &= \Gamma(t) \alpha(t)^{-n}. \end{aligned}$$

So what we must check is

$$\alpha(t)^{-n} = \Gamma(m_\alpha)^{1/2(1 - \alpha(t))}.$$

If $\alpha(t) = 1$, this is true; so suppose $\alpha(t) = -1$. Then it says that

$$(-1)^n = \Gamma(m_\alpha).$$

Since γ does not satisfy the parity condition, Definition 8.3.11 says

$$\Gamma(m_\alpha) = (-1)^n (-\varepsilon_\alpha).$$

So we have to show that $\varepsilon_\alpha = -1$. This is not true in general; but we are assuming now that $\alpha(t) = -1$, so α is type II (Definition 8.3.8). By Lemma 8.3.5, the corresponding non-compact imaginary root $\tilde{\alpha}$ of $\mathfrak{h}_\alpha$ in $\mathfrak{g}$ is type II as well. Now

we can argue exactly as in the proof of Lemma 8.3.7 (b) to conclude that ε_α is -1 . Q.E.D.

Proposition 8.3.18 Let H be a θ -stable Cartan subgroup of G , $\gamma \in \hat{H}$, a regular character with $\bar{\gamma}$ nonsingular, and $\alpha \in R^+(\bar{\gamma})$ a simple root (notation (7.2.16)).

a) If α is compact imaginary, then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

b) If α is type I noncompact imaginary (Definition 8.3.4), then $s_\alpha \times \gamma$ is not conjugate to γ . If γ^α is defined by (8.3.6), and $\tilde{\alpha} \in \mathfrak{h}^{\alpha*}$ is the real root corresponding to α , then (up to conjugacy by K)

$$c\tilde{\alpha}(\gamma^\alpha) = \{\gamma, s_\alpha \times \gamma\}$$

$$(s_\alpha \times \gamma)^\alpha = s_{\tilde{\alpha}} \cdot \gamma^\alpha = s_{\tilde{\alpha}} \times \gamma^\alpha.$$

c) If α is type II noncompact imaginary, then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

With notation as in (b), we have (up to conjugacy by K)

$$\begin{aligned} c\tilde{\alpha}(\gamma_\pm^\alpha) &= \gamma \\ s_{\tilde{\alpha}} \times \gamma_\pm^\alpha &= s_\alpha \cdot \gamma_\mp^\alpha. \end{aligned}$$

d) If α is type I real and satisfies the parity condition (Definition 8.3.8 and 8.3.11), then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

If $\gamma_\alpha^\pm$ is defined by (8.3.16), and $\tilde{\alpha} \in \mathfrak{h}_{\alpha}^*$ is the

noncompact imaginary root corresponding to α , then
(up to conjugacy by K)

$$c^{\tilde{\alpha}}(\gamma_{\alpha}^{\pm}) = \gamma$$

$$s_{\tilde{\alpha}} \times \gamma_{\alpha}^{\pm} = \gamma_{\alpha}^{\mp}.$$

e) If α is type II real and satisfies the parity condition, then

$$s_{\alpha} \times \gamma \neq s_{\alpha} \cdot \gamma.$$

With notation as in (d), we have (up to conjugacy by K)

$$c^{\tilde{\alpha}}(\gamma_{\alpha}) = \{\gamma, s_{\alpha} \times \gamma\}$$

$$(s_{\alpha} \times \gamma)_{\alpha} = s_{\tilde{\alpha}} \cdot \gamma_{\alpha} = s_{\tilde{\alpha}} \times \gamma_{\alpha}.$$

f) If α is real and does not satisfy the parity condition, then

$$s_{\alpha} \times \gamma = s_{\alpha} \cdot \gamma.$$

This is just a summary of Lemmas 8.3.3, 8.3.7, 8.3.15, and 8.3.17. The important thing technically is that there are always [in cases (b) - (e)] three regular characters -- two on one Cartan and a third on another -- all related by Cayley transform and the cross action of $W(\overline{\gamma})$. The important thing philosophically is the formal symmetry of the proposition around its midpoint. We will have more to say about that later.

We conclude this section with a result which will be needed in Chapter 9. The cross action of $W(\overline{\gamma})$ was really defined to make this result true.

Proposition 8.3.19 Suppose $\gamma \in \hat{H}'$, and $\overline{\gamma}$ is non-singular. Fix $w \in W(\overline{\gamma})$, such that w acts trivially

on $\mathcal{X}$. Then

$$w^{-1} \cdot \Theta(\gamma) = \Theta(w \times \gamma)$$

(notation (6.6.5), (7.2.28), and (8.3.1)).

Proof. Choose a parabolic $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$, with $\mathfrak{l}$ the centralizer of $\mathcal{X}$ in $\mathfrak{g}$, in accordance with Corollary 8.1.15. By that result, and Definition 7.2.8,

$$\Theta(\gamma) = \chi_{\mathfrak{g}}(\Theta_L(\gamma_{\mathfrak{g}}))$$

By Corollary 7.2.10 (applied first to θ -stable parabolic induction for G , then to real parabolic induction for L)

$$\Phi(\overline{\gamma} + \lambda) = \chi_{\mathfrak{g}}(\Theta_L(\gamma_{\mathfrak{g}} + \lambda))$$

is a coherent family based on $\overline{\gamma} + \lambda$. Write

$$w\overline{\gamma} = \overline{\gamma} + \sum n_{\alpha} \alpha,$$

with n_{α} an integer, and a sum over real roots. By Definitions 7.2.28 and 8.3.1,

$$\begin{aligned} w^{-1} \cdot \Theta(\gamma) &= \Phi(\overline{\gamma} + (\sum n_{\alpha} \alpha)) \\ &= \chi_{\mathfrak{g}}(\Theta_L(\gamma_{\mathfrak{g}} + \sum n_{\alpha} \alpha)) \\ &= \chi_{\mathfrak{g}}\left(\Theta_L\left((\gamma + \sum n_{\alpha} \alpha)_{\mathfrak{g}}\right)\right) \\ &= \chi_{\mathfrak{g}}\left(\Theta_L((w \times \gamma)_{\mathfrak{g}})\right) \end{aligned}$$

Clearly $\mathfrak{g}$ satisfies the hypotheses of Corollary 8.1.15 for $w \times \gamma$ as well as for γ , since $\overline{\gamma}$ and $(\overline{w \times \gamma})$ agree on $\mathcal{X}$. So this last character is $\Theta(w \times \gamma)$. Q.E.D.

§4. Non-complex root walls.

Our aim is to get analogues of Corollary 8.2.9 for non-complex roots. We will use θ -stable parabolic induction to reduce the problem more or less to split Cartan subgroups. Then we would like to use Theorem 6.6.15 to reduce to $SL(2, \mathbb{R})$, using real parabolic induction. Since we didn't prove that theorem, we will prove here the special case we need (Proposition 8.4.1). Since we didn't prove Theorem 6.6.14 either, the argument we use is rather convoluted. Nevertheless, it could probably be extended to cover the general case of Theorem 6.6.15, with Theorem 6.6.14 appearing as a corollary of the proof. We leave this as an exercise for the ambitious reader.

Proposition 8.4.1 Suppose $H = TA$ is a θ -stable Cartan subgroup of G , and α is a noncompact imaginary root of $\mathfrak{h}$ in $\mathfrak{g}$. Assume that

$$(A) \quad \text{if } \beta \in \Delta(\mathfrak{g}, \mathfrak{h}), \text{ then } \beta|_{\mathfrak{t}} = c\alpha.$$

Suppose $\gamma \in \hat{H}'$, $\bar{\gamma}$ is nonsingular, and α is simple in $R^+(\bar{\gamma})$ (notation (7.2.16)). Let (MA, δ, ν) be the set of cuspidal data associated to (H, γ) (Lemma 6.6.11), and define $P = MA$ as in (6.6.13). Then

$$X(\gamma) \cong \text{Ind}_P^G(\delta \otimes \nu \otimes 1).$$

Proof. Since MA is by definition the centralizer of A in G , assumption (A) gives

$$\begin{aligned} \Delta(m, \mathfrak{t}) &= \{\text{imaginary roots of } \mathfrak{h} \text{ in } \mathfrak{g}\} \\ &= \{\pm\alpha\}. \end{aligned}$$

So M is locally a product of $SL(2, \mathbb{R})$ and a compact torus. We will divide the argument more or less arbitrarily into several parts, for the convenience of the reader who is reading during the commercials only.

Step I. $X(\gamma)$ and $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$ have the same set of lowest K -types. Since M is locally $SL(2, \mathbb{R})$ and center, T has finite index in $M \cap K$; and

$$\begin{aligned} M \cap K/T &= W(M, T) \\ &= \begin{cases} \{1\} & \text{if } \alpha \text{ is type I} \\ \{1, s_\alpha\} & \text{if } \alpha \text{ is type II} \end{cases} \end{aligned}$$

(notation (8.3.4)).

[This is a special case of what we found in Section 1.5.] We want to understand the $(m, M \cap K)$ module

$$\begin{aligned} \delta &= X_M(\gamma|_T) \\ &= R_{\mathfrak{g}_m}^0 \{ (\Gamma - \alpha)|_T \} \end{aligned}$$

(Definition 6.6.11, Proposition 6.6.2, and Definition 6.5.2); here

$$\mathfrak{g}_m = \mathfrak{t} + \mathfrak{g} X_\alpha,$$

with X_α a root vector for α . By definition of $R_{\mathfrak{g}_m}^0$,

$$\begin{aligned} \delta &= \text{Ind}_{m, T}^{m, M \cap K} \left(\text{pro}_{\mathfrak{g}_m, T}^{m, T} \{ \Gamma|_T \} \right) \\ \delta &= \text{Ind}_{m, T}^{m, M \cap K} \left(\text{Hom}_{\mathfrak{g}_m} \left[U(m), \mathbb{C}_T \right]_{T\text{-finite}} \right) \end{aligned}$$

(compare Proposition 1.5.26). By Theorem 6.3.12 or (easy) direct calculation, it follows that

$$\delta \Big|_{\text{MnK}} = \bigoplus_{n=0}^{\infty} \text{Ind}_T^{\text{MnK}} \left(\Gamma \Big|_T + n\alpha \right).$$

By (4.1.13), it follows that

$$\text{Ind}_P^G(\delta \otimes \nu \otimes 1) \Big|_K = \bigoplus_{n=0}^{\infty} \text{Ind}_T^K \left(\Gamma \Big|_T + n\alpha \right).$$

The assertion of Step I now follows from Lemma 6.6.16.

Step II. There is a non-zero map

$$\phi: \text{Ind}_P^G(\delta \otimes \nu \otimes 1) \rightarrow X(\gamma).$$

To prove this, we choose a weight μ of a finite dimensional representation of G satisfying (8.1.17)(a) - (d). Let $\delta' \otimes \nu'$ be the cuspidal data corresponding to $\gamma + \mu$. By applying Proposition 7.4.1 first to P and then to $\mathcal{G}_m$, we find

$$\psi_{\gamma+s+0}^{-s+\mu^s} (\text{Ind}_P^G(\delta \otimes \nu \otimes 1)) = \text{Ind}_P^G(\delta' \otimes \nu' \otimes 1).$$

By Theorem 8.2.1,

$$\psi_{\gamma+s+0}^{-s+\mu^s} (X(\gamma)) = X(\gamma + \mu).$$

By the translation principle (Proposition 7.3.3), it is therefore enough to find a non-zero map

$$\phi: \text{Ind}_P^G(\delta' \otimes \nu' \otimes 1) \rightarrow X(\gamma + \mu).$$

But this follows from Lemma 6.6.18 [applied to $\gamma =$

$\text{Ind}_P^G(\delta' \otimes \nu' \otimes 1)$] using Step I. This completes Step II.

Step III. $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$ and $X(\gamma)$ have the same K -type multiplicities. This assertion does not depend on ν , so we may assume for its proof that ν is unitary and non-singular. By Corollary 6.5.14, $X(\gamma)$ is irreducible; so the map ϕ of Step II is surjective; and its kernel contains none of the lambda-lowest K -types of $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$. Now the proof of Theorem 6.5.9(c) used only a knowledge of the infinitesimal character and lambda-lowest K -types of $X(\gamma)$; so the corresponding result holds for $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$. It follows that the kernel of ϕ is zero, so ϕ is an isomorphism. In particular, the two representations have the same K -structure, as we wished to show.

Now construct H^α as in Definition 8.3.4. In the present case, assumption (A) of Proposition 8.4.1 makes H^α a split Cartan subgroup of G -- that is, T_0^α is central in G . Fix a real root $\tilde{\alpha}$ of $\mathcal{H}^\alpha$ in $\mathcal{G}$, orthogonal to $\alpha \subseteq \alpha^\alpha$, as in Definition 8.3.4. Define

$$P^\alpha = H N^\alpha$$

by

$$\Delta(n^\alpha, \mathcal{H}^\alpha) = \{\tilde{\alpha}\} \cup \{\Delta(n, \mathcal{H}^\alpha)\}.$$

Since α is simple in $R^+(\bar{\gamma})$, it is easy to check that

$$R^+(\bar{\gamma}^\alpha) \subseteq -\Delta(n, \mathcal{H}^\alpha) \cup \{\alpha\} = \Delta^+(g, \mathcal{H}^\alpha)$$

(notation (8.3.6)). After we pass to a covering group,

Lemma 7.3.6 guarantees the existence of a weight $\lambda^\alpha \in \Lambda^\alpha$ such that

$$(8.4.2) \quad \begin{aligned} & \langle \alpha, \bar{\gamma}^\alpha + \lambda^\alpha \rangle = 0 \\ & \operatorname{Re} \langle \beta, \bar{\gamma}^\alpha + \bar{\lambda}^\alpha \rangle > 0, \quad \beta \in \Delta^+(\sigma, h^\alpha) - \{\alpha\}. \end{aligned}$$

Now define ϕ_α, ψ_α as in (8.2.6), using λ^α as λ^S .

Step IV. Structure of various induced representations: if α is type I,

$$\operatorname{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha) = \psi_\alpha[\operatorname{Ind}_P^G(\delta \otimes 1)] \oplus \psi_\alpha[\operatorname{Ind}_P^G(\delta' \otimes 1)].$$

Here $\delta' \otimes 1$ is the set of cuspidal data attached to $s_\alpha \times \gamma$ (notation (8.3.1)), and both terms are non-zero; this is the decomposition (4.2.8). There is a short exact sequence

$$0 \rightarrow \operatorname{Ind}_{P\alpha}^G((\gamma^\alpha) \rightarrow \phi_\alpha \operatorname{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha) \rightarrow \operatorname{Ind}_{P\alpha}^G(s_\alpha(\gamma^\alpha)) \rightarrow 0.$$

If α is type II, then

$$\operatorname{Ind}_{P\alpha}^G(\gamma_\pm^\alpha + \lambda^\alpha) = \psi_\alpha[\operatorname{Ind}_P^G(\delta \otimes 1)]$$

$$0 \rightarrow \operatorname{Ind}_{P\alpha}^G(\gamma_\pm^\alpha) \rightarrow \phi_\alpha[\operatorname{Ind}_{P\alpha}^G(\gamma_\pm^\alpha + \lambda^\alpha)] \rightarrow \operatorname{Ind}_{P\alpha}^G(s_\alpha(\gamma_\mp^\alpha)) \rightarrow 0.$$

For the first and third statements, we can reduce to the case $G = M$: this uses Proposition 7.4.1 and induction by stages for real parabolic subgroups. Write $\lambda \in \Lambda \subseteq \hat{\Gamma}$ for the weight corresponding to λ^α [cf. (8.2.6)(d)]. By Definition 8.3.1, when $G = M$,

$$\psi_\alpha(\delta) = \mathcal{R}_{\gamma m}^0(\Gamma - \alpha + \lambda), \quad \psi_\alpha(\delta') = \mathcal{R}_{s_\alpha(\gamma m)}^0(s_\alpha\Gamma + \alpha + s_\alpha\lambda).$$

Now the first and third statements follows from Proposition 1.5.9, using the description of the components given by Proposition 1.5.26. (Alternatively, one can reduce to

$G = \operatorname{SL}(2, \mathbb{R})$ or $G = \operatorname{SL}^\pm(2, \mathbb{R})$; these two cases are treated

carefully in Sections 1.3 and 1.4, respectively.) For the second and fourth statements, Proposition 7.4.3 gives short exact sequences

$$\begin{aligned} 0 & \rightarrow \operatorname{Ind}_{P\alpha}^G(\gamma^\alpha) \rightarrow \phi_\alpha \operatorname{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha) \rightarrow \operatorname{Ind}_{P\alpha}^G(\gamma^\alpha - n\tilde{\alpha}) \rightarrow 0 \\ 0 & \rightarrow \operatorname{Ind}_{P\alpha}^G(\gamma_\pm^\alpha) \rightarrow \phi_\alpha \operatorname{Ind}_{P\alpha}^G(\gamma_\pm^\alpha + \lambda^\alpha) \rightarrow \operatorname{Ind}_{P\alpha}^G(\gamma_\pm^\alpha - n\tilde{\alpha}) \rightarrow 0. \end{aligned}$$

So the statements we want follow from Lemma 8.3.7.

$$\text{Step V. } \psi_\alpha X(\gamma) = \mathcal{R}_{\sigma\gamma}^S(X_L(\gamma + \lambda))$$

$$\operatorname{Hom}_{\mathfrak{g}, K}(Y, \psi_\alpha X(\gamma)) \cong \operatorname{Hom}_{\ell, L \cap K}(H^R(u, Y), X_L(\gamma\sigma + \lambda))$$

(notation (8.2.6)). Here Y is any $(\sigma\gamma, K)$ module of finite length, $\sigma\gamma = \ell + \gamma u$ is the θ -stable parabolic corresponding to γ (Proposition 6.6.2), $S = \dim u \cap \mathfrak{k}$, and $R = \dim u \cap \mathfrak{p}$. $\psi_\alpha X(\gamma)$ has the same K -multiplicities as $\psi_\alpha[\operatorname{Ind}_P^G(\delta \otimes 1)]$, and any submodule of it contains a fine K -type. The first claim follows from Proposition 7.4.1 and the definition of $X(\gamma)$; this also shows that

$$(*) \quad \mathcal{R}_{\sigma\gamma}^i(X_L(\gamma\sigma + \lambda)) = 0, \quad i \neq S.$$

Now Corollary 6.4.14 gives the second claim. For the K -multiplicities, (*) and Theorem 6.3.12 show that they are independent of ν ; so we may assume ν is unitary and non-singular. [To be precise, this argument applies only if ν is chosen in such a way that (8.4.2) still holds after this modification of ν ; but we can certainly arrange that.]

In that case we saw in Step III that $X(\gamma)$ and $\operatorname{Ind}_P^G(\delta \otimes 1)$ are actually isomorphic, so the claim is obvious. For the last claim, we may as well assume G is semisimple. Then

$\mathcal{X}$ is spanned by α , so (8.4.2) means that

$$(**) \quad \bar{\gamma} + \bar{\lambda} \Big| \mathcal{X} = 0.$$

Let Y be a submodule of $\psi_\alpha X(\gamma)$. By the formula for Hom already proved,

$$\text{Hom}_{\theta, L \cap K} (H^R(\mathcal{U}, Y), X_L^{(\gamma_{\mathcal{G}} + \lambda)}) \neq 0.$$

By Proposition 4.4.1, $H^R(\mathcal{U}, Y)$ contains an $L \cap K$ -type

$$\pi_{\mathcal{G}} \in A_L^{(\gamma_{\mathcal{G}} + \lambda)}.$$

By Lemma 6.5.4, the weight associated to $\pi_{\mathcal{G}}$ by Proposition 5.3.3 is

$$\bar{\gamma}_{\mathcal{G}} + \bar{\lambda} \Big| \mathcal{X}.$$

By Lemma 8.1.7 and Theorem 5.2.2, there is a K -type π of Y with

$$\begin{aligned} ||\pi||_{\text{lambda}} &\leq \langle (\bar{\gamma}_{\mathcal{G}} + \bar{\lambda}) \Big| \mathcal{X} + \rho(\mathcal{U}), (\bar{\gamma}_{\mathcal{G}} + \bar{\lambda}) \Big| \mathcal{X} + \rho(\mathcal{U}) \rangle \\ &= \langle (\bar{\gamma} + \bar{\lambda}) \Big| \mathcal{X}, (\bar{\gamma} + \bar{\lambda}) \Big| \mathcal{X} \rangle \\ &= 0 \end{aligned}$$

by (**). So the weight associated to π by Proposition 5.3.3 is zero; so by Proposition 5.3.26, π is fine.

Step VI. If α is type I, and δ' is as in Step IV, we have

$$\begin{aligned} 0 &\rightarrow \text{Ind}_{\mathcal{P}}^G((\delta \otimes \vee \otimes 1) \oplus (\delta' \otimes \vee \otimes 1)) \rightarrow \text{Ind}_{\mathcal{P}\alpha}^G(\gamma^\alpha) \\ &\quad \text{Ind}_{\mathcal{P}\alpha}^G(s_{\tilde{\alpha}}(\gamma^\alpha)) \rightarrow \text{Ind}_{\mathcal{P}}^G((\delta \otimes \vee \otimes 1) \oplus (\delta' \otimes \vee \otimes 1)) \rightarrow 0. \end{aligned}$$

If α is type II, then

$$0 \rightarrow \text{Ind}_{\mathcal{P}}^G(\delta \otimes \vee \otimes 1) \rightarrow \text{Ind}_{\mathcal{P}\alpha}^G(\gamma_\pm^\alpha)$$

$$\text{Ind}_{\mathcal{P}\alpha}^G(s_\alpha(\gamma_\pm^\alpha)) \rightarrow \text{Ind}_{\mathcal{P}}^G(\delta \otimes \vee \otimes 1) \rightarrow 0.$$

All of these claims can be reduced immediately to the case $G = M$ by induction by stages. If α is type I, then it is easy to verify that

$$G = \text{SL}(2, \mathbb{R}) \cdot (\text{center of } G)$$

(up to cover), so we can reduce to the case of $\text{SL}(2, \mathbb{R})$.

This is treated by Proposition 1.3.3. If α is type II, G need not have a subgroup isomorphic to $\text{SL}^\pm(2, \mathbb{R})$; but nevertheless the proof of Proposition 1.4.8 still applies, and gives what we want.

Step VII.

$$\psi_\alpha X(\gamma) \cong \psi_\alpha [\text{Ind}_{\mathcal{P}}^G(\delta \otimes \vee \otimes 1)],$$

and $\bar{X}(\gamma)$ is the unique irreducible preimage under ψ_α of the corresponding Langlands subrepresentation of $\text{Ind}_{\mathcal{P}\alpha}^G(\gamma^\alpha + \lambda^\alpha)$ [or $\text{Ind}(\gamma_\pm^\alpha + \lambda^\alpha)$]. We will write $\gamma^\alpha + \lambda^\alpha$ throughout, and discuss only the case when α is type I; the other is very similar. By Proposition 4.5.8,

$$\begin{aligned} &\text{Hom}_{\mathcal{G}, K}(\text{Ind}_{\mathcal{P}\alpha}^G(\gamma^\alpha + \lambda^\alpha), \psi_\alpha X(\gamma)) \\ &\cong \text{Hom}_{\mathcal{G}, K}(\phi_\alpha \text{Ind}_{\mathcal{P}\alpha}^G(\gamma^\alpha + \lambda^\alpha), X(\gamma)) \\ &\geq \text{Hom}_{\mathcal{G}, K}(\text{Ind}_{\mathcal{P}\alpha}^G(s_{\tilde{\alpha}}\gamma^\alpha), X(\gamma)) \quad [\text{by step IV}] \\ &\geq \text{Hom}_{\mathcal{G}, K}(\text{Ind}_{\mathcal{P}}^G(\delta \otimes \vee \otimes 1), X(\gamma)) \quad [\text{by step VI}] \\ &\neq 0 \end{aligned}$$

by Step II. So we get a non-zero map

$$\phi_0: \text{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha) \rightarrow \psi_\alpha X(\gamma).$$

The fine K-types in $A(\gamma^\alpha + \lambda^\alpha|_{T\alpha})$ are partitioned by Steps IV and V into those occurring in $\psi_\alpha X(\gamma)$ and those in $\psi_\alpha X(s_\alpha \gamma)$. By Step V, we conclude that ϕ_0 is nonzero on the first set; so we get

$$\phi_1: \psi_\alpha(\text{Ind}_P^G(\delta \otimes \mathbf{1})) \rightarrow \psi_\alpha X(\gamma),$$

a map which is non-zero on the fine K-types on the left. By Theorem 4.2.12 and Proposition 4.4.1, ϕ_1 is injective. By Step V, it is an isomorphism. To prove the last claim, we must show that

$$\begin{aligned} 0 &\neq \text{Hom}_{\mathfrak{g}, K}(\psi_\alpha \bar{X}(\gamma), \psi_\alpha X(\gamma)) \\ &\cong \text{Hom}_{\mathfrak{g}, K}(\bar{X}(\gamma), \phi_\alpha \psi_\alpha X(\gamma)). \end{aligned} \quad (*)$$

By Proposition 7.4.3, there is an exact sequence

$$\begin{aligned} 0 &\rightarrow \mathcal{R}_{\mathfrak{g}}^{S-1}(X_L(\gamma_{\mathfrak{g}} - n\alpha)) \rightarrow \mathcal{R}_{\mathfrak{g}}^S(X_L(\gamma_{\mathfrak{g}})) \rightarrow \phi_\alpha \psi_\alpha \mathcal{R}_{\mathfrak{g}}^S(X_L(\gamma_{\mathfrak{g}})) \\ &\rightarrow \mathcal{R}_{\mathfrak{g}}^S(X_L(\gamma_{\mathfrak{g}} - n\alpha)) \rightarrow 0; \end{aligned} \quad (**)$$

here notation is as in Step V. Furthermore, (by Proposition 7.4.3)

$$\mathcal{R}_{\mathfrak{g}}^i(X_L(\gamma_{\mathfrak{g}} - n\alpha)) = 0 \quad i \neq S, S-1.$$

So what we need to show is that

$$\mathcal{R}_{\mathfrak{g}}^{S-1}(X_L(\gamma_{\mathfrak{g}} - n\alpha)) = 0;$$

then (**) will prove (*). Suppose not. By (**),

$$\text{Hom}_{\mathfrak{g}, K}(\bar{X}(\gamma), \mathcal{R}_{\mathfrak{g}}^{S-1}(X_L(\gamma_{\mathfrak{g}} - n\alpha))) \neq 0.$$

By Corollary 6.4.14, we deduce that

$$\text{Hom}_{\mathfrak{g}, L\cap K}(\bar{X}^{R+1}(\mathfrak{u}, \bar{X}(\gamma)), X_L(\gamma_{\mathfrak{g}} - n\alpha)) \neq 0.$$

But it is easy to check that

$$\gamma_{\mathfrak{g}} - n\alpha = (s_\alpha \times \gamma)_{\mathfrak{g}};$$

so this implies that $\bar{X}_L((s_\alpha \times \gamma)_{\mathfrak{g}})$ occurs in $H^*(\mathfrak{u}, \bar{X}(\gamma))$, contradicting Theorem 8.1.6. (If α is type II, we have to use the degree in which the cohomology occurs to get the contradiction.)

Step VIII. The unique irreducible submodule Y of $\text{Ind}_P^G(\delta \otimes \mathbf{1})$ is the preimage under ψ_α of the appropriate Langlands subrepresentation of $\text{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha)$ [or $\text{Ind}_{P+}^G(\gamma^\alpha + \lambda^\alpha)$]; and Y occurs exactly once in the composition series of $\text{Ind}_P^G(\delta \otimes \mathbf{1})$. Again we do only the type I case. Let Y be any irreducible submodule of $\text{Ind}_P^G(\delta \otimes \mathbf{1})$. Then

$$\begin{aligned} &\text{Hom}_{\mathfrak{g}, K}(\psi_\alpha Y, \text{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha)) \\ &\cong \text{Hom}_{\mathfrak{g}, K}(Y, \phi_\alpha \text{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha)) \quad [\text{by Prop. 4.5.8}] \\ &\cong \text{Hom}_{\mathfrak{g}, K}(Y, \text{Ind}_{P\alpha}^G(\gamma^\alpha)) \quad [\text{by Step IV}] \\ &\cong \text{Hom}_{\mathfrak{g}, K}(Y, \text{Ind}_P^G(\delta \otimes \mathbf{1})) \quad [\text{by Step VI}] \\ &\neq 0 \end{aligned}$$

by hypothesis. So $\psi_\alpha Y$ must be a Langlands subrepresentation of $\text{Ind}_{P\alpha}^G(\gamma^\alpha + \lambda^\alpha)$. So

$$\begin{aligned}
m(Y, \text{Ind}_P^G(\delta \otimes \nu \otimes 1)) &= m(\psi_\alpha Y, \psi_\alpha \text{Ind}_P^G(\delta \otimes \nu \otimes 1)) \\
&\leq m(\psi_\alpha Y, \text{Ind}_{P\alpha}^G(\gamma\alpha + \lambda\alpha)) \\
&= 1
\end{aligned}$$

by Step IV.

To finish the proof, let ϕ be as in Step II; then the image of ϕ contains $\bar{X}(\gamma)$. By Steps VII and VIII, it follows that ϕ is non-zero on the unique irreducible submodule of $\text{Ind}_P^G(\delta \otimes \nu \otimes 1)$; so ϕ is injective. By Step III, it is an isomorphism. Q.E.D.

Proposition 8.4.3 (Hecht-Schmid [15]). In the setting (8.2.6), suppose α is compact imaginary.

Then

$$\psi_\alpha X(\gamma) = 0,$$

so α lies in the τ invariant of each constituent of $X(\gamma)$.

Proof. Choose $\sigma_f = \lambda + u$ for γ as in Corollary 8.1.15; thus λ is split, and

$$\rho_{\sigma_f}^i(X_L(\gamma_{\sigma_f})) = \begin{cases} X(\gamma) & (i = S) \\ 0 & (i \neq S) \end{cases}.$$

By Proposition 7.4.1, and Theorem 8.2.1 applied to L ,

$$\rho_{\sigma_f}^i(X_L(\gamma_{\sigma_f} + \lambda)) = \begin{cases} \psi_\alpha X(\gamma) & (i = S) \\ 0 & (i \neq S) \end{cases}.$$

By Corollary 6.4.14,

$$\begin{aligned}
(8.4.4) \text{ (a)} \quad & \text{Hom}_{g,K}(\psi_\alpha X(\gamma), \psi_\alpha X(\gamma)) \\
& \cong \text{Hom}_{\mathfrak{g}, L \cap K} \left[H^R(u, \psi_\alpha X(\gamma)), X_L(\gamma_{\sigma_f} + \lambda) \right],
\end{aligned}$$

and

$$(8.4.4) \text{ (b)} \quad \bar{X}_L(\gamma_{\sigma_f} + \lambda) \text{ does not occur in } H^{R+1}(u, \psi_\alpha X(\gamma)).$$

We want to show that the left side of (8.4.4) (a) is zero; so we must show that $\bar{X}_L(\gamma_{\sigma_f} + \lambda)$ does not occur in $H^R(u, \psi_\alpha X(\gamma))$. To use Proposition 7.4.7, we must compute

$$\begin{aligned}
& m\left(\bar{X}_L(\gamma_{\sigma_f}), H^i(u, X(\gamma))\right) \\
& m\left(\bar{X}_L(\gamma_{\sigma_f} - n\alpha), H^i(u, X(\gamma))\right).
\end{aligned}$$

and

By Theorem 8.1.6, the first of these is 1 if $i = R$ and zero otherwise; and (using Lemma 8.3.3) the second is 1 if $i = R+1$, and zero otherwise. By (8.4.4) (b) and the long exact sequence (7.4.7) (a), it follows that

$$m\left(\bar{X}_L(\gamma_{\sigma_f} + \lambda), H^R(u, \psi_\alpha X(\gamma))\right) = 0,$$

as we wished to show. Q.E.D.

Proposition 8.4.5 ([45], proof of Theorem 4.12; cf. Hecht-Schmid [15]). In the setting (8.2.6), suppose α is a noncompact imaginary root. Then there is a representation X^α , with character

$$\Theta(X^\alpha) = s_\alpha \cdot \Theta(X(\gamma))$$

(notation (7.2.28)), and a short exact sequence

$$0 \rightarrow X(\gamma) \rightarrow \phi_\alpha \psi_\alpha X(\gamma) \rightarrow X^\alpha \rightarrow 0.$$

If α is type I, then (notation (8.3.6))

parametrized as in Lemma 1.1.3. δ is just the representation called $X_d(n)$ in Definition 1.2.13.) By Propositions 7.4.1 and 7.4.3(a) (and Corollary 6.3.21 -- note that here $S = 0$), there is a short exact sequence

$$0 \rightarrow \delta \rightarrow \phi_\alpha \psi_\alpha(\delta) \rightarrow \mathcal{P}_g^0(s_\alpha \Gamma + \alpha) \rightarrow 0.$$

Now

$$\mathcal{P}_g^0(s_\alpha \Gamma + \alpha) \cong \text{Hom} \left(U(g), \mathbb{C}_{-n+1} \right),$$

which has weights

$$-n+1, -n+3, \dots$$

by Theorem 6.3.12 (or easy calculation). So $\phi_\alpha \psi_\alpha(\delta)$ has weights

$$\begin{aligned} & -n+1, -n+3, \dots, n-1 \quad (\text{with multiplicity one}) \\ & n+1, n+3, \dots \quad (\text{with multiplicity two}). \end{aligned}$$

By Proposition 1.2.14 (that is, "obviously")

$$\Theta \left(\phi_\alpha \psi_\alpha(\delta) \right) = 2 \cdot \Theta(\delta) + \Theta(F_{n-1});$$

here F_{n-1} is the irreducible representation of dimension n . By Corollary 7.3.24, δ is the unique irreducible submodule and the unique irreducible quotient of $\phi_\alpha \psi_\alpha(\delta)$. So if we set

$$X^\alpha = \phi_\alpha \psi_\alpha(\delta) / \delta,$$

then X^α has F_{n-1} as its unique irreducible submodule, and δ as its unique irreducible quotient. By Proposition 1.3.3 and its proof

$$X^\alpha \cong \pi(\gamma^\alpha) / \delta',$$

$$\begin{aligned} \psi_\alpha X(\gamma^\alpha) &\cong \psi_\alpha X(\gamma) \oplus \psi_\alpha X(s_\alpha \gamma) \\ 0 &\rightarrow X^\alpha \rightarrow X(\gamma^\alpha) \rightarrow X(s_\alpha \gamma) \rightarrow 0. \end{aligned}$$

If α is type II,

$$\begin{aligned} \psi_\alpha X(\gamma_+^\alpha) &\cong \psi_\alpha X(\gamma) \\ 0 &\rightarrow X^\alpha \rightarrow X(\gamma_+^\alpha) \oplus X(\gamma_-^\alpha) \rightarrow X(\gamma) \rightarrow 0. \end{aligned}$$

Proof. Suppose first that

$$(A) \quad \text{if } \beta \in \Delta(g, h), \text{ then } \beta|_t \text{ is proportional to } \alpha.$$

With notation defined there, Proposition 8.4.1 says that

$$X(\gamma) \cong \text{Ind}_P^G(\delta \otimes \nu \otimes 1);$$

here $P = M \ltimes N$, and

$$\Delta(m, t) = \{\pm \alpha\}.$$

Using Proposition 7.4.3(b) and induction by stages, we can therefore reduce to the case $G = M$, which is locally isomorphic to $SL(2, \mathbb{R})$ times a compact center. Suppose first that α is type I. As was pointed out in Step VI of the proof of (8.4.1),

$$G = SL(2, \mathbb{R}) \cdot (\text{center of } G)$$

up to covering, so we may assume $M = G = SL(2, \mathbb{R})$. As we saw in Step I of the proof of Proposition 8.4.1,

$$\begin{aligned} \delta &\cong \mathcal{P}_g^0(\Gamma - \alpha) \\ &\cong \text{Hom} \left(U(g), \mathbb{C}_{n+1} \right)^{K\text{-finite}}. \end{aligned}$$

(Here $\mathbb{C}_{n+1}$ denotes the n^{th} weight of $K = T = SO(2)$,

with δ' the discrete series of $SL(2, \mathbb{R})$ "opposite" to δ . So we have exact sequences

$$\begin{aligned} 0 \rightarrow \delta &\rightarrow \phi_\alpha \psi_\alpha(\delta) \rightarrow X^\alpha \rightarrow 0 \\ 0 \rightarrow X^\alpha &\rightarrow X(\gamma^\alpha) \rightarrow \delta' \rightarrow 0; \end{aligned}$$

these are the ones in the lemma. The formula for the character of X^α is (7.3.2)(a), and the one for $\psi_\alpha X(\gamma^\alpha)$ is Proposition 1.3.3(e).

If α is type II, then by definition

$$\begin{aligned} \delta &= \text{Ind}_{\mathcal{G}, T}^{\mathcal{G}, K}(X_{G^\#}(\gamma)) \\ X(\gamma_+^\alpha) \oplus X(\gamma_-^\alpha) &= \text{Ind}_{\mathcal{G}, T}^{\mathcal{G}, K}(X_{G^\#}(\gamma^\alpha)). \end{aligned}$$

So the exact sequences for G are simply induced from those for $G^\#$, if we let X^α be the module induced from X^α for $G^\#$. The formula

$$\psi_\alpha X(\gamma_\pm^\alpha) = \psi_\alpha X(\gamma)$$

follows from Propositions 1.5.9 and 1.5.26.

We turn now to the general case. Choose a parabolic subalgebra

$$\mathcal{G} = \mathfrak{l} + \mathfrak{u}$$

for γ^α (or $\gamma_\pm^\alpha$) in accordance with Corollary 8.1.15; thus $\mathfrak{l}$ is the centralizer in $\mathcal{G}$ of $\mathcal{L}^\alpha$, and L satisfies hypothesis (A) above. By Theorem 8.2.4, if $S = \dim \mathfrak{u} \cap \mathfrak{k}$,

$$\begin{aligned} X(\gamma) &= \mathcal{R}_{\mathcal{G}}^S(X_L(\gamma_{\mathcal{G}})) \\ X(s_\alpha \times \gamma) &= \mathcal{R}_{\mathcal{G}}^S \left[X_L((s_\alpha \times \gamma)_{\mathcal{G}}) \right] \\ X(\gamma^\alpha) &= \mathcal{R}_{\mathcal{G}}^S(X_L(\gamma^\alpha)) \end{aligned}$$

(or $\gamma_\pm^\alpha$ in the type II case); and the lower degree $\mathcal{R}^i$ are all zero. Define X_L^α for $\gamma_{\mathcal{G}}^\alpha$ by the first case, and set

$$X^\alpha = \mathcal{R}_{\mathcal{G}}^S(X_L^\alpha).$$

By the first case, we have a short exact sequence (if α is type I):

$$0 \rightarrow X_L^\alpha \rightarrow X_L(\gamma_{\mathcal{G}}^\alpha) \rightarrow X_L((s_\alpha \times \gamma)_{\mathcal{G}}) \rightarrow 0$$

By the long exact sequence for cohomological induction,

$$\dots \mathcal{R}^{S-2} \left[X_L((s_\alpha \times \gamma)_{\mathcal{G}}) \right] \rightarrow \mathcal{R}^{S-1}(X_L^\alpha) \rightarrow \mathcal{R}^{S-1}(X_L(\gamma_{\mathcal{G}}^\alpha)) \rightarrow \dots$$

We have seen above that the outside groups are zero, so

$$\mathcal{R}^{S-1}(X_L^\alpha) = 0.$$

A similar argument gives the same conclusion for type II.

All the isomorphisms and exact sequences now follow from the corresponding results for L , using the long exact sequence in $\mathcal{R}^*$ and the various vanishing results we have just stated. The character formula for X^α follows from the first exact sequence and Proposition 7.3.2(a). Q.E.D.

Corollary 8.4.6 (Hecht-Schmid [15]). In the setting of Proposition 8.4.5, if α is type I, then

$$s_\alpha \cdot \Theta(\gamma) + \Theta(s_\alpha \cdot \gamma) = \Theta(\gamma^\alpha).$$

If α is type II,

$$s_\alpha \cdot \Theta(\gamma) + \Theta(\gamma) = \Theta(\gamma_+^\alpha) + \Theta(\gamma_-^\alpha).$$

This is immediate from the exact sequences and the formula for $\Theta(X^\alpha)$ in Proposition 8.4.5.

exactly as in the proof of Proposition 8.4.5; and in that case they are given by Step VI of the proof of (8.4.1). Q.E.D.

In Proposition 8.2.7, the case where α and $\theta\alpha$ have opposite signs was treated as a formal consequence of the case when they have the same sign. The same idea handles some real roots.

Corollary 8.4.8 In the setting (8.2.6), suppose α is a real root satisfying the parity condition (notation (8.3.11)). Then $\alpha \in \tau(\bar{X}(\gamma))$, and $X(\gamma)$ is reducible. If α is type I, then there is a surjective map

$$X(\gamma) \rightarrow X(\gamma_{\alpha}^{+}) \oplus X(\gamma_{\alpha}^{-}) \rightarrow 0$$

(notation (8.4.12)). If α is type II, there is a surjective map

$$X(\gamma) \rightarrow X(\gamma_{\alpha}) \rightarrow 0.$$

Proof. If α is type I, then $\bar{X}(\gamma)$ occurs in $U_{\alpha}(\bar{X}(\gamma_{\alpha}^{+}))$ by Corollary 8.4.7; so $\alpha \in \tau(\bar{X}(\gamma))$ by Theorem 7.3.16. Suppose α is type II, and $\alpha \notin \tau(\bar{X}(\gamma))$; that is,

$$\psi_{\alpha}\bar{X}(\gamma) \neq 0.$$

By the exactness of ψ_{α} , $\psi_{\alpha}\bar{X}(\gamma)$ is a submodule of $\psi_{\alpha}X(\gamma)$. But by Proposition 8.4.5

$$\psi_{\alpha}X(\gamma) \cong \psi_{\alpha}X(\gamma_{\alpha}).$$

So

Corollary 8.4.7 In the setting (8.2.6), suppose α is a noncompact imaginary root. Then $\alpha \notin \tau(\bar{X}(\gamma))$. There is a short exact sequence

$$0 \rightarrow Y_1 \rightarrow U_{\alpha}(\bar{X}(\gamma)) \rightarrow Y_2 \rightarrow 0$$

(notation 7.3.17), with

$$Y_1 \subseteq X(\gamma).$$

a) Suppose α is type I (Definition 8.3.4). Then in the notation of (8.3.6), Y_2 is a non-zero submodule of $X(\gamma^{\alpha})$, and $\bar{X}(\gamma^{\alpha})$ occurs exactly once in $U_{\alpha}(\bar{X}(\gamma))$.

b) Suppose α is type II. Then

$$0 \neq Y_2 \subseteq X(\gamma_{+}^{\alpha}) \oplus X(\gamma_{-}^{\alpha}).$$

c) If α is type I, then $X(\gamma^{\alpha})$ is reducible, and contains $\bar{X}(\gamma)$ exactly once as a composition factor.

There is a surjection

$$X(\gamma^{\alpha}) \rightarrow X(\gamma) \oplus X(s_{\alpha} \times \gamma) \rightarrow 0.$$

d) If α is type II, then $X(\gamma_{+}^{\alpha})$ are both reducible, and each contains $\bar{X}(\gamma)$ exactly once as a composition factor. There is a surjection

$$X(\gamma_{+}^{\alpha}) \rightarrow X(\gamma) \rightarrow 0.$$

Proof. Except for the maps in (c) and (d), this is deduced from Proposition 8.4.5 exactly as Corollary 8.2.9 was deduced from Proposition 8.2.7; that can be left to the reader. The maps are obtained by reduction to the case of Proposition 8.4.1.

$$0 \neq \text{Hom}_{\mathcal{G}_K}(\psi_\alpha \bar{X}(\gamma), \psi_\alpha X(\gamma_\alpha)) \\ \cong \text{Hom}_{\mathcal{G}_K}(\phi_\alpha \psi_\alpha \bar{X}(\gamma), X(\gamma_\alpha)).$$

So $\bar{X}(\gamma_\alpha)$ must occur in $\phi_\alpha \psi_\alpha \bar{X}(\gamma)$. Since $\alpha \notin \tau(\bar{X}(\gamma_\alpha))$ (by Corollary 8.4.7), Theorem 7.3.16 says that

$$\bar{X}(\gamma_\alpha) \cong \bar{X}(\gamma),$$

a contradiction. So $\alpha \in \tau(\bar{X}(\gamma))$. The other assertions are contained in (8.4.7)(c), (d). Q.E.D.

Proposition 8.4.9 In the setting (8.2.6), suppose α is a real root not satisfying the parity condition (notation (8.4.8)). Then

$$s_\alpha \gamma = s_\alpha \times \gamma = \gamma - n\alpha$$

and there is a short exact sequence

$$0 \rightarrow X(\gamma) \rightarrow \phi_\alpha \psi_\alpha X(\gamma) \rightarrow X(\gamma) \rightarrow 0.$$

Proof. The first claim is Lemma 8.3.17. For the short exact sequence, we can use Proposition 7.4.3(b) to reduce to the case when H is split. So if $B = HN$ is a Borel subgroup, with N negative for γ ,

$$X(\gamma) = \text{Ind}_B^G(\gamma).$$

By a translation argument and Lemma 7.3.6, we may as well assume $-\alpha$ is simple in $\Delta(\mathfrak{n}, \mathfrak{h})$. Let

$$P_\alpha = M_\alpha A_\alpha N_\alpha$$

be the parabolic subgroup through α ; thus M_α is locally

isomorphic to $SL(2, \mathbb{R})$. By induction by stages and Proposition 7.4.3(b), we can reduce to the case $G = M$. Now Proposition 7.4.3(c) provides a short exact sequence

$$0 \rightarrow \text{Ind}_B^G(\gamma - n\alpha) \rightarrow \phi_\alpha \psi_\alpha X(\gamma) \rightarrow X(\gamma) \rightarrow 0.$$

By the first part of the proposition, this is

$$0 \rightarrow \text{Ind}_B^G(s_\alpha \gamma) \rightarrow \phi_\alpha \psi_\alpha X(\gamma) \rightarrow X(\gamma) \rightarrow 0.$$

By Theorem 4.1.4, the term on the left has the same composition factors and multiplicities as $X(\gamma)$; so it is enough to check that it is irreducible. The induced representation is still a principal series after restriction to $SL(2, \mathbb{R})$ (Lemma 1.5.7), and so is irreducible by Proposition 1.3.3(c). Q.E.D.

Corollary 8.4.10 In the setting (8.2.6), suppose

α is a real root not satisfying the parity condition.

Then $\alpha \notin \tau(\bar{X}(\gamma))$. There is an injection

$$U_\alpha(\bar{X}(\gamma)) \hookrightarrow X(\gamma)/\bar{X}(\gamma).$$

This is deduced from Proposition 8.4.9 just as Corollary 8.2.9 was deduced from Proposition 8.2.7.

These results (Corollaries 8.2.9, 8.4.7, 8.4.8, 8.4.10, and Proposition 8.4.3) completely determine $\tau(\bar{X}(\gamma))$. We will give a complete summary of the information later (Theorem 8.5.18) after sharpening the results on $U_\alpha(\bar{X}(\gamma))$ slightly. The most striking feature of them is a formal duality between various types of roots, which reverses the τ -invariant condition. The correspondences are listed below.

complex root $\alpha, \theta\alpha$ positive	$\Leftrightarrow$
compact imaginary root	$\Leftrightarrow$
noncompact type I imaginary root	$\Leftrightarrow$
noncompact type II imaginary root	$\Leftrightarrow$

complex root $\alpha, \theta\alpha$ negative	$\Leftrightarrow$
real root not satisfying parity condition	$\Leftrightarrow$
real type II root satisfying parity condition	$\Leftrightarrow$
real type I root satisfying parity condition	$\Leftrightarrow$

This duality presumably constitutes the formal framework in which to state a version of Conjecture 2.1.7 (B) for real groups. Proposition 9.2.12 has a similar flavor.

§5. More about U_α .

We want to sharpen our results on U_α . An important technical tool is the notion of dual $(\mathfrak{g}, K)$ modules.

Definition 8.5.1 If X is a $(\mathfrak{g}, K)$ module, set

$$X^C = \{f: X \rightarrow \mathbb{C} \mid \dim U(\mathfrak{k}) \cdot f < \infty\};$$

here $\mathfrak{k}$ acts by the algebraic dual representation,

$$(Z \cdot f)(x) = f(-Z \cdot x) \quad (x \in X, Z \in \mathfrak{k}).$$

X^C is called the dual or contragredient module.

Lemma 8.5.2 Suppose X is a $(\mathfrak{g}, K)$ module. Then X^C is a $\mathfrak{g}$ -stable subspace of the algebraic dual of X . If

$$X = \bigoplus_{\delta \in \hat{K}} \bigoplus_{\alpha \in I_\delta} V_\alpha^\delta,$$

with I_δ an index set and V_α^δ an irreducible

$\mathfrak{k}$ module of type δ , then

$$X^C = \bigoplus_{\delta \in \hat{K}} \bigoplus_{\alpha \in I_\delta} (V_\alpha^\delta)^*,$$

with $(V_\alpha^\delta)^*$ an irreducible $\mathfrak{k}$ module belonging to the contragredient of δ . In particular, X^C is a $(\mathfrak{g}, K)$ module.

This is trivial -- we leave the proof to the reader.

Corollary 8.5.3 The functor $X \mapsto X^C$ is contravariant and exact. If X has finite $\mathfrak{k}$ -multiplicities, then canonically, X is irreducible if and only if X^C is.

$$X^{CC} \cong X$$

The exactness and double dual statements are immediate from the lemma, and the irreducibility condition is a formal consequence.

Definition 8.5.4 Suppose H is a θ -stable Cartan subgroup of G , and $\gamma \in \hat{H}'$ is a regular character; say

$$\gamma = (\Gamma, \overline{\gamma}), \quad \Gamma \in \hat{H}, \overline{\gamma} \in \mathfrak{k}^*$$

(Definition 6.6.1). Put

$$\Gamma^C = \Gamma^{-1}, \quad \overline{\gamma}^C = -\gamma$$

$$\gamma^C = (\Gamma^C, \overline{\gamma}^C),$$

the contragredient or dual regular character of γ .

We leave to the reader the easy verification that γ^C is a regular character.

Proposition 8.5.6 Suppose $H \subseteq G$ is a θ -stable Cartan subgroup, and $\gamma \in \hat{H}'$ is a regular character (notation (6.6.1)). Then

a) for $\pi \in A(\gamma)$ (notation (6.6.3))

$$[\overline{X}(\gamma)(\pi)]^C \cong \overline{X}(\gamma^C)(\pi^*)$$

(notation (6.5.11), (8.4.4)).

b) $X(\gamma)^C$ has the same composition factors and multiplicities as $X(\gamma^C)$.

[In (b), the two representations are not in general isomorphic: the Langlands subrepresentations of $X(\gamma^C)$ occur as quotients of $X(\gamma)^C$.] There are conceptual proofs of this result, but we have so much machinery available that it is easier to give an ad hoc argument. One of the intermediate results, Proposition 8.5.12, is of some independent interest.

Lemma 8.5.7 Suppose X is a $(\mathfrak{g}, K)$ module, of infinitesimal character $\xi \in \mathfrak{h}^*$.

a) X^C has infinitesimal character $-\xi$.

b) $\pi \in \hat{K}$ is a lambda-lowest K -type of X if and only if π^* is a lambda-lowest K -type of X^C .

We leave the elementary proof to the reader.

Corollary 8.5.8 In the setting of Proposition 8.4.6, $X(\gamma)^C$ and $X(\gamma^C)$ have the same infinitesimal character and the same set of lambda-lowest K -types.

[For the second statement, one uses the characterization of $A(\gamma)$ given by Lemma 6.6.15.]

Corollary 8.5.9 In the setting of Definition 4.5.7, suppose X is a $(\mathfrak{g}, K)$ module of finite length. Then

$$\left(\psi_{\xi+\lambda_2}^{\xi+\lambda_1} X \right)^C \cong \psi_{-\xi-\lambda_2}^{-\xi-\lambda_1} (X^C).$$

This is immediate from the definition and Lemma 8.5.7.

Corollary 8.5.10 In the setting of Definition 7.2.5, suppose ϕ is a coherent family of virtual characters based on H and $\xi + \Lambda$. Define

$$\phi^C: -\xi + \Lambda \rightarrow \mathcal{U}(\mathfrak{g}, K)$$

by

$$\phi^C(-\xi + \lambda) = [\phi(\xi - \lambda)]^C.$$

[The contragredient of a virtual character is well-defined, since the contragredient functor is exact.] Then ϕ^C is a coherent family based on H and $-\xi + \Lambda$.

This is also clear from the definition and Lemma 8.5.7.

Corollary 8.5.11 Suppose $H^S \subseteq G$ is a maximally

split Cartan subgroup, and X is an irreducible

$(\mathfrak{g}, K)$ module with nonsingular infinitesimal

character $\xi \in \{\mathfrak{h}^S\}^*$. Fix a simple root $\alpha \in R^+(\xi)$

(notation (7.2.16)), and suppose $\alpha \notin \tau(X)$ (Definition 7.3.8). Suppose that $\lambda \in \Lambda^S$ satisfies

$$\langle \alpha, \xi + \overline{\lambda} \rangle = 0$$

$$\langle \beta, \xi + \overline{\lambda} \rangle > 0 \quad \beta \in R^+(\xi) - \{\alpha\}.$$

Write

$$\begin{aligned}\psi_\alpha &= \psi_{\xi+0}^{\xi+\lambda}, & \phi_\alpha &= \psi_{\xi+\lambda}^{\xi+0} \\ \psi_{-\alpha}^C &= \psi_{-\xi+0}^{-\xi-\lambda}, & \phi_{-\alpha}^C &= \psi_{-\xi-\lambda}^{-\xi-0}.\end{aligned}$$

Then, in the notation of Definition 7.3.17,

- a) $(\phi_\alpha \psi_\alpha X)^C \cong \phi_{-\alpha}^C \psi_{-\alpha}^C (X^C)$
- b) $(Q_\alpha(X))^C \cong K_{-\alpha}(X^C)$
- c) $U_\alpha(X^C) \cong (U_{-\alpha}(X))^C$

Proof. Statement (a) is a special case of Corollary 8.5.9.

The remaining claims are formal consequences of (a) and the definitions. Q.E.D.

Proposition 8.5.12 Suppose H is a θ -stable Cartan subgroup of G , $\xi \in \mathcal{H}^*$, and ϕ_1 and ϕ_2 are two coherent families of virtual characters based on H and $\xi + \Lambda$. Fix a positive root system $R^+ \subseteq R(\xi)$ (notation (7.2.16)). Assume that whenever $\lambda \in \Lambda$ and $\xi + \bar{\lambda}$ is dominant for R^+ , then $\phi_1(\xi + \lambda)$ is the character of a representation (instead of just a virtual character), and

$$\phi_1(\xi + \lambda) \Big|_K = \phi_2(\xi + \lambda) \Big|_K.$$

Then $\phi_1 = \phi_2$.

It is not true that a representation is determined by its infinitesimal character and its restriction to K ; for example if $G = \mathrm{SL}^+(2, \mathbb{R})$, and $\delta \in \hat{M}$ has $\delta(m_\alpha) = -1$, then $X(\delta \otimes \nu)$ and $X(\delta \otimes (-\nu))$ have the same infinitesimal character and

restriction to K , but are inequivalent (cf. Section 1.4). The proposition says that this phenomenon cannot persist under the translation functors.

Proof. By induction on the number of irreducible constituents of $\phi_1(\xi + \lambda)$ [with $\xi + \bar{\lambda}$ dominant and regular], it suffices to find a single such λ so that $\phi_1(\xi + \lambda)$ and $\phi_2(\xi + \lambda)$ have a constituent in common. Fix $\lambda_0 \in \Lambda$ so that $\xi + \bar{\lambda}_0$ is dominant and regular for R^+ . List the irreducible components of $\phi_1(\xi + \lambda_0)$ as

$$\{\bar{X}(\gamma_0^i), \gamma_0^i \in (\hat{H}^i)^\vee \mid 1 \leq i \leq r\};$$

that is,

$$\phi_1(\xi + \lambda_0) = \sum \bar{O}(\gamma_0^i).$$

Here

$$H^i = {}_{\mathbb{T}^i} A^i$$

is a θ -stable Cartan subgroup. Write

$$\bar{\gamma}^i = (\phi_0^i, \nu_0^i) \in (\mathcal{L}^i)^* + (\sigma^i)^*.$$

Let

$$\sigma^i: \mathcal{H}^* \rightarrow (\mathcal{H}^i)^*$$

be the unique σ^i -inner automorphism taking $\xi + \bar{\lambda}_0$ to $\bar{\gamma}_0^i$. If $\bar{\lambda} \in \mathcal{H}^*$, define

$$\begin{aligned}\phi^i(\bar{\lambda}) &= [\sigma^i(\xi + \bar{\lambda})] \Big|_{\mathcal{L}^i} \\ \nu^i(\bar{\lambda}) &= [\sigma^i(\xi + \bar{\lambda})] \Big|_{\sigma^i}.\end{aligned}\tag{8.5.13} \text{ (a)}$$

Suppose $\xi + \bar{\lambda}$ is dominant and regular for R^+ . By Proposition 7.2.22(a), and Theorem 8.2.1, we can find

$$\gamma^i(\lambda) \in (\hat{H}^i)'$$

such that

$$\overline{\gamma^i(\lambda)} = (\phi^i(\bar{\lambda}), \nu^i(\bar{\lambda}))$$

(8.5.13) (b)

$$\phi_1(\xi + \lambda) = \Sigma \bar{\Theta}((\gamma^i(\lambda))).$$

Consider the quadratic polynomial functions on $\mathcal{H}^*$

$$\delta_{i,j}(\bar{\lambda}) = \langle \phi^i(\bar{\lambda}), \phi^i(\bar{\lambda}) \rangle - \langle \phi^j(\bar{\lambda}), \phi^j(\bar{\lambda}) \rangle.$$

Define

$$P(\bar{\lambda}) = \prod_{\substack{i,j \\ \delta_{i,j} \neq 0}} \delta_{i,j}(\bar{\lambda}).$$

Given any $N > 0$, we can obviously find $\lambda_1 \in \Lambda$ so that

- (a) whenever $\lambda \in \Lambda$, and $|\bar{\lambda} - \bar{\lambda}_1| < N$, then $\xi + \bar{\lambda}$ is dominant and regular for R^+ ; and
- (b) whenever $\bar{\lambda} \in \mathcal{H}^*$, and $|\bar{\lambda} - \bar{\lambda}_1| < N$, then $P(\bar{\lambda}) \neq 0$.

[This only uses the fact that P is a polynomial which is not identically zero.] Because of (b), the functions $\delta_{i,j}$ cannot change sign near λ_1 . So we may assume that the γ_0^i are ordered so that

$$(8.5.14) \quad (c) \quad |\phi^{-1}(\bar{\lambda})| \leq |\phi^2(\bar{\lambda})| \leq \dots \leq |\phi^r(\bar{\lambda})|, \\ \text{all } \lambda \in \Lambda \text{ with } |\bar{\lambda} - \bar{\lambda}_1| < N.$$

If N was chosen large enough, then by Lemma 8.1.12 we can choose $\lambda \in \Lambda$ so that $|\lambda - \lambda_1| < N$, and the conditions (a) and (b) of Lemma 6.6.18 hold for $\gamma^1(\lambda)$. [For any large ball

has to contain points on a fixed coset of Λ_0^1 , and outside of the specified subspaces.] By (8.5.13) (b) and (8.5.14) (c), the lambda-lowest K -types of $\bar{X}(\gamma^1(\lambda))$ are lambda-lowest in $\phi_1(\xi + \lambda)$, and hence also in $\phi_2(\xi + \lambda)$. By Lemma 6.6.18, $\bar{X}(\gamma_1(\lambda))$ occurs also in $\phi_2(\xi + \lambda)$. Q.E.D.

Proof of Proposition 8.5.6 Obviously it is enough to prove the second statement. Choose a positive root system $R^+ \subseteq R(\bar{\gamma})$ so that

$$R^+ \supseteq \{\alpha \in R(\bar{\gamma}) \mid \langle \alpha, \bar{\gamma} \rangle > 0, \text{ or } \langle \alpha, \bar{\gamma} \rangle = 0 \text{ and } \langle \theta\alpha, \bar{\gamma} \rangle > 0\};$$

this is obviously possible. Let $(\sigma_{\gamma}, \delta_{\theta\nu})$ be the θ -stable data corresponding to γ (Proposition 6.6.2). Suppose $\lambda \in \Lambda$ is a weight of a finite dimensional representation of G , and $\bar{\gamma} + \bar{\lambda}$ is R^+ -dominant and regular. We claim that σ_{γ} satisfies the hypotheses of Theorem 8.2.4 for $\gamma + \lambda$. To see this, suppose

$$\alpha \in \Delta(\mathcal{U}) \cap R(\bar{\gamma}).$$

By definition of $\mathcal{U}$,

$$\langle \alpha + \theta\alpha, \bar{\gamma} \rangle > 0;$$

so by the choice of R^+ , either α or $\theta\alpha$ lies in R^+ ; so since $\bar{\gamma} + \bar{\lambda}$ is dominant for R^+ , either

$$\langle \alpha, \bar{\gamma} + \bar{\lambda} \rangle > 0 \text{ or } \langle \theta\alpha, \bar{\gamma} + \bar{\lambda} \rangle > 0,$$

as required. So by Theorem 8.2.4,

$$X(\gamma + \lambda) = \mathcal{P}_{\sigma_{\gamma}}^S(X_L(\gamma_{\theta} + \lambda)).$$

Now introduce notation as in Proposition 7.4.1. Let ϕ_0 be the unique coherent family based on H^S and $\bar{\gamma}^S + \Lambda^S$ (cf. Corollary 7.2.27) such that

$$(8.5.15) \text{ (a)} \quad \phi_0(\bar{\gamma}^S + \lambda^S) = \theta(\gamma + \lambda).$$

By Propositions 7.4.1 and 7.2.22(a),

$$\begin{aligned} \phi_0(\bar{\gamma}^S + 0) &= \psi_{\bar{\gamma}^S + \lambda^S}^{\bar{\gamma}^S + 0} \theta(\gamma + \lambda) \\ &= \theta\left(\mathcal{P}_{\mathcal{G}}^S(x_L(\gamma_{\mathcal{G}}))\right) \end{aligned}$$

$$(8.5.15) \text{ (b)} \quad \phi_0(\bar{\gamma}^S + 0) = \theta(\gamma).$$

Define ϕ_1 and ϕ_2 based on $-\bar{\gamma}^S + \Lambda^S$ by

$$\begin{aligned} (8.5.16) \text{ (a)} \quad \phi_1(-\bar{\gamma}^S + \mu^S) &= (\phi_0(\bar{\gamma}^S - \mu^S))^c \quad (\mu^S \in \Lambda^S) \\ \phi_2(-\bar{\gamma}^S - \lambda^S) &= \theta(\gamma^C - \lambda^S). \end{aligned}$$

The first formula defines a coherent family by Corollary 8.5.8, and the second specifies a unique family by Corollary 7.2.27. We have

$$\begin{aligned} (8.5.16) \text{ (b)} \quad \phi_1(-\bar{\gamma}^S) &= \theta(\gamma)^C \\ \phi_2(-\bar{\gamma}^S) &= \theta(\gamma^C). \end{aligned}$$

The first formula is (8.5.15)(b), and the second is proved in the same way. Suppose $\mu^S \in \Lambda$, and $\bar{\gamma}^S + \bar{\mu}^S$ is dominant and regular for R^+ . By Propositions 7.4.1 and 7.2.22(a), applied to (8.5.15)(a) and the second part of (8.5.16)(a),

$$\begin{aligned} (8.5.16) \text{ (c)} \quad \phi_1(-\bar{\gamma} - \mu) &= \theta(\gamma + \mu)^C \\ \phi_2(-\bar{\gamma} - \mu) &= \theta((\gamma + \mu)^C). \end{aligned}$$

By Corollary 8.5.8, $\phi_1(-\bar{\gamma} - \mu)$ and $\phi_2(-\bar{\gamma} - \mu)$ have the same set of lambda-lowest K-types. By the proof of Proposition 8.5.12 (which only needed information about non-lowest K-types to keep the induction going),

$$(8.5.17) \quad \bar{X}(\gamma + \mu)^C = \bar{X}((\gamma + \mu)^C) \quad [\bar{\gamma} + \bar{\mu} \text{ } R^+ \text{-dominant and regular}]$$

Apply this to the case when $\bar{\gamma}$ itself is regular, and $\gamma|_A$ is unitary. In that case $\mu = 0$ satisfies the hypothesis for 8.5.17, and

$$\begin{aligned} \bar{X}(\gamma) &= X(\gamma) \\ \bar{X}(\gamma^C) &= X(\gamma^C) \end{aligned}$$

By Corollary 6.5.15. So by (8.5.17),

$$X(\gamma)^C|_K = X(\gamma^C)|_K.$$

But this statement is independent of $\gamma|_A$, so it holds in general. By (8.5.16)(c), therefore,

$$\phi_1(-\bar{\gamma} - \mu)|_K = \phi_2(-\bar{\gamma} - \mu)|_K$$

whenever $\bar{\gamma} + \bar{\mu}$ is R^+ -dominant and regular. By Proposition 8.5.12,

$$\phi_1 = \phi_2;$$

and by (8.5.16)(b), this proves the result. Q.E.D.

Here is the main result on U_α ; we have already proved most of it.

Theorem 8.5.18 ([45], Theorem 4.12, and [39], Theorem 6.16). Suppose we are in the situation 8.2.6.

a) If α is compact imaginary, then $\alpha \in \tau(\bar{X}(\gamma))$ (Definition 7.3.8).

b) Suppose α is noncompact imaginary, and α is type I (Definition 8.3.4). Define H^α as in Definition 8.3.4, and

$$\gamma^\alpha \in (\hat{H}^\alpha),$$

as in Definition 8.3.6. Then $\alpha \notin \tau(\bar{X}(\gamma))$, and

$$\Theta\left[u_\alpha(\bar{X}(\gamma))\right] = \bar{\Theta}(\gamma^\alpha) + \Theta_0$$

(Definition 7.5.17).

c) Suppose α is noncompact imaginary and α is type II (Definition 8.3.4). Define H^α as in Definition 8.3.4 and

$$\gamma_\pm^\alpha \in (\hat{H}^\alpha),$$

as in Definition 8.3.6. Then $\alpha \notin \tau(\bar{X}(\gamma))$, and

$$\Theta\left[u_\alpha(\bar{X}(\gamma))\right] = \bar{\Theta}(\gamma_+^\alpha) + \bar{\Theta}(\gamma_-^\alpha) + \Theta_0.$$

d) Suppose α is complex, and $\theta\alpha \in R^+(\bar{\gamma})$. Then $\alpha \notin \tau(\bar{X}(\gamma))$, and

$$\Theta\left[u_\alpha(\bar{X}(\gamma))\right] = \bar{\Theta}(\gamma - n\alpha) + \Theta_0.$$

e) Suppose α is complex, and $\theta\alpha \notin R^+(\bar{\gamma})$. Then $\alpha \in \tau(\bar{X}(\gamma))$.

f) Suppose α is real and satisfies the parity condition (Definition 8.3.11). Then $\alpha \in \tau(\bar{X}(\gamma))$.

g) Suppose α is real and does not satisfy the parity condition (Definition 8.3.11). Then

$$\Theta(u_\alpha(x)) = \Theta_0.$$

In every case, Θ_0 denotes the character of a representation, and every constituent of Θ_0 occurs in $X(\gamma)$. More precisely, let Θ_1 be any irreducible constituent of Θ_0 . Then

h) $m(\Theta_1, \Theta_0) \leq 2 \cdot m(\Theta_1, \Theta(\gamma))$ in cases (b), (c), and (d)
 $m(\Theta_1, \Theta_0) \leq m(\Theta_1, \Theta(\gamma))$ in case (g).

proof. The argument will be slightly convoluted. First, we will prove everything except the fact that both $\bar{\Theta}(\gamma_\pm^\alpha)$ occur in U_α in case (c). Then we will deduce that from the rest of the theorem.

All the statements about $\tau(\bar{X}(\gamma))$ have already been proved, as have (a) - (g) and the fact that Θ_0 is the character of a representation (except for the part of (c) mentioned above); this is Corollary 8.2.9, Proposition 8.4.3, Corollary 8.4.7, Corollary 8.4.8, and Corollary 8.4.11. Furthermore, the second statement of (h) is contained in Corollary 8.4.10. So we consider the first statement of (h); we may as well assume we are in case (c), as the other cases are similar and easier. In the setting of Corollary 8.4.7, it suffices to show that

$$m(\Theta_1, Y_2) \leq m(\Theta_1, \Theta(\gamma)).$$

By Proposition 8.5.6, this is equivalent to

$$(*) \quad m(\theta_1^C, Y_2^C) \leq m(\theta_1^C, \theta(Y^C)).$$

The short exact sequence of Corollary 8.4.7 gives

$$0 \rightarrow Y_2^C \rightarrow U_\alpha(\bar{X}(Y))^\mathbb{C} \rightarrow Y_1^C \rightarrow 0.$$

By Corollary 8.5.11(c),

$$U_\alpha(\bar{X}(Y))^\mathbb{C} \cong U_\alpha(\bar{X}(Y^C)).$$

Let

$$0 \rightarrow W_1 \rightarrow U_\alpha(\bar{X}(Y^C)) \rightarrow W_2 \rightarrow 0.$$

denote the short exact sequence of Corollary 8.4.7 applied to Y^1 . Since Y_2^C is a submodule of the middle term, we get a new short exact sequence

$$0 \rightarrow V_1 \rightarrow Y_2^C \rightarrow V_2 \rightarrow 0,$$

with

$$\begin{aligned} V_1 &= Y_2^C \cap W_1 \subseteq W_1 \subseteq X(Y^C)/\bar{X}(Y^C) \\ V_2 &\subseteq W_2 \subseteq X((Y^C)_+^{-\alpha}) \oplus X((Y^C)_-^{-\alpha}). \end{aligned}$$

So (*) will follow if we show that

$$(**) \quad m(\theta_1^C, V_2) = 0.$$

Now by Corollary 8.4.7,

$$Y_2 \subseteq X(Y_+^\alpha) \oplus X(Y_-^\alpha);$$

so the only possible irreducible submodules of Y_2 are $\bar{X}(Y_\pm^\alpha)$,

and they occur at most once in the composition series. So by Proposition 8.5.6, Y_2^C has $\bar{X}((Y^C)_\pm^{-\alpha})$ as its only possible irreducible quotients, and they occur at most once. So the same holds for the quotient V_2 of Y_2^C . But

$$V_2 \subseteq X((Y^C)_+^{-\alpha}) \oplus X((Y^C)_-^{-\alpha});$$

so the same statement holds for V_2 with quotient replaced by subrepresentation. It follows that

$$V_2 \subseteq \bar{X}((Y^C)_+^{-\alpha}) \oplus \bar{X}((Y^C)_-^{-\alpha}),$$

which implies (**).

Finally we prove the remaining part of (c). The argument is a refinement of the proof of Corollary 8.4.10. By Corollary 8.4.6, $\bar{\theta}(Y_+^\alpha)$ (say) occurs in

$$\theta(Y) + s_\alpha \theta(Y).$$

So by Corollary 7.3.20, either $\bar{\theta}(Y_+^\alpha)$ occurs in $\theta(Y)$, or there is a constituent $\bar{\theta}(Y^1) \{Y^1 \in \{\hat{H}^1\}'\}$ of $\theta(Y)$, such that $\alpha \notin \tau(\bar{X}(Y^1))$, and $\bar{\theta}(Y_+^\alpha)$ occurs in $U_\alpha(\bar{X}(Y^1))$. Suppose we are in the first case; or in the second, and $\bar{\theta}(Y_+^\alpha)$ actually occurs in $\theta(Y^1)$. Then we have containments

$$\bar{\theta}(Y_+^\alpha) \subseteq \theta(Y^1), \quad \bar{\theta}(Y^1) \subseteq \theta(Y), \quad \bar{\theta}(Y) \subseteq \theta(Y_+^\alpha),$$

the last being part of Corollary 8.4.7(d). Now Lemma 6.6.5 gives a contradiction. So we are in the second case, and $\bar{\theta}(Y_+^\alpha)$ does not occur in $\theta(Y^1)$. So the part of the theorem already proved limits the possibilities for the relation between Y_+^α and Y^1 . Let

$$R^+(\bar{\gamma}^1) \leftrightarrow R^+(\bar{\gamma})$$

be the unique identification coming from $\text{Ad}(\sigma)$, and let α^1 be the root corresponding to α . If α^1 is complex, then (d) says that

$$\gamma = \gamma^1 - n\alpha^1 \quad (\text{up to conjugacy}),$$

so that α would also be complex; a contradiction. So we must be in case (b) or case (c): α^1 is noncompact imaginary, and

$$\gamma_+^\alpha \in c\alpha^1(\gamma^1) \quad (\text{up to conjugacy})$$

(Definition 8.3.6). If we apply c_α^{-1} to this (Definition 8.3.16), then Proposition 8.3.18 implies that γ^1 is conjugate to γ ; so $\bar{X}(\gamma_+^\alpha)$ occurs in $U_\alpha(\bar{X}(\gamma))$, as we wished to show.

Q.E.D.

§6. Standard irreducible modules.

The machinery developed so far in this chapter provides a useful structure for inductive arguments. However, these arguments have to begin somewhere. To see where, we begin with some elementary facts about root systems.

Lemma 8.6.1 Let R be a root system with an

involutive automorphism θ , and a positive system

R^+ . We call a root $\alpha \in R$ real if $\theta\alpha = -\alpha$,

imaginary if $\theta\alpha = \alpha$, and complex otherwise. The following conditions on R^+ are equivalent.

a) There is a complex root $\alpha \in R^+$ such that

$$\theta\alpha \notin R^+.$$

b) There is a simple complex root $\alpha \in R^+$ such that

$$\theta\alpha \notin R^+.$$

c) The set of non-real roots in R^+ is not θ -stable.

Proof. Trivially (b) $\Rightarrow$ (a) $\iff$ (c). So assume (c). Choose a nonreal root $\delta \in R^+$ such that

$$-\theta\delta \in R^+$$

and $h(\delta)$ [the number of terms in the expression of δ as a sum of simple roots] is minimal. Obviously δ is complex; if it is simple, we are done. So suppose it is not. Write

$$\delta = \alpha + \beta. \quad \alpha, \beta \in R^+$$

In particular,

$$h(\delta) = h(\alpha) + h(\beta)$$

$$-\theta\delta = -\theta\alpha - \theta\beta \in R^+(\gamma).$$

Suppose first that $\theta\alpha$ is positive. Then

$$-\theta\beta = -\theta\delta + \theta\alpha \in R^+(\gamma);$$

and by the minimality assumption on $h(\delta)$, β is necessarily real. So

$$\theta\beta = -\beta, \quad h(\theta\beta) = -h(\beta).$$

So

$$h(-\theta\delta) = h(\beta) - h(\theta\alpha) < h(\beta) + h(\alpha) = h(\delta).$$

So $-\theta\delta$ is a complex positive root, with

$$\theta(-\theta\delta) = -\delta \notin R^+(\gamma),$$

$$h(-\theta\delta) < h(\delta).$$

This contradicts the minimality of δ . So $\theta\alpha$ is negative. Similarly $\theta\beta$ is negative. By the minimality of δ , α and β are both real; so δ is also, a contradiction. Q.E.D.

Lemma 8.6.2 In the setting of Lemma 8.6.1, suppose that the equivalent conditions fail; that is, that the non-real roots in R^+ are stable under δ . Then the real roots are spanned by simple real roots.

Proof. Exactly the same argument applies: let δ be a real root which cannot be expressed as a sum of simple real roots, and having minimal height with respect to this property. Then δ cannot be simple, so we write

$$\delta = \alpha + \beta \quad (\alpha, \beta \in R^+);$$

by the minimality of δ , α and β must be non-real. So by hypothesis

$$\theta\alpha \in R^+, \quad \theta\beta \in R^+,$$

and therefore

$$-\delta = \theta\delta \in R^+$$

a contradiction. Q.E.D.

In the formal duality described after Corollary 8.4.10, the next result corresponds to the symmetric space property of compact and noncompact imaginary roots.

Lemma 8.6.3 In the setting of Definition 8.3.11, $R_0^{\mathbb{R}}(\gamma)$ is a root system; and if the subscripts are

interpreted mod 2,

$$[\mathbb{R}_i^{\mathbb{R}}(\gamma) + \mathbb{R}_{i+j}^{\mathbb{R}}(\gamma)] \cap \mathbb{R}^{\mathbb{R}}(\gamma) \subseteq \mathbb{R}_{i+j}^{\mathbb{R}}(\gamma) \\ \text{for all } i \text{ and } j.$$

Proof. The second formula is immediate from Corollary 4.3.20 and Lemma 8.3.12(b). Taking $i = j = 0$, we see that $R_0^{\mathbb{R}}(\gamma)$ is a root system; and so $R_0^{\mathbb{R}}(\gamma)$ is as well. Q.E.D.

Corollary 8.6.4 In the setting of Definition 8.4.8, let $\Pi^{\mathbb{R}}$ denote a set of simple roots of $R^{\mathbb{R}}(\gamma)$ (with respect to any positive system). Then the following are equivalent:

- a) No root $\alpha \in \Pi^{\mathbb{R}}$ satisfies the parity condition.
- b) No root $\alpha \in R^{\mathbb{R}}(\gamma)$ satisfies the parity condition.

Proof. That (b) $\implies$ (a) is obvious; so assume (a). Then $R_0^{\mathbb{R}}(\gamma)$ contains $\Pi^{\mathbb{R}}$. Since $R_0^{\mathbb{R}}(\gamma)$ is a root system, it follows that $R_0^{\mathbb{R}}(\gamma)$ is all of $R^{\mathbb{R}}(\gamma)$. Q.E.D.

Definition 8.6.5 Suppose $\gamma \in \hat{\mathfrak{h}}'$ is a regular character, and $\bar{\gamma} \in \mathcal{K}^*$ is nonsingular. We say γ is minimal, if either of the following sets of conditions is satisfied. (They are equivalent by Lemmas 8.6.1 and 8.6.2, and Corollary 8.6.4.)

- a) For any non-real root $\alpha \in R^+(\gamma)$, $\theta\alpha \in R^+(\gamma)$; and no real root in $R(\gamma)$ satisfies the parity condition.
- b) For any non-real simple root $\alpha \in R^+(\gamma)$, $\theta\alpha \in R^+(\gamma)$; and no real simple root of $R(\gamma)$ satisfies the parity condition.

Many inductive arguments will lead to minimal regular characters, so we study them separately. The first main result is:

Theorem 8.6.6 ([39], Theorem 6.1.5) Suppose
 $\gamma \in \hat{H}^1$ is a regular character (Definition 6.6.1),
and $\bar{\gamma} \in \hat{h}^*$ is nonsingular. Then the standard
representation $X(\gamma)$ is irreducible if and only
if γ is minimal (Definition 8.6.5).

The proof requires a little preparation. We begin with the following standard structural fact.

Lemma 8.6.7 Suppose σ_0^1 and σ_0^2 are abelian
subalgebras of $\mathfrak{g}_0$ (the -1 eigenspace of θ),
and σ is an inner automorphism of $\mathfrak{g}$ taking
 σ_0^1 onto σ_0^2 . Then there is an element $k \in K$
such that

$$\text{Ad}(k): \sigma_0^1 \rightarrow \sigma_0^2$$

is the same as the map given by σ .

Proof. We may as well assume G is semisimple and connected. Choose a maximally split Cartan subgroup $H^S = T^S A^S$ of G ; thus σ_0^S is maximal abelian in $\mathfrak{g}_0$. After replacing σ_0^1 and σ_0^2 by conjugates under K , we may assume

$$\sigma_0^1 \subseteq \sigma_0^S.$$

Choose an element $x^1 \in \sigma_0^1$, so that

$$(8.6.8) \text{ (a)} \quad \Delta^0 = \{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^S) \mid \alpha(x^1) = 0 \}$$

$$= \{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^S) \mid \alpha|_{\sigma_0^1} = 0 \};$$

that is, a maximally regular element of σ_0^1 . Since σ_0 may be characterized as the subspace of σ on which the roots take real values, we have

$$\sigma \sigma_0^1 = \sigma_0^2;$$

so $x^2 = \sigma x^1$ is a maximally regular element of σ_0^2 . Choose positive systems

$$(\Delta_r^+)^i \subseteq \Delta(\mathfrak{g}, \sigma_0^S) \quad (i = 1, 2)$$

of restricted roots, so that

$$(8.6.8) \text{ (b)} \quad \langle \beta, x^i \rangle \geq 0 \quad (\beta \in (\Delta^+)^i).$$

After modifying σ_0^2 by an element of $W(G, A)$, we may assume

$$(\Delta_r^+)^1 = (\Delta_r^+)^2.$$

Now choose a positive system

$$(\Delta^+)^1 \subseteq \Delta(\mathfrak{g}, \mathfrak{h}^S), \quad (\Delta^+)^1|_{\sigma_0^S} = (\Delta_r^+)^1.$$

Then x^1 and x^2 are both dominant for $(\Delta^+)^1$. Any two dominant weights which are conjugate under a complex semisimple group must coincide; so

$$x^2 = \sigma x^1 = x^1.$$

Any element y^1 near x^1 in σ_0^1 will have the properties (8.6.8); so

$$\sigma y^1 = y^1.$$

So σ is the identity on an open set in σ_0^1 ; so $\sigma_0^1 = \sigma_0^2$.
 Q.E.D.

Lemma 8.6.6.9 Suppose $\gamma \in \hat{H}^*$, $\bar{\gamma}$ is nonsingular, and the set of non-real roots in $R^+(\bar{\gamma})$ is $\emptyset$ -stable.

Let $\sigma_{\bar{\gamma}} = \ell + \mathcal{U}$ be a $\emptyset$ -stable parabolic, $H^1 \subseteq L$ a $\emptyset$ -stable Cartan subgroup, and $\gamma^1 \in (\hat{H}^1)^*$. Suppose

$\bar{X}_L(\gamma^1)$ occurs in $H^*(\mathcal{U}, \bar{X}(\gamma))$. Then, after replacing (H^1, γ^1) by a conjugate under K , we may assume that

$$a) \quad A^1 \subseteq A$$

$$b) \quad T_0^1 \supseteq T_0;$$

and that there is an inner automorphism σ of $\mathfrak{g}$,

such that

$$c) \quad \sigma: \mathfrak{h} \rightarrow \mathfrak{h}^1$$

$$d) \quad \sigma^{-1} \Big|_{\sigma^1} = \text{identity}$$

$$e) \quad t_{\sigma \bar{\gamma}^1} = \bar{\gamma}.$$

Proof. If $\lambda \in \mathfrak{h}^*$, recall that

$$\lambda = \text{RE } \lambda + i \text{ IM } \lambda$$

RE λ , IM λ real-valued on $(\sigma_{\bar{\gamma}} + i\mathfrak{t}_0)$

(Definition 5.4.11). Now $\bar{X}_L(\gamma^1)$ has infinitesimal character

$$\bar{\gamma}_{\mathfrak{g}}^1 = \bar{\gamma}^1 - \rho(\mathcal{U}^1)$$

(Lemmas 6.6.6 and 8.1.2). By Corollary 3.1.6, $\bar{\gamma}^1$ defines the same infinitesimal character as $\bar{\gamma}$. So there is a unique inner automorphism σ of $\mathfrak{g}$ such that

$$\sigma: \mathfrak{h} \rightarrow \mathfrak{h}^1, \quad t_{\sigma \bar{\gamma}^1} = \bar{\gamma}.$$

By Lemma 5.4.12, t_{σ} commutes with RE. Consider the cones

$$(8.6.10) \quad S = \{ \lambda \in \mathfrak{h}^* \mid \text{RE } \lambda \text{ is dominant and regular for } R^+(\bar{\gamma}) \}$$

$$S_{\text{R}} = \{ \lambda \in S \mid \lambda = \text{RE } \lambda \} = \text{RE } S.$$

Consider the function on $\mathfrak{h}^*$

$$(8.6.11) \quad (a) \quad f(\lambda) = \left| \text{RE}(t_{\sigma}^{-1} \lambda) \Big|_{\mathfrak{h}^1} \right| - \left| \text{RE } \lambda \Big|_{\mathfrak{h}^1} \right|.$$

We want to prove that

$$(8.6.11) \quad (b) \quad f(\lambda) \geq 0, \text{ all } \lambda \in \bar{S}.$$

Since f is homogeneous, continuous, and depends only on RE λ , it is enough to prove (8.6.11) (b) for λ belonging to a fixed translate of a lattice in $\text{RE}(\mathfrak{h}^*)$. We will use the lattice of weights of finite dimensional representations, and translate it by $\bar{\gamma}$. So suppose $\mu \in \Lambda$, and $\bar{\gamma} + \bar{\mu} \in S$; we want to prove that $f(\bar{\gamma} + \bar{\mu})$ is non-negative. By (8.6.10), $\bar{\gamma} + \bar{\mu}$ is dominant and regular for $R^+(\bar{\gamma})$; so by Theorem 8.2.1 (with notation defined there)

$$\psi_{\bar{\gamma} + \mu}^{\bar{\gamma} + \mu, S} X(\gamma) = X(\gamma + \mu).$$

By Proposition 7.4.6 and Theorem 8.2.1 applied to L , there is a weight

$$\mu^1 \in \Lambda^1 \subseteq \hat{H}^1$$

satisfying

$$(8.6.12) \quad \mu^1 = t_{\sigma}^{-1} \bar{\mu}.$$

$$\bar{X}_L(\gamma_{\mathfrak{g}}^1 + \mu^1) \text{ occurs in } H^*(\mathcal{U}, X(\gamma + \mu)).$$

By Corollary 8.1.11 and Lemma 6.6.5,

$$\begin{aligned}
|(\bar{\gamma} + \bar{\mu})|_{\mathcal{X}}| &\leq |(\bar{\gamma}^1_{\mathcal{G}} + \rho(\mathcal{W}) + \bar{\mu}^1)|_{\mathcal{X}}| \\
&= |(\bar{\gamma}^1 + \bar{\mu}^1)|_{\mathcal{X}}| \quad (\text{by Lemma 8.1.2}) \\
&= |\mathbf{t}_{\sigma^{-1}}(\bar{\gamma} + \bar{\mu})|_{\mathcal{X}}|
\end{aligned}$$

by (8.6.12) and the definition of σ . But this inequality

says precisely that $f(\bar{\gamma} + \bar{\mu})$ is non-negative, proving (8.6.11) (b).

Write

$$S_{\mathbb{R}}^{\theta} = \{x + \theta x | x \in S_{\mathbb{R}}\} \subseteq \text{RE } \mathcal{X}^* ;$$

this is an open cone, and is contained in $\bar{S}$ by the hypothesis on $R^+(\bar{\gamma})$. If $x \in S_{\mathbb{R}}^{\theta}$, then

$$|x| = |x|_{\mathcal{X}}| \leq |\mathbf{t}_{\sigma^{-1}}x|_{\mathcal{X}}| \leq |\mathbf{t}_{\sigma^{-1}}x| = |x| ;$$

the first inequality is (8.6.11) (b), and the others are clear.

So equality holds everywhere; and we have proved

$$x \in S_{\mathbb{R}}^{\theta} \implies \mathbf{t}_{\sigma^{-1}}x \in (\mathcal{X}^1)^* .$$

Since $S_{\mathbb{R}}^{\theta}$ is open in $\text{RE } \mathcal{X}^*$,

$$\mathbf{t}_{\sigma^{-1}}\mathcal{X}^* \subseteq (\mathcal{X}^1)^*$$

or

$$\sigma\mathcal{X} \subseteq \mathcal{X}^1 .$$

Taking orthogonal complements, we get

$$\sigma^{-1}\sigma\mathcal{X}^1 \subseteq \sigma\mathcal{X} .$$

Apply Lemma 8.6.7, with $\sigma\mathcal{X}^2 = \sigma^{-1}\sigma\mathcal{X}^1$; the conclusion of the lemma is immediate. Q.E.D.

Lemma 8.6.13 In the setting of Theorem 8.5.18, suppose we are in cases (b), (c) or (d). Let $\bar{X}(\gamma^1)$ be one of the constituents of $U_{\alpha}(\bar{X}(\gamma))$ listed which is not in Θ_0 . Then

$$\ell^I(\gamma^1) = \ell^I(\gamma) + 1$$

(notation (8.1.4)).

Proof. Suppose we are in case (b); define H^{α} , γ^{α} as there, and fix a real root $\tilde{\alpha}$ as in Definition 8.3.4. If we identify $\mathcal{H}^{\alpha}$ with $\mathcal{H}$ by the identity map on $\mathcal{H} \cap \mathcal{H}^{\alpha}$, and sending $\tilde{\alpha}$ to α , then the Cartan involution θ on $\mathcal{H}^{\alpha}$ is identified with

$$\theta^{\alpha} = s_{\alpha}\theta = \theta s_{\alpha}$$

on $\mathcal{H}$. Now

$$\theta\alpha = \alpha ;$$

so if $\beta \neq \alpha$ is in $R^+(\bar{\gamma})$, then $s_{\alpha}\theta\beta$ has the same sign as $\theta\beta$, since α is a simple root. So by Definition 8.1.4,

$$\begin{aligned}
\ell^I(\gamma^{\alpha}) &= \frac{1}{2}|\{\beta \in R^+(\bar{\gamma}^{\alpha}) | \theta\beta \notin R^+(\bar{\gamma}^{\alpha})\} + \frac{1}{2}\dim A^{\alpha} - c_0 \\
&= \frac{1}{2}|\{\beta \in R^+(\bar{\gamma}) | s_{\alpha}\theta\beta \notin R^+(\bar{\gamma})\} + \frac{1}{2}(\dim A + 1) - c_0 \\
&= \frac{1}{2}|\{\alpha\}| + \frac{1}{2}|\{\beta \in R^+(\bar{\gamma}) - \{\alpha\} | \theta\beta \notin R^+(\bar{\gamma})\}| \\
&\quad + \frac{1}{2}\dim A + \frac{1}{2} - c_0 \\
&= 1 + \ell^I(\gamma) ,
\end{aligned}$$

as we wished to show. The other cases are similar, and we leave them to the reader. Q.E.D.

Lemma 8.6.14 In the setting of Lemma 8.6.9, suppose that H^1 is not conjugate to H . Then there is a real root $\beta \in R(\bar{\gamma})$ such that $t_{\sigma^{-1}\beta}$ is imaginary; and there is a simple real root $\alpha \in R^+(\bar{\gamma})$ such that $t_{\sigma^{-1}\alpha} = \alpha^1$ is imaginary, or is complex with $\theta\alpha^1 \in R^+(\bar{\gamma}^1)$.

Proof. Assume that H^1 has been conjugated in accordance with Lemma 8.6.9. Let L^1 be the centralizer in G of $A^1T_0 \subseteq H \cap H^1$; then L^1 contains both H and H^1 . By the theorem of Kostant and Sugita ([50], Theorem 1.3.1.10), we can find a set $\{\beta_1, \dots, \beta_r\}$ of orthogonal real roots of $\mathcal{H}$ in $\mathfrak{g}^1$ such that, in the notation of 8.3.8, H^1 is conjugate under $L^1 \cap K$ to

$$\left\{ \dots ((H\beta_1)\beta_2) \dots \right\}_{\beta_r}.$$

We may as well assume that H^1 is actually equal to this new Cartan. Let $\tilde{\beta}_1 \dots \tilde{\beta}_r$ be the imaginary roots of $\mathcal{H}^1$ in $\mathfrak{g}^1$ corresponding to $\{\beta_1 \dots \beta_r\}$. Let

$$\phi: \mathcal{H} \rightarrow \mathcal{H}^1$$

be the map which is the identity on $\mathcal{H} \cap \mathcal{H}^1$, and sends $\tilde{\beta}_i$ to β_i . In particular, ϕ is the identity on σ^1 . Since σ was inner for $\mathcal{G}$, there is an element

$$w \in W(\mathcal{G}, \mathcal{H}^1)$$

such that

$$t_{\sigma} = t_{\phi} \circ w$$

as a map from $(\mathcal{H}^1)^*$ to $\mathcal{H}^*$. Since t_{ϕ} and t_{σ} are both the identity on $(\sigma^1)^*$, w is as well; so by Chevalley's theorem, we can write w as a product of reflections around imaginary roots. In particular, w commutes with θ .

Let

$$\begin{aligned} R^0 &= \{\beta \in R(\bar{\gamma}^1) \mid t_{\sigma\beta} \text{ is real}\} \\ &= \{\beta \in R(\bar{\gamma}^1) \mid w\beta \text{ is supported on } \sigma_0^1 \oplus \Sigma \mathbb{C}\tilde{\beta}_i\}. \end{aligned}$$

Since w commutes with θ , the latter description shows that R^0 is θ -stable, and contains the imaginary roots $\{w^{-1}\tilde{\beta}_i\}$. Obviously, this set is non-empty when $H^1 \neq H$, proving the first claim. Since the real roots in $R(\bar{\gamma})$ are spanned by simple roots, R^0 is spanned by simple roots of $R^+(\bar{\gamma}^1)$. Now apply Lemmas 8.6.1 and 8.6.2 to R^0 and the involution $-\theta$. We conclude that either there is a complex simple root α^1 with

$$\theta\alpha^1 \in R^+(\bar{\gamma}^1);$$

or there is a simple imaginary root $\alpha^1 \in R^0$. (Of course this second step uses the fact that there are imaginary roots.) This is precisely the conclusion of the lemma. Q.E.D.

Proof of Theorem 8.6.6 Suppose first that γ is not minimal. By Definition 8.6.5, there is a simple root $\alpha \in R^+(\bar{\gamma})$, such that either α is complex and $\theta\alpha \notin R^+(\bar{\gamma})$, or α is real and satisfies the parity condition. In either case, $X(\gamma)$ is reducible (Corollary 8.2.9 or Corollary 8.4.8). Conversely, suppose that γ is minimal, but $X(\gamma)$ is reducible. Since $\bar{X}(\gamma)$ is the unique irreducible submodule of $X(\gamma)$, there is an

irreducible quotient $\bar{X}(\gamma^1)$ ($\gamma^1 \in (\hat{H}^1)'$) of $X(\gamma)$, which is distinct from $X(\gamma)$; so (H, γ) is not conjugate to (H^1, γ^1) . Since $\bar{X}(\gamma^1)$ is a submodule of $X(\gamma^1)$, we have

$$(8.6.15) \text{ (a)} \quad \text{Hom}_{\mathcal{G}, K}(X(\gamma), X(\gamma^1)) \neq 0.$$

Choose γ^1 so that (8.6.15) (a) holds, (H^1, γ^1) is not conjugate to (H, γ) , and $\ell^I(\gamma^1)$ is maximal. We want to get a contradiction. Let $\mathcal{G} = \ell + \mathcal{U}$ be the θ -stable parabolic associated to γ^1 (Proposition 6.6.2). By Theorem 6.5.9 (g),

$$(8.6.15) \text{ (b)} \quad \text{Hom}_{\ell, L \cap K} \left[H^R(\mathcal{U}, X(\gamma)), X_L(\gamma^1) \right] \neq 0.$$

By Lemma 8.6.9, we may assume

$$(8.6.16) \quad \begin{matrix} \sigma^1 \subseteq \sigma \\ \mathcal{L}^1 \supseteq \mathcal{L}; \end{matrix}$$

choose σ as in that lemma.

Suppose first that H^1 is not conjugate to H . By Lemma 8.6.14, we can find a simple real root $\alpha \in R^+(\bar{\gamma})$ such that the corresponding root $\alpha^1 \in R^+(\bar{\gamma}^1)$ is imaginary, or is complex, and $\theta\alpha^1 \in R^+(\bar{\gamma}^1)$. We are in the situation (8.2.6), and we use that notation. Notice that by Proposition 8.4.9, there is short exact sequence

$$(8.6.17) \quad 0 \rightarrow X(\gamma) \rightarrow \phi_\alpha \psi_\alpha X(\gamma) \rightarrow X(\gamma) \rightarrow 0.$$

So (8.6.15) (a) implies that

$$(8.6.18) \text{ (a)} \quad 0 \neq \text{Hom}_{\mathcal{G}, K}(\phi_\alpha \psi_\alpha X(\gamma), X(\gamma^1));$$

for we can find a map by first passing to the quotient $X(\gamma)$ of $\phi_\alpha \psi_\alpha X(\gamma)$. Since

$$(8.6.18) \text{ (b)} \quad \begin{aligned} &\text{Hom}(\phi_\alpha \psi_\alpha X(\gamma), X(\gamma^1)) \cong \\ &\text{Hom}(\psi_\alpha X(\gamma), \psi_\alpha X(\gamma^1)) \end{aligned}$$

by Proposition 4.5.8, we conclude that

$$\psi_\alpha X(\gamma^1) \neq 0.$$

By Proposition 8.4.1, α^1 is not compact imaginary. We treat the remaining cases separately. Suppose first that α^1 is noncompact imaginary, of type I. By Proposition 8.4.3,

$$\psi_\alpha X(\gamma^1) \subseteq \psi_\alpha X((\gamma^1)^1, \alpha^1)$$

as a direct summand; so by (8.6.18),

$$\text{Hom}_{\mathcal{G}, K}(\phi_\alpha \psi_\alpha X(\gamma), X((\gamma^1)^1, \alpha^1)) \neq 0.$$

By (8.6.17), this implies that

$$\text{Hom}_{\mathcal{G}, K} \left[X(\gamma), X((\gamma^1)^1, \alpha^1) \right] \neq 0.$$

By Lemma 8.6.13, $\ell^I((\gamma^1)^1, \alpha^1) = \ell^I(\gamma^1) + 1$, so this contradicts the maximality assumption on γ^1 . The case of type II noncompact imaginary α^1 is almost identical, and the case of complex α^1 follows in the same way from Proposition 8.2.7; we leave the verification of this to the reader. Q.E.D.

The next result contains all of Proposition 2.2.10 which we have not already proved. The reader should study the proof very carefully: most of the arguments in Chapter 9 will be refinements of it.

Proposition 8.6.19 Suppose $\gamma \in \hat{H}'$, and $\bar{\gamma} \in \mathcal{H}^*$ is nonsingular. Suppose that $\gamma^1 \in (\hat{H}^1)'$, and

$\bar{X}(\gamma^1)$ is a composition factor of $X(\gamma)$. Then (notation (8.1.4))

$$\ell^I(\gamma^1) \leq \ell^I(\gamma).$$

Equality holds if and only if (H^1, γ^1) is conjugate to (H, γ) ; that is, $\bar{X}(\gamma^1) = \bar{X}(\gamma)$.

Proof. We proceed by induction on $\ell^I(\gamma)$.

Case I. There is a complex simple root $\alpha \in R^+(\gamma)$, such that $\theta\alpha \in R^+(\gamma)$. By Propositions 8.2.7 and 7.2.22,

$$\Theta(\gamma) = s_\alpha \cdot \Theta(\gamma - n\alpha).$$

By Lemma 8.6.13,

$$\ell^I(\gamma - n\alpha) = \ell^I(\gamma) - 1,$$

so the result is available for $X(\gamma - n\alpha)$ by induction. By Corollary 7.3.20, either

- a) $\bar{X}(\gamma^1)$ occurs in $X(\gamma - n\alpha)$; or
- b) There is a composition factor $\bar{X}(\gamma^2)$ of $X(\gamma - n\alpha)$ such that $\alpha \notin \tau(\bar{X}(\gamma^2))$, and $\bar{X}(\gamma^1)$ occurs in $U_\alpha(\bar{X}(\gamma^1))$.

(notation (7.3.17)). In case (a), we have by induction

$$\ell^I(\gamma^1) \leq \ell^I(\gamma - n\alpha) = \ell^I(\gamma) - 1,$$

so that the proposition is true. In case (b),

$$(*) \quad \ell^I(\gamma^2) \leq \ell^I(\gamma - n\alpha) = \ell^I(\gamma) - 1.$$

In the notation of Theorem 8.5.18, suppose first that $\bar{X}(\gamma^1)$ occurs in Θ_0 . By that theorem, $\bar{X}(\gamma^1)$ occurs in $X(\gamma^2)$; so by

induction

$$\ell^I(\gamma^1) \leq \ell^I(\gamma^2) = \ell^I(\gamma) - 1,$$

and the proposition holds. So suppose $\bar{X}(\gamma^1)$ does not occur in Θ_0 ; thus we are in cases (b), (c), or (d), and by Lemma 8.6.13,

$$\ell^I(\gamma^1) = \ell^I(\gamma^2) + 1 \leq [\ell^I(\gamma) - 1] + 1$$

by (*); this is the inequality we want. Suppose equality holds. By (*) and the inductive hypothesis, γ^2 is conjugate to $\gamma - n\alpha$. By Theorem 8.5.18, this means that γ^1 is conjugate to γ , as we wished to show.

Case II. There is a simple type I real root $\alpha \in R^+(\gamma)$ satisfying the parity condition. Write

$$c_\alpha(\gamma) = \{\gamma_\alpha^\pm\} \subseteq (\hat{H}_\alpha)',$$

as in Definition 8.3.16. By Proposition 8.3.18(d) and Corollary 8.4.6,

$$\Theta(\gamma) = s_\alpha \cdot \Theta(\gamma_\alpha^+) + \Theta(\gamma_\alpha^-).$$

By Lemma 8.6.13,

$$\ell^I(\gamma_\alpha^\pm) = \ell^I(\gamma) - 1.$$

Now we argue exactly as in Case I.

Case III. There is a simple type II real root $\alpha \in R^+(\gamma)$ satisfying the parity condition. Define γ_α as in (8.4.12). By Proposition 8.3.18(e) and Corollary 8.4.6,

$$\Theta(\gamma) + \Theta(\gamma - n\alpha) = \Theta(\gamma_\alpha) + s_\alpha \Theta(\gamma_\alpha).$$

The argument of Case I shows that

$$\varrho^I(\gamma^1) \leq \varrho^I(\gamma),$$

and that equality holds only if

$$\bar{X}(\gamma^1) = \bar{X}(\gamma) \text{ or } \bar{X}(\gamma - n\alpha).$$

Now

$$\bar{\gamma} - n\alpha = s_\alpha \bar{\gamma},$$

so

$$||\gamma||_{\text{lambda}} = ||\gamma - n\alpha||_{\text{lambda}}$$

(notation (6.6.4)). By Lemma 6.6.6, $\bar{X}(\gamma - n\alpha)$ does not occur in $X(\gamma)$; so we are done in this case.

Case IV. We are not in Cases I - III. By Definition 8.6.5, γ is minimal and the result is obvious (from Theorem 8.6.6). Q.E.D.

Corollary 8.6.20 In the setting of Theorem 8.5.18, suppose $\alpha \notin \tau(\bar{X}(\gamma))$, and $\bar{X}(\gamma^1)$ occurs in $U_\alpha(\bar{X}(\gamma))$.

Then either

a) $\varrho^I(\gamma^1) \leq \varrho^I(\gamma) - 1$, or

b) We are in cases (8.5.18) (b) - (d), $\bar{X}(\gamma^1)$ is

one of the special constituents of U_α described

there, and

$$\varrho^I(\gamma^1) = \varrho^I(\gamma) + 1.$$

Proof. This follows from (8.5.18) (h), Proposition 8.6.19, and Lemma 8.6.13. Q.E.D.

§7. Singular infinitesimal characters.

The problem of computing composition series can be "reduced" (logically, not in the sense of "simplified") to the case of nonsingular infinitesimal character. We will now describe briefly how to do this. The method is taken from [39]. Suppose $\gamma \in \hat{H}'$ is a regular character. Choose a weight μ of a finite dimensional representation of G , in such a way that

(a) $\bar{\gamma} + \bar{\mu}$ is nonsingular

(8.7.1)

(b) if $\alpha \in R(\bar{\gamma})$ (notation (7.2.16)), and $\langle \alpha, \bar{\gamma} \rangle$ is positive, then

$$\langle \alpha, \bar{\gamma} + \bar{\mu} \rangle > 0.$$

With notation as in Proposition 8.2.12, write

$$\psi = \psi_{\bar{\gamma} + \bar{\mu}}^{\bar{\gamma} + 0}$$

$$(8.7.2) \quad R^+ = R^+(\bar{\gamma} + \bar{\mu}) \subseteq R(\bar{\gamma})$$

$$\{\alpha_1, \dots, \alpha_t\} = \text{simple roots of } R^+ \text{ such that } \langle \alpha, \bar{\gamma} \rangle = 0.$$

Using Theorem 6.5.12, list the (finitely many) inequivalent irreducible $(\mathfrak{g}, K)$ modules with infinitesimal character $\bar{\gamma} + \bar{\mu}$. With the help of Theorem 8.5.18, we can list them as

$$\{X_1, \dots, X_r\} = \text{those with } \alpha_i \notin \tau(X), \text{ any } i$$

(8.7.3) (a)

$$\{Y_1, \dots, Y_s\} = \text{those with some } \alpha_{i_j} \in \tau(Y).$$

By Corollaries 7.3.22 and 7.3.23,

ψX_i is irreducible

$\psi X_i \neq \psi X_j$ if $i \neq j$

(8.7.3) (b)

$\{\psi X_i\} = \{\text{irreducible } (\mathfrak{g}, K) \text{ modules with infinitesimal character } \bar{\gamma}\}$

$$\psi Y_j = 0$$

By Proposition 8.2.12,

$$(8.7.4) \quad \psi X(\gamma + \mu) = X(\gamma).$$

If we know the composition series of $X(\gamma + \mu)$, we can write

$$(8.7.5) (a) \quad \Theta(\gamma + \mu) = \sum n_i \Theta(X_i) + \sum m_j \Theta(Y_j).$$

By (8.7.3) (b) and (8.7.4),

$$(8.7.5) (b) \quad \Theta(\gamma) = \sum n_i \Theta(\psi X_i).$$

This gives the "reduction" to nonsingular infinitesimal character. The most unsatisfactory feature of this reduction is that the ψX_i are not specified in the Langlands classification. That can be done, with a little bookkeeping effort. The nature of the problem is apparent already for $SL(2, \mathbb{R})$: if X is a discrete series representation and $\bar{\gamma} = 0$, then ψX is a component of the reducible unitary principal series representation. In particular, it is not associated to the same Cartan subgroup any longer. This problem is dealt with partially in [39], and the ideas needed to do the bookkeeping may be found in [27].

Another problem is that this method does not allow one to decide whether $X(\gamma)$ is reducible simply from knowing

whether $X(\gamma + \mu)$ is; one needs to know something about the composition series. The best easy result is

Proposition 8.7.6 Suppose $\gamma \in \hat{H}'$ is a regular character. Then $X(\gamma)$ is reducible only if either there is a complex root $\alpha \in R(\bar{\gamma})$ such that

$$\begin{aligned} \langle \alpha, \bar{\gamma} \rangle &> 0 \\ \langle \theta \alpha, \bar{\gamma} \rangle &< 0; \end{aligned}$$

or there is a real integral root satisfying the parity condition.

Proof. Suppose neither of these conditions is satisfied. The set

$$R_0 = \{\beta \in R(\bar{\gamma}) \mid \langle \beta, \bar{\gamma} \rangle = \langle \theta \beta, \bar{\gamma} \rangle = 0\}$$

is θ -stable, so we can choose R_0^+ in such a way that the set of non-real roots in R_0^+ is θ -stable. Define

$$R^+ = \{\alpha \in R(\bar{\gamma}) \mid \langle \alpha, \bar{\gamma} \rangle > 0, \text{ or } \langle \theta \alpha, \bar{\gamma} \rangle > 0, \text{ or } \alpha \in R_0^+\};$$

this makes sense by the hypothesis that no root satisfies the first condition in the proposition. Clearly the set of non-real roots in R^+ is θ -stable. Choose $\mu \in \hat{H}$ a weight of a finite dimensional representation of G , so that $\bar{\gamma} + \bar{\mu}$ is dominant for R^+ and nonsingular. Then, by Proposition 8.2.12,

$$\psi_{\bar{\gamma} + \bar{\mu}}^{\bar{\gamma} + 0}(X(\gamma + \mu)) = X(\gamma);$$

so by Corollary 7.3.23, it suffices to show that $X(\gamma + \mu)$ is irreducible. Since the set of non-real roots in

$$R^+ = R^+(\bar{\gamma} + \bar{\mu})$$

is θ -stable, Theorem 8.6.6 says that we need only make sure that no real root satisfies the parity condition for $\gamma + \mu$. The parity condition is invariant under translation by a weight of a finite dimensional representation [by inspection of (8.3.11)]; so this follows from the corresponding assumption on γ . Q.E.D.

A more careful analysis, which we will not reproduce here, gives:

Proposition 8.7.7 ([39]). Suppose $\gamma \in \hat{H}'$. Then

$X(\gamma)$ is reducible only if either

a) it has more than one Langlands subrepresentation

[see (6.5.10)]; or

b) there is a complex integral root α with

$$\langle \alpha, \bar{\gamma} \rangle > 0$$

$$\langle \theta\alpha, \bar{\gamma} \rangle < 0;$$

or

c) there is a real integral root α satisfying the parity condition, with

$$\langle \alpha, \bar{\gamma} \rangle \neq 0.$$

Thus the improvement is in replacing the weak condition that there exist a singular real root α satisfying the parity condition, with the much stronger one that there be more than one Langlands subrepresentation. Proposition 8.7.6 fails to predict the irreducibility of all the unitary principal series for $SL^+(2, \mathbb{R})$ (see Proposition 1.4.8); but Proposition 8.7.7 does not fail. In fact, a still more careful

analysis shows that the conditions of Proposition 8.7.7 are also sufficient for reducibility when there are no compact imaginary roots for any Cartan in G . (This is the case for complex groups and $SL(n, \mathbb{R})$, for example; and one recovers the theorems of Zhelobenko (see [8]) and Speh [38], respectively.) They are not sufficient in general; perhaps the simplest example is $SO(4,1)$, when they do not agree with Kostant's criterion for the reducibility of the spherical principal series.

In the other direction, one can write down sufficient conditions for reducibility of $X(\gamma)$. One way to do this [in the setting (8.7.1) - (8.7.5)] is the following: one lists the composition factors of $X(\gamma + \mu)$ which are provided by Corollaries 8.2.9 and 8.4.8, and then checks whether more than one of them is among the X_i [cf. (8.7.3)(a)]. This does not give a simply stated condition, so we will leave the formulation to the interested reader.

The main point to be made is that the conditions one gets are not necessary for reducibility. The simplest example occurs for $G = Sp(3,1)$. In the notation of Section 6 of [45], the standard character called $\theta(2,8)$ is reducible after translating to the wall $e_3 - e_4 = 0$; but its only additional composition factor there is the translate of $\bar{\theta}(1,4)$, which is not detected by the procedure above. The problem of finding necessary and sufficient conditions for the reducibility of $X(\gamma)$ is an important one for the theory of complementary series, and should not be completely intractable. Some reductions of the problem may be found in [39].

Chapter 9. The Kazhdan-Lusztig Conjecture

In this chapter we will describe an algorithm which, assuming Conjecture 7.3.25 (complete reducibility of all U_q), will allow us to compute the composition series of the standard representations. This algorithm is a generalization of the one given by Kazhdan and Lusztig in [24] for Verma modules. The idea is to compute at the same time all of the cohomology groups

$$H^i(\mathcal{U}, \bar{X}(\gamma))$$

($\mathcal{U} = \mathfrak{l} + \mathcal{U}$ a θ -stable parabolic) as $(\mathfrak{l}, \text{LnK})$ modules, using the technique in the proof of Proposition 8.6.19. The formal part of this calculation is not very hard; most of the effort will be devoted to proving the non-occurrence of certain representations in the cohomology, to keep the algorithm going. This is accomplished by a study of "blocks" in the category of $(\mathcal{U}, K)$ modules of finite length. These are the subcategories generated by equivalence classes of irreducibles, in which the equivalence relation is generated by $X \sim Y$ if X and Y occur in a common indecomposable module. We will essentially determine these blocks (Theorem 9.2.11). Almost all of the results in this chapter are taken from [46]. As an introduction, the reader could look at Section 3 of [46], which treats the much simpler case of Verma modules.

§1. Commutativity of certain diagrams.

We are going to do diagram chasing in the long exact sequence of Proposition 7.4.7. This requires some natural

diagrams to be commutative; and I know of no straightforward proof that this is so. We will use instead a refinement of the proof of Proposition 7.3.14. The result we want is Corollary 9.1.17; and this section is perhaps best read backwards, beginning with that Corollary and seeing what was needed to prove it. With luck, the reader may discover that the result is much easier than this proof of it indicates.

Lemma 9.1.1 Suppose R is a ring with 1, and ϕ is a natural correspondence which associates to any R -module M , a map

$$\phi_M: M \rightarrow M$$

of abelian groups. Then there is an element $r \in R$ such that for all M ,

$$\phi_M(m) = rm \quad m \in M.$$

Proof. Regard R as an R -module, and define

$$r = \phi_R(1).$$

The proof that this r works is formally identical to the one given for Lemma 7.3.15. Q.E.D.

Lemma 9.1.2 Suppose R is a $\mathbb{C}$ algebra with 1, and V is a finite dimensional vector space. Then any module for $R \otimes_{\mathbb{C}} (\text{End } V)$ is naturally isomorphic to

$$M \otimes_{\mathbb{C}} V$$

for a natural R -module M .

Proof. Let X be a module for $R \otimes_{\mathbb{C}} \text{End } V$. Define

$$M = \text{Hom}_{\text{End } V} (V, X),$$

which we make into an R module by

$$(r \cdot m)(v) = (r \otimes \text{Id}) \cdot (m(v)) \quad (m \in M, v \in V).$$

Then there is a natural map

$$M \otimes_{\mathbb{C}} V \rightarrow X, \quad m \otimes v \mapsto m(v).$$

We leave the verification that this is an isomorphism to the reader. Q.E.D.

Lemma 9.1.3 Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, and

a weight $\gamma \in \mathfrak{h}^*$. Let

$$\mathcal{L}_{\gamma} \subseteq Z(\mathfrak{g})$$

be the kernel of the infinitesimal character γ
(notation (0.2.9)), and let

$$I_{\gamma} = \mathcal{L}_{\gamma} U(\mathfrak{g}) \subseteq U(\mathfrak{g})$$

be the two-sided ideal it generates.

- a) The category of $\mathfrak{g}$ modules with infinitesimal character γ is the category of modules over $U(\mathfrak{g})/I_{\gamma}$.
- b) The center of $U(\mathfrak{g})/I_{\gamma}$ is one dimensional.
- c) Recall the group G^2 constructed by (7.1.2) - (7.1.8), and the $(\mathfrak{g}^2, K^2)$ module structure on quotients of $U(\mathfrak{g})$. Then the $\mathfrak{g}$ module $U(\mathfrak{g})/I_{\gamma}$ is isomorphic to the $\mathfrak{g}^R$ module $U(\mathfrak{g})/I_{\gamma}$.

(cf. Proposition 7.1.10); here we identify $\mathfrak{g}$ and $\mathfrak{g}^R$
by the map ϕ^R of (7.1.5).

d) There is a unique regular character

$$-\gamma^2 \in (\hat{H}^2),$$

(notation (6.6.1), (7.1.8)), such that

$$-\gamma^2 = (\gamma, \gamma) \in (\mathfrak{h}^2)^* \cong \mathfrak{h}^* \oplus \mathfrak{h}^*.$$

There is a $(\mathfrak{g}^2, K^2)$ module injection (notation 8.5.1)

$$[U(\mathfrak{g})/\mathfrak{I}_{\gamma}]^{\mathbb{C}} \hookrightarrow X_{G^2}(-\gamma^2).$$

Proof. Part (a) is obvious. Part (b) says that the map

$$Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}} \rightarrow (U(\mathfrak{g})/I_{\gamma})^{\mathfrak{g}}$$

is surjective; and this is a consequence of the fact that $\mathfrak{g}$ acts reductively on $U(\mathfrak{g})$ by the adjoint action (compare the end of the proof of Lemma 7.3.15). For (c), use the isomorphism

$$u + I_{\gamma} \rightarrow u + \mathfrak{I}_{\gamma}$$

of $U(\mathfrak{g})/I_{\gamma}$ onto $U(\mathfrak{g})/\mathfrak{I}_{\gamma}$; this takes left multiplication by $\mathfrak{g}$ (the $\mathfrak{g}$ module action) to right multiplication by $\mathfrak{g}$, twisted by hat (the $\mathfrak{g}^R$ action -- see Proposition 7.1.10). For (d), the first claim is obvious since H^2 is connected (recall that G^2 is a complex reductive group). For the second, $(U(\mathfrak{g})/\mathfrak{I}_{\gamma})^{\mathbb{C}}$ contains the trivial representation π_0 of K^2 exactly once by (b); for $\mathfrak{h}^2 \cong \mathfrak{g}$ acts on $U(\mathfrak{g})/\mathfrak{I}_{\gamma}$ by the adjoint action (Proposition 7.1.10). So π_0 is a $\text{lambda-lowest } K^2\text{-type of } (U(\mathfrak{g})/\mathfrak{I}_{\gamma})^{\mathbb{C}}$. Choose

$$\delta \in (\hat{H}^2)', \quad \bar{\delta} = (\bar{\delta}_1, \bar{\delta}_2)^* \in (\mathcal{H}^2)^*$$

so that $\bar{X}_{G2}(\delta)$ is isomorphic to the constituent of $(U(\mathfrak{g})/\mathbb{I}_\gamma)^c$ containing π_0 . By Proposition 7.1.10(c) and Lemmas 6.6.6 and 8.5.7, the equality of the infinitesimal characters gives

$$\bar{\delta}_1 = -w_1 \gamma, \quad w_1 \in W(\mathfrak{g}, \mathcal{h}).$$

Now the maximal torus $\mathcal{X}^2$ of $\mathcal{H}^2$ sits in $\mathcal{H}^2$ as

$$\mathcal{X}^2 = \{(X, -X) \mid X \in \mathcal{h}\}$$

[cf. (7.1.7) and (7.1.8)]; and Lemma 6.6.16 guarantees that $\bar{\delta}|_{\mathcal{X}^2}$ occurs in $\pi_0|_{\mathcal{X}^2}$. Since π_0 is trivial, this means that

$$\bar{\delta}_1 - \bar{\delta}_2 = 0,$$

or $w_1 = w_2$; so

$$\bar{\delta} = (-w\gamma, -w\gamma) \quad (w \in W(\mathfrak{g}, \mathcal{h})).$$

It is easy to check that the Weyl group $W(G^2, H^2)$ acts diagonally in $(\mathcal{H}^2)^*$, since it is realized by elements of K^2 ; so δ is conjugate to $-\gamma^2$. Thus $\bar{X}_{G2}(\gamma^2)$ occurs in $(U(\mathfrak{g})/\mathbb{I}_\gamma)^c$. By Corollary 6.4.13, there is a non-zero map

$$\phi: (U(\mathfrak{g})/\mathbb{I}_\gamma)^c \rightarrow X_{G2}(\gamma^2).$$

Now ϕ must be non-zero on the trivial K^2 -type of $(U(\mathfrak{g})/\mathbb{I}_\gamma)^c$, since π_0 occurs in the unique irreducible submodule of $X_{G2}(\gamma^2)$. But any submodule of $(U(\mathfrak{g})/\mathbb{I}_\gamma)^c$ contains this K^2 -type (since $1+\mathbb{I}_\gamma$ generates $U(\mathfrak{g})/\mathbb{I}_\gamma$ as a $\mathfrak{g}^2$ module); so ϕ is injective. Q.E.D.

[It can be shown that ϕ is onto, but we will not need this.]

Proposition 9.1.4 Suppose $H^S \subseteq G$ is a maximally split Cartan subgroup, $\gamma \in (\mathcal{H}^S)^*$ is nonsingular, $\lambda \in \Lambda^S$ (the lattice of weights of finite dimensional representations of G), and

$$\langle \gamma + \bar{\lambda}, \alpha \rangle \geq 0 \quad (\alpha \in R^+(\gamma) \text{ -- notation (7.2.16)})$$

Write ψ for the translation functor

$$\psi_{\gamma+0}^{\gamma+\lambda}$$

on $Z(\mathfrak{g})$ -finite $\mathfrak{g}$ modules (Definition 4.5.7). Let ϕ be a natural correspondence which assigns to each $\mathfrak{g}$ module X of infinitesimal character γ , a $\mathfrak{g}$ module map

$$\phi_X: \psi X \rightarrow \psi X.$$

Then there is a complex number c such that for all such X ,

$$\phi_X(x) = cx \quad (x \in \psi X).$$

Proof. We will write R for the ring $U(\mathfrak{g})/\mathbb{I}_\gamma$ of Lemma 9.1.3. Let F be the finite dimensional $\mathfrak{g}$ module of extremal weight λ . By Definition 4.5.7, ψX is a direct summand of $X \otimes F$, so we have natural maps

$$\psi X \xrightarrow{i} X \otimes F \xrightarrow{\pi} \psi X.$$

So ϕ defines in a natural way

$$\bar{\phi}_X: X \otimes F \rightarrow X \otimes F, \quad \bar{\phi}_X = i \circ \phi_X \circ \pi.$$

By Lemmas 9.1.1.1 and 9.1.1.2, we can find elements $r_i \in R$, $A_i \in \text{End}(F)$, such that for any R-module X ,

$$(9.1.1.5) \quad \bar{\phi}_X(x \otimes f) = \sum r_i x \otimes A_i f \quad (x \in X, f \in F).$$

We will prove the proposition using the following reductions. Any R-module is a quotient of a free R module; so by the naturality of ϕ and the exactness of ψ , it suffices to prove the result for the R-module R ; that is, to show that

$$\phi_R: \psi R \rightarrow \psi R$$

is a scalar. By Lemma 9.1.1.3, R is a quotient of the $\mathcal{G}^2$ module $[X(-\gamma^2)]^C$, which we regard as an R module by the action of the second factor of $\mathcal{G}^2$; so it is enough to prove the result for $[X(-\gamma^2)]^C$. This we will do in two steps.

First, we show how to regard $\psi[X(-\gamma^2)]^C$ as a $\mathcal{G}^2$ module, and show that $\phi_{[X(-\gamma^2)]^C}$ is a $\mathcal{G}^2$ module map. Next, we calculate $\psi[X(-\gamma^2)]^C$ as a $\mathcal{G}^2$ module, and observe that any $\mathcal{G}^2$ -module map

$$\psi[X(-\gamma^2)]^C \rightarrow \psi[X(-\gamma^2)]^C$$

is a scalar. This will prove the proposition.

For the first step, let Y be any $\mathcal{G}^2$ module which has infinitesimal character γ as a module for the second factor $\mathcal{G}^R$. Regard F as a $\mathcal{G}^2$ module by making the first factor $\mathcal{G}^L$ of $\mathcal{G}^2$ act trivially. Then

$$Y \otimes F$$

is a $\mathcal{G}^2$ module, and the usual $\mathcal{G}$ action is still that of the $\mathcal{G}^R$ factor. Now ψY is a certain generalized eigenspace of $Z(\mathcal{G}^R)$

in $Y \otimes F$, and this is obviously invariant by $\mathcal{G}^L$; so ψY is a $\mathcal{G}^2$ module. That this action commutes with ϕ_Y is clear from the formula (9.1.1.5) for $\bar{\phi}$. This completes the first step.

For the second, we will interpret the action of ψ on $\mathcal{G}^2$ modules (just defined) as a translation functor for G^2 . To do this, we must choose K^2 so that the $\mathcal{G}^2$ module F defined above is actually a G^2 module. This is clearly possible (see Section 7.1). Then if Y is any $\mathcal{G}^2$ module with infinitesimal character (γ, γ) , we obviously have

$$\psi Y \cong \psi_{(\gamma, \gamma) + (0, 0)}^{(\gamma, \gamma) + (0, \lambda)} Y,$$

the translation functor on the right being the one for G^2 .

So

$$\begin{aligned} \psi[X(-\gamma^2)]^C &\cong \psi_{(\gamma, \gamma) + (0, 0)}^{(\gamma, \gamma) + (0, \lambda)} [X(-\gamma^2)]^C \\ &\cong \left[\psi_{(-\gamma, -\gamma) + (0, 0)}^{(-\gamma, -\gamma) + (0, -\lambda)} X(-\gamma^2) \right]^C, \end{aligned}$$

the second step being Corollary 8.5.9. By Proposition 8.2.12, this is

$$[X(-\gamma, -\gamma - \lambda)]^C.$$

For formal reasons,

$$\text{Hom}_{\mathcal{G}, K}(\mathcal{Y}^C, Z^C) \cong \text{Hom}_{\mathcal{G}, K}(Z, Y)$$

whenever Y and Z have finite length. So to complete the second step, we must show that

$$\text{Hom}_{\mathcal{G}^2}(X(-\gamma, -\gamma - \lambda), X(-\gamma, -\gamma - \lambda))$$

is one dimensional. But this follows from the formal argument given for Proposition 4.2.26. Q.E.D.

Corollary 9.1.6 Under the hypotheses of Proposition 9.1.4, suppose X is a $\mathfrak{g}$ module of infinitesimal character γ . Then $Z(\mathfrak{g})$ acts by scalars on ψX .

Proof. If $z \in Z(\mathfrak{g})$, then

$$\phi_X = \text{action of } z \text{ on } \psi X$$

is a construction of the sort considered in the proposition.

Q.E.D.

It is critical here that the translation functor is going from a nonsingular infinitesimal character to a (possibly) singular one. The result is always false for translation from a singular infinitesimal character to a nonsingular one; verification of this is left as an interesting exercise for the reader.

To use these ideas on cohomology, we need a simple technical fact, the proof of which is due to Harish-Chandra (cf. [50], Theorem 2.1.3.6).

Lemma 9.1.7 Suppose $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{h} \subseteq \mathfrak{l}$ is a Cartan subalgebra, and X is $\mathfrak{g}$ module of nonsingular infinitesimal character γ . Then $Z(\mathfrak{l})$ acts semisimply on $H^*(\mathfrak{u}, X)$; that is, it acts by scalars on each generalized eigenspace of $Z(\mathfrak{l})$.

Proof. Write

$$\begin{aligned} \xi: Z(\mathfrak{g}) &\rightarrow S(\mathfrak{h}, W(\mathfrak{g}, \mathfrak{h})) \\ \xi^\ell: Z(\mathfrak{l}) &\rightarrow S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})) \end{aligned}$$

$$\begin{aligned} \xi^\ell &= T_{\rho(\mathfrak{u})} \circ \xi^\ell: Z(\mathfrak{l}) \rightarrow S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})) \\ \gamma: Z(\mathfrak{g}) &\rightarrow Z(\mathfrak{l}); \end{aligned}$$

the first two maps are Harish-Chandra homomorphisms, $T_{\rho(\mathfrak{u})}$ is translation by $\rho(\mathfrak{u})$ (cf. (0.2.7)), and γ is the map of (3.1.4). By (3.1.7),

$$\xi = \xi^\ell \circ \gamma.$$

Identify $Z(\mathfrak{g})$ and $Z(\mathfrak{l})$ with subalgebras of $S(\mathfrak{h})$ using ξ and ξ^ℓ . Let $\mathcal{M}_\gamma \subseteq S(\mathfrak{h})$ be the kernel of evaluation at γ . By Theorem 3.1.5, the action of $Z(\mathfrak{l})$ on $H^*(\mathfrak{u}, X)$ factors through

$$\begin{aligned} Z(\mathfrak{l}) &\cong S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})) \rightarrow S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})) / \\ &\quad (\mathcal{M}_\gamma \cap S(\mathfrak{h}, W(\mathfrak{g}, \mathfrak{h})))_{S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h}))}. \end{aligned}$$

So what we must show is that the finite dimensional commutative ring

$$R = S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})) / (\mathcal{M}_\gamma \cap S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})))_{S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h}))}$$

is semisimple. We claim that

$$\begin{aligned} (*) \quad &\left[(\mathcal{M}_\gamma \cap S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})))_{S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h}))} \right] \cap S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})) = \\ &(\mathcal{M}_\gamma \cap S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})))_{S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h}))}. \end{aligned}$$

That the left side contains the right is obvious; so suppose

$$p = \sum p_i q_i \quad (p_i \in \mathcal{M}_\gamma \cap S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})), q_i \in S(\mathfrak{h}, W(\mathfrak{l}, \mathfrak{h})))$$

is in the left side. Then

$$p = |W(\mathfrak{l}, \mathfrak{h})|^{-1} \sum_{x \in W(\mathfrak{l}, \mathfrak{h})} x \cdot p$$

$$\begin{aligned}
&= |W(\ell, \mathcal{h})|^{-1} \sum_{x, i} P_i(x \cdot q_i) \\
&= \sum_i P_i \left(|W(\ell, \mathcal{h})|^{-1} \sum_x x \cdot q_i \right),
\end{aligned}$$

which is in the right side of (*). By (*)

$$R \subseteq S(\mathcal{h}) / (\mathcal{M}_\gamma \cap S(\mathcal{h})^W) S(\mathcal{h}) = R'.$$

Now R' has dimension $|W|$ (by the harmonic decomposition of $S(\mathcal{h})$) -- cf. [50], Proposition 2.1.3.5). So

$$R' \cong \{\text{functions on } W \cdot \gamma\}$$

by the evaluation map, since $|W \cdot \gamma| = |W|$. (Here is where the hypothesis that γ is nonsingular is used.) So R' is a sum of copies of $\mathfrak{C}$ (as a ring), so its subring R is as well.

Q.E.D.

Lemma 9.1.8 Suppose $\mathfrak{g} = \ell + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$, $\mathcal{h} \subseteq \ell$ is a Cartan subalgebra, and Y is an ℓ module. Suppose that the center of ℓ acts by scalars on Y . Then there is a natural isomorphism

$$\text{Hom}_{\mathfrak{g}}(\text{pro}_{\mathfrak{g}}^{\mathfrak{g}} Y, \text{pro}_{\mathfrak{g}}^{\mathfrak{g}} Y) \cong \text{Hom}_{\ell}(Y, Y).$$

Proof. Fix an element x of the center of ℓ such that $\text{ad}(x)$ has positive eigenvalues on $\mathfrak{u}$; and let c_0 be the eigenvalue of x on Y . Then

$$\begin{aligned}
\text{pro}_{\mathfrak{g}}^{\mathfrak{g}} Y &= \text{Hom}_{\mathfrak{g}}(U(\mathfrak{g}), Y) \\
&\cong \text{Hom}_{\mathbb{C}}(U(\overline{\mathfrak{u}}), Y)
\end{aligned}$$

as an ℓ module. So $\text{pro}(Y)$ is a direct product of eigenspaces

of x , with eigenvalues greater than or equal to c_0 . The c_0 eigenspace is isomorphic to Y , by the map

$$f: Y \rightarrow \text{pro}(Y)$$

$$[f(Y)] = 0 \text{ on } U(\mathfrak{g})\overline{\mathfrak{u}}$$

$$[f(Y)](q) = qY \quad q \in U(\mathfrak{g}).$$

So the induced map

$$\text{Hom}_{\ell}(\text{pro } Y, Y) \rightarrow \text{Hom}_{\ell}(Y, Y)$$

is an isomorphism. Since the action of $\mathfrak{u}$ on $\text{pro}(Y)$ raises the eigenvalue of x , these maps from $\text{pro } Y$ to Y are zero on $\mathfrak{u} \cdot \text{pro}(Y)$; so they are $\mathfrak{g}$ module maps. So

$$\text{Hom}_{\ell}(Y, Y) \cong \text{Hom}_{\mathfrak{g}}(\text{pro } Y, Y).$$

The lemma follows from (6.1.23). Q.E.D.

Lemma 9.1.9 Suppose R is a ring, and U and V are two R modules. Let $\phi_{U,V}$ be a natural correspondence which attaches to any R module M a map

$$\phi_{U,V}(M): \text{Hom}_R(M, U) \rightarrow \text{Hom}_R(M, V).$$

Then there is a unique R module map

$$\psi: U \rightarrow V$$

such that for any R module M , $\phi_{U,V}(M)$ is composition with ψ .

Proof. To define ψ , fix $u \in U$, and set

The proof of this is virtually identical to that of Lemma 6.4.2, so we leave the details to the reader.

Corollary 9.1.11 ([6]) In the setting of Lemma 9.1.7, fix $\gamma \in \mathcal{h}^*$ and define $I_\gamma \subseteq U(\mathfrak{g})$ as in Lemma 9.1.3. Let $\mathcal{U} \subseteq \mathcal{V}$ be any subalgebra. Then $U(\mathfrak{g})/I_\gamma$ is a free $U(\mathcal{U})$ module.

Theorem 9.1.12 ([46], Theorem 4.2) Let $H^S \subseteq G$ be a maximally split Cartan subgroup, and X a

$\mathfrak{g}$ module of nonsingular infinitesimal character $\gamma^S \in (\mathcal{h}^S)^*$. Fix a simple root $\alpha^S \in R^+(\gamma^S)$ (notation (7.2.16)), and a weight $\lambda^S \in \Lambda^S$ such that

$$\begin{aligned} \langle \alpha^S, \gamma^S + \lambda^S \rangle &= 0 \\ \langle \beta^S, \gamma^S + \lambda^S \rangle &> 0 \end{aligned} \quad \beta^S \in R^+(\gamma^S) - \{\alpha^S\}.$$

Define

$$\begin{aligned} \psi_\alpha &= \psi_{\gamma^S + \lambda^S + 0}^{\gamma^S + \lambda^S} \\ \phi_\alpha &= \psi_{\gamma^S + \lambda^S + 0}^{\gamma^S + 0} \end{aligned}$$

Fix a parabolic subalgebra $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ with real Levi factor; a maximally split Cartan subgroup H of L ; and a weight $\gamma \in \mathcal{h}^*$ conjugate to γ^S by an $\text{ad}(\mathfrak{g})$ inner automorphism

$$\psi: (\mathcal{h}^S)^* \rightarrow \mathcal{h}^*.$$

Let $\lambda \in \Lambda \subseteq \hat{H}$ be the image of λ^S under the map of Lemma 7.2.6. Define translation functors for $\mathfrak{l}$

$$\delta_u: R \rightarrow U, \delta_u(r) = ru.$$

Then $(\phi_{U,V}^{(R)})(\delta_u)$ is a map from R to V ; and to have the conclusion of the lemma, we are forced to set

$$\psi(u) = [\phi_{U,V}^{(R)}(\delta_u)](1).$$

The proof that ψ is an R module map with the desired properties is a straightforward verification which we leave to the reader. Q.E.D.

To do dimension shifting while keeping track of infinitesimal characters requires the following technical idea of Casselman and Osborne.

Lemma 9.1.10 Suppose $\mathcal{h} = \mathcal{h} + \mathcal{V}$ is a Borel subalgebra of $\mathfrak{g}$, and $\overline{\mathcal{V}}$ is the opposite nil radical.

Write

$$\begin{aligned} \Delta^+ &= \Delta^+(\mathfrak{g}, \mathcal{h}) = \Delta(\mathcal{V}) \\ W &= W(\mathfrak{g}, \mathcal{h}) \\ \rho &= \rho(\Delta^+). \end{aligned}$$

Fix a homogeneous linear complement $\mathcal{W}$ for $U(\mathcal{h})W$ in $U(\mathcal{h})$. Then

$$U(\mathfrak{g}) \cong U(\mathcal{V}) \otimes_{\mathbb{C}} \mathcal{W} \otimes_{\mathbb{C}} Z(\mathfrak{g}) \otimes_{\mathbb{C}} U(\overline{\mathcal{V}});$$

here

$$\mathcal{W} = T_{-\rho} \mathcal{W}$$

(notation (0.2.7)), the translate of $\mathcal{W}$ by $-\rho$.

$$\begin{aligned}\psi_\alpha^L &= \psi_{\gamma-\rho(\mathcal{U})}^{\gamma-\rho(\mathcal{U})+\lambda} + \psi_{\gamma-\rho(\mathcal{U})}^{\gamma-\rho(\mathcal{U})+0} \\ \phi_\alpha^L &= \psi_{\gamma-\rho(\mathcal{U})}^{\gamma-\rho(\mathcal{U})+0} + \psi_{\gamma-\rho(\mathcal{U})}^{\gamma-\rho(\mathcal{U})+\lambda} \\ \psi_\alpha^L &= \psi_{S_\alpha\gamma-\rho(\mathcal{U})}^{S_\alpha\gamma-\rho(\mathcal{U})+s_\alpha\lambda} + \psi_{S_\alpha\gamma-\rho(\mathcal{U})}^{S_\alpha\gamma-\rho(\mathcal{U})+0}.\end{aligned}$$

Finally, let $\alpha = \psi(\alpha^S)$ be the root in $R^+(\gamma)$ corresponding to α^S . Let

$$f: \phi_\alpha \psi_\alpha X \rightarrow X, \quad \tilde{f}: X \rightarrow \phi_\alpha \psi_\alpha X$$

be the map induced by the identity on $\psi_\alpha X$ and

Proposition 4.5.8, and

$$g^i: \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X) \rightarrow H^i(\mathcal{U}, X), \quad \tilde{g}^i$$

the analogous maps for ℓ . Let

$$f^i: H^i(\mathcal{U}, \phi_\alpha \psi_\alpha X) \rightarrow H^i(\mathcal{U}, X), \quad \tilde{f}^i$$

be the maps on cohomology induced by f and $\tilde{f}$.

a) Suppose $\alpha \in \Delta(\mathcal{U})$. Then there is a natural long exact sequence

$$\begin{aligned}\dots \rightarrow \phi_\alpha^L \psi_\alpha^L H^{i-1}(\mathcal{U}, X) \rightarrow \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X) \rightarrow P_{\gamma-\rho(\mathcal{U})} H^i(\mathcal{U}, \phi_\alpha \psi_\alpha X) \\ \xrightarrow{h^i} \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X) \rightarrow \dots\end{aligned}$$

b) Suppose $\alpha \in \Delta(\ell)$. Then there are natural isomorphisms

$$P_{\gamma-\rho(\mathcal{U})} H^i(\mathcal{U}, \phi_\alpha \psi_\alpha X) \xrightarrow{\tilde{h}^i} \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X).$$

c) Suppose $-\alpha \in \Delta(\mathcal{U})$. Then the long exact sequence of (a) exists, with ψ_α^L and $\tilde{\psi}_\alpha^L$ reversed; we write $\tilde{h}^i$ for the map analogous to h^i .

d) In cases (a) and (b), there is a commutative

diagram

$$\begin{array}{ccc} & & \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X) \\ & \nearrow h^i & \\ P_{\gamma-\rho(\mathcal{U})} H^i(\mathcal{U}, \phi_\alpha \psi_\alpha X) & \xrightarrow{f^i} & P_{\gamma-\rho(\mathcal{U})} H^i(\mathcal{U}, X) \\ & \searrow \tilde{f}^i & \\ & & \downarrow \tilde{g}^i \\ & & \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X)\end{array}$$

e) In cases (b) and (c), there is a commutative

diagram

$$\begin{array}{ccc} & & \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X) \\ & \nearrow \tilde{h}^i & \\ P_{\gamma-\rho(\mathcal{U})} H^i(\mathcal{U}, \phi_\alpha \psi_\alpha X) & \xrightarrow{\tilde{f}^i} & H^i(\mathcal{U}, X) \\ & \searrow \tilde{f}^i & \\ & & \downarrow \tilde{g}^i \\ & & \phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X)\end{array}$$

Proof. Parts (a) - (c) are restatements of Propositions 7.4.6 and 7.4.7. [In order to arrange (d) and (e), we are going to scale all the h^i (or $\tilde{h}^i$) by a nonzero constant depending only on $\mathcal{U}$, γ and λ^S ; this does not affect (a) - (c).]

Consider (d) for case (a). We first treat $i = \dim \mathcal{U}$. It is clear from the complex (3.1.1) that

$$H^{\dim \mathcal{U}}(\mathcal{U}, X) \cong (\wedge^{\dim \mathcal{U}} \mathcal{U})^* \otimes X/\mathcal{U}X,$$

and therefore that

$$(9.1.13) \quad P_{\gamma-\rho(\mathcal{U})} H^{\dim \mathcal{U}}(\mathcal{U}, X) \cong P_{\gamma+\rho(\mathcal{U})}(X/\mathcal{U}X)$$

in a natural way [which twists the action of ℓ by $2\rho(\mathcal{U})$]. So we consider the space $\mathcal{S}$ of correspondences π which assign to each module X of infinitesimal character γ , an

ℓ -module map

$$\pi_X: P_{\gamma+\rho}(\mathcal{U})[\phi_\alpha \psi_\alpha X/\mathcal{U} \cdot (\phi_\alpha \psi_\alpha X)] \rightarrow P_{\gamma+\rho}(\mathcal{U}) X/\mathcal{U} X.$$

We want to show that $\mathcal{S}$ is at most one dimensional. We do this by exhibiting a map from $\mathcal{S}$ to another space of natural correspondences which we already know to be one dimensional; and showing that the map is injective. So suppose $\pi \in \mathcal{S}$, and Y is an ℓ -module of infinitesimal character $\gamma+\rho(\mathcal{U})$. Then π_X induces a map

$$\mathrm{Hom}_\ell(P_{\gamma+\rho}(\mathcal{U})(X/\mathcal{U}X), Y) \rightarrow \mathrm{Hom}_\ell(P_{\gamma+\rho}(\mathcal{U})(\phi_\alpha \psi_\alpha X/\mathcal{U} \cdot \phi_\alpha \psi_\alpha X), Y).$$

Since Y has infinitesimal character $\gamma+\rho(\mathcal{U})$, this is the same as

$$\mathrm{Hom}_\ell(X/\mathcal{U}X, Y) \rightarrow \mathrm{Hom}_\ell(\phi_\alpha \psi_\alpha X/\mathcal{U} \cdot \phi_\alpha \psi_\alpha X, Y).$$

If we regard Y as a trivial $\mathcal{U}$ module, this is

$$\mathrm{Hom}_{\mathcal{U}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{U}}(\phi_\alpha \psi_\alpha X, Y)$$

or by (6.1.23)

$$\mathrm{Hom}_{\mathcal{U}}(X, \mathrm{pro} Y) \rightarrow \mathrm{Hom}_{\mathcal{U}}(\phi_\alpha \psi_\alpha X, \mathrm{pro} Y).$$

By Proposition 4.5.8, this gives a natural map

$$\mathrm{Hom}_{\mathcal{U}}(X, \mathrm{pro} Y) \rightarrow \mathrm{Hom}_{\mathcal{U}}(X, \phi_\alpha \psi_\alpha \mathrm{pro} Y).$$

By Lemma 9.1.9, such a correspondence comes from a natural map

$$\mathrm{pro} Y \rightarrow \phi_\alpha \psi_\alpha \mathrm{pro} Y.$$

By Proposition 4.5.8 again, this gives a natural map

$$\psi_\alpha \mathrm{pro} Y \rightarrow \psi_\alpha \mathrm{pro} Y.$$

By the (easy) analogue of Proposition 7.4.1 for pro , this is

$$\mathrm{pro}(\psi_\alpha^L Y) \rightarrow \mathrm{pro}(\psi_\alpha^L Y).$$

By Corollary 9.1.6, $\psi_\alpha^L Y$ has an infinitesimal character; so by Lemma 9.1.8, we have attached to Y (using π) a natural map

$$\phi_Y: \psi_\alpha^L Y \rightarrow \psi_\alpha^L Y.$$

By Proposition 9.1.4, the space of such correspondences is one dimensional. To prove that $\mathcal{S}$ is one dimensional, we must show that

$$\pi \rightarrow \phi$$

is injective. So suppose ϕ is zero. Reversing the construction of ϕ from π , we conclude that if Y is any ℓ module of infinitesimal character $\gamma - \rho(\mathcal{U})$, then the map

$$\begin{aligned} \rho_Y: \mathrm{Hom}_\ell(P_{\gamma+\rho}(\mathcal{U})(X/\mathcal{U}X), Y) &\rightarrow \\ &\rightarrow \mathrm{Hom}_\ell(P_{\gamma+\rho}(\mathcal{U})(\phi_\alpha \psi_\alpha X/\mathcal{U} \cdot \phi_\alpha \psi_\alpha X), Y) \end{aligned}$$

induced by π is zero. Apply this to

$$Y = P_{\gamma+\rho}(\mathcal{U})(X/\mathcal{U}X)$$

(which has infinitesimal character $\gamma+\rho(\mathcal{U})$ by Lemma 9.1.7). Clearly ρ_Y applied to the identity is the original map π_X , which is therefore zero. So $\pi \rightarrow \phi$ is injective, and $\mathcal{S}$ has dimension at most one.

Set $t = \dim \mathcal{U}$. The two constructions f^t and $g^t \circ h^t$ of (d) are natural, and so (by (9.1.13)) define elements of $\mathcal{S}$. So there is a constant c depending only on $\mathcal{G}, \mathcal{F}$, γ , and λ so that

$$f^t = c \cdot g^t \circ h^t$$

(or vice versa). So to prove the result for $i = t$, we only have to show that f^t and $g^t h^t$ are not always zero; then we can scale all the maps h^i by c^{-1} without affecting (a) - (c), as pointed out earlier. We will construct the necessary examples after the proof, in order not to interrupt the argument.

Suppose now [still in case (a) of (d)] that $i < t$, and that the commutativity has been established for $i+1$. Choose a free $U(\mathcal{G})/I_\gamma$ module M of which X is a quotient:

$$0 \rightarrow K \rightarrow M \rightarrow X \rightarrow 0.$$

By Corollary 9.1.11, M is a free $U(\mathcal{U})$ module; so, as was pointed out in the proof of Proposition 6.1.31,

$$H^i(\mathcal{U}, M) = 0$$

(since $i < \dim \mathcal{U}$). By the long exact sequence (9.1.12) (a) applied to M (or more directly), it follows that

$$H^i(\mathcal{U}, \phi_\alpha \psi_\alpha M) = 0$$

as well. So the long exact sequences in cohomology give natural maps

$$\begin{aligned} 0 &\rightarrow H^i(\mathcal{U}, \phi_\alpha \psi_\alpha X) \rightarrow H^{i+1}(\mathcal{U}, \phi_\alpha \psi_\alpha K) \\ 0 &\rightarrow H^i(\mathcal{U}, X) \rightarrow H^{i+1}(\mathcal{U}, K). \end{aligned}$$

All the maps f^* , g^* , h^* commute with connecting homomorphisms; so the diagram of (d) for X is contained in the one with X replaced by K and i by $i+1$. This diagram is commutative by induction; so that for X is as well. This proves (d) in case (a). Case (b) is similar. Case (c) can be deduced from (a) by taking dual modules everywhere and applying (a); or it can be proved in the same way as (a), by reducing to the case $i = 0$ and using ind instead of pro . Q.E.D.

We will now describe the example of non-zero maps f^t and $g^t h^t$ needed above. Let Y be any ℓ module of infinitesimal character $\gamma - \rho(\mathcal{U})$. Consider the parabolic $\overline{\mathcal{G}} = \ell + \overline{\mathcal{U}}$ opposite to $\mathcal{G}$. Set

$$\tilde{Y} = Y \otimes (\wedge^{\dim \mathcal{U}} \mathcal{U})^*$$

$$X = \text{ind}_{\overline{\mathcal{G}}}^{\mathcal{G}} \tilde{Y} = U(\mathcal{G}) \otimes_{\overline{\mathcal{G}}} \tilde{Y}$$

which is easily seen to have infinitesimal character γ . As a $\mathcal{U}$ module,

$$X \cong U(\mathcal{U}) \otimes_{\mathbb{C}} \tilde{Y};$$

so (by (9.1.13))

$$H^i(\mathcal{U}, X) \cong \begin{cases} Y; i = \dim \mathcal{U} \\ 0; i < \dim \mathcal{U} \end{cases}$$

as ℓ modules. Suppose we are in case (a) of Theorem 9.1.12. Then the long exact sequence degenerates to a short exact sequence

$$0 \rightarrow \phi_\alpha^L \psi_\alpha^L Y \rightarrow H^t(\mathcal{U}, \phi_\alpha \psi_\alpha X) \xrightarrow{h^t} \phi_\alpha^L \psi_\alpha^L Y \rightarrow 0.$$

Since Y has infinitesimal character $\gamma_{-\rho}(\mathcal{U})$ [and not $s_\alpha \gamma_{-\rho}(\mathcal{U})$], $\tilde{\psi}_\alpha^{\mathcal{U}} Y$ is zero; so h^t is an isomorphism

$$H^t(\mathcal{U}, \phi_\alpha \psi_\alpha X) \stackrel{h^t}{\cong} \psi_\alpha^{\mathcal{U}} \psi_\alpha^{\mathcal{U}} X.$$

If Y is a Harish-Chandra module of some sort for ℓ , then

$$g^t: \psi_\alpha^{\mathcal{U}} \psi_\alpha^{\mathcal{U}} Y \rightarrow Y$$

is also an isomorphism by Proposition 7.3.3; so $g^t \circ h^t$ is an isomorphism. In particular, it can be non-zero. To study f^t , we have to understand $\phi_\alpha \psi_\alpha X$. Define

$$\tilde{\phi}_\alpha^{\mathcal{U}} = \psi_{s_\alpha \gamma_{-\rho}(\mathcal{U}) + 0}^{\mathcal{U}} = \psi_{s_\alpha \gamma_{-\rho}(\mathcal{U}) + s_\alpha \lambda}^{\mathcal{U}}$$

(cf. Theorem 9.1.12). The proof of Proposition 7.4.3 provides a short exact sequence

$$0 \rightarrow \text{ind}_{\tilde{\mathcal{G}}}^{\mathcal{G}} \left((\tilde{\phi}_\alpha^{\mathcal{U}} \psi_\alpha^{\mathcal{U}} Y)^\sim \right) \rightarrow \phi_\alpha \psi_\alpha X \overset{\mathbb{T}}{\rightarrow} X \rightarrow 0.$$

Call the first module $\tilde{X}$. By the same computation as for X , $\tilde{X}$ has no cohomology of infinitesimal character $\gamma_{-\rho}(\mathcal{U})$; so the induced map

$$\pi^t: P_{\gamma_{-\rho}(\mathcal{U})} H^t(\mathcal{U}, \phi_\alpha \psi_\alpha X) \rightarrow H^t(\mathcal{U}, X) \simeq Y$$

is an isomorphism. The only difficulty with this as an example of non-vanishing f^t is that it is not obvious that π coincides with f . But if we choose Y to be an irreducible Harish-Chandra module, then

$$\begin{aligned} \text{Hom}_{\mathcal{G}}(\phi_\alpha \psi_\alpha X, X) &\cong \text{Hom}_{\mathcal{G}}(\psi_\alpha X, \psi_\alpha X) \\ &\cong \text{Hom}_{\mathcal{G}}(\text{ind}(\psi_\alpha^{\mathcal{U}} Y)^\sim, \text{ind}(\psi_\alpha^{\mathcal{U}} Y)^\sim) \\ &\cong \text{Hom}_{\mathcal{Q}}(\psi_\alpha^{\mathcal{U}} Y, \psi_\alpha^{\mathcal{U}} Y) = \mathbb{C} \end{aligned}$$

here we have used analogues of Proposition 7.4.1 and Lemma 9.1.8. So for such Y , π must be a multiple of f .

We now describe the context in which we will be working.

Situation 9.1.14 Suppose H is a θ -stable Cartan subgroup of G , and $\gamma \in \hat{H}'$ is a regular character (notation (6.6.1)) of nonsingular infinitesimal character $\bar{\gamma} \in \mathcal{H}^*$. Let $\mathcal{G} = \ell + \mathcal{U}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$, and $H^1 \subseteq L$ a θ -stable Cartan subgroup. Let $\gamma^1 \in (H^1)'$ be a regular character, such that $\bar{\gamma}^1$ is conjugate $\bar{\gamma}$ by an inner automorphism of $\mathfrak{g}$. We will study the occurrence of $\bar{X}_L(\gamma^1)$ in $H^1(\mathcal{U}, \bar{X}(\gamma))$.

Often we will have even more structure.

Situation 9.1.15 Suppose we are in the situation (9.1.14) and $\alpha \in R^+(\bar{\gamma})$ is a simple root. We invoke the situation 8.2.6: thus we have

$$n = \langle \alpha, \bar{\gamma} \rangle,$$

and translation functors ϕ_α, ψ_α , among other things. Write $\alpha^1 \in R^+(\bar{\gamma}^1)$ for the root corresponding to α . Choose a maximally split Cartan subgroup $H^{1,s}$ of ℓ , and an ℓ -inner automorphism

$$\psi^1: (\mathcal{H}^{1,s})^* \rightarrow \mathcal{H}^*.$$

Use this to get

$$\bar{\gamma}^{1,s} (\mathcal{H}^{1,s})^*, \quad \lambda^{1,s} \in \wedge^{1,s}$$

corresponding to the data in (8.2.6), and define

four translation functors for L

$$\begin{aligned}\psi_{\alpha}^L &= \psi_{\gamma}^{\bar{1},s}_{\bar{1},s_{-\rho}(\mathcal{U})+\lambda^1,s} \\ \phi_{\alpha}^L &= \psi_{\gamma}^{\bar{1},s}_{\bar{1},s_{-\rho}(\mathcal{U})+\lambda^1,s} \\ \tilde{\psi}_{\alpha}^L &= \psi_{s_{\alpha}\gamma}^{\bar{1},s}_{\bar{1},s_{-\rho}(\mathcal{U})+\lambda^1,s} \\ \tilde{\phi}_{\alpha}^L &= \psi_{s_{\alpha}\gamma}^{\bar{1},s}_{\bar{1},s_{-\rho}(\mathcal{U})+\lambda^1,s}.\end{aligned}$$

Lemma 9.1.16 In the setting (9.1.15), suppose $\alpha \notin \Delta(\ell)$. Then

$$\phi_{\alpha}^L \tilde{\psi}_{\alpha}^L \bar{X}_L^1(\gamma^1 - n\alpha^1) = \bar{X}_L^1(\gamma^1)_{\mathcal{G}}.$$

This is a special case of Theorem 8.2.1.

Corollary 9.1.17 In the setting (9.1.15), suppose $\alpha \notin \tau(\bar{X}(\gamma))$. Define Q_{α} , K_{α} [attached to $\bar{X}(\gamma)$] by (7.3.17), and write $X = \bar{X}(\gamma)$.

a) If $\alpha^1 \in \Delta(\mathcal{U})$, then the $(\ell, L \cap K)$ modules

$$P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, K_{\alpha}), \quad \phi_{\alpha}^L \tilde{\psi}_{\alpha}^L H^i(\mathcal{U}, X)$$

have the same irreducible composition factors and multiplicities.

b) If $-\alpha^1 \in \Delta(\mathcal{U})$, then

$$P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, Q_{\alpha}), \quad \phi_{\alpha}^L \tilde{\psi}_{\alpha}^L H^i(\mathcal{U}, X)$$

have the same composition factors and multiplicities.

Proof. Consider (a). By definition of K_{α} , there is a short exact sequence

$$0 \rightarrow K_{\alpha} \rightarrow \phi_{\alpha} \psi_{\alpha} X \rightarrow X \rightarrow 0.$$

We will write the corresponding long exact sequence in cohomology, and (9.1.12)(a). This gives

$$\begin{aligned}\dots &\rightarrow \phi_{\alpha}^L \tilde{\psi}_{\alpha}^L H^i(\mathcal{U}, X) \rightarrow P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, \phi_{\alpha} \psi_{\alpha} X) \xrightarrow{h^i} \phi_{\alpha}^L \tilde{\psi}_{\alpha}^L H^i(\mathcal{U}, X) \rightarrow \\ &\rightarrow \dots \quad P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, K_{\alpha}) \rightarrow P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, \phi_{\alpha} \psi_{\alpha} X) \xrightarrow{f^i} g^i \\ &\quad \quad \quad P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, X) \rightarrow\end{aligned}$$

The maps f^i and g^i are those in Theorem 9.1.12; so by (9.1.12)(d), this last square commutes. Furthermore, g^i is an isomorphism by Proposition 7.3.3. So

$$\ker h^i \cong \ker f^i$$

$$\phi_{\alpha}^L \tilde{\psi}_{\alpha}^L H^{i-1}(\mathcal{U}, X) / \text{im } h^{i-1} \cong P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^{i-1} H^{i-1}(\mathcal{U}, X) / \text{im } f^{i-1}.$$

By the exactness of the rows, the two $(\ell, L \cap K)$ modules on the left fit into exact sequences

$$\begin{aligned}0 &\rightarrow \phi \psi H^{i-1} / \text{im } h^{i-1} \rightarrow \phi \tilde{\psi} H^i \rightarrow \ker h^i \rightarrow 0 \\ 0 &\rightarrow PH^{i-1} / \text{im } f^{i-1} \rightarrow PH^i(K_{\alpha}) \rightarrow \ker f^i \rightarrow 0.\end{aligned}$$

Since the ends are isomorphic, the middle terms have the same character. This is (a). Case (b) is identical; or it can be deduced from (a) by Corollary 8.5.11. Q.E.D.

Corollary 9.1.18 In the setting (9.1.15), suppose $\alpha \in \tau(\bar{X}(\gamma))$.

a) If $\alpha^1 \in \Delta(\mathcal{U})$, then

$$P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, \bar{X}(\gamma)) \cong \phi_{\alpha}^L \tilde{\psi}_{\alpha}^L H^{i+1}(\mathcal{U}, \bar{X}(\gamma)).$$

b) If $-\alpha^1 \in \Delta(\mathcal{U})$, then

$$P_{\bar{\gamma}^{1-\rho}(\mathcal{U})}^i H^i(\mathcal{U}, \bar{X}(\gamma)) \cong \phi_{\alpha}^L \tilde{\psi}_{\alpha}^L H^{i-1}(\mathcal{U}, \bar{X}(\gamma)).$$

Proof. By Corollary 7.3.9(a) and Proposition 7.3.2,

$$\phi_{\alpha}\psi_{\alpha}\bar{X}(\gamma) = 0.$$

By Proposition 7.3.3,

$$\phi_{\alpha}\psi_{\alpha}H^i(\mathcal{U}, \bar{X}(\gamma)) \cong P_{\gamma-1-\rho}^1(\mathcal{U})H^i(\mathcal{U}, \bar{X}(\gamma)).$$

The corollary is immediate from these two facts and Theorem 9.1.12(a) and (c). Q.E.D.

§2. Blocks in $\mathcal{K}(\mathcal{O}, K)$.

The definitions in this section were suggested by G. Zuckerman.

Definition 9.2.1 Block equivalence of irreducible $(\mathcal{O}, K)$ modules is the equivalence relation generated by

$$X \sim Y \text{ if } \text{Ext}_{\mathcal{O}, K}^1(X, Y) \neq 0$$

The equivalence classes are called blocks.

This definition is designed to study indecomposable $(\mathcal{O}, K)$ modules. To see the relation, we need a very special case of the Yoneda interpretation of Ext.

Lemma 9.2.2 Suppose X and Y are $(\mathcal{O}, K)$ modules. Then $\text{Ext}_{\mathcal{O}, K}^1(X, Y)$ is non-zero if and only if there is a $(\mathcal{O}, K)$ module E , not isomorphic to $X \oplus Y$, and a short exact sequence

$$(*) \quad 0 \rightarrow Y \xrightarrow{i} E \xrightarrow{j} X \rightarrow 0.$$

Proof. Suppose first that E exists. In the exact sequence

$$0 \rightarrow \text{Hom}(X, Y) \rightarrow \text{Hom}(X, E) \xrightarrow{\pi} \text{Hom}(X, X) \xrightarrow{\delta} \text{Ext}^1(X, Y) \rightarrow \dots,$$

we claim that the identity map on X is not in the image of π . For if it were, a preimage

$$\phi: X \rightarrow E$$

would be a splitting of $(*)$. So π is not onto, so δ is not zero.

Conversely, suppose $\text{Ext}^1(X, Y)$ is non-zero. Choose an injective module I containing Y :

$$0 \rightarrow Y \rightarrow I \rightarrow Q \rightarrow 0.$$

Since I is injective, $\text{Ext}^1(X, I)$ is zero; so the long exact sequence is

$$\dots \text{Hom}(X, I) \rightarrow \text{Hom}(X, Q) \rightarrow \text{Ext}^1(X, Y) \rightarrow 0.$$

Pick a map

$$\phi: X \rightarrow Q$$

not in the image of $\text{Hom}(X, I)$. Let $\bar{E}_0$ be the image of X under ϕ , and E_0 the preimage of $\bar{E}_0$ in I . Then there is a short exact sequence

$$0 \rightarrow Y \rightarrow E_0 \xrightarrow{\pi_0} \bar{E}_0 \rightarrow 0$$

and a surjection

$$X \xrightarrow{\delta_0} \bar{E}_0.$$

Set

$$E_1 = X \oplus E_0$$

$$E = \{(x, e) \in E_1 \mid \delta_0(x) = \pi_0(e)\}$$

$$i: Y \rightarrow E, \quad i(y) = (0, i_0(y))$$

$$\pi: E \rightarrow X, \quad \pi(x, e) = x.$$

We leave the easy verification that this is a non-split exact sequence to the reader. Q.E.D.

Lemma 9.2.3 Suppose X is a $(\mathcal{G}, K)$ module of finite length. Then (cf. (9.2.1))

$$X = \sum_{\text{blocks } B} X_B,$$

and every irreducible constituent of X_B belongs to the block B .

This is a formal consequence of the definition and Lemma 9.2.2 by induction on the number of irreducible composition factors of X . We leave the proof to the reader.

Lemma 9.2.4 Suppose X and Y are $(\mathcal{G}, K)$ modules.

Then there is a natural isomorphism (notation (8.5.1))

$$\text{Hom}_{\mathcal{G}, K}(X, Y^C) \cong \text{Hom}_{\mathcal{G}, K}(Y, X^C).$$

Proof. Suppose

$$f: X \rightarrow Y^C$$

is a $(\mathcal{G}, K)$ module map. Define

$$g: Y \rightarrow X^C$$

$[g(y)](x) = [f(x)](y) \quad (x \in X, y \in Y).$

The correspondence we want identifies f and g ; we leave the trivial verifications to the reader. Q.E.D.

Corollary 9.2.5 Let X and Y be $(\mathcal{G}, K)$ modules.

- a) Suppose $P_* \rightarrow X \rightarrow 0$ is a projective resolution of X (Definition 6.1.9). Then $0 \rightarrow X^C \rightarrow P_*^C$ is an injective resolution of X^C .
- b) There is a natural isomorphism

$$\text{Ext}_{\mathcal{G}, K}^i(X, Y^C) \cong \text{Ext}^i(Y, X^C).$$

Proof. For (a), we only have to show that P_i^C is an injective $(\mathcal{G}, K)$ module. The functor

$$A \rightarrow \text{Hom}(A, P_i^C)$$

is equivalent to

$$A \rightarrow \text{Hom}(P_i, A);$$

and this is exact since P_i is projective. For (b), the left side is the cohomology of the complex (Definition 6.1.11)

$$\text{Hom}_{\mathcal{G}, K}(P_*, Y^C) \cong \text{Hom}_{\mathcal{G}, K}(Y, P_*^C).$$

But the cohomology of the complex on the right is $\text{Ext}^i(Y, X^C)$ (Corollary 6.1.24). Q.E.D.

Lemma 9.2.6 Suppose $\gamma^i \in (\hat{H}^i)$, ($i = 1, 2$) are regular characters of nonsingular infinitesimal characters $\bar{\gamma}^i \in (\bar{h}^i)^*$. Suppose that $\bar{\gamma}^1$ and $\bar{\gamma}^2$ are conjugate by an inner automorphism

$$\psi: (\mathfrak{h}^1)^* \rightarrow (\mathfrak{h}^2)^*$$

for $\mathfrak{g}$. Assume that for every weight $\lambda^1 \in \Lambda^1 \subseteq \hat{\mathfrak{H}}^1$ of a finite dimensional representation of G , such that $\bar{\gamma}^1 + \lambda^1$ is $R^+(\bar{\gamma}^1)$ dominant and nonsingular, we have

$$|(\bar{\gamma}^1 + \lambda^1)|_{\mathfrak{t}^1} = |(\bar{\gamma}^2 + \psi(\bar{\lambda}^1))|_{\mathfrak{t}^2}.$$

Then $\mathfrak{H}^1$ is conjugate by K to $\mathfrak{H}^2$. If $\mathfrak{H}^1 = \mathfrak{H}^2$, there is an element $w \in W(\mathfrak{g}, \mathfrak{h}^1)$, commuting with θ , such that

$$w\bar{\gamma}^1 = \bar{\gamma}^2.$$

Proof. Since the intersection of $\bar{\Lambda}^1$ with an open cone is Zariski dense in $(\mathfrak{h}^1)^*$, the hypothesis means that

$$|\mu|_{\mathfrak{t}^1} = |\psi(\mu)|_{\mathfrak{t}^2} \quad \mu \in (\mathfrak{h}^1)^*.$$

In particular, ψ maps $(\sigma\mathfrak{t}^1)^*$ onto $(\sigma\mathfrak{t}^2)^*$. By Lemma 8.6.7, Λ^1 and Λ^2 are conjugate under K ; so $\mathfrak{H}^1$ and $\mathfrak{H}^2$ are as well. If $\mathfrak{H}^1 = \mathfrak{H}^2$, ψ is given by a Weyl group element w , which must preserve $\sigma\mathfrak{t}^1$. So w preserves the orthogonal complement $\mathfrak{t}^1$ of $\sigma\mathfrak{t}^1$, and therefore commutes with θ . Q.E.D.

Lemma 9.2.7 Suppose $\gamma^i \in (\hat{\mathfrak{H}}^i)^*$ ($i = 1, 2$) are non-conjugate regular characters, of nonsingular infinitesimal character. Suppose that either

$$\text{Ext}_{\mathfrak{g}, K}^1(\bar{X}(\gamma^1), \bar{X}(\gamma^2)) \neq 0$$

or

$$\text{Ext}_{\mathfrak{g}, K}^1(\bar{X}(\gamma^2), \bar{X}(\gamma^1)) \neq 0.$$

Then either

$$\bar{X}(\gamma^1) \text{ occurs in } X(\gamma^2)$$

or

$$\bar{X}(\gamma^2) \text{ occurs in } X(\gamma^1).$$

Proof. By the translation principle (in the form of Theorem 8.2.1) and Lemma 9.2.6, and possibly by interchanging γ^1 and γ^2 , we can arrange for either

$$(9.2.8) \text{ (a)} \quad |\gamma^1|_{\mathfrak{t}^1} < |\gamma^2|_{\mathfrak{t}^2}$$

or

$$(9.2.8) \text{ (b)} \quad \mathfrak{H}^1 = \mathfrak{H}^2, \quad \bar{\gamma}^1 = w\bar{\gamma}^2 \quad \text{for some } w \in W(\mathfrak{g}, \mathfrak{h})^\theta.$$

In either case, since γ^1 and γ^2 are assumed to be non-conjugate, $\bar{X}(\gamma^1)$ does not occur in $X(\gamma^2)$ (Lemma 6.6.6). Assume first that

$$(9.2.9) \quad \text{Ext}^1(\bar{X}(\gamma^2), \bar{X}(\gamma^1)) \neq 0.$$

Set

$$Q^1 = X(\gamma^1)/\bar{X}(\gamma^1),$$

and consider the corresponding long exact sequence for Ext .

We get

$$\dots \text{Hom}(\bar{X}(\gamma^2), Q^1) \rightarrow \text{Ext}^1(\bar{X}(\gamma^2), \bar{X}(\gamma^1)) \rightarrow \text{Ext}^1(\bar{X}(\gamma^2), X(\gamma^1)) \rightarrow \dots$$

So either

$$\text{Hom}(\bar{X}(\gamma^2), Q^1) \neq 0$$

or

$$\text{Ext}^1(\bar{X}(\gamma^2), X(\gamma^1)) \neq 0.$$

In the first case, $\bar{X}(\gamma^2)$ occurs in $X(\gamma^1)$, and we are done. For the second, let $\mathcal{G} = \mathfrak{L} + \mathcal{U}$ be the θ -stable parabolic associated to γ^1 (Proposition 6.6.2). By Theorem 6.5.9(f), either

$$\text{Ext}_{\mathfrak{L}, L\cap K}^1 \left(H^R(\mathcal{U}, \bar{X}(\gamma^2)), X_{\mathfrak{L}}(\gamma_{\mathcal{G}}^1) \right) \neq 0$$

or

$$\text{Hom}_{\mathfrak{L}, L\cap K} \left(H^{R-1}(\mathcal{U}, \bar{X}(\gamma^2)), X_{\mathfrak{L}}(\gamma_{\mathcal{G}}^1) \right) \neq 0.$$

So

$$\bar{X}_{\mathfrak{L}}(\gamma_{\mathcal{G}}^1) \text{ occurs in } H^*(\mathcal{U}, X(\gamma^2));$$

this is obvious in the second case, and follows from Corollary 6.4.13 in the first. By Lemma 8.1.7, this cannot happen under assumption (9.2.8)(a). Assuming (9.2.8)(b), Theorem 8.1.6 forces γ^1 and γ^2 to be conjugate under K , a contradiction. This proves the lemma under assumption (9.2.9).

Next, assume that (9.2.9) fails; so by hypothesis

$$\text{Ext}^1(\bar{X}(\gamma^1), \bar{X}(\gamma^2)) \neq 0.$$

By Corollary 9.2.5(b), and Proposition 8.5.6,

$$\text{Ext}^1 \left[\bar{X}((\gamma^2)^c), \bar{X}((\gamma^1)^c) \right] \neq 0.$$

By the first case, $\bar{X}((\gamma^2)^c)$ occurs in $X((\gamma^1)^c)$. By Proposition 8.5.6 again, $\bar{X}(\gamma^2)$ occurs in $X(\gamma^1)$. Q.E.D.

Proposition 9.2.10 Among $(\mathcal{G}, K)$ modules with non-singular infinitesimal character, block equivalence

(Definition 9.2.1) is the equivalence relation generated by

$X \sim Y$ if X and Y occur in a common standard representation.

Proof. Since a standard representation with nonsingular infinitesimal character is indecomposable (since it has a unique subrepresentation), Lemma 9.2.3 implies that all its constituents lie in the same block. So the present equivalence relation is finer than block equivalence. But Lemma 9.2.7 guarantees that block equivalence is finer than the present relation; so they coincide. Q.E.D.

The same proposition holds for singular infinitesimal character, but is somewhat harder to prove (mainly because the standard representations may be decomposable). Since we will not need such a result, we omit the proof.

Theorem 9.2.11 Consider the equivalence relation

$\tilde{c}$ on irreducible $(\mathcal{G}, K)$ modules with nonsingular infinitesimal character, generated by the following

relations: fix $\gamma \in \hat{H}'$, with $\bar{\gamma}$ nonsingular, and a

simple root $\alpha \in R^+(\bar{\gamma})$ (notation (7.2.16)).

a) If α is noncompact imaginary, and $\gamma' \in c\alpha(\gamma)$ (notation (8.3.4)(c)), then

$$\bar{X}(\gamma) \tilde{c} \bar{X}(\gamma').$$

b) If α is complex, and $\langle \alpha, \bar{\gamma} \rangle = n$, then

$$\bar{X}(\gamma) \tilde{c} \bar{X}(\gamma - n\alpha).$$

Then $\tilde{c}$ coincides with block equivalence (Definition 9.2.1) on $(\mathcal{G}, K)$ modules with nonsingular infinitesimal character.

Proof. By Proposition 9.2.10, Corollary 8.2.9, and Corollary 8.4.8, $\tilde{c}$ is finer than block equivalence. So by Proposition 9.2.10, it is enough to prove the following claim: if $\gamma \in \hat{H}'$, $\gamma^1 \in (\hat{H}^1)'$, $\bar{\gamma}$ is nonsingular, and $\bar{X}(\gamma^1)$ occurs in $X(\gamma)$, then $\bar{X}(\gamma) \tilde{c} \bar{X}(\gamma^1)$. This we prove by induction on $\ell^I(\gamma)$ (notation 8.1.4); the argument is exactly the one given for Proposition 8.6.19, but we repeat it for emphasis. We will prove simultaneously that if γ is any irreducible constituent of $U_\alpha(\bar{X}(\gamma))$, for any simple $\alpha \in R^+(\bar{\gamma})$, then $\gamma \tilde{c} \bar{X}(\gamma)$.

Case I. There is a complex simple root $\alpha \in R^+(\gamma)$, such that $\theta\alpha \notin R^+(\gamma)$. By Propositions 8.2.7 and 7.2.22,

$$\Theta(\gamma) = s_\alpha \cdot \Theta(\gamma - n\alpha).$$

By Lemma 8.6.13,

$$\ell^I(\gamma - n\alpha) = \ell^I(\gamma) - 1,$$

so the claims are known for $X(\gamma - n\alpha)$. By (9.2.11)(b), $\bar{X}(\gamma) \tilde{c} \bar{X}(\gamma - n\alpha)$. By Corollary 7.3.20, either

- a) $\bar{X}(\gamma^1)$ occurs in $X(\gamma - n\alpha)$, or
- b) there is a γ^2 , $\alpha \notin \tau(\bar{X}(\gamma^2))$, such that $\bar{X}(\gamma^1)$ occurs in $U_\alpha(\bar{X}(\gamma^2))$, and $\bar{X}(\gamma^2)$ occurs in $X(\gamma - n\alpha)$.

In case (a), we have by induction

$$\bar{X}(\gamma^1) \tilde{c} \bar{X}(\gamma - n\alpha) \tilde{c} \bar{X}(\gamma).$$

In case (b),

$$\ell^I(\gamma^2) \leq \ell^I(\gamma - n\alpha) = \ell^I(\gamma) - 1$$

by Proposition 8.6.19; so by applying the first inductive hypothesis to $(\gamma - n\alpha, \gamma^2)$, and the second to (γ^2, α) , we find

$$\bar{X}(\gamma^1) \tilde{c} \bar{X}(\gamma^2) \tilde{c} \bar{X}(\gamma - n\alpha) \tilde{c} \bar{X}(\gamma).$$

This proves the first claim in this case.

Case II. There is a simple type I real root $\alpha \in R^+(\gamma)$ satisfying the parity condition. Write

$$c_\alpha(\gamma) = \{\gamma_\alpha^\pm\}$$

as in Definition 8.3.16. By Proposition 8.3.18(d) and Corollary 8.4.6,

$$\Theta(\gamma) = s_\alpha \Theta(\gamma_\alpha^+) + \Theta(\gamma_\alpha^-).$$

By Lemma 8.6.13,

$$\ell^I(\gamma_\alpha^\pm) = \ell^I(\gamma) - 1;$$

and by (9.2.11)(b), $\bar{X}(\gamma_\alpha^\pm) \tilde{c} \bar{X}(\gamma)$. Now argue as in Case I.

Case III. There is a simple type II real root $\alpha \in R^+(\gamma)$ satisfying the parity condition. By Proposition 8.3.18(e) and Corollary 8.4.6,

$$\Theta(\gamma) + \Theta(\gamma - n\alpha) = \Theta(\gamma_\alpha) + s_\alpha \Theta(\gamma_\alpha);$$

here γ_α is as in (8.3.16). So $\bar{X}(\bar{\gamma}^1)$ occurs in $\Theta(\gamma_\alpha)$ or in $s_\alpha \Theta(\gamma_\alpha)$. By Lemma 8.6.13,

$$\varrho^I(\gamma_\alpha) = \varrho^I(\gamma) - 1;$$

and by (9.2.11)(b), $\bar{X}(\gamma_\alpha) \tilde{c} \bar{X}(\gamma)$. Now argue as in Case I.

Case IV. Otherwise. By Definition 8.6.5, γ is minimal; so by Theorem 8.6.6, $\bar{X}(\gamma^1) = \bar{X}(\gamma)$. So the first claim is trivial.

In all cases, the second claim (for γ) follows from the first by Theorem 8.5.18. This completes the induction, and proves the theorem. Q.E.D.

Properly interpreted (by adding the assumption that all Langlands subrepresentations of singular standard representations are $\tilde{c}$ equivalent), this theorem is also true for singular infinitesimal characters. This follows fairly easily from the singular version of Proposition 9.2.10 (which, as we already remarked, is not trivial).

Theorem 9.2.11 allows us to apply purely structural (as opposed to representation-theoretic) techniques to study blocks. The first result like that which we will need is the following.

Proposition 9.2.12 ([46], Proposition 6.19). Suppose $\gamma \in \hat{H}'$ and $\gamma^1 \in (\hat{H}^1)'$ are regular characters, $\bar{\gamma}$ is nonsingular, and $\bar{X}(\gamma)$ and $\bar{X}(\gamma^1)$ lie in the same block. Let

$$\psi: \mathcal{H}^* \rightarrow (\mathcal{H}^1)^*$$

be the unique ψ -inner automorphism taking γ to γ^1 .
Let

$$R(\bar{\gamma}) = {}^1R(\bar{\gamma}) \cup \dots \cup {}^rR(\bar{\gamma})$$

be the decomposition of the integral roots (notation (7.2.16)) into simple θ -stable subsystems; (see remarks after the statement); and let

$$R(\bar{\gamma}^1) = {}^1R(\bar{\gamma}^1) \cup \dots \cup {}^rR(\bar{\gamma}^1)$$

be the corresponding decomposition of $R(\bar{\gamma}^1)$; that is,

$${}^iR(\bar{\gamma}^1) = \psi({}^iR(\bar{\gamma})).$$

Define

$${}^iR_0^{\text{IR}}(\gamma), {}^iR_1^{\text{IR}}(\gamma), {}^iR_0^{\text{IR}}(\gamma^1), {}^iR_1^{\text{IR}}(\gamma^1)$$

the real roots which do not (for subscript zero) or do (for subscript one) satisfy the parity condition (cf. (8.4.8)). Using Lemma 8.6.3, regard

$${}^iV_R^{\text{IR}}(\gamma) \supseteq {}^iV_{R_0}^{\text{IR}}(\gamma)$$

as the root system of a compact Cartan subgroup ${}^iT^C$ in an equal rank semisimple group iG , and ${}^iV_{R_0}^{\text{IR}}(\gamma)$ as the roots of ${}^iT^C$ in a maximal compact subgroup iK . Similarly, define ${}^iG^1$ and ${}^iK^1$ using ${}^1\gamma$. Put

$${}^i\mathcal{H}^* = \text{span } {}^iR(\bar{\gamma}) = {}^i\mathcal{H}^* + {}^i\mathcal{H}^*$$

$$({}^i\mathcal{H}^1)^* = \text{span } {}^iR(\bar{\gamma}^1) = ({}^i\mathcal{H}^1)^* + ({}^i\mathcal{H}^1)^*$$

$${}^i_{\text{C}}(\gamma) = \text{real rank of } {}^iG/{}^iK + \dim {}^i\mathcal{H}$$

$${}^i_{\text{C}}(\gamma^1) = \text{real rank of } {}^iG^1/{}^iK^1 + \dim {}^i\mathcal{H}^1$$

Then

$${}^i_{\text{C}}(\gamma) = {}^i_{\text{C}}(\gamma^1) \quad \text{for all } i.$$

The point of this result is that it provides some easily computed invariants of a block, which we will use for theoretical purposes. The algorithm of Theorem 9.2.11 is easy to apply in examples, but it is difficult to see by inspection when two regular characters can be connected by a chain of the sort described by (9.2.11)(a) - (b). The problem of giving a "closed form" characterization of blocks is an interesting one. Beyond what is contained in Proposition 9.2.12, it is entirely open.

A small explanation of the term " θ -stable simple subsystems" is required. We have seen that θ preserves $R(\bar{\gamma})$ (Lemma 8.2.5); so it acts as a permutation of order 2 on the set of simple factors. We lump together as a single $i_R(\bar{\gamma})$ any pair of simply subsystems interchanged by θ . Implicit in the proposition is the claim that ψ preserves this kind of decomposition -- a fact which is not obvious, since ψ need not commute with θ .

Proof of Proposition 9.2.12 By Theorem 9.2.11, we may assume that γ^1 is related to γ as in (9.2.11)(a) or (9.2.11)(b). In the second case, ψ is just reflection about α , and

$$i_{R_j^{\text{IR}}}(\gamma) = i_{R_j^{\text{IR}}}(\gamma), \quad i_{\mathcal{T}}^0 = i_{\mathcal{T}}^1,$$

as is essentially obvious from Definition 8.4.8. So

$$i_G \cong i_G^1,$$

and the conclusion is immediate. So assume we are in the first case; that is, that γ^1 is obtained from γ by Cayley transform through a noncompact imaginary root α . We write

α^1 for the real root of $\mathcal{h}^1$ in $\mathcal{g}$ corresponding to α . Then (cf. (8.3.2))

$$\text{kernel of } \alpha^1 = \mathcal{h}^1 \cap \mathcal{h} = \text{kernel of } \alpha.$$

The map ψ is the identity on $(\mathcal{h}^1 \cap \mathcal{h})^*$, and sends α to α^1 . Clearly

real roots of $\mathcal{h}$ in $\mathcal{g} = \psi^{-1}(\text{real roots of } \mathcal{h}^1 \text{ in } \mathcal{g} \text{ orthogonal to } \alpha)$

$$(9.2.13) \text{ (a)} \quad R^{\text{IR}}(\gamma) = \psi^{-1}(\{\beta \in R^{\text{IR}}(\gamma^1) \mid \langle \beta, \alpha \rangle = 0\}).$$

By Proposition 8.3.18, α^1 satisfies the parity condition for γ^1 ; so

$$(9.2.13) \text{ (b)} \quad \forall_{\alpha}^1 \in \Delta \left[\mathcal{g}_{\alpha}^1, (\mathcal{T}_{\alpha}^c)^1 \right] \text{ is noncompact.}$$

[Here j_R^1 is the simple factor containing α^1 .] We are going to use Proposition 5.3.1 to calculate the real rank of j_G^1/j_K^1 . In the notation of that Proposition (what is called β there is our $\forall_{\alpha}^1$), it is clear that we can find a maximal abelian subalgebra of $\mathfrak{p}$ by adjoining to $\langle Z_{\beta} \rangle$ a maximal abelian subalgebra of $\mathcal{g}_{\alpha}^{\beta} \cap \mathfrak{p}$. So

$$(9.2.14) \quad \text{real rank } \mathcal{g}_0^{\beta} = \text{real rank } \mathcal{g}_0^{\beta} + 1.$$

To apply this in our case, we have to identify $i_R^{\text{IR}}(\gamma)$ with the root system of $\mathcal{h}^{\beta}$ in $\mathcal{g}^{\beta}$. Formally this is done by (9.2.13)(a) [cf. (5.3.1)]; we have to see that this identification preserves the notions of compact and noncompact. Because of Proposition 5.3.1, what we have to show is this.

Suppose δ^1 is a real integral root of $\mathcal{h}^1$ in $\mathfrak{g}$ orthogonal to α^1 ; let δ be the corresponding root of $\mathfrak{g}$ in $\mathcal{h}$.

a) Suppose δ^1 and α^1 are strongly orthogonal.

(9.2.15) Then δ satisfies the parity condition (for γ) if and only if δ^1 does (for γ^1).

b) Suppose δ^1 and α^1 are not strongly orthogonal. Then δ satisfies the parity condition if and only if δ^1 does not.

Now $\gamma^1|_{T_1 \cap T}$ is just the restriction of γ (Definition 8.3.4); so (9.2.15) will follow from

Under the hypotheses of (9.2.15), and in the notation of (8.4.8),

$$(9.2.16) \quad \begin{array}{l} \text{a) } m_\delta = m_{\delta^1} \\ \text{b) } \varepsilon_\delta = \begin{cases} \varepsilon_{\delta^1} & \text{if } \delta^1 \pm \alpha^1 \text{ are not roots} \\ -\varepsilon_{\delta^1} & \text{if } \delta^1 \pm \alpha^1 \text{ are roots} \end{cases} \end{array}$$

Assume this for a moment. Then, as was already explained, (9.3.14) gives

$$\text{real rank } (j_G/j_K) = \text{real rank } (j_G^1/j_K^1) - 1.$$

It is clear that

$$j\mathcal{X}^1 = \text{kernel of } \alpha \text{ in } j\mathcal{X}$$

and so

$$\dim j\mathcal{X}^1 = \dim j\mathcal{X} - 1.$$

These two equations give

$$j_{\mathbf{c}(\gamma^1)} = j_{\mathbf{c}(\gamma)}.$$

Obviously

$$i_{G/i_K} \cong i_{G^1/i_K^1}, \quad i\mathcal{X}^1 = i\mathcal{X}$$

for $i \neq j$, so

$$i_{\mathbf{c}(\gamma^1)} = i_{\mathbf{c}(\gamma)}$$

in these cases as well.

It remains to verify (9.2.16). Let ℓ be the centralizer of $\mathcal{X}^1$ in $\mathfrak{g}$, and $\mathfrak{g} = \ell + \mathcal{U}$ a θ -stable parabolic such that $\langle \beta, \bar{\gamma}^1 | \mathcal{X}^1 \rangle \geq 0$ ($\beta \in \Delta(\mathcal{U}, \mathcal{h}^1)$).

By Lemma 8.3.12, we may assume that $G = L$. Let

$$\begin{aligned} \sigma_0^2 &= \text{kernel of } \delta \text{ in } \sigma_{\mathcal{U}_0} \\ M_A^2 &= \text{Langlands decomposition of } G^{A^2}. \end{aligned}$$

Using d_1 to define the parity condition (see Lemma 8.3.9 and Definition 8.3.11), we reduce to the case $G = M_0^2$. Of course, we may assume G is semisimple; so G is a split, rank 2 semisimple group, with a split Cartan subgroup spanned by the orthogonal roots α^1 and δ^1 . So G is (up to covering) $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, $Sp(2, \mathbb{R})$, or (split group of type G_2). We begin with (9.2.16)(a). Remember that m_δ and m_{δ^1} were defined using maps $\phi_\delta, \phi_{\delta^1}$ of $SL(2, \mathbb{R})$ into G . If δ^1 and α^1 are strongly orthogonal, then the images of ϕ_{δ^1} and ϕ_{α^1} commute; so we may take $\phi_\delta = \phi_{\delta^1}$. In that case (9.2.16)(a) is obvious. Otherwise (by inspection of the list of cases) α^1

and δ^1 are short roots in $\text{Sp}(2, \mathbb{R})$. Using Lemma 4.3.19(e), one checks that m_δ and m_{δ^1} are annihilated by all roots; so they are both equal to the (unique) non-trivial element of the center of $\text{Sp}(2, \mathbb{R})$. This proves (9.2.16)(a). For (9.2.16)(b), Lemma 8.3.12(b) shows that ε_{δ^1} is always -1. By Lemma 8.3.9, ε_δ can be computed as follows. Let T^2 be a compact Cartan subgroup of G , obtained by Cayley transform through δ [cf. (8.3.6)]; and let δ^2 be the noncompact root of $\mathcal{L}^2$ in $\mathfrak{g}$ corresponding to δ . Choose a positive root system $\Delta^+(\mathcal{L}, \mathcal{L}^2)$, and set

$$2\rho_{\mathcal{C}} = 2\rho(\Delta^+(\mathcal{L}, \mathcal{L}^2)).$$

Then

$$\varepsilon_\delta = -(-1)^{\sum \langle \delta, 2\rho_{\mathcal{C}} \rangle}.$$

Because of this, (9.2.16) is equivalent to the following assertion.

Let T^2 be a compact Cartan subgroup of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, $\text{Sp}(2, \mathbb{R})$, or G_2 , and $\Delta^+(\mathcal{L}, \mathcal{L}^2)$ a positive system for $\mathcal{L}^2$ in $\mathcal{L}$. Set

$$2\rho_{\mathcal{C}} = 2\rho(\Delta^+(\mathcal{L}, \mathcal{L}^2))$$

Let δ^2 be a noncompact root of $\mathcal{L}^2$ in

$\mathfrak{g}$. Then

$$\sum_{\langle \delta^2, 2\rho_{\mathcal{C}} \rangle} \text{ is } \begin{cases} \text{odd if } G = \text{Sp}(2, \mathbb{R}) \text{ and } \delta \text{ is long} \\ \text{even otherwise.} \end{cases}$$

This assertion is easily verified by direct computation.

Q.E.D.

Definition 9.2.17 Suppose $\gamma \in \hat{H}^1$, and $\bar{\gamma}$ is non-singular. Define $c(\gamma)$, the compactness of γ by

$$c(\gamma) = \sum_{i=1}^r i c(\gamma_i)$$

(notation (9.2.12)). If $\bar{X}(\gamma)$ belongs to a block B (Definition 9.2.1), we define $c(B)$, the compactness of B by

$$c(B) = c(\gamma);$$

this makes sense by Proposition 9.2.12.

The remaining "abstract nonsense" fact about blocks is that if X and Y lie in distinct blocks, then

$$\text{Ext}_{\mathfrak{g}, K}^i(X, Y) = 0.$$

This is due to Casselman. Unfortunately, the abstract proof is rather unpleasant [since $\mathcal{M}(\mathfrak{g}, K)$ does not have projectives of finite length]; so we will prove this [for nonsingular infinitesimal character, which is the only case we need] using Theorem 9.2.11.

Lemma 9.2.18 Suppose $\gamma^i \in (\hat{H}^1)^*$ ($i = 1, 2$) are regular characters of nonsingular infinitesimal character $\bar{\gamma}^i \in (\mathcal{H}^1)^*$.

a) If γ^1 is not conjugate to γ^2 , then (notation (8.5.1), (8.5.4))

$$\text{Ext}_{\mathfrak{g}, K}^* \left[X((\gamma^1)^c)^c, X(\gamma^2) \right] = 0.$$

b) $\text{Ext}_{\mathfrak{g}, K}^p \left[X((\gamma^1)^c)^c, X(\gamma^1) \right] \cong \Lambda^p(\sigma^1)^*$; here σ^1 is the split part of $\mathcal{H}^1$.

[Part (a) is the only one we will use. It generalizes a result of Schmid [37], who proved it for H^1 compact and Ext^1 -- that is, for discrete series representations (see Definition 6.6.7 and Theorem 6.6.8). For higher Ext and H^1 compact the result is due to Zuckerman (unpublished).

Zuckerman also computed the Euler characteristics

$$\sum (-1)^i \text{Ext}_{\mathfrak{g}, K}^i(X(\gamma^1), X(\gamma^2)),$$

obtaining the result implicit in the lemma.]

Proof. By Proposition 6.1.34, we may assume that $X(\gamma^1)$ and $X(\gamma^2)$ have the same infinitesimal character. By Corollary 9.2.5(b), the assertions are symmetric in γ^1 and γ^2 . So by Lemma 9.2.6, we may assume that either

- 1) $H^1 = H^2$, and there is an element $w \in W(\mathfrak{g}, \mathfrak{h}^{1, \theta})$ such that $w\gamma^1 = \gamma^2$; or
- 2) there is a weight $\lambda^1 \in \Lambda^1$ of a finite dimensional representation of G , such that $\bar{\gamma}^1 + \bar{\lambda}^1$ is $R^+(\bar{\gamma}^1)$ dominant and nonsingular,

and (notation (9.2.6))

$$|(\bar{\gamma}^1 + \bar{\lambda}^1)|_{\mathfrak{t}^1}| > |(\bar{\gamma}^2 + \psi(\bar{\lambda}^1))|_{\mathfrak{t}^2}|.$$

Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic associated to γ^2 (Proposition 6.6.2). By Theorem 6.5.9(f), there is a spectral sequence

$$(9.2.20) \quad \text{Ext}_{\mathfrak{l}, L \cap K}^p \left(H^{R-q}(\mathfrak{u}, X((\gamma^1)^C)^C), X_L(\gamma^2) \right) \implies \text{Ext}_{\mathfrak{g}, K}^{p+q} \left(X((\gamma^1)^C)^C, X(\gamma^2) \right).$$

For (a), suppose that

$$\text{Ext}_{\mathfrak{g}, K}^* \left(X((\gamma^1)^C)^C, X(\gamma^2) \right) \neq 0.$$

By (9.2.20) and Corollary 6.4.13,

$$(9.2.21) \quad \bar{X}_L(\gamma^2) \text{ occurs in } H^{R-q} \left(\mathfrak{u}, X((\gamma^1)^C)^C \right).$$

Suppose first that we are in case (9.2.19)(1). By Corollary 8.1.21, we conclude that γ^1 is conjugate to γ^2 , as desired. So suppose that we are in case (9.2.19)(2). Since the condition (9.2.21) behaves well under the translation functors (Theorem 8.2.1 and Proposition 7.4.6), we may assume that

$$|\bar{\gamma}^1|_{\mathfrak{t}^1}| > |\bar{\gamma}^2|_{\mathfrak{t}^2}|.$$

But then (9.2.21) contradicts Corollary 8.1.11.

For (b), Corollary 6.4.13 says that the E_2^{pq} term of (9.2.20) is non-zero only if $\bar{X}_L(\gamma^1)$ occurs in

$$H^{R-q}(\mathfrak{u}, X((\gamma^1)^C)^C).$$

By Corollary 8.1.21, this happens if and only if $q = 0$; so the spectral sequence degenerates to

$$\text{Ext}_{\mathfrak{g}, K}^p \left(X((\gamma^1)^C)^C, X(\gamma^1) \right) \cong \text{Ext}_{\mathfrak{l}, L \cap K}^p \left(H^R(\mathfrak{u}, X((\gamma^1)^C)^C), X_L(\gamma^1) \right).$$

By Corollary 8.1.21, $\bar{X}_L(\gamma^1)$ occurs in H^R exactly once. By Corollary 6.4.13 and inspection of the long exact sequences in Ext, the right side above is isomorphic to

$$\text{Ext}_{\mathfrak{l}, L \cap K}^p \left(\bar{X}_L(\gamma^1), X_L(\gamma^1) \right) \cong \text{Ext}_{\mathfrak{l}}^p \left(H^t(\mathfrak{n}, \bar{X}_L(\gamma^1))^{\tilde{\delta}, \nu - \rho}, \mathcal{C}_{\nu - \rho} \right)$$

[see Corollary 6.4.12, where the notation is explained]. By Lemma 9.1.7, the cohomology here is a semisimple $\mathfrak{a}^1$ module;

so

$$\cong \Lambda^p(\sigma_L^1) \otimes \text{Hom}_{\sigma_L^1} \left[H^t(\tau, \bar{X}_L(\gamma^1)), \phi_{v-\rho} \right].$$

[This well-known fact is a consequence of Proposition 6.1.16, for example.] So we have shown that

$$\text{Ext}_{\mathfrak{g},K}^p \left[X((\gamma^1)^C)^C, X(\gamma^1) \right] \cong \Lambda^p(\sigma_L^1) * \otimes \text{Hom}_{\mathfrak{g},K} \left[X((\gamma^1)^C)^C, X(\gamma^1) \right].$$

By Theorem 6.5.10, $\bar{X}(\gamma^1)$ is the unique irreducible submodule of $X(\gamma^1)$; and by Proposition 8.5.6, $\bar{X}(\gamma^1)$ occurs exactly once in $X((\gamma^1)^C)^C$. So the Hom is one dimensional, proving (b).

Q.E.D.

Lemma 9.2.22 Suppose B_1 and B_2 are distinct blocks of $(\mathfrak{g}, K)$ modules of nonsingular infinitesimal character. Let X_1 be a $(\mathfrak{g}, K)$ module of finite length, such that every irreducible constituent of X_1 belongs to B_1 . Fix $\gamma^2 \in (\hat{H}^2)$, so that $\bar{X}(\gamma^2)$ belongs to B_2 . Then

$$\text{Ext}_{\mathfrak{g},K}^* (X_1, X(\gamma^2)) = 0.$$

Proof. We consider the lemma with the following additional condition.

$$(9.2.23) \quad \text{If } \bar{X}(\gamma) \text{ occurs in } X_1, \text{ then } \varrho^I(\gamma) \leq m$$

(notation (8.1.4)). This we will prove by induction on m ; for large m , the condition is empty, and we get the lemma.

So suppose the result is known for $m' < m$. Fix $\gamma^1 \in (\hat{H}^1)$, with

$$\varrho^I(\gamma^1) = m, \quad \bar{X}(\gamma^1) \in B_1.$$

Since B_1 and B_2 are distinct, γ^1 is not conjugate to γ^2 ; so

by Lemma 9.2.18(a),

$$\text{Ext}_{\mathfrak{g},K}^* \left[X((\gamma^1)^C)^C, X(\gamma^2) \right] = 0.$$

There is a short exact sequence (by Proposition 8.5.6)

$$0 \rightarrow Y \rightarrow X((\gamma^1)^C)^C \rightarrow \bar{X}(\gamma^1) \rightarrow 0.$$

The corresponding long exact sequence in Ext gives

$$\text{Ext}_{\mathfrak{g},K}^p (\bar{X}(\gamma^1), X(\gamma^2)) \cong \text{Ext}_{\mathfrak{g},K}^{p-1} (Y, X(\gamma^2)).$$

By Proposition 8.6.19, Y satisfies the hypothesis (9.2.23) for $m-1$; so by induction, the right side is zero. So

$$\text{Ext}_{\mathfrak{g},K}^* (\bar{X}(\gamma^1), X(\gamma^2)) = 0.$$

Filtering X , and using the long exact sequence in Ext repeatedly, now proves the result for m . Q.E.D.

Corollary 9.2.24 (Casselman) Suppose B_1 and B_2 are distinct blocks of $(\mathfrak{g}, K)$ modules of nonsingular infinitesimal character. Let X_1, X_2 be $(\mathfrak{g}, K)$ modules of finite length, such that every irreducible constituent of X_1 belongs to

B_1 . Then

$$\text{Ext}_{\mathfrak{g},K}^* (X_1, X_2) = 0.$$

This is deduced from Lemma 9.2.22 exactly as that was deduced from Lemma 9.2.18(a); we leave the details to the reader.

§3. Vanishing theorems for cohomology.

Proposition 9.3.1 In the situation (9.1.14), suppose $\bar{X}(\gamma^1)$ and $\bar{X}(\gamma)$ belong to distinct blocks (Definition 9.2.1). Then $\bar{X}_L(\gamma^1)$ does not occur in $H^p(\mathcal{U}, \bar{X}(\gamma))$, for any p .

Proof. Suppose that this is false. Define

$$(9.3.1) \quad \begin{aligned} (a) \quad \bar{R}^+(\bar{\gamma}^1) &= [R^+(\bar{\gamma}^1) \cap \Delta(\ell, \mathcal{H}^1)] \cup \\ &\cup [R(\bar{\gamma}^1) \cap \Delta(\mathcal{U}, \mathcal{H}^1)]. \end{aligned}$$

We are going to get a contradiction by induction on

$$(9.3.1) \quad (b) \quad m = |\bar{R}^+(\bar{\gamma}^1) - R^+(\bar{\gamma}^1)|$$

(notation (7.2.16)). Suppose first that $m = 0$. We will proceed by downward induction on p . By Lemma 9.2.3 and Corollary 8.1.3, we can find $H^2 \subseteq L$, and $\gamma^2 \in \{\hat{H}^2\}'$, so that

$$(9.3.2) \quad \begin{aligned} (a) \quad \bar{X}_L(\gamma^2) &\text{ is a quotient of } H^p(\mathcal{U}, \bar{X}(\gamma)) \\ (b) \quad \bar{X}_L(\gamma^2) &\text{ lies in the same block (for } L) \text{ as } \bar{X}_L(\gamma^1). \end{aligned}$$

By the definition of m , every simple root of $\mathcal{H}^1$ in

$$R^+(\bar{\gamma}^1) \cap \Delta(\ell, \mathcal{H}^1)$$

is simple in $R^+(\bar{\gamma}^1)$. By Theorem 9.2.11 and (9.3.2) (b), we conclude that $\bar{X}(\gamma^2)$ and $\bar{X}(\gamma^1)$ lie in the same block for G . So we may as well assume $\gamma^1 = \gamma^2$. So

$$(9.3.3) \quad \text{Hom}_{\ell, L \cap K} \left[H^p(\mathcal{U}, \bar{X}(\gamma)), X_L(\gamma^1) \right] \neq 0.$$

By Theorem 9.3.1 and Theorem 8.2.4,

$$\mathcal{R}_{\mathcal{G}}^i(X_L(\gamma^1)) = \begin{cases} 0 & i \neq s = \dim \mathcal{U} \cap \mathcal{H} \\ X(\gamma^1) & i = s \end{cases}$$

By Corollary 6.3.4, there is a spectral sequence (with $R =$

$$\dim \mathcal{U} \cap \mathcal{H}) \quad \text{Ext}_{\ell, L \cap K}^p \left[H^{R-q}(\mathcal{U}, \bar{X}(\gamma)), X_L(\gamma^1) \right] \implies \text{Ext}_{\mathcal{G}, K}^{p+q}(\bar{X}(\gamma), X(\gamma^1)).$$

By Lemma 9.2.22, the E_∞ term is zero; but by (9.3.3),

$$E_2^{0, R-p} \neq 0.$$

Since the differential on E_n has bidegree $(n, 1-n)$, we conclude that for some $n \geq 2$,

$$\text{Ext}_{\ell, L \cap K}^n (H^{p+(n-1)}(\mathcal{U}, \bar{X}(\gamma)), X_L(\gamma^1)) \neq 0.$$

By Lemma 9.2.22, we conclude that there is a constituent $\bar{X}_L(\gamma^2)$ of

$$H^{p+(n-1)}(\mathcal{U}, \bar{X}(\gamma))$$

belonging to the same block for L as $X_L(\gamma^1)$. Exactly as at the beginning of the proof, we conclude that $\bar{X}(\gamma^1)$ and $\bar{X}(\gamma^2)$ belong to the same block; so this is impossible by inductive hypothesis on p . This completes the case $m = 0$.

Now suppose $m > 0$ (cf. (9.3.1)). We may therefore choose a simple root $\alpha \in R^+(\bar{\gamma}^1)$ which is not in $\bar{R}^+(\bar{\gamma}^1)$.

By (9.3.1) (a),

$$(9.3.4) \quad -\alpha \in \Delta(\mathcal{U}, \mathcal{H}^1).$$

Define

$$\gamma = \begin{cases} \bar{X}(\gamma) & \text{if } \alpha \in \tau(\bar{X}(\gamma)) \\ Q_\alpha(\bar{X}(\gamma)) & \text{(notation (7.3.17)) if } \alpha \notin \tau(\bar{X}(\gamma)). \end{cases}$$

In both cases, Y is an indecomposable module containing $\bar{X}(\gamma)$ (see Corollary 7.3.24 for the second case). By Corollaries 9.1.17 and 9.1.18, and Lemma 9.1.16, $\bar{X}_L(\gamma_{\mathcal{G}}^1 - n\alpha)$ occurs in $H^*(\mathcal{U}, Y)$; and so also in $H^*(\mathcal{U}, Y_1)$, for some irreducible Y_1 in the same block as $\bar{X}(\gamma)$. Clearly,

$$\begin{aligned} R^+(\bar{\gamma}^1 - n\alpha) &= s_\alpha R^+(\bar{\gamma}^1) \\ &= (R^+(\bar{\gamma}^1) - \{\alpha\}) \cup \{-\alpha\}. \end{aligned}$$

By (9.3.4) and (9.3.1),

$$\tilde{R}^+(\bar{\gamma}^1) = \tilde{R}^+(\bar{\gamma}^1 - n\alpha)$$

$$|\tilde{R}^+(\bar{\gamma}^1 - n\alpha) - R^+(\gamma^1 - n\alpha)| = m - 1.$$

So we get a contradiction by inductive hypothesis. Q.E.D.

Corollary 9.3.5 In the situation (9.1.14), suppose $\bar{X}_L(\gamma_{\mathcal{G}}^1)$ occurs in $H^*(\mathcal{U}, X(\bar{\gamma}))$. Then (notation (9.2.17))

$$c(\gamma^1) = c(\gamma).$$

This follows from Propositions 9.2.12 and 9.3.1, and Definition 9.2.17.

Corollary 9.3.6 ([46], Theorem 6.13). Suppose $\gamma \in \hat{H}'$ is minimal (Definition 8.6.5). Then all the cohomology of $\bar{X}(\gamma)$ with respect to θ -stable parabolics is described by Theorem 8.1.6. More precisely, in the situation (9.1.14), assume that $\bar{X}_L(\gamma_{\mathcal{G}}^1)$ occurs in $H^*(\mathcal{U}, \bar{X}(\gamma))$. Then (H^1, γ^1) is conjugate to (H, γ) ; and $\bar{X}_L(\gamma_{\mathcal{G}}^1)$ occurs exactly once, in degree $\ell_{\mathcal{G}}(\gamma)$ (notation (8.1.5)).

Proof. By Lemma 8.6.9, we may assume after replacing $\mathcal{h}$ by a conjugate that

$$\begin{aligned} A^1 &\subseteq A \\ T_0^1 &\supseteq T_0 \end{aligned}$$

there is an inner automorphism σ of $\mathcal{G}$, such that

$$\begin{aligned} (9.3.7) \quad \sigma(\mathcal{h}) &= \mathcal{h}^1 \\ \sigma^{-1}|_{\mathcal{A}^1} &= \text{identity} \\ t_{\sigma^{-1}} &= \bar{\gamma}. \end{aligned}$$

By the definition of minimal, there are no real roots satisfying the parity condition for γ . So in the setting of Proposition 9.2.12, all the groups i_G are compact and therefore have real rank zero. So if we set

$$\mathcal{h}^\gamma = \text{span of } R(\gamma) \text{ in } \mathcal{h}^*,$$

then

$$\begin{aligned} c(\gamma) &= \dim(\mathcal{h}^\gamma \cap \mathcal{t}^*) \\ &= \dim(\mathcal{h}^\gamma / (\mathcal{h}^\gamma \cap \sigma\mathcal{t}^*)) \end{aligned}$$

(see (9.2.12), (9.2.17)). On the other hand,

$$\begin{aligned} c(\gamma^1) &\geq \dim(\mathcal{h}^{\gamma^1} \cap (\mathcal{t}^1)^*) \\ &= \dim(\mathcal{h}^{\gamma^1} / \mathcal{h}^{\gamma^1} \cap (\sigma\mathcal{t}^1)^*) \\ &= \dim\left((t_{\sigma^{-1}}^{-1}\mathcal{h}^\gamma) / (t_{\sigma^{-1}}^{-1}\mathcal{h}^\gamma \cap (\sigma\mathcal{t}^1)^*)\right) \\ &= \dim(\mathcal{h}^\gamma / \mathcal{h}^\gamma \cap t_\sigma(\sigma\mathcal{t}^1)^*). \end{aligned}$$

Since t_σ is the identity on $(\sigma\mathcal{t}^1)^*$ (cf. (9.3.7)), this is

$$\begin{aligned} &= \dim(\mathcal{h}^\gamma / \mathcal{h}^\gamma \cap (\sigma\mathcal{t}^1)^*) \\ &\geq \dim(\mathcal{h}^\gamma / \mathcal{h}^\gamma \cap \sigma\mathcal{t}^*) \\ &= c(\gamma). \end{aligned}$$

By Lemma 8.6.14, the last inequality is strict unless $H = H^1$.

So

$$c(\gamma) \leq c(\gamma^1)$$

with equality only if $H = H^1$. By Corollary 9.3.5, $H = H^1$.

Now σ can be regarded as a Weyl group element; and (9.3.7) guarantees that it commutes with θ . So γ^1 satisfies the hypotheses of Theorem 8.1.6, and the corollary follows. Q.E.D.

Our next goal is a sort of local version of Proposition

9.3.1: we will put more precise conditions on γ , and deduce more information about γ^1 . In the geometric setting of Chapter 2, this can be regarded as an analogue of the fact that certain Schubert varieties for a big group are isomorphic to Schubert varieties for smaller groups. Of course, this is explained in terms of parabolic subgroups.

Situation 9.3.8 Suppose H is a θ -stable Cartan

subgroup of G , and $\gamma \in (\hat{H})^\bullet$ is a regular character, with $\bar{\gamma}$ nonsingular. Define

a) $\mathcal{L}^{\mathbb{R}} = \text{centralizer of } \mathcal{L} \text{ in } \mathfrak{g} (\mathcal{L} = \mathcal{L} + \sigma\mathcal{L})$.

[We use the $\mathbb{R}$ since the roots of $\mathcal{L}$ in $\mathcal{L}^{\mathbb{R}}$ are the real roots of $\mathcal{L}$ in $\mathfrak{g}$.]

Assume that

b) $R^+(\bar{\gamma}) - (R^+(\bar{\gamma}) \cap \Delta(\mathcal{L}^{\mathbb{R}}, \mathcal{L}))$ is θ -stable.

Define

c) $\mathbb{R}^{\mathbb{R}} = \mathbb{R}^{\mathbb{R}}(\gamma) = R(\bar{\gamma}) \cap \Delta(\mathcal{L}^{\mathbb{R}}, \mathcal{L})$ (cf. (8.4.8))

d) $\mathbb{W}^{\mathbb{R}} = \mathbb{W}(\mathbb{R}^{\mathbb{R}})$

e) $\mathbb{H}^{\mathbb{R}} = \mathbb{H}^{\mathbb{R}}(\gamma) = \text{simple roots of } (\mathbb{R}^{\mathbb{R}})^+$.

Recall from Definition 8.3.1 the cross action of

$W(\bar{\gamma})$ on γ . Set

- f) $\mathcal{G}(\gamma) = \mathbb{W}^{\mathbb{R}} \times \gamma = \{w \times \gamma | w \in \mathbb{W}^{\mathbb{R}}\}$
 g) $\mathcal{H}(\gamma) = \{\gamma^1 \in (\hat{H}^1)^\bullet | H^1 \subseteq L^{\mathbb{R}}, \text{ and } \bar{\gamma}^1 \text{ differs from } \bar{\gamma} \text{ by an inner automorphism of } \mathcal{L}^{\mathbb{R}}\}$
 h) $\mathcal{K}(\gamma) = \{\gamma^1 \in \mathcal{H}(\gamma) | \bar{X}(\gamma^1) \text{ occurs in } X(w \times \gamma), \text{ some } w \in \mathbb{W}^{\mathbb{R}}\}$

We will sometimes use a fixed choice of a θ -stable parabolic subalgebra $\mathfrak{g}^{\mathbb{R}} = \mathcal{L}^{\mathbb{R}} + \mathcal{U}^{\mathbb{R}}$, satisfying

i) $R^+(\gamma) - \mathbb{R}^{\mathbb{R}} \subseteq \Delta(\mathcal{U}^{\mathbb{R}}, \mathcal{L})$.

The set $\mathcal{K}(\gamma)$ will be a sort of local analogue of the block containing $\bar{X}(\gamma)$. To prove the result which makes this precise, we need a simple structural fact.

Lemma 9.3.9 In the setting (9.3.8), suppose

$\gamma^1 \in \mathcal{H}(\gamma)$.

a) The simple roots of $\mathcal{L}^1$ in $R^+(\bar{\gamma}^1) \cap \Delta(\mathcal{L}^{\mathbb{R}}, \mathcal{L}^1)$ are simple in $R^+(\bar{\gamma}^1)$.

b) The set

$$R^+(\bar{\gamma}^1) - (R^+(\bar{\gamma}^1) \cap \Delta(\mathcal{L}^{\mathbb{R}}, \mathcal{L}^1))$$

is θ -stable.

Proof. By the definition (9.3.8)(g), it is clearly enough to prove (a) with γ^1 replaced by γ . In that case it is Lemma 8.6.2. For (b), notice that an inner automorphism of $\mathcal{L}^{\mathbb{R}}$ extends naturally to $\mathfrak{g}$, and the extension normalizes $\mathcal{U}^{\mathbb{R}}$ (cf. (9.3.8)(j)). So

$$R^+(\bar{\gamma}^1) - (R^+(\bar{\gamma}^1) \cap \Delta(\mathcal{L}^{\mathbb{R}}, \mathcal{L}^1)) = R(\bar{\gamma}^1) \cap \Delta(\mathcal{U}^{\mathbb{R}}, \mathcal{L}^1),$$

which is obviously θ -stable.

Q.E.D.

Lemma 9.3.10 Suppose we are in the setting (9.3.8).

a) If $\gamma^1 \in \mathcal{H}(\gamma)$ and γ is an irreducible constituent of $X(\gamma^1)$, then there is a $\gamma^2 \in \mathcal{H}(\gamma)$ such that

$$\gamma \cong \bar{X}(\gamma^2).$$

b) Define an equivalence relation $\tilde{\sim}$ on $\mathbb{L}^{\mathbb{R}} \cap K$ -conjugacy classes in $\mathcal{H}(\gamma)$ as in Theorem 9.2.11, except that only roots of $\mathcal{H}^1$ in $\mathbb{L}^{\mathbb{R}}$ are allowed as possible α . If $\gamma^1 \in \mathcal{H}(\gamma)$ ($i = 1, 2$) and $\bar{X}(\gamma^2)$ occurs in $X(\gamma^1)$, then

$$\gamma^1 \tilde{\sim} \gamma^2.$$

Proof. We will prove (a) and (b) simultaneously, by induction on $\ell^1(\gamma^1)$. The proof is exactly like that of Proposition 8.6.19 or Theorem 9.2.11; so we will only sketch it. Suppose first that there is a complex simple root $\alpha \in R^+(\bar{\gamma}^1)$, such that $\theta\alpha$ is negative. By Lemma (9.3.9)(b), α is a root of $\mathcal{H}^1$ in $\mathbb{L}^{\mathbb{R}}$; so we argue as in Case I of Theorem 9.2.11. Next, suppose that there is a simple real root $\alpha \in R^+(\bar{\gamma}^1)$ which satisfies the parity condition. Then $\theta\alpha = -\alpha$; so by Lemma 9.3.9(b), α is a root of $\mathcal{H}^1$ in $\mathbb{L}^{\mathbb{R}}$, and we can argue as in Cases II and III of Theorem 9.2.11. If neither of these possibilities occurs, then γ^1 is minimal, so γ^2 is conjugate to γ^1 (Theorem 8.6.6). Q.E.D.

Lemma 9.3.11 In the setting (9.3.8), suppose

$\gamma^1 \in \mathcal{H}(\gamma)$, $\alpha^1 \in R^+(\bar{\gamma}^1) \cap \Delta(\mathbb{L}^{\mathbb{R}}, \mathcal{H}^1)$ is simple and $\alpha^1 \notin \tau(\bar{X}(\gamma^1))$. Let γ be an irreducible constituent of $U_{\alpha^1}[\bar{X}(\gamma^1)]$ (notation (7.3.17)).

Then there is a $\gamma^2 \in \mathcal{H}(\gamma)$ such that

$$\gamma \cong \bar{X}(\gamma^2).$$

Proof. Suppose $\bar{X}(\gamma^1)$ occurs in $X(w \times \gamma)$, with $w \in W^{\mathbb{R}}$. We may as well assume γ does not occur in $X(w \times \gamma)$, or we are done by Lemma 9.3.10(a). Let $\alpha \in R^+(w \cdot \bar{\gamma})$ be the root corresponding to α . By definition of $\mathcal{H}(\gamma)$, α is a root of $\mathcal{H}$ in $\mathbb{L}^{\mathbb{R}}$, so α is real. By Proposition 8.3.19,

$$s_{\alpha} \cdot \theta(w \times \gamma) = \theta(s_{\alpha} w \times \gamma).$$

We use Corollary 7.3.20 to compute the multiplicity of γ in $s_{\alpha} \cdot \theta(w \times \gamma)$. Since we assume that X does not occur in $\theta(w \times \gamma)$, but it does occur in $U_{\alpha}[\bar{X}(\gamma^1)]$, this multiplicity is positive. So the result follows from Lemma 9.3.10(a). Q.E.D.

Lemma 9.3.12 In the setting (9.3.8), suppose

$\gamma^1 \in \mathcal{H}(\gamma)$ and γ is an irreducible constituent of $X(\gamma^1)$. Then there is a $\gamma^2 \in \mathcal{H}(\gamma)$ such that

$$\gamma \cong \bar{X}(\gamma^2).$$

Proof. We proceed by downward induction on $\ell^1(\gamma^1)$. By change of notation (replacing γ by some $w \times \gamma$), we may as well assume $\bar{X}(\gamma^1)$ occurs in $X(\gamma)$. Suppose first that $\mathcal{H}^1$ is split in $\mathbb{L}^{\mathbb{R}}$. Then $\mathcal{H}^1$ is conjugate to $\mathcal{H}$ under $\mathbb{L}^{\mathbb{R}} \cap K$, so we may as well assume that $\mathcal{H}^1 = \mathcal{H}$. By definition of $\mathcal{H}(\gamma)$, there is a $w \in W(\mathbb{L}^{\mathbb{R}}, \mathcal{H})$ such that

$$w\bar{\gamma} = \bar{\gamma}^1.$$

Since w is a product of real roots,

$$|\bar{\gamma}| \chi = |\bar{\gamma}^1| \chi.$$

Since $\bar{X}(\gamma^1)$ occurs in $X(\gamma)$, Lemma 6.6.6 implies that γ is conjugate to γ^1 ; so $X(\gamma) = X(\gamma^1)$, and we are done by Lemma 9.3.10(a).

So we may assume H^1 is not split in L^R . Then there is a noncompact imaginary root in $R^+(\bar{\gamma}^1) \cap \Delta(\ell^R, \ell^1)$, which we may take to be simple among the imaginary roots. Apply Lemmas 8.6.1 and 8.6.2 to $-\theta$. We conclude that there is a simple root α in $R^+(\bar{\gamma}^1) \cap \Delta(\ell^R, \ell^1)$, such that either

- 1) α is noncompact imaginary, or
- 2) α is complex, and $\theta\alpha$ is positive.

By Lemma 9.3.9(a), α is simple in $R^+(\bar{\gamma}^1)$. In case (1), choose

$$\gamma^2 \in c^{\alpha}(\gamma^1)$$

(notation (8.3.6)), and in case (2), set

$$\gamma^2 = \gamma^1 - n\alpha \quad (n = \langle \alpha, \bar{\gamma}^1 \rangle).$$

By Lemma 8.6.13,

$$\ell^I(\gamma^2) = \ell^I(\gamma^1) + 1.$$

By Theorem 8.5.18, $\bar{X}(\gamma^2)$ occurs in $U_{\alpha}(\bar{X}(\gamma^1))$; so by Lemma 9.3.11, $\gamma^2 \in \mathcal{F}(\gamma)$. We will apply the inductive hypothesis to γ^2 . In case (1), Corollary 8.4.7 (c) - (d) guarantees that every irreducible constituent of $X(\gamma^1)$ occurs in $X(\gamma^2)$; so we are done. In case (2), Proposition 8.2.7 gives

$$\Theta(\gamma^1) = s_{\alpha} \cdot \Theta(\gamma^2).$$

By Corollary 7.3.20, either γ occurs in $\Theta(\gamma^2)$ (and we are done by induction), or there is an irreducible constituent

z of $X(\gamma^2)$ such that γ occurs in $U_{\alpha}(z)$. In the latter case, we are done by Lemma 9.3.11. Q.E.D.

Proposition 9.3.13 In the setting (9.3.8), $\mathcal{F}(\gamma)$ is the equivalence class of γ in $\mathcal{H}(\gamma)$ under the equivalence relation $\tilde{c}$ of Lemma 9.3.10(b).

Proof. We first show that $\mathcal{L}(\gamma)$ is contained in the equivalence class of γ . Clearly it suffices to show that if $\alpha \in \Pi^R$ (cf. (9.3.8)(e)), then

$$s_{\alpha} \times \gamma \tilde{c} \gamma.$$

By Proposition 8.3.18, $s_{\alpha} \times \gamma$ is conjugate to γ unless α is type II and satisfies the parity condition. In that case, (8.3.18)(e) (and case (a) of the definition in Theorem 9.2.11) give

$$s_{\alpha} \times \gamma \tilde{c} \gamma_{\alpha} \tilde{c} \gamma.$$

This proves the claim. Now Lemma 9.3.10 guarantees that $\mathcal{F}(\gamma)$ is contained in the equivalence class of γ . So suppose $\gamma^1 \in \mathcal{F}(\gamma)$, ($i = 1, 2$), and $\gamma^1 \tilde{c} \gamma^2$; we want to show that $\gamma^2 \in \mathcal{F}(\gamma)$. We may as well assume that γ^1 and γ^2 are related as in Theorem 9.2.11, via a simple root α in $R^+(\bar{\gamma}^1) \cap \Delta(\ell^R, \ell^1)$. If $\theta\alpha$ is positive, then Theorem 8.5.18 says that $\bar{X}(\gamma^2)$ occurs in $U_{\alpha}(\bar{X}(\gamma^1))$, and Lemma 9.3.11 says that $\gamma^2 \in \mathcal{F}(\gamma)$. If $\theta\alpha$ is negative, then Corollaries 8.2.9 and 8.4.5 say that $\bar{X}(\gamma^2)$ occurs in $X(\gamma^1)$, so $\gamma^2 \in \mathcal{F}(\gamma)$ by Lemma 9.3.12. Q.E.D.

The value of this proposition is that it describes $\mathcal{F}(\gamma)$ simply in terms of the structure of the group L , rather than

in terms of the representation theory of G . The following result illustrates how this will be used.

Corollary 9.3.14 In the setting (9.3.8), define

$1^{\mathbb{R}} =$ product of the simple factors of $\mathbb{R}^{\mathbb{R}}$
which contain a root satisfying the
parity condition.

Suppose $\gamma^1 \in \mathcal{F}(\gamma)$, and α^1 is an integral root of $\mathcal{H}^1$ in $\ell^{\mathbb{R}}$. Let α be the root of $\mathcal{H}$ in $\ell^{\mathbb{R}}$ corresponding to α . Then either

a) α^1 is real and does not satisfy the parity condition, or

b) $\alpha \in 1^{\mathbb{R}}$.

Proof. Choose $\mathcal{Q}_{\mathcal{F}}^{\mathbb{R}} = \ell^{\mathbb{R}} + \mathcal{U}^{\mathbb{R}}$ as in (9.3.8)(i). By Proposition 9.3.13,

$$\mathcal{F}(\gamma) = \{\gamma^1 \in \mathcal{H}(\gamma) \mid \gamma_{\mathcal{Q}_{\mathcal{F}}^{\mathbb{R}}}^1 \in \mathcal{F}(\gamma_{\mathcal{Q}_{\mathcal{F}}^{\mathbb{R}}})\}.$$

So (using Lemma 8.3.12(a)), we may as well assume that $\mathcal{Q}_{\mathcal{F}} = \ell^{\mathbb{R}}$. We now apply Proposition 9.2.12. Let $i_{\mathbb{R}}^{\mathbb{R}}$ be the simple factor of $\mathbb{R}^{\mathbb{R}}$ containing α , and $i_{\mathbb{R}(\bar{\gamma})}^1$ the corresponding simple factor of $\mathbb{R}(\bar{\gamma})^1$. Suppose (b) fails. Then $i_{\mathbb{R}}^{\mathbb{R}}$ consists of real roots not satisfying the parity condition, so $i_G^{\mathbb{R}}$ is compact, and

$$i_{\mathbb{C}(\gamma)}^{\mathbb{R}} = 0$$

(cf. (9.2.12)). By Proposition 9.2.12, $i_{\mathbb{C}(\gamma^1)}^1$ is also zero; and therefore $i_{\mathbb{R}(\bar{\gamma})}^1$ consists of real roots not satisfying the parity condition. So (a) holds. Q.E.D.

Lemma 9.3.15 In the setting (9.3.8), suppose $\mathcal{Q}_{\mathcal{F}} = \ell + \mathcal{U}$ is a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathcal{F}}$, with $\ell \subseteq \ell^{\mathbb{R}}$. Fix θ -stable Cartan subgroups

$$\mathfrak{h}^i \subseteq \mathbb{L} \quad (i = 1, 2),$$

and regular characters

$$\gamma^i \in (\hat{H}^i)',$$

Assume that $\gamma^1 \in \mathcal{F}(\gamma)$, and $\bar{X}_{\mathbb{L}}(\gamma^2)_{\mathcal{Q}_{\mathcal{F}}}$ occurs in $X_{\mathbb{L}}(\gamma^1)_{\mathcal{Q}_{\mathcal{F}}}$. Then $\gamma^2 \in \mathcal{F}(\gamma)$.

Proof. We proceed by induction on $\ell_{\mathbb{L}}^1(\gamma_{\mathcal{Q}_{\mathcal{F}}})$; and then by induction on $\ell^1(\gamma^1)$. By consideration of infinitesimal characters, we see that $\bar{\gamma}^1$ and $\bar{\gamma}^2$ differ by an inner automorphism of ℓ .

Case I. There is a complex simple root $\alpha \in \mathbb{R}^+(\bar{\gamma}^1)$, such that $\theta\alpha \notin \mathbb{R}^+(\bar{\gamma}^1)$. By Lemma 9.3.9(b), α is a root of $\mathcal{H}^1$ in $\ell^{\mathbb{R}}$. Set

$$n = \langle \bar{\gamma}, \bar{\gamma}^1 \rangle$$

$\alpha^2 =$ root of $\mathcal{H}^2$ in $\ell^{\mathbb{R}}$ corresponding to α .

By definition (see Lemma 9.3.10 and Theorem 9.2.11),

$$\gamma^1 - n\alpha \in \gamma^1,$$

so $\gamma^1 - n\alpha$ lies in $\mathcal{F}(\gamma)$ by Proposition 9.3.13. Suppose first that α is not a root in ℓ . By the translation principle for $\mathbb{L}$ (Proposition 7.3.3 and Theorem 8.2.1),

$$\bar{X}_{\mathbb{L}}(\gamma_{\mathcal{Q}_{\mathcal{F}}}^2 - n\alpha^2) \text{ occurs in } X_{\mathbb{L}}(\gamma_{\mathcal{Q}_{\mathcal{F}}}^1 - n\alpha).$$

It is easy to check that

$$\ell_L^I(\gamma^1 - \alpha) = \ell_L^I(\gamma^1);$$

and by Lemma 8.6.13,

$$\ell_L^I(\gamma^1 - \alpha) = \ell_L^I(\gamma^1) - 1.$$

So we can apply the inductive hypothesis. Since

$$\gamma^2 - \alpha^2 = (s_{\alpha^2} \times \gamma^2)\gamma$$

(Definition 8.3.1 -- it is simplest to consider the cases when α^2 is imaginary or not separately to verify this), we conclude that

$$s_{\alpha^2} \times \gamma^2 \in \mathcal{F}(\gamma).$$

By Proposition 8.3.18 and the definition in Lemma 9.3.10,

either $s_{\alpha^2} \times \gamma^2$ is conjugate to γ^2 , or $s_{\alpha^2} \times \gamma^2 \sim \gamma^2$. So by Proposition 9.3.13, γ^2 lies in $\mathcal{F}(\gamma)$.

So we may assume that α is a root of $\mathcal{H}^1$ in $\mathcal{L}$. By Lemma 8.6.13,

$$\ell_L^I(\gamma^1 - \alpha) = \ell_L^I(\gamma^1) - 1,$$

so we can apply the inductive hypothesis to $\gamma^1 - \alpha$. By Proposition 8.2.7,

$$s_{\alpha} \cdot \ell_L^I(\gamma^1 - \alpha) = \ell_L^I(\gamma^1).$$

By Corollary 7.3.20, either $\bar{X}_L(\gamma^2)$ occurs in $X_L(\gamma^1 - \alpha)$ (in which case we have done by induction); or there is a constituent $\bar{X}_L(\gamma^3)$ of $X_L(\gamma^1 - \alpha)$ such that $\bar{X}_L(\gamma^2)$ occurs in $U_{\alpha}^L(\bar{X}_L(\gamma^3))$. By induction, $\gamma^3 \in \mathcal{F}(\gamma)$. By Proposition 8.6.19,

$$\ell_L^I(\gamma^3) \leq \ell_L^I(\gamma^1 - \alpha) = \ell_L^I(\gamma^1) - 1.$$

So the inductive hypothesis applies to γ^3 . So if $\bar{X}_L(\gamma^2)$ occurs in $X_L(\gamma^3)$, we are done by induction. Otherwise, Theorem 8.5.18 implies that $\bar{X}_L(\gamma^2)$ is one of the special constituents of $U_{\alpha}(\bar{X}_L(\gamma^3))$; so by definition

$$\gamma^3 \sim \gamma^2.$$

Since γ^3 belongs to $\mathcal{F}(\gamma)$, γ^2 does as well by Proposition 9.3.13.

Case II. There is a real simple root $\alpha \in R^+(\bar{\gamma}^1)$ satisfying the parity condition. Essentially the same argument applies, with $c_{\alpha}(\gamma^1)$ (Definition 8.3.16) playing the role of $\gamma^1 - \alpha$. (In fact α is necessarily a root in $\mathcal{L}$, so the argument is even a little shorter.) We leave the details to the reader.

Case III. Otherwise. If there are no roots as in Cases I and II, then γ^1 is minimal (Definition 8.6.5); and it follows that γ^1 is minimal for $\mathcal{L}$. So γ^1 is conjugate to γ^2 by Theorem 8.6.6 (applied to $\mathcal{L}$). Q.E.D.

Here is the second cohomology vanishing theorem.

Proposition 9.3.16 In the setting (9.3.8), let

$$\gamma = \ell + \mathbf{u} \text{ be a } \theta\text{-stable parabolic subalgebra, with } \ell \subseteq \mathcal{L}^R. \text{ Fix } \gamma^1 \in \mathcal{F}(\gamma) \text{ and } \gamma^2 \in (\hat{H}^2), \text{ with } H^2 \subseteq L. \text{ Assume that}$$

$$\bar{X}_L(\gamma^2) \text{ occurs in } H^*(\mathbf{u}, \bar{X}(\gamma^1)).$$

Then γ^2 is conjugate under K to be an element of

$$\mathcal{F}(\gamma).$$

Proof. We proceed by induction on $\ell(\gamma^1)$. Suppose first that there is a simple type II real root $\alpha \in R^+(\gamma^1)$ satisfying the parity condition. By Lemma 9.3.9(a), α is a root of $\mathcal{H}^1$ in $\mathbb{R}$. Define γ_α^1 by (8.3.16); then by Lemma 8.6.13,

$$\ell^I(\gamma_\alpha^1) = \ell^I(\gamma^1) - 1;$$

By Proposition 9.3.13, γ_α^1 belongs to $\mathcal{F}(\gamma)$; so we can apply the inductive hypothesis to γ_α^1 . Put

$$X = \bar{X}(\gamma^1) \oplus \bar{X}(s_\alpha \times \gamma^1);$$

then it is enough to prove the proposition with $\bar{X}(\gamma^1)$ replaced by X . By Corollary 8.4.7(b), X is a subquotient of

$$Z = \phi_\alpha \psi_\alpha(\bar{X}(\gamma_\alpha^1));$$

and by Theorem 8.5.18, every other constituent of Z occurs in $X(\gamma_\alpha^1)$ [and so is subject to the inductive hypothesis]. So there are short exact sequences of the form

$$0 \rightarrow Y_1 \rightarrow Z \rightarrow \tilde{Y} \rightarrow 0$$

$$0 \rightarrow Y \rightarrow Y_1 \rightarrow X \rightarrow 0.$$

If $\bar{X}_L(\gamma_g^2)$ occurs in the cohomology of Y or $\tilde{Y}$, we are done by induction; so assume that it does not. By the long exact sequences in cohomology, we deduce that $\bar{X}_L(\gamma_g^2)$ occurs in the cohomology of Z . This we study using Theorem 9.1.12. Suppose we are in cases (a) or (c) of that theorem. By the long exact sequences there, and Lemma 9.1.16, we conclude that either $\bar{X}_L(\gamma_g^2)$ occurs in $H^*(\mathcal{U}, \bar{X}(\gamma_\alpha^1))$ [and we are done by induction]; or $\bar{X}_L(\gamma_g^2 - n\alpha^2)$ occurs in $H^*(\mathcal{U}, \bar{X}(\gamma_\alpha^1))$. Arguing exactly as in the first part of case I of Lemma 9.3.15, we

conclude that γ^2 is conjugate to an element of $\mathcal{F}(\gamma)$. In case (b) of Theorem 9.1.12, there is an irreducible constituent $\bar{X}_L(\gamma_g^3)$ of $H^*(\mathcal{U}, \bar{X}(\gamma_\alpha^1))$, such that $\bar{X}_L(\gamma_g^2)$ occurs in $\phi_\alpha \psi_\alpha \bar{X}_L(\gamma_g^3)$.

By induction, we may as well assume that $\gamma^3 \in \mathcal{F}(\gamma)$. By Theorem 8.5.18, either $\bar{X}_L(\gamma_g^2)$ occurs in $X_L(\gamma_g^3)$ [and we are done by Lemma 9.3.15]; or $\gamma^2 \tilde{c} \gamma^3$ [and we are done by Proposition 9.3.13].

The other cases when γ^1 is not minimal are entirely similar; we singled out this one because of the small subtlety involved in defining X . So we may assume that γ^1 is minimal. By Corollary 9.3.6, γ^2 is conjugate to γ^1 . Q.E.D.

Our last cohomology vanishing theorem (Proposition 9.3.23) is needed at the very end of the algorithm for computing cohomology.

Situation 9.3.17 In the setting 9.3.8, fix a

θ -stable Cartan subgroup H^0 of $\mathbb{R}$. Let ℓ be the centralizer of $\mathcal{X}^0$ in $\mathbb{R}$ (or, equivalently, in $\mathcal{G}$). Fix a regular character $\gamma^0 \in (\hat{H}^0)'$ in $\mathcal{F}(\gamma)$; and assume that

- a) the set of non-imaginary positive integral roots of $\mathcal{H}^0$ in $\mathbb{R}$ (with respect to $\bar{\gamma}^0$) is stable under $-\theta$. Define
- b) $M A^{0,0} =$ Langlands decomposition of centralizer of A^0 in G ($H^0 = T A^0$).
- c) $R^I = R^I(\gamma^0) \cap R(\bar{\gamma}^0) \cap \Delta(m^0 + \sigma^0, h^0) = \Delta(m^0 + \sigma^0, h^0)$

- d) $w^I = w(R^I) = w(m^0, t^0)$
 - e) $\pi^I = \pi^I(\gamma^0) = \text{simple roots of } (R^I)^+$
- Recall from Definition 8.3.1 the cross action of $w(\gamma^0)$ on γ^0 . Set

- f) $\mathcal{E}(\gamma, \gamma^0) = w^I \times \gamma^0$
- g) $\mathcal{H}(\gamma, \gamma^0) = \{\gamma^1 \in (\hat{H}^1)^\vee \mid H^1 \subseteq L^{\mathbb{R}} \cap M^0 A^0, \text{ and } \bar{\gamma}^1 \text{ differs from } \bar{\gamma} \text{ by an inner automorphism of } \mathcal{L}^{\mathbb{R}} \cap (m^0 + \sigma^0)\}.$

Define an equivalence relation $\tilde{\Delta}$ on $\mathcal{H}(\gamma, \gamma^0)$ as in Theorem 9.2.11, except that only simple roots of $\mathcal{L}^1$ in $\mathcal{L}^{\mathbb{R}} \cap (m^0 + \sigma^0)$ are allowed. Put

- h) $\mathcal{F}(\gamma, \gamma^0) = \text{equivalence class of } \gamma^0 \text{ in } \mathcal{H}(\gamma, \gamma^0) \text{ under } \tilde{\Delta}$
- i) $\mathcal{J}(\gamma, \gamma^0) = \mathcal{F}(\gamma, \gamma^0) \cap \mathcal{S}(\gamma).$

For a more perfect analogy with (9.3.8), we should pick a θ -stable parabolic subalgebra $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ (with $\mathfrak{l}$ as above), and define

$$\mathcal{F}(\gamma, \gamma^0) = \{\gamma^1 \in \mathcal{H}(\gamma) \mid \text{for some } w \in W^I, \bar{X}_L\{(w \times \gamma^0)\mathfrak{g}\} \text{ occurs in } H^*(\mathfrak{u}, \bar{X}(\gamma^1))\}.$$

One can show that this is equivalent to the definition given above; but for our purposes, (9.3.17) (h) is a little more convenient.

- Lemma 9.3.18 In the setting (9.3.17), suppose $\gamma^1 \in \mathcal{H}(\gamma, \gamma^0)$
- a) The simple roots of $\mathcal{L}^1$ in $\mathcal{L}^{\mathbb{R}} \cap (m^0 + \sigma^0)$ are simple in $R^+(\bar{\gamma}^1)$.
 - b) The set $R^+(\bar{\gamma}^1) \cap \Delta(\mathcal{L}^{\mathbb{R}}, \mathcal{L}^1) - \Delta(\mathcal{L}^{\mathbb{R}} \cap (m^0 + \sigma^0), \mathcal{L}^1)$ is stable under $-\theta$.

The proof is formally identical to that of Lemma 9.3.9, and we leave it to the reader.

Lemma 9.3.19 In the setting (9.3.17), suppose $\gamma^1 \in \mathcal{H}(\gamma), \gamma^2 \in \mathcal{H}(\gamma, \gamma^0)$ and $\bar{X}(\gamma^2)$ occurs in $X(\gamma^1)$. Then [up to conjugacy by $L^{\mathbb{R}} \cap K$], $\gamma^1 \in \mathcal{H}(\gamma, \gamma^0)$.

Proof. We imitate the proof of Lemma 8.6.9. Let

$$\psi: (\mathcal{L}^2)^* \rightarrow (\mathcal{L}^1)^*$$

be the $\mathcal{L}^{\mathbb{R}}$ -inner automorphism taking γ^2 to γ^1 . Let $\lambda^2 \in \Lambda^2$ be a weight of a finite dimensional representation of G , such that

$$\bar{\gamma}^2 + \bar{\lambda}^2 \text{ is dominant and nonsingular for } R^+(\bar{\gamma}^2).$$

By the translation principle, $\bar{X}(\gamma^2 + \lambda^2)$ occurs in $X(\gamma^1 + \lambda^1)$, for some $\lambda^1 \in \Lambda^1$ with

$$\bar{\lambda}^1 = \psi(\bar{\lambda}^2).$$

By Lemma 6.6.6,

$$|(\bar{\gamma}^2 + \bar{\lambda}^2)|_{\sigma^0}| \leq |(\bar{\gamma}^2 + \bar{\lambda}^2)|_{\sigma^2}| \leq |(\bar{\gamma}^1 + \bar{\lambda}^1)|_{\sigma^1}|;$$

that is,

$$|(\bar{\gamma}^2 + \bar{\lambda}^2)|_{\sigma^0}| \leq |\psi(\bar{\gamma}^2 + \bar{\lambda}^2)|_{\sigma^1}|.$$

A limit argument like the one in the proof of Lemma 8.6.9 now shows that for any $R^+(\bar{\gamma}^2)$ -dominant weight $\mu \in (\mathcal{L}^2)^*$

$$|\mu|_{\sigma^0}| \leq |\psi(\mu)|_{\sigma^1}|.$$

Choose $\sigma_{\mathcal{F}}^{\mathbb{R}}$ as in (9.3.8) (i), and set

$$\mu_0 = \rho(\mu^{\mathbb{R}}) \in (\mathcal{H}^2)^*$$

then μ_0 is $R^+(\bar{\gamma}^{-2})$ -dominant, singular exactly for the roots in $\mathcal{K}^{\mathbb{R}}$, and

$$|\mu_0|_{\sigma^0} = |\psi(\mu_0)|_{\sigma^1} = 0$$

since ψ is $\mathcal{K}^{\mathbb{R}}$ -inner. Let $\mu_1 \in (\sigma^0)^*$ be any weight which is dominant for

$$R^+(\bar{\gamma}^{-2}) \cap \Delta(\mathcal{K}^{\mathbb{R}});$$

by Lemma 9.3.18, there is an open cone of such μ_1 . Then for large N , $N\mu_0 + \mu_1$ is $R^+(\bar{\gamma}^{-2})$ -dominant; so

$$\begin{aligned} |\mu_1| &= |(N\mu_0 + \mu_1)|_{\sigma^0} \leq |\psi(N\mu_0 + \mu_1)|_{\sigma^1} \\ &= |\psi(\mu_1)|_{\sigma^1} \\ &\leq |\psi(\mu_1)|. \end{aligned}$$

So equality holds everywhere, and $\psi(\mu_1)$ lies in σ^1 . Since this holds for an open cone of values of μ_1 ,

$$\psi(\sigma^0) \subseteq \sigma^1.$$

By Lemma 8.6.7, we may replace H^1 by an $L^{\mathbb{R}} \cap K$ conjugate, and arrange

$$\sigma^0 \subseteq \sigma^1, \quad \psi|_{\sigma^0} = \text{identity}.$$

So H^1 centralizes A^0 , so $H^1 \subseteq L^{\mathbb{R}} \cap M_A^{0,0}$, and ψ acts trivially; on σ^0 , so ψ is inner for $\mathfrak{m}^0 + \sigma^0$. Q.E.D.

Lemma 9.3.20 In the setting (9.3.17), suppose γ^1 and γ^2 belong to $\mathcal{H}(\gamma, \gamma^0)$ and $\bar{X}(\gamma^2)$ occurs in $X(\gamma^1)$. Then $\gamma^1 \tilde{d} \gamma^2$.

Proof. This is essentially the same as Lemma 9.3.12. We proceed by downward induction on $\ell^I(\gamma^2)$. Suppose first that there is a simple root α in

$$R^+(\bar{\gamma}^{-2}) \cap \Delta(\mathcal{K}^{\mathbb{R}} \cap (\mathfrak{m}^0 + \sigma^0), \mathcal{H}^1)$$

such that α is complex and $\theta\alpha$ is positive, or α is noncompact imaginary. By Theorem 8.5.18 and Lemma 8.6.13, there is a $\gamma^3 \in \mathcal{H}(\gamma, \gamma^0)$, such that

$$\begin{aligned} \gamma^3 \tilde{d} \gamma^2 \\ \ell^I(\gamma^3) = \ell^I(\gamma^2) + 1 \end{aligned}$$

$\bar{X}(\gamma^3)$ occurs in $U_{\alpha}(\bar{X}(\gamma^2))$.

[If α is complex, γ^3 is $\gamma^2 - n\alpha$; if α is real, γ^3 is obtained from γ^2 by Cayley transform.] So it is enough to show that $\gamma^3 \tilde{d} \gamma^1$. If $\bar{X}(\gamma^3)$ occurs in $X(\gamma^1)$, we are done by induction; so suppose that it does not. By Corollary 7.3.20, $\bar{X}(\gamma^3)$ occurs in

$$s_{\alpha} \cdot \theta(\gamma^1)$$

with positive multiplicity. Let $\alpha^1 \in R^+(\bar{\gamma}^{-1})$ be the root corresponding to α . If α^1 is complex, then

$$s_{\alpha} \cdot \theta(\gamma^1) = \theta(\gamma^1 - n\alpha^1)$$

by Proposition 8.2.7. Furthermore, $\gamma^1 \tilde{d} \gamma^1 - n\alpha^1$; so we are done by induction. If α^1 is real,

$$s_{\alpha} \cdot \theta(\gamma^1) = \theta(s_{\alpha}^1 \times \gamma^1)$$

by Proposition 8.3.19; and

$$\gamma^1 \tilde{d} s_{\alpha}^1 \times \gamma^1$$

by Proposition 8.3.18. So we are done by induction. If α^1 is compact imaginary,

$$s_\alpha \cdot \theta(\gamma^1) = -\theta(\gamma^1)$$

by Proposition 8.4.3; and this is impossible, since $\bar{X}(\gamma^3)$ was to occur with positive multiplicity. Finally, if α^1 is noncompact imaginary, then by Corollary 8.4.6, there is a $\gamma^4 \in c^{\alpha^1}(\gamma^1)$ such that $\bar{X}(\gamma^3)$ occurs in $X(\gamma^4)$; and we can again apply the inductive hypothesis.

So we may suppose that no such root α exists. By Lemma 8.6.2 (applied to $-\theta$), the imaginary roots of $\mathcal{R}^1$ in $\mathcal{C}^{\mathbb{R}}$ are spanned by simple roots of $\mathcal{R}^1$ in $\mathbb{R}^+(\bar{\gamma}^1)$. Since there are no simple noncompact imaginary roots, $\mathcal{R}^1$ is split in $\mathcal{C}^{\mathbb{R}}$; so H^1 is conjugate to H . By the argument at the beginning of the proof of Lemma 9.3.12, it follows that γ^1 is conjugate to γ^2 . Q.E.D.

Proposition 9.3.21 In the setting (9.3.17),

let $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra. (Recall that $\mathfrak{l} \subseteq \mathcal{C}^{\mathbb{R}}$ is the centralizer

of $\mathcal{L}^1$). Suppose $\gamma^1 \in \mathcal{F}(\gamma)$, $\gamma^2 \in \mathcal{E}(\gamma, \gamma^0)$ and

$\bar{X}_L(\gamma^2_{\mathfrak{g}})$ occurs in $H^1(\mathfrak{u}, \bar{X}(\gamma^1))$

a) Up to conjugacy by $\mathbb{P}^{\mathbb{R}} \cap K$, $\gamma^1 \in \mathcal{H}(\gamma, \gamma^0)$.

b) If $\gamma^1 \in \mathcal{F}(\gamma, \gamma^0)$, then

$$|i - \ell_{\mathfrak{g}}(\gamma^2)| \leq \ell^I(\gamma^1) - \ell^I(\gamma^0).$$

Proof. Part (a) is proved in exactly the same way as

Lemma 9.3.19, using Lemma 8.1.7 in place of Lemma 6.6.6.

We leave the argument to the reader. The proof of (b) is very much like that of Proposition 9.3.16, so we will be

very sketchy. We proceed by induction on $\ell^I(\gamma^1)$. Notice first that all elements of $\mathcal{E}(\gamma, \gamma^0)$ have the same ℓ^I . Suppose there is a simple type II real root α of $\mathcal{R}^1$ in $\mathcal{C}^{\mathbb{R}} \cap (\mathfrak{m}^0 + \mathfrak{o}^0)$ satisfying the parity condition. Define γ^1_α and X as in the proof of (9.3.16); then $\gamma^1_\alpha \in \mathcal{F}(\gamma, \gamma^0)$,

$$\ell^I(\gamma^1_\alpha) = \ell^I(\gamma^1) - 1.$$

Let W be any irreducible constituent of $X(\gamma^1_\alpha)$, and γ^3 any element of $\mathcal{E}(\gamma, \gamma^0)$. Suppose $\bar{X}_L(\gamma^3_{\mathfrak{g}})$ occurs in

$$H^{i+\varepsilon}(\mathfrak{u}, W),$$

with $\varepsilon = -1, 0$, or 1 . By (a), W is of the form $\bar{X}(\gamma^4)$, for some $\gamma^4 \in \mathcal{H}(\gamma, \gamma^0)$. By Lemma 9.3.20, $\gamma^4 \in \mathcal{F}(\gamma, \gamma^0)$; so by induction

$$\begin{aligned} |i+\varepsilon - \ell_{\mathfrak{g}}(\gamma^3)| &\leq \ell^I(\gamma^4) - \ell^I(\gamma^0) \\ &\leq \ell^I(\gamma^1_\alpha) - \ell^I(\gamma^0) \\ &= \ell^I(\gamma^1) - \ell^I(\gamma^0) - 1 \end{aligned}$$

(Proposition 8.6.19 and Lemma 8.6.13). An inspection of the proof of (9.3.16) shows that it produces just such a W , γ^3 , and ε : if $\varepsilon = \pm 1$, $\gamma^3 = \gamma^2$, and if $\varepsilon = 0$, $\gamma^3 = s_{\alpha^2} \times \gamma^2$ or γ^2 . In the latter case,

$$\ell_{\mathfrak{g}}(\gamma^3) = \ell_{\mathfrak{g}}(\gamma^2) + \delta \quad (\delta = 0, 1, \text{ or } -1);$$

so in both cases,

$$|i \pm 1 - \ell_{\mathfrak{g}}(\gamma^2)| \leq \ell^I(\gamma^1) - \ell^I(\gamma^0) - 1.$$

This implies the inequality in (b). Exactly the same argument (with simplifications) applies whenever there is a simple root

α of $\mathcal{h}^1$ in $\mathcal{L}^{\mathbb{R}} \cap (\mathcal{m}^0 + \sigma\mathcal{L}^0)$, such that α is type I real and satisfies the parity condition, or α is complex and $\theta\alpha$ is negative. So we may suppose that there are no such roots. By Lemma 8.6.2, the real roots of $\mathcal{h}^1$ in $\mathcal{L}^{\mathbb{R}} \cap (\mathcal{m}^0 + \sigma\mathcal{L}^0)$ are spanned by simple roots not satisfying the parity condition. By assumption, γ^1 is equivalent to γ^0 under $\tilde{\alpha}$. So if we define $c'(\gamma^0)$ and $c'(\gamma^1)$ as in Definition 9.2.17, using only the roots in $\mathcal{L}^{\mathbb{R}} \cap (\mathcal{m}^0 + \sigma\mathcal{L}^0)$, then Proposition 9.2.12 [applied to $\mathcal{L}^{\mathbb{R}} \cap (\mathcal{M}_A^0)$] says that

$$c'(\gamma^0) = c'(\gamma^1).$$

Arguing as in the proof of Corollary 9.3.6, we conclude that H^1 is conjugate to H^0 and γ^0 is conjugate to γ^1 . By Theorem 8.1.6, (b) is true in that case. Q.E.D.

Lemma 9.3.22 Suppose we are in the setting (9.3.17)

and $\mathcal{O}\mathcal{F} = \mathcal{L} + \mathcal{U}$ is a θ -stable parabolic subalgebra.

a) There is an element $\gamma' \in \mathcal{L}(\gamma, \gamma^0)$ such that

every simple root of $\mathcal{h}$ in $\mathcal{L}^{\mathbb{R}} \cap (\mathcal{m}^0 + \sigma\mathcal{L}^0)$ satisfies the parity condition.

b) Let $\gamma^2 \in \mathcal{L}(\gamma, \gamma^0)$ be the unique element which is dominant for $R^{\mathbb{I}} \cap \Delta(\mathcal{U}, \mathcal{h}^0)$. Write ℓ for the length function on $W_{\mathbb{I}}$, and fix $w \in W_{\mathbb{I}}$. If γ' is as in (a), and $\bar{X}_{\mathbb{I}}[(w \times \gamma^2)_{\mathcal{O}\mathcal{F}}]$ occurs in $H^{\mathbb{I}}[\mathcal{U}, \bar{X}(\gamma')]$, then $\bar{X}_{\mathbb{I}}(\gamma^2)$ occurs in $H^{i-\ell(w)}(\mathcal{U}, \bar{X}(\gamma'))$.

c) Let $\gamma^3 \in \mathcal{L}(\gamma, \gamma^0)$ be the unique element such that $-\gamma^3$ is dominant for $R^{\mathbb{I}} \cap \Delta(\mathcal{U}, \mathcal{h}^0)$. If γ' is as in (a), and $\bar{X}_{\mathbb{I}}[(w \times \gamma^3)_{\mathcal{O}\mathcal{F}}]$ occurs in $H^{\mathbb{I}}[\mathcal{U}, \bar{X}(\gamma')]$, then $\bar{X}_{\mathbb{I}}(\gamma^3)$ occurs in $H^{i+\ell(w)}(\mathcal{U}, \bar{X}(\gamma'))$.

Proof. Part (a) is a purely structural statement, involving only $\mathcal{L}^{\mathbb{R}} \cap (\mathcal{m}^0 + \sigma\mathcal{L}^0)$. To prove it, we may therefore assume that G is simple, and

$$G = L^{\mathbb{R}} = M^0;$$

that is, that H is a split Cartan subgroup of G , and H^0 is compact. Since the elements of $\mathcal{L}(\gamma, \gamma^0)$ lie in the same block as γ^0 , they have compactness equal to the dimension of H^0 ; that is, the rank of G . Fix $\gamma^1 \in \mathcal{L}(\gamma, \gamma^0)$ and set

R_0 = roots not satisfying the parity condition for γ^1

R_1 = roots satisfying the parity condition

$$R = R_0 \cup R_1$$

Let $\check{G} \supseteq \check{K}$ be the group with a compact Cartan subgroup $\check{H}$ having root system $\check{K} \supseteq \check{K}_0$ (compare Proposition 9.2.12). By definition of $c(\gamma^1)$, the rank of $\check{G}/\check{K}$ is the dimension of $\check{H}$; that is, $\check{G}/\check{K}$ is split. If we replace γ^1 by $w \times \gamma^1$, for $w \in W(G, H)$, we do not leave $\mathcal{L}(\gamma, \gamma^0)$ or change R ; and R^+ is replaced by wR^+ . So the assertion of (a) can be formulated in terms of $\check{G}/\check{K}$ as follows: if $\check{G}$ is split and has compact Cartan subgroup $\check{H}$, then there is a positive root system for $\check{K}$ in $\check{\mathcal{O}}\mathcal{F}$ making all the simple roots noncompact. This is well known, and can be verified directly for each split simple group with a compact Cartan.

For (b), we proceed by induction on $\ell(w)$. If $w = 1$, the statement is trivial; so suppose w is not 1. Then we can find a simple root α of $\mathcal{h}^0$ in

$$w(R^{\mathbb{I}} \cap \Delta(\mathcal{U}, \mathcal{h}^0)),$$

such that $-\alpha$ belongs to $\Delta(\mathcal{U}, \mathcal{H}^0)$. Let α' be the simple root of $\mathcal{H}$ in $R^+(\bar{\gamma}')$ corresponding to α . Then α' satisfies the parity condition for γ' , so $\alpha \in \tau(\bar{X}(\gamma'))$. By Corollary 9.1.18 and Lemma 9.1.16,

$$\bar{X}_L\left[\left((w \times \gamma^2) - n\alpha\right)\sigma_{\mathcal{H}}\right] = \bar{X}_L\left[\left((s_{\alpha}w) \times \gamma^2\right)\sigma_{\mathcal{H}}\right]$$

occurs in $H^{i-1}(\mathcal{U}, \bar{X}(\gamma'))$. Since

$$\ell(s_{\alpha}w) = \ell(w) - 1,$$

we are done by induction. The proof of (c) is identical. Q.E.D.

Proposition 9.3.23 Suppose we are in the setting (9.3.17) and $\sigma_{\mathcal{H}} = \ell + \mathcal{U}$ is a θ -stable parabolic subalgebra. Define

$$N = |\{\alpha \in \Delta(\mathcal{U}, \mathcal{H}^0) \mid \alpha \text{ is not a root of } \mathcal{H}^0 \text{ in } \ell^{\mathbb{R}} \cap (\mathcal{M}^0 + \sigma \mathcal{C}^0), \alpha \text{ is imaginary, and either } \alpha \text{ is compact and } \alpha \notin R^+(\bar{\gamma}^0), \text{ or } \alpha \text{ is noncompact and } \alpha \in R^+(\bar{\gamma}^0)\}| + \frac{1}{2}|\{\alpha \in \Delta(\mathcal{U}, \mathcal{H}^0) \mid \theta\alpha \neq \alpha\}|.$$

Suppose $\gamma' \in \mathcal{L}(\gamma, \gamma^0)$ is as in (9.3.22) (a), $\gamma^2 \in \mathcal{E}(\gamma, \gamma^0)$ is as in (9.3.22) (b), and $w \in W^I$. If $\bar{X}_L(w \times \gamma_{\mathcal{H}}^2)$ occurs in $H^i(\mathcal{U}, \bar{X}(\gamma'))$, then

$$i = N + \ell(w).$$

This result is useful because it allows a multiplicity in cohomology to be computed as an Euler characteristic.

Proof. Let M_n be half the number of noncompact roots of $\mathcal{H}^0$ in $\ell^{\mathbb{R}} \cap (\mathcal{M}^0 + \sigma \mathcal{C}^0)$. We will show first that

$$i \geq N + \ell(w).$$

By inspection of the definitions (cf. (8.1.5)) one sees that

$$\ell_{\sigma_{\mathcal{H}}}(\gamma^2) = N + M_n.$$

We claim that

$$\ell^I(\gamma') - \ell^I(\gamma^0) = M_n.$$

By inspection of Definition 8.1.4, one sees that this is really just a statement about $\ell^{\mathbb{R}} \cap (\mathcal{M}^0 + \sigma \mathcal{C}^0)$; so to prove it, we may assume that

$$G = \ell^{\mathbb{R}} = \mathcal{M}^0.$$

In that case, $\ell^I(\gamma^0)$ is zero. Let $G = KAN$ be an Iwasawa decomposition. Since $G = \ell^{\mathbb{R}}$ is split,

$$\begin{aligned} \ell^I(\gamma') &= \frac{1}{2} \dim (NA) \\ &= \frac{1}{2} \dim (G/K) \\ &= \frac{1}{2} \dim \mathfrak{p} \\ &= M_n, \end{aligned}$$

as we wished to show. By Lemma 9.3.22 (b), $\bar{X}_L(\gamma_{\mathcal{H}}^2)$ occurs in $H^{i-\ell(w)}(\mathcal{U}, \bar{X}(\gamma'))$. By Proposition 9.3.21 (b) and the calculations above,

$$|i - \ell(w) - (N + M_n)| \leq M_n.$$

In particular,

$$-i + \ell(w) + N + M_n \leq M_n$$

or

$$i \geq N + \ell(w),$$

as we wished to show. Similarly, (using (9.3.22)(c)), one shows that

$$i \leq N + \ell(w).$$

Q.E.D.

We will also need the following structural result.

Lemma 9.3.24 In the setting (9.3.8), suppose

$\gamma^1 \in \mathcal{R}(\gamma)$. Then at least one of the following is true.

- a) There is a simple root $\alpha^1 \in R^+(\bar{\gamma}^1) \cap \Delta(\ell^R, h^1)$, such that $\alpha^1 \notin \tau(\bar{X}(\gamma^1))$; and an element γ' of $\mathcal{G}(\gamma)$ such that the simple root $\alpha' \in R^+(\bar{\gamma}')$ corresponding to α^1 satisfies the parity condition; or
- b) There are a complex simple root $\alpha^1 \in R^+(\bar{\gamma}^1) \cap \Delta(\ell^R, h^1)$, a simple root $\beta^1 \in R^+(\bar{\gamma}^1 - n\alpha^1) \cap \Delta(\ell^R, h^1)$, and an element γ' of $\mathcal{G}(\gamma)$, such that if (α', β') are the simple roots in $R^+(\bar{\gamma}')$ corresponding to $(-\alpha^1, \beta^1)$ in $R^+(\bar{\gamma} - n\alpha^1)$, then

θ_{α^1} is positive for γ^1

$$\beta^1 \notin \tau(\bar{X}(\gamma^1 - n\alpha^1))$$

β' satisfies the parity condition for γ' ;

or

$$c) \quad \gamma^1 \in \mathcal{G}(\gamma).$$

Proof. Suppose first that there is a complex simple root $\alpha^1 \in R^+(\bar{\gamma}^1) \cap \Delta(\ell^R, h^1)$ such that θ_{α^1} is positive, but that (a) fails. Let $\alpha \in R^+(\bar{\gamma})$ be the simple root corresponding to α^1 , and $0_K^R \subseteq R^R$ the simple factor of R^R containing α

(notation (9.3.8)(c)). Let

$0_{R_0}^R =$ roots not satisfying the parity condition
for γ

$0_{R_1}^R =$ roots satisfying the parity condition.

By Corollary 9.3.14, $0_{R_1}^R$ is non-empty. If $w \in W^R$, then $0_{R_0}^R$ is still the set of roots not satisfying the parity condition for $w \times \gamma$; but the root corresponding to α^1 for $w \times \gamma$ is $w\alpha$. So the hypothesis that (a) fails means that

$$(9.3.25) \quad W^R \cdot \alpha \subseteq 0_{R_0}^R.$$

Since $0_{R_1}^R$ is non-empty, and

$$0_K^{VR} \supseteq 0_{R_0}^{VR}$$

is the root system for a symmetric space (compare Proposition 9.2.12), it follows (say by the classification of symmetric spaces, although one can give a more direct argument)

$$(9.3.26) \quad 0_R^R = B_n, C_n, \text{ or } F_4$$

$$0_{R_0}^R = B_p \times B_{n-p}, D_n, C_4, \text{ respectively.}$$

In particular, $0_{R_0}^R$ contains all the short roots. By (9.3.25), α is short. We treat these three possibilities separately.

Case 1. $0_K^R = B_n$. Let

$$1_R \subseteq R(\bar{\gamma}^1 - n\alpha^1)$$

be the root system corresponding to 0_R^R . Since it contains the complex root α^1 , but sits in the split group ℓ^R , we can find another simple root β^1 in 1_R^+ such that either β^1 is complex and $\theta\beta^1$ is positive, or β^1 is noncompact imaginary

(cf. Lemma 8.6.2). Since $\theta(-\alpha^1)$ is negative, β^1 is not $-\alpha^1$. Since B_n has only one short simple root, β^1 is long. So we can choose $w \in \tilde{W}^R$ so that

$$w\beta \in {}^0R_1^R;$$

that is, $w\beta$ satisfies the parity condition for $w \times \gamma$. So (b) of the lemma holds.

Case 2. $\frac{{}^0R^R}{n} = C_n$. The symmetric space corresponding to ${}^0\sqrt{R}^R$ is $SO(2n,1)/SO(2n)$, which has rank one; so the contribution of ${}^0R^R$ to $c(\gamma)$ [defined for ℓ^R -- compare the proof of Corollary 9.3.14] is one. This allows us to keep track of the effect of $\tilde{c}$ (Lemma 9.3.10) on ${}^0R^R$. The conclusion is that after a single Cayley transform, there are no more real roots satisfying the parity condition. So if ${}^1R \subseteq R(\bar{\gamma}^1 - n\alpha^1)$ is the root system corresponding to ${}^0R^R$, then

$$\theta|_{{}^1R} = -[\text{reflection around one noncompact positive long root } \delta] \quad (9.3.27)$$

no real root in 1R satisfies the parity condition

Now choose $\beta^1 \in {}^1R^+$ to be the unique long simple root. Then either $\beta^1 = \delta$ is noncompact imaginary; or β^1 is orthogonal to δ , and therefore is real and does not satisfy the parity condition. So by Theorem 8.5.18,

$$\beta^1 \notin \tau(\bar{X}(\gamma^1 - n\alpha^1))$$

is either case. Just as in Case 1, (b) holds.

Case 3. $\frac{{}^0R^R}{n} = F_4$. In this case the symmetric space is $F_4/\text{Spin}(9)$, which also has rank one; so (9.3.27) holds. By inspection of the roots of F_4 , it is easy to check that if δ

is any positive long root, then there is a simple positive long root β^1 such that

$$\langle \beta^1, \delta \rangle \geq 0.$$

We consider such a β^1 in ${}^1R^+$. If the inner product is zero, or $\beta^1 = \delta$, we argue as in Case 2. So suppose β^1 is complex.

Then

$$\begin{aligned} \theta\beta^1 &= -s_\delta(\beta_1^1) \\ &= -\beta_1^1 + \langle \beta_1^1, \delta \rangle \delta, \end{aligned}$$

which is positive since δ is positive and β^1 is simple. So $\beta^1 \notin \tau(\bar{X}(\gamma^1 - n\alpha^1))$, and we can finish the argument as in Case 1.

So we may as well assume that there is no simple complex root α^1 with $\theta\alpha^1$ positive. We are then in the situation (9.3.17), with γ^1 playing the role of γ^0 . So (in that notation) the roots of $\mathcal{H}^1$ in $\mathcal{M}^0 + \alpha\mathcal{C}^0$ are spanned by simple roots in $R^+(\bar{\gamma}^1)$. If $\mathcal{M}^0$ is abelian, then $\mathcal{H}^1 = \mathcal{H}$, and (c) of the lemma holds. So suppose $\mathcal{M}^0$ is not abelian. Then there is a simple noncompact imaginary root $\alpha^1 \in R^+(\bar{\gamma}^1)$. By Lemma 9.3.22(a), we can find $\gamma' \in \mathcal{G}(\gamma)$ with α' satisfying the parity condition; so (a) holds. Q.E.D.

§4. Euler characteristics, character formulas, and Zuckerman's theorem.

Theorem 9.4.1 Suppose $\gamma \in \hat{H}'$ is a regular character with $\bar{\gamma}$ nonsingular, and $\alpha\gamma = \ell + \nu$ is a θ -stable parabolic subalgebra of $\alpha\mathfrak{g}$.

a) If $H \subseteq L$, then

$$\frac{1}{|W(L, H)|} \prod_{\gamma \in \hat{H}} \sum_{\gamma^1 \in \hat{H}} (-1)^i \left(\prod_{\gamma^1 \in \hat{H}} \gamma^1 \right) \left(\prod_{\gamma^1 \in \hat{H}} \gamma^1 \right) =$$

b) If H is not conjugate to a Cartan subgroup of L , then

$$\sum_i (-1)^i \left(\prod_{\gamma^1 \in \hat{H}} \gamma^1 \right) \left(\prod_{\gamma^1 \in \hat{H}} \gamma^1 \right) = 0.$$

The proof requires some preparation.

Lemma 9.4.2 The equalities in Theorem 9.4.1 hold after restriction to $L \cap K$.

Proof. Both sides depend only on $\gamma|_T$, as follows from Theorems 6.3.12 and 5.2.2. So it is enough to prove that the result for $\gamma|_A$ "generic". For such γ , $R(\bar{\gamma})$ consists only of imaginary roots; so γ is minimal (Definition 8.6.5), and the result follows from Corollary 9.3.6. Q.E.D.

Lemma 9.4.3 Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{h} \subseteq \mathfrak{l}$ a Cartan subalgebra. Write $\mathfrak{x}^0$ for the compact part of the center of $\mathfrak{l}$. There is a finite set $\{V_i\}$ of proper subspaces of $\mathfrak{h}^*$, such that if $w \in W(\mathfrak{g}, \mathfrak{h})$,

$$\mu \in \mathfrak{h}^* - \cup V_i.$$

and

$$\mu|_{\mathfrak{x}^0} = w\mu|_{\mathfrak{x}^0},$$

then $w \in W(\mathfrak{l}, \mathfrak{h})$.

Proof. Fix $w \in W(\mathfrak{g}, \mathfrak{h})$, and define

$$V = \{\mu \in \mathfrak{h}^* \mid \mu|_{\mathfrak{x}^0} = w\mu|_{\mathfrak{x}^0}\}.$$

Suppose $V = \mathfrak{h}^*$; to prove the lemma, we must show that $w \in W(\mathfrak{l}, \mathfrak{h})$. Set

$$\rho(u) = \rho(\Delta(u, \mathfrak{h})).$$

Since u is θ -stable, $\rho(u)$ is supported on the compact part of $\mathfrak{h}$; so by Corollary 3.2.5, $\rho(u)$ is supported on $\mathfrak{x}^0$. By hypothesis,

$$\rho(u) = \rho(u)|_{\mathfrak{x}^0} = w\rho(u)|_{\mathfrak{x}^0}.$$

Taking lengths, we find that

$$w\rho(u) \in (\mathfrak{x}^0)^*.$$

So

$$\rho(u) = w\rho(u).$$

By Chevalley's theorem, w is a product of reflections around roots orthogonal to $\rho(u)$; so by Corollary 3.2.5, $w \in W(\mathfrak{l}, \mathfrak{h})$. Q.E.D.

Definition 9.4.4 Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of G , and $H \subseteq L$ a θ -stable Cartan subgroup. Let $\Lambda \subseteq \hat{H}$ be the weight lattice for G and $\xi \in \mathfrak{h}^*$ a weight. A weak coherent family of virtual $(\mathfrak{l}, L \cap K)$ modules, based on $\xi + \Lambda$, is a function

$$\phi: \xi + \Lambda \rightarrow \mathcal{V}(\mathfrak{l}, L \cap K)$$

such that

- a) $\phi(\xi + \lambda)$ has infinitesimal character $\xi + \bar{\lambda} \in \mathfrak{h}^*$
 b) If F is any finite dimensional representation of L whose weights lie in Λ , then

$$\phi(\xi + \lambda) \otimes F = \sum_{\mu \in \Delta(F)} \phi(\xi + (\lambda + \mu)).$$

Most of the theory of coherent families depended only on the fact that the lattice of weights contains the root lattice; so this carries over to weak coherent families. In particular, there are existence and uniqueness theorems, and Proposition 7.2.22 still works.

Lemma 9.4.5 In the setting of Definition 9.4.4, suppose ϕ_1 and ϕ_2 are two weak coherent families of virtual $(\ell, L_n K)$ modules. Fix a positive system $R^+ \subseteq R(\xi)$ (for $\mathfrak{g}$), and a finite set $\{V_i\}$ of proper subspaces of $\mathfrak{h}^*$. Assume that whenever $\lambda \in \Lambda$, $\xi + \bar{\lambda}$ is dominant and nonsingular for R^+ , and

$$\xi + \bar{\lambda} \notin \cup V_i,$$

then

$$\phi_1(\xi + \lambda) \Big|_{L_n K} = \phi_2(\xi + \lambda) \Big|_{L_n K}.$$

Then $\phi_1 = \phi_2$.

Proof. Assume first $\phi_1(\xi + \lambda)$ is the character of a representation whenever $\xi + \bar{\lambda}$ is dominant. If $\ell = \mathfrak{g}$ and $\{V_i\}$ is empty, this is Proposition 8.5.12. Exactly the same proof applies in the present case, almost word for word. We leave the verification of this to the reader. In the general case, write

$$\phi_i = \phi_i^+ - \phi_i^-,$$

with ϕ_i^+ having the property above. By hypothesis

$$(\phi_1^+(\xi + \lambda) - \phi_1^-(\xi + \lambda)) \Big|_{L_n K} = (\phi_2^+(\xi + \lambda) - \phi_2^-(\xi + \lambda)) \Big|_{L_n K}$$

This means that

$$(\phi_1^+(\xi + \lambda) + \phi_2^-(\xi + \lambda)) \Big|_{L_n K} = (\phi_2^+(\xi + \lambda) + \phi_1^-(\xi + \lambda)) \Big|_{L_n K}.$$

By the first case,

$$\phi_1^+ + \phi_2^- = \phi_2^+ + \phi_1^-$$

or

$$\phi_1^+ - \phi_1^- = \phi_2^+ - \phi_2^-.$$

Q.E.D.

Proof of Theorem 9.4.1 Let $H^0 \subseteq L$ be a maximally split Cartan subgroup, and

$$\psi: \mathfrak{h}^* \rightarrow (\mathfrak{h}^0)^*$$

a $\mathfrak{g}$ -inner automorphism. Set $\gamma^0 = \psi(\bar{\gamma})$. Let ϕ_1 be the unique weak coherent family based on $\{\gamma^0 - \rho(\mathfrak{u})\} + \Lambda^0$, with

$$\phi_1(\gamma^0 - \rho(\mathfrak{u}) + 0) = \sum_i (-1)^i p_{\gamma^0 - \rho(\mathfrak{u})} \ominus \left[H^1(\mathfrak{u}, X(\gamma)) \right].$$

In case (a), define a second weak coherent family by

$$\phi_2(\gamma^0 - \rho(\mathfrak{u}) + 0) = \frac{1}{|W(L, H)|} \sum_{\gamma^1 \in \hat{H}^+} (-1)^\ell \mathfrak{g}^{(\gamma^1)} \ominus_{L^+}(\gamma^1).$$

γ_1^1 conjugate to γ under K
 $\bar{\gamma}_1^1$ conjugate to γ^0 under
 an inner automorphism of ℓ

In case (b), define

$$\phi_2 = 0.$$

By Theorem 8.2.1 and Proposition 7.4.6, ϕ_1 and ϕ_2 behave in the obvious way on the $R^+(\gamma^0)$ -dominant part of $\gamma^0 + \Lambda^0$. If we define $\{V_i\}$ for $\mathcal{h}^0$ by Lemma 9.4.3, then Lemma 9.4.2 guarantees that the hypotheses of Lemma 9.4.5 are satisfied. So $\phi_1 = \phi_2$. Evaluating at $\gamma^0 - \rho(\mathcal{u})$, we get the conclusion of Theorem 9.4.1. Q.E.D.

Lemma 9.4.6 Let Θ be a virtual $(\mathcal{g}, K)$ module, all of whose constituents have nonsingular infinitesimal character. Write

$$\Theta = \sum n_i \Theta(\gamma^i),$$

with $\gamma^i \in \{\hat{H}^i\}$ distinct (up to conjugacy by K) regular characters, and $n_i \in \mathbb{Z}$. [This is possible by Proposition 6.6.6.] Suppose that the Θ -stable parabolic associated to γ^1 is $\mathcal{g}$ (Proposition 6.2.2). Then

$$n_1 = \text{multiplicity of } \bar{X}(\gamma^1) \text{ in } \Theta.$$

Proof. The claim amounts to this: if γ is not conjugate to γ^1 , then $\bar{X}(\gamma^1)$ does not occur in $X(\gamma)$. So suppose $\bar{X}(\gamma^1)$ occurs in $X(\gamma)$. We may as well assume G is semisimple. By Lemma 6.6.4,

$$|\bar{\gamma}^{-1}|_{\mathcal{X}}|_{\mathcal{X}}| \geq |\bar{\gamma}|_{\mathcal{X}}|_{\mathcal{X}}|.$$

But by Proposition 6.6.2 and the assumption that γ^1 is attached to $\mathcal{g}$, the left side is zero. So equality holds. By Lemma 6.6.4, γ is conjugate to γ^1 . Q.E.D.

Proposition 9.4.7 Let X be a $(\mathcal{g}, K)$ module, all of whose constituents have nonsingular infinitesimal character. Write

$$\Theta(X) = \sum n_i \Theta(\gamma^i),$$

with $\gamma^i \in \{\hat{H}^i\}$ distinct (up to conjugacy by K) regular characters, and $n_i \in \mathbb{Z}$. Let $\tilde{\mathcal{g}}^i = \tilde{\mathcal{X}}^i + \tilde{\mathcal{U}}^i$ be the Θ -stable parabolic subalgebra associated to γ^i (Proposition 6.6.2); and let $\mathcal{g}^i = \mathcal{X}^i + \mathcal{U}^i$ satisfy

$$\mathcal{h}^i \subseteq \mathcal{X}^i \subseteq \tilde{\mathcal{X}}^i.$$

Then

$$n_i = (-1)^{\ell(\mathcal{g}^i(\gamma^i))} \sum_j (-1)^j m(\bar{X}_{L^i}(\mathcal{g}^i), H^j(\mathcal{U}^i, X)).$$

Proof. This is immediate from Theorem 9.4.1 and Lemma 9.4.6 (applied to ℓ^i). Q.E.D.

The rest of this section is devoted to a proof of Zuckerman's theorem (Proposition 2.2.10(d)).

Lemma 9.4.8 Suppose G is quasisplit, $H = \text{TA}$ is a maximally split Cartan subgroup of G , $\gamma \in \hat{H}$, and $\gamma|_T$ is fine with respect to G (Definition 4.3.8). Suppose further that $\bar{\gamma}$ is nonsingular, and set

$$\Delta^+(\mathcal{g}, \mathcal{h}) = \{\alpha \in \Delta(\mathcal{g}, \mathcal{h}) \mid \langle \alpha, \bar{\gamma} \rangle > 0\}$$

$$\rho = \rho(\Delta^+(\mathcal{g}, \mathcal{h})).$$

$$\text{a) } \rho|_{\mathcal{X}} = 0.$$

Let $\Phi_{-\rho}$ denote the character of H on which T acts trivially, and A by $-\rho$. Set

$$\lambda = \gamma \otimes \mathbb{C}_{-\rho} \in \hat{H}.$$

b) $\bar{X}_G(\gamma)$ is finite dimensional if and only if λ lies in the weight lattice for G ; and in that case it has highest weight λ .

Proof. For (a), we may assume G is semisimple. In that case $\bar{\gamma}|_T = 0$ by definition of fine, so $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is stable under $-\theta$; and (a) follows. For (b), let $P = T \cap N$ be the parabolic subgroup defined by $-\bar{\gamma}|_N$ (Definition 4.2.1); then

$$X(\gamma) = \text{Ind}_P^G(\gamma).$$

If Y is any $(\mathfrak{g}, K)$ module, then the last statement of (6.3.5) (c) is the [Jacquet-Casselman] isomorphism

$$\text{Hom}_{\mathfrak{g}, K}(Y, X(\gamma)) \cong \text{Hom}_H(Y/\pi_Y, \mathbb{C}_\lambda).$$

For the "only if" part of (b), take $Y = \bar{X}_G(\gamma)$, and assume it is finite dimensional. We conclude that Y has lowest weight λ with respect to N , and therefore highest weight λ with respect to $\Delta^+(\mathfrak{g}, \mathfrak{h})$. In particular, λ belongs to the weight lattice. Conversely, suppose λ belongs to the weight lattice. Then $\bar{\lambda}$ is integral, so $\bar{\gamma}$ is as well; and since $\bar{\gamma}$ is dominant and nonsingular, $\bar{\lambda} = \bar{\gamma} - \rho$ is dominant. So there is a finite dimensional irreducible representation F of highest weight λ . So $F/\pi F$ has weight λ , and therefore

$$\text{Hom}_{\mathfrak{g}, K}(F, X(\gamma)) \neq 0.$$

So F must be $\bar{X}_G(\gamma)$.

Q.E.D.

Lemma 9.4.9 Let $H \subseteq G$ be a θ -stable Cartan subgroup, $\Delta^+(\mathfrak{g}, \mathfrak{h})$ a positive root system, and $\gamma \in \hat{H}$ a regular character with $\bar{\gamma} = \rho[\Delta^+(\mathfrak{g}, \mathfrak{h})]$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{u}$ be the θ -stable parabolic associated to γ (Proposition 6.6.2). Define

$$\chi: H/H_0 \rightarrow \mathbb{C}$$

by (6.7.3). Then $\bar{X}_L(\gamma)$ is finite dimensional with highest weight in the root lattice if and only if χ is trivial.

Proof. Define $\rho_{\mathfrak{k}}$ for $\mathfrak{k}$ as in Lemma 9.4.8. By that lemma, we must find the condition for

$$\Gamma_{\mathfrak{g}} \otimes \mathbb{C}_{-\rho_{\mathfrak{k}}}$$

to be a sum of roots. Recall that (Proposition 6.6.2)

$$(9.4.10) \quad \Gamma_{\mathfrak{g}}|_T = \Gamma|_T - 2\rho(\mathfrak{u} \cap \mathfrak{p});$$

here $2\rho(\mathfrak{u} \cap \mathfrak{p})$ denotes the sum of the roots of T on $\mathfrak{u} \cap \mathfrak{p}$. Choose $B \subseteq \Delta^+$ in accordance with Lemma 6.7.1; since $\Delta^+(\mathfrak{k}, \mathfrak{h})$ is stable under $-\theta$, B is contained in $\Delta(\mathfrak{u}, \mathfrak{h})$.

Set

$$C = \{\alpha \in \Delta(\mathfrak{u}, \mathfrak{h}) \mid \alpha \notin \Delta^+\};$$

then

$$(9.4.11) \quad \Delta(\mathfrak{u}, \mathfrak{h}) =$$

$$\{\text{imaginary roots}\} \cup (B \cup \theta B) \cup (C \cup \theta C),$$

a disjoint union. Clearly,

(9.4.11) (b)

$$\Delta(\mathcal{U} \cap \mathcal{P}, \mathcal{L}) \leftrightarrow$$

{noncompact imaginary roots} $\cup B \cup C$,

the correspondence on the level of root vectors being

$$X_{\alpha} - \theta X_{\alpha} \leftrightarrow X_{\alpha}.$$

So

$$2\rho(\mathcal{U} \cap \mathcal{P}) = 2\rho(\mathcal{M} \cap \mathcal{P}) + 2\rho(B) \Big|_{\mathcal{T}} + 2\rho(C) \Big|_{\mathcal{T}}.$$

So (9.4.10) can be written

$$\begin{aligned} \Gamma_{\mathcal{P}} \Big|_{\mathcal{T}} &= \Gamma \Big|_{\mathcal{T}} - 2\rho(\mathcal{M} \cap \mathcal{P}) - 2\rho(B) \Big|_{\mathcal{T}} - 2\rho(C) \Big|_{\mathcal{T}} \\ (9.4.12) \quad (a) \quad &= \chi \Big|_{\mathcal{T}} - 2\rho(C) \Big|_{\mathcal{T}} \end{aligned}$$

in light of (6.7.3). An argument formally identical to the proof of Lemma 6.7.1 shows that

$$[\rho + 2\rho(C)] \Big|_{\sigma} = \rho(\mathcal{U}) + \rho_{\ell}.$$

In light of Proposition 6.6.2, we deduce that

$$\begin{aligned} (9.4.12) \quad (b) \quad (\Gamma_{\mathcal{P}} \otimes \mathbb{C}_{-\rho_{\ell}}) \Big|_{\sigma} &= (\rho - \rho(\mathcal{U}) - \rho_{\ell}) \Big|_{\sigma} \\ &= -2\rho(C) \Big|_{\sigma}. \end{aligned}$$

We have therefore shown that

$$(9.4.13) \quad \Gamma_{\mathcal{P}} \otimes \mathbb{C}_{-\rho_{\ell}} = [-2\rho(C)] \otimes \chi,$$

and so the lemma is obvious.

Q.E.D.

Lemma 9.4.14 Recall the element γ_0 of $\mathcal{A}$ defined in (2.2.8). Then $\ell(\gamma_0)$ may be computed as follows:

choose a Cartan subalgebra $\mathcal{L}^C$ of $\mathfrak{h}$; then $\ell(\gamma_0)$ is half the number of non-zero roots of $\mathcal{L}^C$ in $\mathcal{P}$.

Proof. Let $G = KAN$ be an Iwasawa decomposition, and

$\mathfrak{h}^C = \mathcal{L}^C + \sigma^C$ a fundamental Cartan. By Definition 2.2.8,

$$\begin{aligned} \ell(\gamma_0) &= \frac{1}{2} \dim N + \frac{1}{2} \dim A - \frac{1}{2} \dim A^C \\ &= \frac{1}{2} \dim NA - \frac{1}{2} \dim A^C. \end{aligned}$$

By the Cartan decomposition,

$$\dim NA = \dim \mathcal{P};$$

so

$$\ell(\gamma_0) = \frac{1}{2} \dim (\mathcal{P} / \sigma^C). \quad \text{Q.E.D.}$$

Lemma 9.4.15 With notation as in Lemmas 9.4.9 and

9.4.14, and B and C defined in the proof of Lemma

9.4.9, we have (notation (8.1.4), (8.1.5))

- a) $\ell^I(\gamma) = |C| + \ell_{\mathcal{L}}^I(\gamma_{\mathcal{P}})$
- b) $\ell_{\mathcal{P}}(\gamma) = |\{\text{noncompact imaginary roots of } \mathfrak{h} \text{ in } \mathcal{U}\}| + |B| + |C|$
- c) $\ell^I(\gamma_0) = |B| + |C| + \ell_{\mathcal{L}}^I(\gamma_{\mathcal{P}}) + |\{\text{noncompact imaginary roots of } \mathfrak{h} \text{ in } \mathcal{U}\}|$
- d) If χ is trivial, $\bar{X}_{\mathcal{L}}(\gamma_{\mathcal{P}})$ occurs in

$$H^*(\mathcal{U}, \mathbb{C})$$

exactly once, in degree $|C|$.

Proof. By Definition 8.1.4,

$$\ell^I(\gamma) = \frac{1}{2} |\{\alpha \in \Delta^+ | \theta\alpha \neq \Delta^+\}| + \frac{1}{2} \dim A - c_0$$

$$\begin{aligned}
&= \frac{1}{2} | -C \cup \theta C | + \frac{1}{2} \{ \alpha \in \Delta^+(\lambda) \mid \theta \alpha \notin \Delta^+(\lambda) \} + \frac{1}{2} \dim A - c_0 \\
&= |C| + \ell_L^I(\gamma_{\mathcal{G}}).
\end{aligned}$$

This is (a). Part (b) is clear from (9.4.11) (a) and the definition of $\ell_{\mathcal{G}}(\gamma)$. For (c), choose a Cartan $\mathcal{L}^C$ in $\ell \cap \mathcal{L}$. By (9.4.11) (b), there are

$$2|B| + 2|C| + 2|\{\text{noncompact imaginary roots of } \mathcal{L} \text{ in } \mathcal{U}\}|$$

roots of $\mathcal{L}^C$ in $\mathcal{U} \cap \mathcal{P} + \overline{\mathcal{U}} \cap \mathcal{P}$. By Lemma 9.4.14, there are

$$2\ell_L^I(\gamma_{\mathcal{G}})$$

non-zero roots of $\mathcal{L}^C$ in $\ell \cap \mathcal{P}$. So (c) follows from Lemma 9.4.14 and the triangular decomposition

$$\mathcal{P} = \mathcal{U} \cap \mathcal{P} \oplus \ell \cap \mathcal{P} \oplus \overline{\mathcal{U}} \cap \mathcal{P}.$$

For (d), suppose χ is trivial. By Lemma 9.4.8, $\overline{X}_L(\gamma_{\mathcal{G}})$ is finite dimensional, of highest weight $-2\rho(C)$. By the definition of C , Corollary 3.2.16, and (3.2.9), (d) follows.

Q.E.D.

We can now prove Proposition 2.2.10(d). For convenience we restate it in our current notation.

Proposition 9.4.16 (Zuckerman) Choose a set $\{(\mathcal{H}^i, (\Delta^+)^i)\}$ of representatives of K -conjugacy classes of pairs consisting of a θ -stable Cartan subgroup of G , and a positive root system for $\mathcal{H}^i$ in $\mathcal{G}$. For each i , let $\gamma^i \in (\hat{\mathcal{H}}^i)^i$ be the unique regular character such that

$$\overline{\gamma}^i = \rho((\Delta^+)^i),$$

and the character χ of $\mathcal{H}^i/(\mathcal{H}^i)_0$ defined by (6.7.3) is trivial. Arrange the notation so that $\mathcal{H}^0$ is a maximally split Cartan subgroup, and $(\Delta^+)^0$ is compatible with an Iwasawa decomposition of G . Then:

- a) $\overline{X}_G(\gamma^0) = \mathbb{C}$, the trivial $(\mathcal{G}, K)$ module;
- b) There is an identity in the Grothendieck group

$$\Theta(\mathbb{C}) = \Theta(\overline{X}_G(\gamma^0)) = \sum_i (-1)^{\ell^I(\gamma^i)} - \ell^I(\gamma^0) \Theta(\gamma^i).$$

Proof. For (a), it is trivial to check from (6.7.3) that

$$\chi|_{\mathbb{T}} = \Gamma^0|_{\mathbb{T}};$$

so $\Gamma^0|_{\mathbb{T}}$ is trivial. By Lemma 6.6.15, $\overline{X}_G(\gamma^0)$ contains the trivial representation of K . Since it has the same infinitesimal character as the trivial representation, they coincide.

For (b), we know that a formula of the form

$$\Theta(\mathbb{C}) = \sum_j n_{\gamma} \Theta(\gamma),$$

with γ as in Lemma 9.4.9, exists (cf. Proposition 6.6.6); we only have to calculate n_{γ} . With $\mathcal{G}$ as in Lemma 9.4.9, n_{γ} is

$$(-1)^{\ell_{\mathcal{G}}(\gamma)} \sum_j (-1)^j m(\overline{X}_L(\gamma_{\mathcal{G}}), H^j(\mathcal{U}, \mathbb{C})).$$

This can be non-zero only if $\overline{X}_L(\gamma_{\mathcal{G}})$ is finite dimensional and has its highest weight in the root lattice; so by Lemma 9.4.9, we may assume $\gamma = \gamma^i$ for some i . By Lemma 9.4.15, n_{γ^i} is -1 to the power

$$\begin{aligned} & \ell_{\mathcal{G}}^i(\gamma^i) - |C^i| \\ &= |\{\text{noncompact imaginary roots in } \mathcal{W}^i\} + |B^i| \\ &= \ell^I(\gamma^0) - \ell^I(\gamma^i). \end{aligned} \quad \text{Q.E.D.}$$

§5. Vanishing theorems using Conjecture 7.3.25.

One of the main features of Conjecture 7.3.25 is that it imposes a very rigid parity structure on irreducible $(\mathcal{G}, K)$ modules. In this section we will explain that structure, and draw a few technical consequences. We will then be in a position to describe the Kazhdan-Lusztig conjecture.

Theorem 9.5.1 Suppose Conjecture 7.3.25 holds.

In the setting (9.1.14),

- a) $\bar{X}_L(\gamma^1)$ occurs in $H^i(\mathcal{W}, \bar{X}(\gamma))$ only if
- $$\ell^I(\gamma^1) - \ell^I(\gamma) \equiv \ell_{\mathcal{G}}(\gamma^1) - i \pmod{2}$$
- b) If γ and γ^1 are not conjugate, and $\ell^I(\gamma) \geq \ell^I(\gamma^1)$,

then

$$\text{Ext}_{\mathcal{G}, K}^1(\bar{X}(\gamma), \bar{X}(\gamma^1)) \neq 0$$

only if

$$\ell^I(\gamma^1) - \ell^I(\gamma) \equiv 1 \pmod{2}.$$

- c) $H^i(\mathcal{W}, \bar{X}(\gamma))$ is completely reducible as an $(\mathcal{L}, L \cap K)$ module.

- d) In the setting (9.1.15), write $X = \bar{X}(\gamma)$.

- 1) If $\alpha^1 \in \Delta(\mathcal{W})$, then

$$m\left(\bar{X}_L(\gamma_{\mathcal{G}}^1), H^{i+1}(\mathcal{W}, X)\right) + m\left(\bar{X}_L(\gamma_{\mathcal{G}}^1 - n\alpha^1), H^i(\mathcal{W}, X)\right).$$

- 2) If $\alpha^1 \in \Delta(\mathcal{L})$, the $\bar{X}_L(\gamma_{\mathcal{G}}^1)$ occurs in $H^i(\mathcal{W}, U_{\alpha}(X)$ only if $\alpha \in \tau(\bar{X}_L(\gamma_{\mathcal{G}}^1))$. In that case,

$$m\left(\bar{X}_L(\gamma_{\mathcal{G}}^1), H^i(\mathcal{W}, U_{\alpha}(X))\right) = m\left(\bar{X}_L(\gamma_{\mathcal{G}}^1), \phi_{\alpha}^L \psi_{\alpha}^H(\mathcal{W}, X) \oplus H^{i-1}(\mathcal{W}, X) \oplus H^{i+1}(\mathcal{W}, X)\right).$$

- 3) If $-\alpha^1 \in \Delta(\mathcal{W})$, then the formula of (1) holds with $i+1$ replaced by $i-1$.

- e) In the setting (9.1.15), $\bar{X}(\gamma^1)$ occurs in $U_{\alpha}(\bar{X}(\gamma))$ only if

$$\ell^I(\gamma^1) - \ell^I(\gamma) \equiv 1 \pmod{2}.$$

The proof will essentially be another incarnation of the proof of Proposition 8.6.19; but it requires a few technical ideas which have not been treated yet.

Lemma 9.5.2 Suppose X is an irreducible $(\mathcal{G}, K)$ module of nonsingular infinitesimal character.

Let E be an extension of X by itself; that is, assume that there is a short exact sequence

$$0 \rightarrow X \rightarrow E \rightarrow X \rightarrow 0.$$

Assume that $Z(\mathcal{G})$ acts by scalars on E . Then

$$E \cong X \oplus X.$$

Proof. Write $X = \bar{X}(\gamma)$, $\gamma \in \hat{H}'$. Clearly it is enough to show that

$$\text{Hom}_{\mathcal{G}, K}(E, X(\gamma))$$

is two-dimensional. Let $\mathcal{G}$ be the associated θ -stable parabolic, and $HN \subseteq L$ the real parabolic defined by $-\bar{\gamma}_{\mathcal{G}}|_{\alpha}$ (Definition 4.2.1). By Theorem 6.5.9(g),

$$\text{Hom}_{\mathcal{G}, K}(E, X(\gamma)) \cong \text{Hom}_{\mathcal{L}, L \otimes K}(H^R(\mathcal{U}, E), X_L(\gamma_{\mathcal{G}})).$$

By 6.3.5(c), this is

$$\cong \text{Hom}_{\mathcal{H}, T} \left[H^{\dim} \mathcal{V}(\mathcal{U}, H^R(\mathcal{U}, E)), \mathcal{C}_{\mathcal{V}} \right];$$

here $\tilde{\gamma}$ is a certain twist of $\gamma_{\mathcal{G}}$, whose precise form is not important. By Lemma 9.1.7, the term on the left in the last Hom is a semisimple $(\mathcal{H}, T)$ module; so

$$\cong \left(\tilde{\gamma}\text{-primary subspace of } H^{\dim} \mathcal{V}(\mathcal{U}, H^R(\mathcal{U}, E)) \right)^*.$$

By Corollary 6.4.12(a), this space has dimension equal to the multiplicity of $\bar{X}_L(\gamma_{\mathcal{G}})$ in $H^R(\mathcal{U}, E)$. By the long exact sequence in cohomology attached to

$$0 \rightarrow X \rightarrow E \rightarrow X \rightarrow 0$$

and Theorem 8.1.6, this multiplicity is two. Q.E.D.

Lemma 9.5.3 In the setting of Definition 7.3.17, $Z(\mathcal{G})$ acts by scalars on $K_{\alpha}(X)$, $Q_{\alpha}(X)$, and $U_{\alpha}(X)$.

Proof. Fix $z \in Z(\mathcal{G})$; after changing it by adding a constant we may assume that z acts by zero on X . So z defines a $(\mathcal{G}, K)$ module map

$$m_z: W/X \rightarrow W.$$

Now X is the unique irreducible submodule of W (Corollary 7.3.24). By Corollary 7.3.24 again, X occurs exactly once in W/X , as the unique irreducible quotient. So m_z must factor as

$$W/X \rightarrow X \rightarrow W.$$

So m_z annihilates $K_{\alpha}(X)$; and its composition with the quotient map

$$W \rightarrow Q_{\alpha}(X)$$

is zero. This proves the first two statements, and the third follows. Q.E.D.

Lemma 9.5.4 Suppose X and Y are $(\mathcal{G}, K)$ modules and F is a finite dimensional representation of

G. Then there is a natural isomorphism

$$\text{Ext}_{\mathcal{G}, K}^i(X, Y \otimes F) \cong \text{Ext}_{\mathcal{G}, K}^i(X \otimes F^*, Y).$$

Proof. Let

$$0 \rightarrow Y \rightarrow I_*$$

be an injective resolution of Y . By Lemma 7.2.2(a)

$$0 \rightarrow Y \otimes F \rightarrow I_* \otimes F$$

is an injective resolution of $Y \otimes F$. By Corollary 6.1.24,

$$\begin{aligned} \text{Ext}_{\mathcal{G}, K}^i(X, Y \otimes F) &\cong H^i(\text{Hom}_{\mathcal{G}, K}(X, I_* \otimes F)) \\ &\cong H^i(\text{Hom}_{\mathcal{G}, K}(X \otimes F^*, I_*)) \\ &\cong \text{Ext}_{\mathcal{G}, K}^i(X \otimes F^*, Y). \end{aligned}$$

The second isomorphism is Lemma 4.5.9, and the last is Corollary 6.1.24. Q.E.D.

Lemma 9.5.5 Suppose X and Y are $(\mathcal{G}, K)$ modules of finite length, $\mathcal{H} \subseteq \mathcal{G}$ is a Cartan subalgebra, and $\gamma \in \mathcal{H}^*$. Then

$$\text{Ext}_{\mathcal{G},K}^i(P_Y X, Y) \cong \text{Ext}_{\mathcal{G},K}^i(Y, P_Y Y) \\ \text{(notation (0.3.18))}.$$

Proof. By Proposition 0.3.19(c), this is a special case of Proposition 6.1.34. Q.E.D.

Corollary 9.5.6 (Zuckerman) Suppose X and Y are $(\mathcal{G}, K)$ modules of finite length. In the setting of Definition 4.5.7,

$$\text{Ext}_{\mathcal{G},K}^i(X, \psi_{\xi+\lambda_2}^{\xi+\lambda_1} Y) \cong \text{Ext}_{\mathcal{G},K}^i(\psi_{\xi+\lambda_1}^{\xi+\lambda_2} X, Y).$$

The proof is formally identical to that of Proposition 4.5.8 (using the two lemmas above), and we leave it to the reader.

Corollary 9.5.7 In the setting of Theorem 7.3.16, let Y be a $(\mathcal{G}, K)$ module of finite length such that α lies in the τ invariant of every irreducible constituent of Y . Then there are natural isomorphisms

$$\text{Hom}_{\mathcal{G},K}(Y, U_\alpha(X)) \cong \text{Ext}_{\mathcal{G},K}^1(Y, X) \\ \text{Hom}_{\mathcal{G},K}(U_\alpha(X), Y) \cong \text{Ext}_{\mathcal{G},K}^1(X, Y).$$

Proof. Define ϕ_α and ψ_α as in (8.2.6). The hypothesis means

$$\psi_\alpha Y = 0.$$

So

$$\text{Ext}_{\mathcal{G},K}^i(Y, \phi_\alpha \psi_\alpha X) \cong \text{Ext}_{\mathcal{G},K}^i(\psi_\alpha Y, \psi_\alpha X) \\ = 0$$

by Corollary 9.5.6. Consider the long exact sequence in Ext

attached to

$$0 \rightarrow X \rightarrow \phi_\alpha \psi_\alpha X \rightarrow Q_\alpha(X) \rightarrow 0.$$

By the vanishing result above, it gives an isomorphism

$$\text{Hom}_{\mathcal{G},K}(Y, Q_\alpha(X)) \cong \text{Ext}_{\mathcal{G},K}^1(Y, X).$$

Since X cannot occur in Y , a map of X into $Q_\alpha(X)$ must land in the subspace $U_\alpha(X)$. So

$$\text{Hom}_{\mathcal{G},K}(Y, U_\alpha(X)) \cong \text{Hom}_{\mathcal{G},K}(Y, Q_\alpha(X)).$$

This gives the first claim; the second is similar. Q.E.D.

Corollary 9.5.8 In the setting of Theorem 7.3.16, let Y be another irreducible $(\mathcal{G}, K)$ module of infinitesimal character ξ , and β another simple root in $R^+(\xi)$. Assume that

$$\alpha \notin \tau(X), \quad \beta \in \tau(X)$$

$$\alpha \in \tau(Y), \quad \beta \notin \tau(X).$$

Then there is a natural isomorphism

$$\text{Hom}_{\mathcal{G},K}(Y, U_\alpha(X)) \cong \text{Hom}_{\mathcal{G},K}(U_\beta(Y), X).$$

If Conjecture 7.3.25, holds

$$m(Y, U_\alpha(X)) = m(X, U_\beta(Y)).$$

Proof. For the first claim, Corollary 9.5.7, shows that both sides are isomorphic to $\text{Ext}_{\mathcal{G},K}^1(Y, X)$. The second is obvious. Q.E.D.

The last statement can be proved (with substantially more work) without Conjecture 7.3.25 ([46], Theorem 4.14).

It is a powerful tool for computing composition series for small groups. For Verma modules it is due to Jantzen; that result is one of the main technical tools for the study of primitive ideals in $U(\mathfrak{g})$.

Lemma 9.5.9 Let W be a $(\mathfrak{g}, K)$ module of finite length, on which $Z(\mathfrak{g})$ acts by scalars according to a nonsingular infinitesimal character. Assume that whenever X and Y are distinct irreducible constituents of W , we have

$$\text{Ext}_{\mathfrak{g}, K}^1(X, Y) = \text{Ext}_{\mathfrak{g}, K}^1(Y, X) = 0.$$

Then W is completely reducible.

Proof. One argues by induction on the number of irreducible constituent of W , using Lemmas 9.2.2 and 9.5.2. We leave the details to the reader. Q.E.D.

Lemma 9.5.10 In the setting (9.1.14), let $H^2 \subseteq L$ be another θ -stable Cartan subgroup, and $\gamma^2 \in (\hat{H}^2)'$ a regular character. Assume that $\bar{\gamma}^1$ and $\bar{\gamma}^2$ are conjugate by an inner automorphism of ℓ . Then (notation (8.1.4), (8.1.5)):

$$\ell_{(\gamma^1)}^I - \ell_{\mathfrak{g}(\gamma^1)}^I - \ell_L^I(\gamma^1) = \ell^I(\gamma^2) - \ell_{\mathfrak{g}(\gamma^2)}^I - \ell_L^I(\gamma^2) \pmod{2}$$

Proof. Put

$$N = |\{\alpha \in R^+(\bar{\gamma}^1) \mid -\alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^1)\}|.$$

Since $\bar{\gamma}^1$ and $\bar{\gamma}^2$ are conjugate by an inner automorphism of ℓ ,

N is well defined. We will proceed by induction on N . Suppose first that $N = 0$. Thus

$$R^+(\bar{\gamma}^1) = (R^+(\bar{\gamma}^1) \cap \Delta(\mathfrak{u}, \mathfrak{h}^1)) \cup R_L^+(\bar{\gamma}_{\mathfrak{g}}^1).$$

The first of these is θ -stable; so

$$\ell^I(\gamma^1) = \ell_L^I(\gamma_{\mathfrak{g}}^1).$$

On the other hand, one checks easily that

$$\ell_{\mathfrak{g}(\gamma^1)}^I = \dim \mathfrak{u} \cap \mathfrak{p}.$$

So both sides of the equation in the lemma are $-\dim \mathfrak{u} \cap \mathfrak{p}$.

Suppose next that $N > 0$. We can find a simple root $\alpha^1 \in R^+(\bar{\gamma}^1)$, such that $-\alpha^1 \in \Delta(\mathfrak{u}, \mathfrak{h}^1)$. If α^2 is the corresponding root for $\bar{\gamma}^2$, then $-\alpha^2 \in \Delta(\mathfrak{u}, \mathfrak{h}^1)$. We will apply the inductive hypothesis to the pair (notation (8.3.1)).

$$(s_{\alpha^1} \times \gamma^1)^1.$$

Clearly $s_{\alpha^1} \bar{\gamma}^1$ are conjugate by an inner automorphism of ℓ , and $N(s_{\alpha^1} \times \gamma^1)$ is $N-1$. To prove the lemma, it is enough to show that

$$\begin{aligned} \ell^I(s_{\alpha^1} \times \gamma^1) - \ell_{\mathfrak{g}(s_{\alpha^1} \times \gamma^1)}^I - \ell_L^I((s_{\alpha^1} \times \gamma^1)_{\mathfrak{g}}^1) = \\ \ell^I(\gamma^1) - \ell_{\mathfrak{g}(\gamma^1)}^I - \ell_L^I(\gamma_{\mathfrak{g}}^1) \pm 1; \end{aligned}$$

for this applies to γ^2 as well. Since α^1 is not a root in ℓ ,

$$\ell_L^I((s_{\alpha^1} \times \gamma^1)_{\mathfrak{g}}^1) = \ell_L^I(\gamma_{\mathfrak{g}}^1).$$

If α^1 is imaginary,

$$\ell^I(s_{\alpha^1} \times \gamma^1) = \ell^I(\gamma^1)$$

$$\ell_{\mathfrak{g}(s_{\alpha^1} \times \gamma^1)}^I = \ell_{\mathfrak{g}(\gamma^1)}^I \pm 1$$

by Definitions 8.1.4 and 8.1.5. If α^1 is complex,

$$\begin{aligned}\ell^I(s_{\alpha}1 \times \gamma^1) &= \ell^I(\gamma^1) \pm 1 \\ \ell_{\mathcal{G}}(s_{\alpha}1 \times \gamma^1) &= \ell_{\mathcal{G}}(\gamma^1).\end{aligned}$$

This proves the claim, and so the lemma. Q.E.D.

Proof of Theorem 9.5.1 We proceed by induction on the dimension of $\mathcal{G}$; and for fixed $\mathcal{G}$, by induction on $\ell^I(\gamma)$. For (a), if γ is minimal, this is a weak version of Corollary 9.3.6. So suppose γ is not minimal. By Definition 8.6.5, there is a simple root $\alpha \in R^+(\bar{\gamma})$ such that $\theta\alpha \notin R^+(\bar{\gamma})$; and if α is real, then α satisfies the parity condition. By Theorem 8.5.18 and Lemma 8.6.13, we can choose $\gamma^2 \in (\hat{H}^2)'$ with

$$\begin{aligned}\ell^I(\gamma^2) &= \ell^I(\gamma) - 1 \\ \bar{X}(\gamma) &\text{ occurs in } U_{\alpha}(\bar{X}(\gamma^2)).\end{aligned}$$

By Conjecture 7.3.25, we can replace $\bar{X}(\gamma)$ in the statement of (a) by $U_{\alpha}(\bar{X}(\gamma^2))$. Now apply the inductive hypothesis to γ^2 , using (9.5.1)(d) to calculate the cohomology of $U_{\alpha}(\bar{X}(\gamma^2))$. [In case (d2), we apply (9.5.1)(e) to the group L to investigate

$$\phi_{\alpha}^{L, \psi_{\alpha}^L, i}(\mathcal{U}, \bar{X}(\gamma^2)).]$$

By a simple case-by-case argument, one sees that the desired parity condition for γ follows from the one for γ^2 ; we leave this to the reader. [Helpful hints may be found in the proof of (d) below.]

Consider next (b). By Proposition 8.6.19, $\bar{X}(\gamma)$ does not occur in $X(\gamma^1)$. Thus

$$\text{Hom}_{\mathcal{G}, K}(\bar{X}(\gamma), X(\gamma^1)/\bar{X}(\gamma^1)) = 0.$$

By the long exact sequence in cohomology,

$$\text{Ext}_{\mathcal{G}, K}^1(\bar{X}(\gamma), X(\gamma^1)) \neq 0.$$

Let $\mathcal{O}_{\mathcal{G}}^1 = \ell^1 + \mathcal{U}^1$ be the parabolic associated to γ^1 . By Theorem 6.5.9(f), either

- 1) $\text{Ext}_{\ell^1, L^1 nK}^1(H^R(\mathcal{U}^1, \bar{X}(\gamma)), X(\gamma^1)) \neq 0$, or
- 2) $\text{Hom}_{\ell^1, L^1 nK}(\mathcal{U}^{R-1}(\mathcal{U}^1, \bar{X}(\gamma)), X(\gamma^1)) \neq 0$.

In case (2), since $R = \ell_{\mathcal{G}}^1(\gamma^1)$, (a) implies that

$$\ell^I(\gamma^1) - \ell^I(\gamma) \text{ is odd,}$$

as we wished to show. In case (1), Corollary 6.4.13 implies that

$$\text{Hom}_{\ell, L nK}(H^R(\mathcal{U}^1, \bar{X}(\gamma)), X(\gamma^1)) \neq 0.$$

By Theorem 6.5.9(g), this implies

$$\text{Hom}_{\mathcal{G}, K}(\bar{X}(\gamma), X(\gamma^1)) \neq 0;$$

and thus that γ is conjugate to γ^1 , a contradiction.

For (c), we will use Lemma 9.5.9. By Lemma 9.1.7, $Z(\ell)$ acts semisimply on the cohomology. By Lemma 9.5.9 and the inductive hypothesis (b) applied to ℓ , we must show that if $\bar{X}_L(\gamma_{\mathcal{G}}^2)$ and $\bar{X}_L(\gamma_{\mathcal{G}}^1)$ both occur in $H^*(\mathcal{U}, \bar{X}(\gamma))$ and have the same infinitesimal character, then

$$\ell_L^I(\gamma_{\mathcal{G}}^2) - \ell_L^I(\gamma_{\mathcal{G}}^1) \equiv 0 \pmod{2}.$$

This follows from (a) of the theorem and Lemma 9.5.10.

For (d1), consider the short exact sequence

$$0 \rightarrow X \rightarrow K_\alpha \rightarrow U_\alpha \rightarrow 0,$$

and the corresponding long exact sequence in cohomology.

If $\bar{X}_L(\gamma^1_{\mathcal{Y}})$ occurs in $H^i(\mathcal{U}, X)$, then by (a),

$$(9.5.11) \text{ (a)} \quad i \equiv \ell^I(\gamma^1) - \ell^I(\gamma) - \ell_{\mathcal{Y}}(\gamma^1) \pmod{2}.$$

If it occurs in $H^i(\mathcal{U}, K_\alpha)$, then by Corollary 9.1.17 and Lemma 9.1.16, $\bar{X}_L(\gamma^1 - n\alpha^1)$ occurs in $H^i(\mathcal{U}, X)$. By (a)

$$(9.5.11) \text{ (b)} \quad i \equiv \ell^I(s_\alpha^1 \times \gamma^1) - \ell^I(\gamma) - \ell_{\mathcal{Y}}(s_\alpha^1 \times \gamma^1) \pmod{2}.$$

As we saw in the proof of Lemma 9.5.10, this means that

$$(9.5.11) \text{ (c)} \quad i+1 \equiv \ell^I(\gamma^1) - \ell^I(\gamma) - \ell_{\mathcal{Y}}(\gamma^1) \pmod{2}.$$

So the long exact sequence in cohomology degenerates (as far as multiplicities of $\bar{X}_L(\gamma^1_{\mathcal{Y}})$: $\bar{X}_L(\gamma^1_{\mathcal{Y}})$ cannot occur in both

$$H^i(\mathcal{U}, X) \text{ and } H^i(\mathcal{U}, K_\alpha).$$

The formula in (d1) follows. The proof of (d3) is identical, and we leave it to the reader. For (d2), (9.1.12) (b) gives [since $\psi_\alpha U_\alpha = 0$]

$$\phi_\alpha^L \psi_\alpha^L H^i(\mathcal{U}, X) = 0.$$

This proves the first claim. For the second, suppose $\alpha^1 \in \tau_L(\bar{X}_L(\gamma^1_{\mathcal{Y}}))$. Write (using (c))

$$P_{\bar{\gamma}}^{-1}(H^i(\mathcal{U}, X)) = Y_n^i \oplus Y_s^i = Y^i.$$

Here Y_n^i is the direct sum of all the irreducible constituents with α^1 not in their τ invariants, and Y_s^i the sum of those with α^1 in their τ invariants. By Theorem 7.3.16, the image of the natural map

$$\phi_\alpha^L \psi_\alpha^L Y^i \rightarrow Y^i$$

is just Y_n^i . Write $K(Y_n^i)$ for its kernel. Arguing as in the proof of Corollary 9.1.17(a), we find a short exact sequence

$$(9.5.12) \quad 0 \rightarrow Y_s^{i-1} \rightarrow P_{\bar{\gamma}}^{-1}(H^i(\mathcal{U}, K_\alpha)) \rightarrow K(Y_n^i) \rightarrow 0.$$

Set $U(Y_n^i) = K(Y_n^i)/Y_n^i$. The long exact sequence in cohomology attached to

$$0 \rightarrow X \rightarrow K_\alpha \rightarrow U_\alpha \rightarrow 0$$

gives

$$\begin{aligned} \dots \xrightarrow{\phi^i} P_{\bar{\gamma}}^{-1} H^i(\mathcal{U}, K_\alpha) &\xrightarrow{\psi^i} P_{\bar{\gamma}}^{-1} H^i(\mathcal{U}, U_\alpha) \xrightarrow{\rho^i} Y_n^{i+1} \oplus Y_s^{i+1} \\ &\xrightarrow{\phi^{i+1}} P_{\bar{\gamma}}^{-1} (H^{i+1}(\mathcal{U}, K_\alpha)) \xrightarrow{\psi^{i+1}} \dots \end{aligned}$$

By the first claim of (d2), the image of ρ^i is contained in Y_s^{i+1} . We claim that it is actually equal to Y_s^{i+1} . Suppose not. Then Y_s^{i+1} has some constituent $\bar{X}_L(\delta^1_{\mathcal{Y}})$ in common with $H^{i+1}(\mathcal{U}, K_\alpha)$. By (9.5.12), $\bar{X}_L(\delta^1_{\mathcal{Y}})$ occurs in Y_s^i or in $U(Y_n^{i+1})$. The first possibility is ruled out by (a). In the second case, there would be a constituent $\bar{X}_L(\delta^2_{\mathcal{Y}})$ of Y_n^{i+1} such that [by (e) applied to L]

$$\ell_L^I(\delta_{\mathcal{Y}}^2) - \ell_L^I(\delta_{\mathcal{Y}}^1) \equiv 1 \pmod{2}.$$

This is impossible by Lemma 9.5.10 and (a). So

$$\text{Im } \rho^i = Y_s^{i+1}.$$

It follows that

$$\text{Im } \phi^i \cong Y_n^i$$

By (9.5.12), there is a short exact sequence

$$0 \rightarrow Y_S^{i-1} \rightarrow \text{Im } \psi^i \rightarrow U(Y_n^i) \rightarrow 0.$$

So

$$(9.5.13) \quad P_{Y_n^1}^{-1}(H^1(\mathcal{U}, U_\alpha)) = Y_S^{i-1} \oplus Y_S^{i+1} \oplus U(Y_n^i),$$

at least on the level of characters. This is (d2).

For (e), we know that $\bar{X}(\gamma^1) \neq \bar{X}(\gamma)$; and Corollary 9.5.- gives

$$\text{Ext}_{\mathcal{G}, K}^1(\bar{X}(\gamma), \bar{X}(\gamma^1)) \neq 0.$$

So this is a special case of (b).

Q.E.D.

Corollary 9.5.14 Let $\mathcal{G} = \ell + \mathcal{U}$ be a θ -stable parabolic subalgebra of $\mathcal{G}$, $\mathcal{G}^1 = \ell^0 + \mathcal{U}^1$ a θ -stable parabolic subalgebra of ℓ , and X an irreducible $(\mathcal{G}, K)$ module of nonsingular infinitesimal character. Set

$$\begin{aligned} \mathcal{U}^0 &= \mathcal{U}^1 + \mathcal{U} \\ \mathcal{G}^0 &= \ell^0 + \mathcal{U}^0. \end{aligned}$$

Suppose Conjecture 7.3.25 holds. Then

$$H^i(\mathcal{U}^0, X) \cong \bigoplus_{p+q=i} H^p(\mathcal{U}^1, H^q(\mathcal{U}, X))$$

as $(\ell, L_n K)$ modules.

Proof. By Corollary 6.1.30, there is a spectral sequence

$$H^p(\mathcal{U}^1, H^q(\mathcal{U}, X)) \Rightarrow H^{p+q}(\mathcal{U}^0, X).$$

By Theorem 9.5.1(a), it is easy to check that the E_2^{pq} and $E_2^{p'q'}$ term can have a common constituent only if

$$p + q \equiv p' + q' \pmod{2}.$$

So the differentials are all zero, and the spectral sequence collapses. The isomorphism follows from Theorem 9.5.1(c).

Q.E.D.

§6. The Kazhdan-Lusztig conjecture.

Let $\gamma \in \hat{H}$ be a regular character, with $\bar{\gamma}$ nonsingular. We are going to describe an algorithm for computing the following things, assuming Conjecture 7.3.25.

- (a) The composition series of $X(\gamma)$ -- that is, the irreducible constituents and their multiplicities.

- (9.6.1) (b) The cohomology $H^*(\mathcal{U}, \bar{X}(\gamma))$ as an $(\ell, L_n K)$ module, for each θ -stable parabolic

$$\mathcal{G} = \ell + \mathcal{U} \text{ of } \mathcal{G}.$$

- (c) For each simple root $\alpha \in R^+(\bar{\gamma})$ such that $\alpha \notin \tau(\bar{X}(\gamma))$, the composition series of $U_\alpha(\bar{X}(\gamma))$.

This algorithm is the Kazhdan-Lusztig conjecture. It proceeds by induction on $\ell^1(\gamma)$. We will treat (a) and (b) first, then show at the end how to get (c) from (b).

Suppose first that there is a simple complex root $\alpha \in R^+(\bar{\gamma})$ such that $\theta\alpha \notin R^+(\bar{\gamma})$. Set $n = \langle \alpha, \bar{\gamma} \rangle$. By Lemma 8.6.13,

$$\ell^1(\gamma - n\alpha) = \ell^1(\gamma) - 1;$$

so we know (9.6.1)(a) - (c) for $\gamma - n\alpha$. Thus we can write

$$(9.6.2)(a) \quad \Theta(\gamma - n\alpha) = \sum_i n_i \bar{\Theta}(\gamma^i),$$

with n_i and γ^i known. By Proposition 8.6.19,

$$(9.6.2)(b) \quad \ell^I(\gamma^i) \leq \ell^I(\gamma - n\alpha) \quad \text{when } n_i \neq 0;$$

so we know $U_\alpha(\bar{X}(\gamma^i))$ whenever $\alpha \notin \tau(\bar{X}(\gamma^i))$ by induction. By Proposition 8.2.7,

$$\Theta(\gamma) = s_\alpha \cdot \Theta(\gamma - n\alpha).$$

Now Corollary 7.3.20 computes the irreducible constituents of $\Theta(\gamma)$ and their multiplicities. This is (9.6.1)(a).

For 9.6.1(b), Theorem 9.5.1(d) allows to write

$$(9.6.3)(a) \quad H^p\left[\mathcal{U}, U_\alpha(\bar{X}(\gamma - n\alpha))\right] = \sum m_j \bar{X}_L(\gamma_j^j)$$

with m_j and γ_j^j known; for we know the cohomology of $\bar{X}(\gamma - n\alpha)$. By inductive hypothesis (c), we can also write

$$(9.6.3)(b) \quad U_\alpha(\bar{X}(\gamma - n\alpha)) = \sum \tilde{p}_i \bar{X}(\gamma^i).$$

By Theorem 8.5.18(d), this can be rewritten as

$$(9.6.3)(b)' \quad U_\alpha(\bar{X}(\gamma - n\alpha)) = \bar{X}(\gamma) + \sum p_i \bar{X}(\gamma^i),$$

with

$$\ell^I(\gamma^i) < \ell^I(\gamma) \quad \text{when } p_i \neq 0.$$

So we also know by induction formulas like

$$(9.6.3)(c) \quad H^p(\mathcal{U}, \bar{X}(\gamma^i)) = \sum q_{ij} \bar{X}_L(\gamma_j^j).$$

So we deduce that

$$(9.6.4) \quad H^p(\mathcal{U}, \bar{X}(\gamma)) = \sum_j (m_j - \sum_i p_i q_{ij}) \bar{X}_L(\gamma_j^j).$$

This is (9.6.1)(b).

So we may assume that

$$(9.6.5) \quad \text{if } \alpha \in R^+(\bar{\gamma}) \text{ is simple and } \theta\alpha \notin R^+(\bar{\gamma}),$$

then α is real.

So we are in the situation (9.3.8), and we will use the notation of (9.3.8) freely. As a practical computational matter, it is worth pointing out that if there is a simple type I real root satisfying the parity condition for γ , then we can imitate the argument above. In general there need not be such a root, however, so we will do something much more complicated. The problems arise from Theorem 8.5.18(c): we can get information about $\bar{X}(\gamma) + \bar{X}(s_\alpha \times \gamma)$, but we cannot separate the two pieces easily.

So suppose we are in the setting (9.1.14): that is, we have $\gamma^1 \in (\hat{H}^1)$, and we want to compute either

$$m(\bar{X}(\gamma^1), X(\gamma))$$

or

$$m\left[\bar{X}_L(\gamma_{\mathcal{G}}^1), H^i(\mathcal{U}, \bar{X}(\gamma))\right].$$

We begin with the former; that is, with (9.6.1)(a). By

Lemma 9.3.12, we may assume $\gamma^1 \in \mathcal{K}(\gamma)$ (notation (9.3.8)(h)).

The problem we are going to solve is

$$(9.6.6) \quad \gamma^1 \in \mathcal{K}(\gamma), \text{ find the multiplicity of } \bar{X}(\gamma^1) \text{ in } X(\gamma').$$

for each $\gamma' \in \mathcal{L}(\gamma)$ [cf. (9.3.8)(f)] and

This is a strengthened version of (9.6.1)(a). To solve it, we need the following observation.

Observation 9.6.7 Suppose $\gamma^1 \in \mathcal{H}(\gamma)$, $\gamma' \in \mathcal{L}(\gamma)$, and that

$$m(\bar{X}(\gamma^1), X(\gamma'))$$

is known. Then

$$m(\bar{X}(\gamma^1), X(\gamma''))$$

is computable for any other $\gamma'' \in \mathcal{L}(\gamma)$.

Proof. By the definition of $\mathcal{L}(\gamma)$, it is clearly enough to prove this for $\gamma'' = s_\alpha \times \gamma'$, with $\alpha \in R^+(\bar{\gamma}')$ simple. If α does not satisfy the parity condition, or α is type I, then

$$X(\gamma') \cong X(\gamma'')$$

(Proposition 8.3.18); so there is nothing to prove. If α is type II and does not satisfy the parity condition, then

$$\Theta(\gamma') + \Theta(\gamma'') = \Theta(\gamma_\alpha) + s_{\tilde{\alpha}} \times \Theta(\gamma_\alpha)$$

(Corollary 8.4.4). The decomposition of $\Theta(\gamma_\alpha)$ into irreducibles is known by induction, and that of $s_{\tilde{\alpha}} \times \Theta(\gamma_\alpha)$ can be computed as outlined at (9.6.2). So we know the multiplicity of $\bar{X}(\gamma^1)$ in

$$\Theta(\gamma') + \Theta(\gamma'').$$

Clearly this establishes the claim.

Q.E.D.

We turn now to the algorithm for (9.6.6). By Lemma 9.3.24, we may separate three cases.

Case I. There is a simple root $\alpha^1 \in R^+(\bar{\gamma}^1) \cap \Delta(\mathcal{K}^R, \mathcal{H}^1)$, such that $\alpha^1 \notin \tau(\bar{X}(\gamma^1))$; and an element γ' of $\mathcal{L}(\gamma)$ such that the root $\alpha' \in R^+(\bar{\gamma}')$ corresponding to α^1 satisfies the parity condition. Assume for definiteness that α' is type II; the other case is similar. By Corollary 8.4.7(d), $X(\gamma')$ and $X(s_{\alpha'} \times \gamma')$ both contain $X(\gamma_\alpha)$ as a quotient. Write K', K'' for the corresponding submodules. Then

$$\begin{aligned} \Theta(K' + K'') &= \Theta(\gamma') + \Theta(s_{\alpha'} \times \gamma') - 2\Theta(\gamma_\alpha) \\ &= s_{\tilde{\alpha}} \cdot \Theta(\gamma_\alpha') - \Theta(\gamma_\alpha') \end{aligned}$$

(by Corollary 8.4.6). The left side transforms by -1 under the action of $s_{\tilde{\alpha}}$; so by Corollary 7.3.19, every irreducible constituent of $K' + K''$ has α in its τ invariant. In particular, $\bar{X}(\gamma')$ itself does not occur; so

$$m(\bar{X}(\gamma^1), X(\gamma')) = m(\bar{X}(\gamma^1), X(\gamma_\alpha')),$$

which is known by induction.

Case II. There are a complex simple root $\alpha^1 \in R^+(\bar{\gamma}^1) \cap \Delta(\mathcal{K}^R, \mathcal{H}^1)$, a simple root $\beta^1 \in R^+(\bar{\gamma}^1 - n\alpha^1) \cap \Delta(\mathcal{K}^R, \mathcal{H}^1)$, and an element $\gamma' \in \mathcal{L}(\gamma)$ such that θ_{α^1} is positive; and if α', β' are the roots in $R^+(\bar{\gamma}')$ corresponding to $(-\alpha^1, \beta^1)$ in $R^+(\bar{\gamma}^1 - n\alpha^1)$, we have

$$\begin{aligned} \beta^1 &\notin \tau(\bar{X}(\gamma^1 - n\alpha^1)) \\ \beta' &\in \tau(\bar{X}(\gamma')) \end{aligned}$$

[that is, β' satisfies the parity condition for γ']. By Theorem 8.5.18, $-\alpha^1 \in \tau(\bar{X}(\gamma^1 - n\alpha^1))$. If $\alpha' \in \tau(\bar{X}(\gamma'))$, we are in case I; so assume it is not. By Corollary 9.5.8,

$$(9.6.8) \quad m\left[\bar{X}(\gamma^1 - n\alpha^1), U_\alpha(\bar{X}(\gamma'))\right] = m\left[\bar{X}(\gamma'), U_\beta(\bar{X}(\gamma^1 - n\alpha^1))\right].$$

Since γ^1 lies in $\mathcal{F}(\gamma)$ and $H^1 \neq H$ [since $\mathcal{H}^1$ has a complex root in $\mathcal{L}^R$],

$$\ell^I(\gamma^1 - n\alpha^1) < \ell^I(\gamma').$$

By Theorem 8.5.18 and Proposition 8.6.19, the right side of (9.6.8) is therefore computable. Now write

$$(9.6.9) \quad \begin{aligned} \Theta(\gamma') &= \bar{\Theta}(\gamma') + x_1 \bar{\Theta}(\gamma^1) + m_0 \bar{\Theta}(\gamma^1 - n\alpha^1) \\ &\quad + \sum x_i \bar{\Theta}(\gamma^i) + \sum Y_j \bar{\Theta}(\gamma^j). \end{aligned}$$

Here the x_i and Y_j are unknown multiplicities; m_0 is known by inductive hypothesis [since $\ell^I(\gamma^1 - n\alpha^1) = \ell^I(\gamma^1) + 1$].

We choose notation so that

$$(9.6.9) \quad \begin{aligned} (a) \quad x_i &\neq 0 \iff \alpha \notin \tau(\bar{X}(\gamma^i)) \\ (b) \quad Y_j &\neq 0 \iff \alpha \in \tau(\bar{X}(\gamma^j)). \end{aligned}$$

By Propositions 8.3.19 and 8.3.18(f)

$$s_\alpha \cdot \Theta(\gamma') = \Theta(\gamma').$$

By Corollary 7.3.20 and (9.6.9), comparing multiplicities of $\bar{X}(\gamma^1 - n\alpha^1)$ gives

$$(9.6.10) \quad \begin{aligned} m_0 &= m\left[\bar{X}(\gamma^1 - n\alpha^1), U_\alpha(\bar{X}(\gamma'))\right] + x_1 - m_0 \\ &\quad + \sum x_i m\left[\bar{X}(\gamma^1 - n\alpha^1), U_\alpha(\bar{X}(\gamma^i))\right]. \end{aligned}$$

Suppose some term in the last sum is non-zero. By Theorem 8.5.18, $\bar{X}(\gamma^1 - n\alpha^1)$ occurs in $X(\gamma^i)$; so

$$\ell^I(\gamma^1 - n\alpha^1) \leq \ell^I(\gamma^i),$$

and x_i is known by induction. Furthermore, $U_\alpha(\bar{X}(\gamma^i))$ is known by induction, since

$$\ell^I(\gamma^i) < \ell^I(\gamma') = \ell^I(\gamma).$$

So all those terms are known. The first term on the left of (9.6.10) is known by (9.6.8) and the remarks after it; so everything is known but x_1 . So we can solve for x_1 , which is the multiplicity we want.

Case III. $\gamma^1 \in \mathcal{J}(\gamma)$. In this case $\ell^I(\gamma^1) = \ell^I(\gamma)$, so

$$m(\bar{X}(\gamma^1), X(\gamma')) = \begin{cases} 1, & \gamma^1 \text{ conjugate to } \gamma' \\ 0, & \gamma^1 \text{ not conjugate to } \gamma'. \end{cases}$$

This completes the algorithm for solving (9.6.6), and therefore (9.6.1) (a). Next, we describe the algorithm for 9.6.1(b). So we are in the setting (9.1.14). We proceed (for fixed γ) by downward induction on $\ell^I(\gamma^1)$, then by downward induction on $\dim \mathcal{U}$, and finally by downward induction on p (the degree of cohomology under consideration). Choose a θ -stable parabolic subalgebra

$$\mathcal{J}_\theta^1 = \mathcal{L}^0 + \mathcal{U}^1$$

of $\mathcal{L}$, with $\mathcal{L}^0$ the centralizer of $\mathcal{X}^1$ in $\mathcal{L}$, satisfying the hypotheses of Corollary 8.1.15 for γ_θ^1 . In particular,

$$\mathcal{R}_{\mathcal{J}_\theta^1}^i \left[X_{\mathcal{L}^0}((\gamma_\theta^1)_1) \right] = \begin{cases} x_{\mathcal{L}^0}(\gamma_\theta^1) & (i = s) \\ 0 & (i \neq s) \end{cases}$$

Set

$$\begin{aligned} \mathcal{U}^0 &= \mathcal{U}^1 + \mathcal{U} \\ \mathcal{U}^0 &= \ell^0 + \mathcal{U}^0. \end{aligned}$$

Suppose first that $\ell \neq \ell^0$, so that $\dim \mathcal{U}^0 > \dim \mathcal{U}$. Now $H^p(\mathcal{U}, \bar{X}(\gamma))$ is a completely reducible ℓ module by Theorem 9.5.1(c); so

$$\begin{aligned} m[\bar{X}_L(\gamma^1_{\mathcal{U}}), H^p(\mathcal{U}, \bar{X}(\gamma))] &= \dim \operatorname{Hom}_{\ell, L \cap K} \left[H^p(\mathcal{U}, \bar{X}(\gamma)), X_L(\gamma^1_{\mathcal{U}}) \right] \\ &= \dim \operatorname{Hom}_{\ell^0, L \cap K} \left[H^R(\mathcal{U}^1, H^p(\mathcal{U}, \bar{X}(\gamma))) \right], \\ &\quad X_L 0((\gamma^1_{\mathcal{U}})_{\mathcal{U}}) \\ &= m[\bar{X}_L 0(\gamma^1_{\mathcal{U}}), H^R(\mathcal{U}^1, H^p(\mathcal{U}, \bar{X}(\gamma)))] ; \end{aligned}$$

the second equality is Corollary 6.4.14(b). By Corollary 6.4.14(a), $\bar{X}_L(\gamma^1_{\mathcal{U}})$ occurs in $H^q(\mathcal{U}^1, *)$ only if $q \leq R$. So

$$\begin{aligned} &= m[\bar{X}_L 0(\gamma^1_{\mathcal{U}})], \\ &\quad \sum_{i+j=p+R} H^i[\mathcal{U}^1, H^j(\mathcal{U}, \bar{X}(\gamma))] - \\ &\quad - \sum_{j=p+1}^{\dim \mathcal{U}} m[\bar{X}_L 0(\gamma^1_{\mathcal{U}}), \\ &\quad \quad H^{p+R-j}(\mathcal{U}^1, H^j(\mathcal{U}, \bar{X}(\gamma)))] \\ &= m[\bar{X}_L 0(\gamma^1_{\mathcal{U}}), H^{p+R}(\mathcal{U}^0, \bar{X}(\gamma))] - \\ &\quad - \sum_{j=p+1}^{\dim \mathcal{U}} m[\bar{X}_L 0(\gamma^1_{\mathcal{U}}), \\ &\quad \quad H^{p+R-j}(\mathcal{U}^1, H^j(\mathcal{U}, \bar{X}(\gamma)))] . \end{aligned}$$

The last equality is Corollary 9.5.14. By inductive hypothesis, all of these multiplicities are known.

So we may suppose that $\ell = \ell^0$; that is,

$$(9.6.11) \quad \ell = \text{centralizer of } \mathcal{U}^1 \text{ in } \mathcal{G}.$$

To continue, we need

Observation 9.6.12 Suppose $\alpha \in (R^+)$ (notation (9.3.8) -- recall (9.6.5)) is a simple root not satisfying the parity condition. Then there is a computable expression

$$U_{\alpha}(\bar{X}(\gamma)) = \sum n_i \bar{X}(\gamma^i),$$

with

$$\ell^I(\gamma^i) < \ell^I(\gamma)$$

whenever $n_i \neq 0$.

Proof. We argue more or less as at (9.6.9) above. Since (9.6.1) (a) is already solved for γ , we know a formula

$$\Theta(\gamma) = \bar{\Theta}(\gamma) + \sum p_i \bar{\Theta}(\gamma^i) + \sum q_j \bar{\Theta}(\gamma^j),$$

with

$$\begin{aligned} \alpha \notin \tau(\bar{X}(\gamma^i)) & \quad \text{if } p_i \neq 0 \\ \alpha \in \tau(\bar{X}(\gamma^j)) & \quad \text{if } q_j = 0. \end{aligned}$$

By Propositions 8.3.19 and 8.3.18(f),

$$s_{\alpha} \cdot \Theta(\gamma) = \Theta(\gamma).$$

By Corollary 7.3.20, we deduce that

$$\begin{aligned} \bar{\Theta}(\gamma) + \sum p_i \bar{\Theta}(\gamma^i) + \sum q_j \bar{\Theta}(\gamma^j) &= \bar{\Theta}(\gamma) + \sum p_i \bar{\Theta}(\gamma^i) - \sum q_j \bar{\Theta}(\gamma^j) \\ &\quad + \Theta \left[U_{\alpha}(\bar{X}(\gamma)) \right] + \\ &\quad + \sum p_i \Theta \left[U_{\alpha}(\bar{X}(\gamma^i)) \right] . \end{aligned}$$

So

$$\Theta\left[U_{\alpha}(\overline{X}(\gamma))\right] = \Sigma 2q_j \overline{\Theta}(\gamma^j) - \Sigma p_i \Theta\left[U_{\alpha}(\overline{X}(\gamma^i))\right].$$

All of the U_{α} appearing on the right are known by induction, so we can compute the left side. The last assertion is Theorem 8.5.18(g).

Q.E.D.

We return now to the algorithm for computing

$$m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1), {}^{H^P}(\mathcal{U}, \overline{X}(\gamma))\right],$$

under the assumption (9.6.11). By Proposition 9.3.16, we may assume that $\gamma^1 \in \mathcal{F}(\gamma)$. Suppose first that there is a simple complex root

$$\alpha^1 \in R^+(\overline{\gamma}^1) \cap \Delta(\mathcal{K}^R, \mathcal{K}^1),$$

with $\theta\alpha^1$ positive. Let $\alpha \in R^+(\overline{\gamma})$ be the corresponding simple root; α is necessarily real, by (9.6.5) and the definition of $\mathcal{F}(\gamma)$. By (9.6.11),

$$(9.6.13) \text{ (a)} \quad \alpha^1 \notin \Delta(\mathcal{K}, \mathcal{K}^1).$$

Define $\varepsilon = \pm 1$ so that

$$(9.6.13) \text{ (b)} \quad \varepsilon\alpha^1 \in \Delta(\mathcal{U}, \mathcal{K}^1).$$

By Lemma 8.6.13,

$$(9.6.13) \text{ (c)} \quad \varrho^I(\gamma^1 - n\alpha^1) = \varrho^I(\gamma^1) + 1.$$

Suppose first that α satisfies the parity condition for γ . By Corollary 9.1.18 and (9.6.13) (b),

$$m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1), {}^{H^P}(\mathcal{U}, \overline{X}(\gamma))\right] = m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1 - n\alpha^1), {}^{H^{P+\varepsilon}}(\mathcal{U}, \overline{X}(\gamma))\right].$$

By (9.6.13) (c), the right side is known by induction. Next, suppose that α does not satisfy the parity condition for γ . Using Observation 9.6.12 and Theorem 9.5.1(d), we compute

$$\begin{aligned} m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1), {}^{H^P}(\mathcal{U}, \overline{X}(\gamma))\right] &= m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1), {}^{H^{P-\varepsilon}}(\mathcal{U}, U_{\alpha}(\overline{X}(\gamma)))\right] \\ &\quad - m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1 - n\alpha^1), {}^{H^{P-\varepsilon}}(\mathcal{U}, \overline{X}(\gamma))\right] \\ &= \Sigma n_i m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1), {}^{H^{P-\varepsilon}}(\mathcal{U}, \overline{X}(\gamma^i))\right] \\ &\quad - m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1 - n\alpha^1), {}^{H^{P-\varepsilon}}(\mathcal{U}, \overline{X}(\gamma))\right]. \end{aligned}$$

The right side is known by induction.

So we may assume

$$(9.6.14) \quad \begin{aligned} &\text{if } \alpha^1 \text{ is a simple complex root of } \mathcal{K}^1 \\ &\text{in } R^+(\overline{\gamma}^1) \cap \Delta(\mathcal{K}^R, \mathcal{K}^1), \text{ then } \theta\alpha^1 \text{ is} \\ &\quad \text{negative.} \end{aligned}$$

This is exactly the situation (9.3.17) (in light of Lemma 8.6.1), with γ^1 playing the role of γ^0 . So we can use the notation there. By Proposition 9.3.21, we may as well assume that $\gamma \in \mathcal{H}(\gamma, \gamma^1)$ (by replacing it by a conjugate); otherwise the multiplicity in question is zero. Now we need

Observation 9.6.15 Suppose $\gamma' \in \mathcal{G}(\gamma)$, and

$$m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1), {}^{H^P}(\mathcal{U}, \overline{X}(\gamma'))\right]$$

is known. Then

$$m\left[\overline{X}_L(\gamma_{\mathcal{G}}^1), {}^{H^P}(\mathcal{U}, \overline{X}(\gamma''))\right]$$

is computable, for any $\gamma'' \in \mathcal{G}(\gamma)$.

This is proved in precisely the same way as Observation 9.6.7, and we leave it to the reader. So we may replace γ by any

$\gamma' \in \mathcal{A}(\gamma)$. We choose γ' as in (9.3.22) (a). By Proposition 9.3.23, $\bar{X}_L(\gamma'_1)$ can occur in $H^*(\mathcal{U}, \bar{X}(\gamma'))$ only in one (computation) degree; so it is enough to compute

$$m\left(\bar{X}_L(\gamma'_1), \sum_p (-1)^p H^p(\mathcal{U}, \bar{X}(\gamma'))\right).$$

We have already computed

$$\theta(\gamma') = \bar{\theta}(\gamma') + \sum_i n_i \bar{\theta}(\gamma^i);$$

so the Euler characteristic above is

$$\begin{aligned} & \sum (-1)^p m\left(\bar{X}_L(\gamma'_1), H^p(\mathcal{U}, X(\gamma'))\right) - \\ & - \sum_{i,p} (-1)^p n_i m\left(\bar{X}_L(\gamma'_1), H^p(\mathcal{U}, \bar{X}(\gamma^i))\right). \end{aligned}$$

The first term is computed by Theorem 9.4.1, and the second is known by induction. This completes the algorithm for (9.6.1) (b).

For (9.6.1) (c), we drop all special hypotheses. Let $\gamma^1 \in (\hat{H}^1)$, and let $\mathcal{G} = \mathfrak{l} + \mathcal{U}$ be the associated θ -stable parabolic. We want to compute

$$m\left(\bar{X}(\gamma^1), U_\alpha(\bar{X}(\gamma))\right).$$

By Theorem 6.5.9 (g), this is

$$\begin{aligned} \dim \operatorname{Hom}_{\ell, L \cap K} \left[H^R(\mathcal{U}, U_\alpha(\bar{X}(\gamma))) \right]_{X_L(\gamma^1_{\mathcal{G}})} \\ = m\left(\bar{X}_L(\gamma^1_{\mathcal{G}}), H^R(\mathcal{U}, U_\alpha(\bar{X}(\gamma)))\right). \end{aligned}$$

This is computable from (9.6.1) (b), Theorem 9.5.1 (d), and the inductive hypothesis except when $\mathcal{G} = \mathcal{G}$. In that case, G is quasisplit, and $X(\gamma^1)$ is a principal series representation corresponding to a fine character of T^1 . By Lemma 6.6.6,

it follows that $\bar{X}(\gamma^1)$ cannot occur in $X(\gamma')$ unless γ' and γ^1 are conjugate; so the multiplicity of $\bar{X}(\gamma^1)$ in $U_\alpha(\bar{X}(\gamma))$ is computed by Theorem 8.5.18. This completes the algorithm.

§7. Open problems.

There is not much to add to the conjectures scattered through the book, but we will collect some of them here.

Problem 1. (Conjecture 7.3.25). Show that if X is an irreducible $(\mathcal{G}, K)$ module of nonsingular infinitesimal character $\bar{\gamma}$, $\alpha \in R^+(\bar{\gamma})$ is simple (notation (7.2.16)), and $\alpha \notin \tau(X)$ (notation (7.3.8)), then $U_\alpha(X)$ is completely reducible (notation (7.3.17)).

More loosely stated: study U_α as a functor on the category of $\mathcal{G}$ modules with infinitesimal character $\bar{\gamma}$, and try to say anything at all about it. How is it related to U_β for $\beta \neq \alpha$? At least for type A_n , the primitive ideals in $U(\mathcal{G})$ can be classified by studying such relationships; can they provide better information about composition series?

Problem 2 is Conjecture 6.5.17 on the unitarizability of certain cohomologically induced representations. Unfortunately, it does not seem to lend itself to simplification.

Problem 3. Suppose $H = TA$ is a θ -stable Cartan subgroup, and $\dim A = 1$. If $\gamma \in \hat{H}$, compute the composition series of $X(\gamma)$. Do the composition factors have multiplicity one?

This has been solved when G has rank one and in a very few other cases. It is quite hard, but should be amenable to existing techniques. Even to treat one family of cases [with rank (G) unbounded] would be interesting. The problem of reducibility of standard representations can be reduced to the case of Problem 3 (see [39]).

Problem 4. Suppose $\gamma \in \hat{H}'$, $\gamma^1 \in \{\hat{H}^1\}'$, and $\bar{\gamma}, \bar{\gamma}^1$ are nonsingular. If (notation (8.1.4))

$$\ell^1(\gamma) = \ell^1(\gamma^1) + 1,$$

compute the multiplicity of $\bar{X}(\gamma^1)$ in $X(\gamma)$. Is it always zero or one?

This problem is completely solved for Verma modules, and is not very hard. For real groups there are some difficulties associated with non-complex simple roots, but they may not be serious.

Problem 5. In the setting of Problem 4, suppose α is a simple root in $R^+(\bar{\gamma})$, $\alpha \notin \tau(\bar{X}(\gamma))$, and $\alpha \in \tau(\bar{X}(\gamma^1))$. Show that, except in the cases described by Theorem 8.5.18,

$$m(\bar{X}(\gamma^1), U_\alpha(\bar{X}(\gamma))) \leq m(\bar{X}(\gamma^1), X(\gamma)).$$

This would be a very useful fact for computing composition series [in the absence of a solution to Problem 1].

Problem 6. Investigate the cohomological induction functors of Section 6.3. More precisely, suppose $\gamma \in \hat{H}'$, and $\bar{\gamma}$ is dominant for a Borel subalgebra

$\ell = h + n$ of $\mathfrak{g}$. Prove

$$\theta(\mathcal{R}_{\xi}^i(\gamma \otimes \mathbb{C}_{\xi})) = \begin{cases} \theta(\gamma) & i = i_0 \\ 0 & i \neq i_0 \end{cases};$$

here $\mathbb{C}_{\xi}$ is some sort of fixed twist, and i_0 is to be computed.

Of course this can be weakened in many ways; we might ask only for an Euler characteristic result. On the other hand, we might try to arrange matters so that $X(\gamma)$ itself appears. The construction specified in the problem gives $X(\gamma)$ if H is compact, and $X(\gamma^C)$ if H is split.

Problem 7. Explain Theorem 9.5.1; or prove any of the statements in it without using Conjecture 7.3.25; or deduce the conjecture of Section 9.6 from (9.5.1)(a) alone [without further use of Conjecture 7.3.25].

This is a rather hazy area. It seems likely that (9.5.1)(a) is almost as strong as Conjecture 7.3.25, but this has not been proved. [The multiplicity formulas of Section 9.6 are equivalent to Conjecture 7.3.25, incidentally.] Of course, one way to get some insight would be to generalize (9.5.1)(a) in some way.

Problem 8. Let $\prec$ be the ordering on the set of regular characters with nonsingular infinitesimal character generated by

$$\gamma \prec \gamma' \text{ if } \bar{X}(\gamma) \text{ occurs in } X(\gamma').$$

Describe $\prec$ in some simple way, analogous to the Bruhat ordering on a Weyl group.

This ordering is computable by the proof of Proposition 8.6.19, so this is entirely a structural problem.

Problem 9. Order the set $\mathcal{A}$ (Definition 2.2.3) as in Problem 8 [cf. Theorem 2.2.4]. Map $\mathcal{A}$ to the set of orbits of $K_{\mathbb{Q}}$ on $\mathcal{B}$ [see Chapter 2] by Proposition 2.3.4. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are such orbits, show that

$$\mathcal{O}_1 \subseteq \overline{\mathcal{O}_2}$$

if and only if we can find elements γ^1 and γ^2 in $\mathcal{A}$, corresponding to $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively, such that

$$\gamma^1 \prec \gamma^2.$$

This would be a consequence of Conjecture 2.3.11. Better still would be a complete geometric description of $\prec$, such as one has for the Bruhat ordering.

Problem 10. Show that the conjectures of Section 9.6 are compatible with (or, better, equivalent to) Conjecture 2.3.11.

This is (judging from Kazhdan and Lusztig's proof in [25] of the analogous fact for Verma modules) a deep algebraic geometry problem; of course, it is an enormously stronger version of Problem 9.

Problem 11. Find a category of $(\mathcal{O}, K)$ modules in which there are vanishing theorems like Theorem 9.5.1(a) for

$$\mathrm{Ext}^i(\overline{X}(\gamma^1), X(\gamma)).$$

As is pointed out in [24], this is (conjecturally) true for the category $\mathcal{O}$ of Verma modules. Perhaps one only needs to work with $(\mathcal{O}, K)$ modules of finite length having a fixed infinitesimal character [in the strong sense that $Z(\mathcal{O})$ actually acts by scalars].

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$\Gamma, \Gamma^* = (\Gamma_{\mathcal{G}, \mathcal{B}}^{\mathcal{K}, \mathcal{K}})^*$

$\gamma_{\mathcal{G}}, \Gamma_{\mathcal{G}}$

$\gamma_{\alpha}^{\alpha}, \gamma_{\pm}^{\alpha}$

$\gamma_{\alpha}, \gamma_{\alpha}^{\pm}$

$\gamma^{\mathcal{C}}$

$\Delta^+(\mathcal{K}, \mathcal{L}^{\mathcal{C}})$

$\Delta^+(w)$

ε_{α}

θ

$\Theta(\quad)$

$\Theta(\gamma), \overline{\Theta}(\gamma), \overline{\Theta}(\gamma)(\pi), \Theta(\gamma)(\pi)$

Λ

$\Lambda^{\mathcal{S}}$

$\pi(\delta \emptyset \nu)$

$\overline{\pi}(\delta \emptyset \nu)$

$\overline{\pi}(\delta \emptyset \nu)(\mu)$

$\rho(\quad)$

σ_{α}

$\phi_{\alpha}, \phi_{\alpha}$

ϕ_{α}

ψ_{α}

$\psi_{\xi+\lambda 1}^{\xi+\lambda 1}$

$\psi_{\xi+\lambda 2}^{\xi+\lambda 2}$

$<, >$

$|| \cdot ||$

lambda

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