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A Generalized τ-Invariant for the Primitive Spectrum of a Semisimple Lie Algebra

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1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $U(\mathfrak{g})$ its universal enveloping algebra. Write $X = \operatorname{Prim} U(\mathfrak{g})$ for the set of primitive ideals in $U(\mathfrak{g})$; an ideal is called *primitive* if it is the annihilator of a (possibly infinite dimensional) irreducible representation of $\mathfrak{g}$. The problem we consider is the determination of X as an ordered set. So far this has been done only in some low rank cases (cf. [2], for example); if $\mathfrak{g}$ is of type A_n, Joseph has determined X as a set. The problem seems to have two parts: the use of algebraic methods to reduce to questions about Weyl groups, and then combinatorics to study these questions. As Joseph remarks in [10], 5.4, even the first part has not yet been carried far enough to describe X for $\mathfrak{g}$ not of type A_n. In the present paper we describe invariants which we hope are sufficient to describe X as an ordered set (i.e. we hope we have solved the first part of the problem above). More precisely, fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ containing $\mathfrak{h}$ (with nil radical $\mathfrak{n}$). Let R be the root system of $\mathfrak{h}$ in $\mathfrak{g}$, and W its Weyl group. For $\lambda \in \mathfrak{h}^*$, let

$$M(\lambda) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\varrho}$$

denote the Verma module with highest weight $\lambda - \varrho$, and $L(\lambda)$ its unique irreducible quotient. Since $L(\lambda)$ is irreducible, $I_\lambda = \operatorname{Ann}(L(\lambda))$ is a primitive ideal in $U(\mathfrak{g})$. By a deep theorem of Duflo [5] the map from $\mathfrak{h}^*$ to X given by $\lambda \mapsto I_\lambda$ is surjective. To describe X as a set, we need only describe the fibers of this map. Joseph has given a very good sufficient condition for $I_\lambda = I_{\lambda'}$ in [8], Theorem 5.1; it seems possible that this condition is necessary when the Dynkin diagram of R (or rather R_λ – cf. Sect. 2) is unbranched and classical. Joseph has proved this for $\mathfrak{g}$ of type A_n (cf. [10]). In Theorem 4.1 below, we give a slight sharpening of Joseph's condition for the case of a branched diagram; this result is probably known to the experts. The proof given here is a very complicated calculation, which is only outlined.

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Finally there is the problem of finding necessary conditions for $I_\lambda = I_{\lambda'}$. The first tool for this is of course of the center $\mathfrak{z}(\mathfrak{g})$ of $U(\mathfrak{g})$; $I_\lambda \cap \mathfrak{z}(\mathfrak{g})$ is a maximal ideal in $\mathfrak{z}(\mathfrak{g})$, and by a theorem of Harish-Chandra one knows exactly when $I_\lambda \cap \mathfrak{z}(\mathfrak{g}) = I_{\lambda'} \cap \mathfrak{z}(\mathfrak{g})$ in terms of λ and λ'. Next there is the Borho-Jantzen-Duflo τ-invariant, which will be described in Sect. 2; again it is trivial to determine when $\tau(I_\lambda) = \tau(I_{\lambda'})$ in terms of I_λ and $I_{\lambda'}$. Unfortunately these two invariants are far from separating points in X. The work of Borho, Jantzen, and Joseph has therefore relied on the introduction of two more complicated invariants: the Gelfand-Kirillov dimension and Goldie rank of $U(\mathfrak{g})/I_\lambda$. While these invariants are not very hard to define, there is no known general procedure for computing them in terms of λ. Thus their use requires enormous amounts of ingenuity and hard work.

To circumvent this, we return to the fundamental technique of Borho and Jantzen: a study of the composition series of the Verma modules $M(\lambda)$. The result we want is Theorem 4.14 of [14]. (It is stated there for Harish-Chandra modules. The result for Verma modules is similar but much easier; it is stated as Theorem 3.2 below.) To describe the consequences for primitive ideals, suppose for simplicity that $\lambda \in \mathfrak{h}^*$ is integral and regular (i.e. the stabilizer of λ in W consists of the identity alone), that α and β are adjacent simple roots of the same length, that $\alpha \in \tau(I_\lambda)$, and $\beta \notin \tau(I_\lambda)$. Then in Definition 3.4 we associate to I_λ a new primitive ideal $T_{\alpha\beta}(I_\lambda)$. Furthermore we describe an element $\lambda_1 \in \mathfrak{h}^*$ such that $T_{\alpha\beta}(I_\lambda) = I_{\lambda_1}$. If $I_\lambda = I_{\lambda'}$, then $T_{\alpha\beta}(I_\lambda) = T_{\alpha\beta}(I_{\lambda'})$; writing $T_{\alpha\beta}(I_{\lambda'}) = I_{\lambda'_1}$, we have in particular $\tau(I_{\lambda_1}) = \tau(I_{\lambda'_1})$; a new necessary condition for $I_\lambda = I_{\lambda'}$, computable in terms of λ and λ'. Continuing in this way (i.e. applying various $T_{\gamma\delta}$ to I_{λ_1} and $I_{\lambda'_1}$), we obtain a family of conditions – the "generalized τ-invariant" of the title. The operators $T_{\alpha\beta}$ are not defined on all of X, but each operator defines an order isomorphism of its domain of definition onto the domain of $T_{\beta\alpha}$, with inverse $T_{\beta\alpha}$. This result is stated in more detail as Corollaries 3.6 and 3.9.

The question of whether the results of this paper describe X completely as an ordered set is purely combinatorial; some low rank examples are mentioned at the end of Sects. 4 and 5. That they describe X as a set for $\mathfrak{g}$ of type A_n was discovered independently by Jantzen (first) and by the author; the proof, using combinatorial results of Joseph, is given in Sect. 6. (This amounts to the part of the Jantzen conjecture proved by Joseph in [10]; Joseph's methods involved much deeper ring-theoretic ideas.)

There is one further advantage of the present approach to the study of X. One of the original motivations for the study of Prim $U(\mathfrak{g})$ was its relevance to the representation theory of real semisimple Lie groups. That theory is now much better understood, but it would still be interesting to know the annihilators of irreducible Harish-Chandra modules in terms of the classification of these modules given by Langlands in [11]. Theorem 4.14 of [14] computes the generalized τ invariant of these annihilators; so if this invariant in fact separates points on X, the annihilators themselves are determined. In particular this is the case for the various real forms of $\mathfrak{sl}(n)$.

Since this paper was written, I have learned that most of the results on Verma modules which are used (notably Theorem 3.2 and Proposition 5.1) are due to Jantzen in [15]. I would like to apologize for not pointing this out in the body of the paper. The references to [14] have been retained as that paper may be more ...

2. Notation and Preliminary Results

We retain the notation of the introduction. A weight $\lambda \in \mathfrak{h}^*$ is called *dominant* if $\frac{2\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle}$ is never a negative integer, for any α in the system R^+ of positive roots defined by $\mathfrak{b}$. (Throughout, $\langle , \rangle$ denotes the standard bilinear form on $\mathfrak{h}^*$ dual to the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$.) λ is called *regular* or *nonsingular* if $\langle\alpha,\lambda\rangle$ is not zero for any $\alpha \in R$. Define

$$R_\lambda = \left\{\alpha \in R \,\middle|\, \frac{2\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}\right\},$$

$$R_\lambda^+ = R^+ \cap R_\lambda .$$

R_λ is a root system, and its Weyl group W_λ may be regarded as a subgroup of W. Write Σ_λ for the set of elements of order 2 in W_λ. We order W_λ as usual, by $w_1 < w_2$ if and only if $l_\lambda(w_2) = l_\lambda(w_1) + l_\lambda(w_2 w_1^{-1})$; here l_λ is the length function on W_λ. (Thus 1 is the smallest element of W_λ.)

Proposition 2.1 (Harish-Chandra – cf. [6], 23.3).

$I_\lambda \cap \mathfrak{z}(\mathfrak{g}) = I_{\lambda'} \cap \mathfrak{z}(\mathfrak{g})$ *if and only if* $\lambda' \in W \cdot \lambda$.

Put $\hat{\lambda} = W \cdot \lambda$, $X_{\hat{\lambda}} = \{I \in X \mid I \cap \mathfrak{z}(\mathfrak{g}) = I_\lambda \cap \mathfrak{z}(\mathfrak{g})\}$. To determine X as an ordered set, it suffices to determine each $X_{\hat{\lambda}}$ as an ordered set. By [2], 2.12, we may assume $-\lambda$ is dominant and regular.

Proposition 2.2 (Duflo [5], Proposition 9 and Corollary 1 of Proposition 10). *Suppose* $-\lambda \in \mathfrak{h}^*$ *is regular and dominant. Then the map* $f_\lambda : W_\lambda \to X_{\hat{\lambda}}$ *by* $w \mapsto I_{w\cdot\lambda}$ *is surjective and order preserving; in fact the restriction of* f_λ *to* Σ_λ *is already surjective.*

Let B_λ be the set of simple roots of R_λ^+. For $w \in W_\lambda$, define

$$S_\lambda(w) = \{\alpha \in R_\lambda^+ : w\alpha \notin R_\lambda^+\},$$
$$\tau_\lambda(w) = S_\lambda(w) \cap B_\lambda .$$

Fix a regular element $\lambda \in \mathfrak{h}^*$ with $-\lambda$ dominant. For each $\alpha \in B_\lambda$, choose an integral element $\mu_\alpha \in \mathfrak{h}^*$ such that $\mu_\alpha - \lambda$ is dominant, and $\langle\beta, \mu_\alpha - \lambda\rangle = 0$ if and only if $\beta = \pm\alpha$ (for $\beta \in R$). (Of course μ_α depends on λ, but we suppress this in the notation.) Let F_α denote the finite dimensional $\mathfrak{g}$ module of extremal weight μ_α, and F_α^* its contragredient module (which has extremal weight $-\mu_\alpha$). If M is a $\mathfrak{g}$ module, we write

$$P_\lambda(M) = \{m \in M \mid \text{for all } x \in \mathfrak{z}(\mathfrak{g}) \cap I_\lambda \text{ there exists } r \in \mathbb{N} \text{ with } x^r \cdot m = 0\}$$

$$\varphi_\alpha(M) = P_\lambda((P_{\lambda-\mu_\alpha}(M)) \otimes F_\alpha)$$

$$\psi_\alpha(M) = P_{\lambda-\mu_\alpha}((P_\lambda(M)) \otimes F_\alpha^*).$$

Lemma 2.3. *Suppose* M_1 *and* M_2 *are* $\mathfrak{g}$ *modules annihilated by ideals in* $\mathfrak{z}(\mathfrak{g})$ *of finite codimension, and suppose* $\mathrm{Ann}(M_1) \subsetneqq \mathrm{Ann}(M_2)$. *Then* $\mathrm{Ann}(\varphi_\alpha(M_1)) \subsetneqq \mathrm{Ann}(\varphi_\alpha(M_2))$, *and* $\mathrm{Ann}(\psi_\alpha(M_1)) \subsetneqq \mathrm{Ann}(\psi_\alpha(M_2))$.

This is essentially obvious; a proof may be found in [14], Lemmas 2.7 and 2.8.

If $I \in X_{\lambda}$, choose an irreducible $\mathfrak{g}$ module M such that $I = \mathrm{Ann}(M)$, and define $\psi_\alpha(I) = \mathrm{Ann}(\psi_\alpha(M))$. By Lemma 2.3, $\psi_\alpha(I)$ is well defined.

Theorem 2.4 ([2], 2.12). *Fix $-\lambda \in \mathfrak{h}^*$ dominant and regular. If $w \in W_\lambda$, then $\psi_\alpha(I_{w\cdot\lambda}) = U(\mathfrak{g})$ if and only if $\alpha \in \tau_\lambda(w)$. ψ_α is an order isomorphism of*

$$\{I_{w\cdot\lambda} | \alpha \notin \tau_\lambda(w)\}$$

onto $X_{(\lambda-\mu_\alpha)}$.

This is a special case of the Borho-Jantzen translation principle. Notice in particular that $\tau_\lambda(w)$ depends only on $I_{w\cdot\lambda}$; so we define $\tau(I_{w\cdot\lambda}) = \tau_\lambda(w)$ for $-\lambda \in \mathfrak{h}^*$ dominant and regular; this is the Borho-Jantzen-Duflo τ invariant.

We conclude this section with Joseph's sufficient condition for equality of primitive ideals.

Theorem 2.5 ([8], Theorem 5.1). *Suppose $-\lambda \in \mathfrak{h}^*$ is dominant and regular. Fix $w \in W_\lambda$ and $\alpha \in \tau_\lambda(w^{-1})$. Suppose $\tau_\lambda((s_\alpha w)^{-1}) \nsubseteq \tau_\lambda(w^{-1})$. Then*

$$I_{(s_\alpha w)\cdot\lambda} = I_{w\cdot\lambda}.$$

Recall that this is proved by a computation when $\dim \mathfrak{h} = 2$, and Joseph's „characteristic variety" is used to prove the result in general. We will extend the result by a computation when $\dim \mathfrak{h} = 4$ (cf. Sect. 4). The possibility of such an extension is pointed out by Joseph in [8]. The reference to "branched Dynkin diagrams" in the title of Sect. 4 reflects the hope that the extension is more than the first in an infinite sequence of "particular" results.

3. Definition of the Maps $T_{\alpha\beta}$ and the Generalized τ-Invariant

If $\mathfrak{g}$ is the direct sum of ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$, then any $I \in \mathrm{Prim}\, U(\mathfrak{g})$ is of the form $U(\mathfrak{g}_1) \otimes I_2 + I_1 \otimes U(\mathfrak{g}_2)$, with $I_i \in \mathrm{Prim}(U(\mathfrak{g}_i))$. The I_i are unique, and this defines an order isomorphism of $\mathrm{Prim}\, U(\mathfrak{g})$ with $\mathrm{Prim}(U(\mathfrak{g}_1)) \times \mathrm{Prim}(U(\mathfrak{g}_2))$. Accordingly there would be no serious loss of generality in assuming $\mathfrak{g}$ to be simple. While this will not be necessary, it will often be convenient to exclude simple factors of type G_2. Since $\mathrm{Prim}(U(\mathfrak{g}))$ is known in that case (see for example [2], 2.20), we may do so without loss of generality.

We fix for this section $-\lambda \in \mathfrak{h}^*$ dominant and regular. Suppose α and β are *adjacent* elements of B_λ (i.e. $\langle \alpha, \beta \rangle < 0$).

Definition 3.1. The domain of $T_{\alpha\beta}$, $D(T_{\alpha\beta})$, consists of those ideals $I \in X_{\lambda}$ such that $\alpha \notin \tau(I)$ and $\beta \in \tau(I)$.

Suppose $I \in D(T_{\alpha\beta})$; say $I = I_{w\cdot\lambda}$, for some $w \in W_\lambda$. Recall that $L(w\cdot\lambda)$ is the irreducible highest weight module of highest weight $w\cdot\lambda - \varrho$; and recall from Sect. 2 the functors φ_α, ψ_α, φ_β, and ψ_β. Clearly $\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$ has a finite composition series as a $\mathfrak{g}$ module.

Theorem 3.2. *With notation as above, suppose $\mathfrak{g}$ has no simple factors of type G_2.*

a) *If α and β have the same length, then $\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$ has precisely one irreducible composition factor M with $\beta \notin \tau(\mathrm{Ann}(M))$. M occurs with multiplicity one in $\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$, $\alpha \in \tau(\mathrm{Ann}(M))$, and we have either*

i) *$M \cong L(ws_\alpha\cdot\lambda)$, and $\beta \notin \tau(ws_\alpha)$, or*

ii) *$M \cong L(ws_\beta\cdot\lambda)$, and $\alpha \in \tau(ws_\beta)$ and these cases are mutually exclusive. $L(w\cdot\lambda)$ is a composition factor of $\varphi_\beta\psi_\beta(M)$.*

b) *If α and β have different lengths, then $\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$ has one or two composition factors M_1 and (possibly) M_2 with $\beta \notin \tau(\mathrm{Ann}(M_i))$. The M_i occur with multiplicity one in $\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$, $\alpha \in \tau(\mathrm{Ann}(M_i))$, and*

$$\{M_i\} = \{L(w'\cdot\lambda) | w' \in \{ws_\alpha, ws_\beta\},\ \alpha \in \tau(w'),\ \beta \notin \tau(w')\}.$$

$L(w\cdot\lambda)$ is a composition factor of $\varphi_\beta\psi_\beta(M_i)$.

Proof. An analogous result for Harish-Chandra modules is proved in [14], (Theorem 4.14). The result for Verma modules is proved in the same way, with many technical simplifications. Q.E.D.

Corollary 3.3. *With notation and hypotheses as in Theorem 3.2. we have in case a)*

$$\psi_\beta\varphi_\alpha\psi_\alpha(L(w\cdot\lambda)) \cong \psi_\beta(M)$$

and in case b)

$$\psi_\beta\varphi_\alpha\psi_\alpha(L(w\cdot\lambda)) \cong \psi_\beta(M_1) \oplus \psi_\beta(M_2)$$

(where the second factor on the right may fail to occur). The factors on the right in both cases are irreducible, and have the same Gelfand-Kirillov dimension as $L(w\cdot\lambda)$ and M (or either M_i).

Proof. The functor ψ_β is exact, and by Theorem 2.4, if M' is irreducible and has infinitesimal character λ, then $\psi_\beta(M') = 0$ if and only if $\beta \in \tau(\mathrm{Ann}(M'))$. So $\psi_\beta\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$ has the composition series indicated. The modules $\psi_\beta(M)$ and $\psi_\beta(M_i)$ are irreducible by [2], proof of Lemma 2.9; and the complete reducibility of $\psi_\beta\varphi_\alpha\psi_\alpha$ then follows as in Corollary 3.19 of [14]. By Lemma 2.2 of [13], the functors φ and ψ do not increase Gelfand-Kirillov dimension; since M (say) occurs in $\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$ and $L(w\cdot\lambda)$ in $\varphi_\beta\psi_\beta(M)$, M and $L(w\cdot\lambda)$ have the same Gelfand-Kirillov dimension. Finally M occurs in $\varphi_\beta(\psi_\beta(M))$ by formula 3.6 of [14], so M and $\psi_\beta(M)$ have the same Gelfand-Kirillov dimension; and similarly for the M_i. Q.E.D.

Definition 3.4. Suppose $I \in D(T_{\alpha\beta})$; say $I = I_{w\cdot\lambda}$, with notation as above, and suppose $\mathfrak{g}$ has no simple factors of type G_2.

a) If α and β have the same length, we define

$$T_{\alpha\beta}(I) = \mathrm{Ann}(M) \in D(T_{\beta\alpha}) \subseteq X_{\lambda}.$$

b) If α and β have different lengths, we define $T_{\alpha\beta}(I)$ to be the *unordered* pair $(\mathrm{Ann}(M_1), \mathrm{Ann}(M_2))$ or (if M_2 does not exist) $(\mathrm{Ann}(M_1), \mathrm{Ann}(M_1))$.

Since our definition of M or M_i dependened on choosing an element $w \in W$, with $I = I_{w\cdot\lambda}$, we must show that $T_{\alpha\beta}(I)$ is independent of that choice. In case a) this is easy; and in fact we can prove a little more.

Lemma 3.5. *Suppose α and β have the same length, $I_{w\cdot\lambda}, I_{w'\cdot\lambda}\in D(T_{\alpha\beta})$, and $I_{w\cdot\lambda}\subsetneqq I_{w'\cdot\lambda}$. Define M, M' corresponding to w, w' as in Theorem 3.2. Then* $\mathrm{Ann}(M)\subsetneqq\mathrm{Ann}(M')$.

Proof. By Corollary 3.3, $\psi_\beta(M)=\psi_\beta\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))$, and similarly for M'. By Lemma 2.3, $\mathrm{Ann}(\psi_\beta(M))\subsetneqq\mathrm{Ann}(\psi_\beta(M'))$. By the Borho-Jantzen translation principle (Theorem 2.4), $\mathrm{Ann}(M)\subsetneqq\mathrm{Ann}(M')$.

Corollary 3.6. *If α and β have the same length, then $T_{\alpha\beta}$ is a well defined order isomorphism of $D(T_{\alpha\beta})$ onto $D(T_{\beta\alpha})$, with inverse $T_{\beta\alpha}$.*

Proof. $T_{\alpha\beta}$ is well defined and order preserving by Lemaa 3.5; and $T_{\alpha\beta}:D(T_{\alpha\beta})\to D(T_{\beta\alpha})$ by definition. To complete the proof we need only show that $T_{\beta\alpha}\cdot T_{\alpha\beta}=\mathrm{Id}|_{D(T_{\alpha\beta})}$. So fix $I_{w\cdot\lambda}\in D(T_{\alpha\beta})$, and define M as in Theorem 3.2; $T_{\alpha\beta}(I_{w\cdot\lambda})=\mathrm{Ann}(M)$. By Theorem 3.2, $\varphi_\beta\psi_\beta(M)$ has a unique composition factor N with $\alpha\notin\tau(\mathrm{Ann}(N))$; and $T_{\beta\alpha}(\mathrm{Ann}(M))=\mathrm{Ann}(N)$ by definition. But by Theorem 3.2, $L(w\cdot\lambda)$ occurs in $\varphi_\beta\psi_\beta(M)$, and $\alpha\notin\tau(\mathrm{Ann}(L(w\cdot\lambda)))$ by hypothesis. So

$$\begin{aligned}T_{\beta\alpha}(T_{\alpha\beta}(I_{w\cdot\lambda}))&=T_{\beta\alpha}(\mathrm{Ann}(M))\\&=\mathrm{Ann}(N)\\&=\mathrm{Ann}(L(w\cdot\lambda))=I_{w\cdot\lambda}.\quad\text{Q.E.D}\end{aligned}$$

If α and β have different lengths, the situation is not quite so pleasant, but one can say a certain amount.

Lemma 3.7. *Let (I_1, I_2) and $I'_1, I'_2)$ be unordered pairs of primitive ideals in $U(\mathfrak{g})$. Then*

$$I_1\cap I_2\subsetneqq I'_1\cap I'_2$$

if and only if for $j=1$ and 2, we have

$$I_1\subsetneqq I'_j \quad or \quad I_2\subsetneqq I'_j.$$

In particular, if $U(\mathfrak{g})/I_i$ and $U(\mathfrak{g})/I'_j$ all have the same Gelfand-Kirillov dimension, then

$$I_1\cap I_2=I'_1\cap I'_2$$

if and only if $(I_1, I_2)=(I'_1, I'_2)$.

Proof. The first assertion follows from the fact that primitive ideals are prime. The second is an immediate consequence of the first, together with the following fact ([3], Korollar 3.6): if $I\subsetneqq J$ are primitive ideals, and $U(\mathfrak{g})/I$ and $U(\mathfrak{g})/J$ have the same Gelfand-Kirillov dimension, then $I=J$. Q.E.D.

Lemma 3.8 (Joseph [10], Proposition 6.5). *The Gelfand-Kirillov dimension of $U(\mathfrak{g})/I_{w\cdot\lambda}$ is twice that of $L(w\cdot\lambda)$.*

Corollary 3.9. *If α and β have different lengths, then $T_{\alpha\beta}$ is well defined.*

Proof. Fix $I=I_{w\cdot\lambda}=I_{w'\cdot\lambda}\in D(T_{\alpha\beta})$, and define M_i, M'_i associated to w and w' as in Theorem 3.2. Put $I_i=\mathrm{Ann}(M_i)$ (or $I_2=I_1$ if M_2 is not defined) and similarly $I'_1=\mathrm{Ann}(M'_i)$. Let ψ_β be the order isomorphism from

$$\{I\in X_\lambda|\beta\notin\tau(I)\}$$

to $X_{(\lambda-\mu_\beta)}$ given by Theorem 2.4. By Corollary 3.3 and Lemma 2.3,

$$\begin{aligned}\psi_\beta(I_1)\cap\psi_\beta(I_2)&=\mathrm{Ann}(\psi_\beta\varphi_\alpha\psi_\alpha(L(w\cdot\lambda))\\&=\mathrm{Ann}(\psi_\beta\varphi_\alpha\psi_\alpha(L(w'\cdot\lambda))\\&=\psi_\beta(I'_1)\cap\psi_\beta(I'_2).\end{aligned}$$

Furthermore, $U(\mathfrak{g})/\psi_\beta(I_i)$ and $U(\mathfrak{g})/\psi_\beta(I'_i)$ all have the same Gelfand-Kirillov dimension as $U(\mathfrak{g})/I$ by Corollary 3.3 and Lemma 3.8. By Lemma 3.7,

$$(\psi_\beta(I_1),\psi_\beta(I_2))=\psi_\beta(I'_1),\psi_\beta(I'_2)).$$

By Theorem 2.4,

$$(I_1,I_2)=(I'_1,I'_2).\quad\text{Q.E.D}$$

Lemma 3.7 gives information about how $T_{\alpha\beta}$ is related to the ordering in X_λ when α and β have different lengths. We leave the formulation of such a result to the reader.

We now formulate a little more precisely the notion of "generalized τ-invariant".

Definition 3.10. Suppose $-\lambda\in\mathfrak{h}^*$ is dominant and regular, and that $\mathfrak{g}$ has no simple factors of type G_2.

If I_1 and I_2 are primitive ideals in X_λ, we say that I_1 and I_2 are *equivalent to order zero*, $I_1\underset{0}{\approx}I_2$, if $\tau(I_1)=\tau(I_2)$. Inductively we define equivalence to order n: for $n\geqq1$, we say I_1 and I_2 are *equivalent to order n*, $I_1\underset{n}{\approx}I_2$, if $I_1\underset{n-1}{\approx}I_2$; and for every pair α, β of adjacent elements of B_λ with $\{I_1,I_2\}\subsetneqq D(T_{\alpha\beta})$, we have

a) if α and β have the same length, then

$$T_{\alpha\beta}(I_1)\underset{n-1}{\approx}T_{\alpha\beta}(I_2),$$

b) if α and β have different lengths, and $T_{\alpha\beta}(I_i)=(J_i,J'_i)$, then either

$$J_1\underset{n-1}{\approx}J_2 \quad\text{and}\quad J'_1\underset{n-1}{\approx}J'_2$$

or

$$J_1\underset{n-1}{\approx}J'_2 \quad\text{and}\quad J'_1\underset{n-1}{\approx}J_2.$$

We say that I_1 and I_2 are *equivalent to infinite order*, or that I_1 and I_2 *have the same generalized τ-invariant* if $I_1\underset{n}{\approx}I_2$ for every non-negative integer n.

Obviously $I_1=I_2$ implies that I_1 and I_2 have the same generalized τ invariant. Quite a bit of evidence suggests that the converse also holds, namely

Conjecture 3.11. *Suppose $-\lambda\in\mathfrak{h}^*$ is dominant and regular, and that $\mathfrak{g}$ has no simple factors of type G_2. If I_1 and I_2 are primitive ideals in X_λ, then $I_1=I_2$ if and only if I_1 and I_2 have the same generalized τ invariant.*

The usefulness of such a result is greatly enhanced by the following proposition.

Proposition 3.12. *With notation and hypotheses as in Definition 3.10, suppose $I_i = I_{w_i \cdot \lambda}$, $w_i \in W_\lambda$. Then one can compute whether or not $I_1 \underset{n}{\approx} I_2$ in terms of w_1 and w_2.*

Proof. Recall from Sect. 2 that $\tau(I_{w \cdot \lambda}) = \tau_\lambda(w)$; so one can compute whether $I_1 \underset{0}{\approx} I_2$ in terms of w_1 and w_2. Now proceed by induction on n, and suppose α and β are as in Definition 3.10. Theorem 3.2 provides *explicit* elements σ_i (if α and β have the same length) or σ_i, σ_i' (if they have different lengths) in W_λ, defined in terms of w_i alone, such that $T_{\alpha\beta}(I_i) = I_{\sigma_i \cdot \lambda}$ or $(I_{\sigma_i \cdot \lambda}, I_{\sigma_i' \cdot \lambda})$ respectively. The proposition is now obvious. Q.E.D.

Because of Proposition 3.12, we may sometimes speak of the generalized τ invariant of $w \in W_\lambda$.

An analogous result holds for Harish-Chandra modules: if I_i is the annihilator of the irreducible Harish-Chandra module J_{ϱ_i} (in the notation of Langland's classification [11]), then Theorem 4.14 of [14] allows one to compute whether $I_1 \underset{n}{\approx} I_2$ in terms of ϱ_1 and ϱ_2, and also whether $I_1 \underset{n}{\approx} I_{w_2 \cdot \lambda}$ in terms of ϱ_1 and w_2. Unfortunately (and this objection applies to Proposition 3.12 as well) the computation is in the form of an algorithm; it would be nice to have something more direct.

If I_1 and I_2 are primitive ideals in X_λ and $I_1 \subsetneqq I_2$, then $\tau(I_2) \subsetneqq \tau(I_1)$; this result of Borho, Jantzen, and Duflo is a trivial consequence of Theorem 2.4. We could therefore define a "containment" relation analogous to "equivalence" exactly as in Definition 3.10, and conjecture that $I_1 \subsetneqq I_2$ if and only if I_1 is "contained in I_2 to infinite order". We have not done this, because it is more difficult to collect evidence for such a conjecture – even in rank three or four, there are too many possible containment relations to check easily. (This would be an excellent project for someone with time or a computer on his hands.) We have determined all containment relations in various low rank cases, however, using in addition Gelfand-Kirillov dimension. This is discussed in more detail in Sect. 5.

4. Branched Dynkin Diagrams

Our goal in this section is another result along the lines of Theorem 2.5.

Theorem 4.1. *Suppose $-\lambda \in \mathfrak{h}^*$ is dominant and regular. Suppose there are four roots α_0, α_1, α_2, and α_3 in B_λ spanning a subsystem of R_λ of type D_4, with α_0 the central root. Set $A = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$. Fix $w \in W_\lambda$ with*

$$\tau_\lambda(w^{-1}) \cap A = \{\alpha_0\},$$

$$\tau_\lambda((s_{\alpha_0} w)^{-1}) \cap A = \{\alpha_1, \alpha_2, \alpha_3\}.$$

Suppose we are given a subset B' of $A - \{\alpha_0\}$ consisting of two or three elements, with the following property: for every two element subset $\{\alpha, \alpha'\}$ of B', $\alpha_0 \notin \tau_\lambda((s_\alpha s_{\alpha'} w)^{-1})$.

Put $s_{B'} = \prod_{\alpha \in B'} s_\alpha$. Then $I_{(s_{B'} w) \cdot \lambda} = I_{w \cdot \lambda}$.

Proof. Using Joseph's characteristic variety exactly as in the proof of Theorem 5.1 of [8], we may reduce to the case $\dim \mathfrak{h} = 4$. Since R contains a simple subsystem (namely R_λ) of rank 4, R itself is simple. Obviously A_4 does not contain a subsystem of type D_4, so R is of type D_4, C_4, B_4, or F_4. Although B_4 and F_4 contain subsystems of type D_4, these are not of the form R_λ for any $\lambda \in \mathfrak{h}^*$ (as one easily checks). So we may assume $\mathfrak{g} \cong \mathfrak{so}(8)$ or $\mathfrak{sp}(4)$.

The Weyl group W_λ of R_λ has order 192, and it is not difficult to compute its fully contracted graph ([8], Sect. 6), which has order 40 ([8], Remark 2 after 7.12). Only 38 of these classes contain involutions; so by [8], X_λ has at most 38 elements. The generalized τ invariant separates the 38 points of the fully contracted graph containing involutions into 36 equivalence classes, exactly two of which have two elements; choose representatives w_1, w_1' and w_2, w_2' for the corresponding classes in W_λ. By Duflo's ordering principle (Proposition 2.2) and the computation, we can make these choices so that $I_{w_i \cdot \lambda} \subseteq I_{w_i' \cdot \lambda}$.

By direct computation, one can verify that under the hypotheses of the theorem (and when $\dim \mathfrak{h} = 4$) $I_{(s_{B'} w) \cdot \lambda}$ and $I_{w \cdot \lambda}$ have the same generalized τ invariant. (Probably there is a conceptual proof of this fact.) So to prove the theorem, it is enough to show that the generalized τ invariant separates points in this case; i.e. that $|X_\lambda| = 36$, or equivalently that $I_{w_i \cdot \lambda} = I_{w_i' \cdot \lambda}$. When $\mathfrak{g} \cong \mathfrak{so}(8)$, this has been proved by Borho and Jantzen (unpublished). The author is not familiar with their methods, but presumably they can be made to work in case $\mathfrak{g} \cong \mathfrak{sp}(4)$ as well. For completeness, however, we will outline (extremely unpleasant) computational proofs in both cases.

First of all, the computation of w_i and w_i' shows that $T_{\alpha_0 \alpha_1}(I_{w_1 \cdot \lambda}) = I_{w_2 \cdot \lambda}$, and $T_{\alpha_0 \alpha_1}(I_{w_1' \cdot \lambda}) = I_{w_2' \cdot \lambda}$, at least after changing notation slightly if necessary. So it suffices to show that $I_{w_1 \cdot \lambda} = I_{w_1' \cdot \lambda}$. Furthermore (again by the computation) we may take $w_1 = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3}$, $w_1' = w_1 s_{\alpha_0} w_1$. By [9], Corollary 3.5, the Gelfand-Kirillov dimension of $U(\mathfrak{g})/I_{w_1 \cdot \lambda}$ is $|R| - 6$; since $I_{w_1 \cdot \lambda} \subseteq I_{w_1' \cdot \lambda}$, it suffices to show that $U(\mathfrak{g})/I_{w_1' \cdot \lambda}$ also has Gelfand-Kirillov dimension $|R| - 6$. This we do by a consideration of principal series representations.

Let $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$. Fix involutory antiautomorphisms $u \to {}^t u$ and $u \to \check{u}$ of $U(\mathfrak{g})$ satisfying ${}^t H = H$ for $H \in \mathfrak{h}_1$ and $\check{Y} = -Y$ for $Y \in \mathfrak{g}$. Put $\mathfrak{k} = \{(Y, -{}^t Y) : Y \in \mathfrak{g}\} \subseteq \bar{\mathfrak{g}}; \mathfrak{k} \cong \mathfrak{g}$. Given μ, $\nu \in \mathfrak{h}^*$ such that $\mu - \nu$ is integral, we define the $\bar{\mathfrak{g}}$ module $L(\mu, \nu)$ to be the space of $\mathfrak{k}$-finite vectors in $(M(-\mu) \otimes M(-\nu))^*$; $L(\mu, \nu)$ is a principal series representation. [Recall that $M(-\mu)$ is the Verma module of highest weight $-\mu - \varrho$.] For $w \in W_\lambda$, a theorem of Zelobenko (cf. [4]) asserts that $L(-w\lambda, -\lambda)$ has a unique irreducible quotient $V(-w\lambda, -\lambda)$. A theorem of Joseph ([7], Theorem 5.2) asserts that $\mathrm{Ann}(V(-w\lambda, -\lambda)) = \check{I}_{-w\lambda} \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \check{I}_{-w^{-1}\lambda}$. (We are using the fact that $-1 \in W_\lambda$ in this case.) Now the Gelfand-Kirillov dimension of $V(-w\lambda, -\lambda)$ is just half that of $U(\mathfrak{g})/\mathrm{Ann}(V(-w\lambda, -\lambda))$ ([9], Remark after 3.3); so the Gelfand-Kirillov dimension of $V(w_1 \cdot \lambda, -\lambda)$ is $|R| - 6$, and we want to show that the same holds for $V(w_1' \cdot \lambda, -\lambda)$. Put $d = |R| - 6$. In terms of the Bernstein degree or multiplicity ([13], 2.1), we must show that

$$c_d(V(w_1' \cdot \lambda, -\lambda)) \neq 0.$$

By [13], Corollary 4.5, there is a polynomial function p on $\mathfrak{h}^* \times \mathfrak{h}^*$ such that if $-\mu$

... are dominant and regular, and $\lambda-\mu$, $\lambda-\nu$ are integral, then

$c_d(V(w_1\cdot\mu, -\nu))=p(\mu,\nu)$.

Since $\tau(V(w_1\cdot\lambda, -\lambda))=\{(\alpha_i,0),(0,\alpha_i)|i=1,2,3\}$, the discussion preceding 4.8 of [13] implies that p is divisible by $\prod_{i=1}^{3}(\mu,\alpha_i)(\nu,\alpha_i)$. We claim that p is actually a multiple of $\prod(\mu,\alpha_i)(\nu,\alpha_i)$. It suffices to show that p has degree at most six; we sketch an argument for this. By well-known intertwining operator considerations, one can construct a non-degenerate pairing of $V(w_1\cdot\mu, -\nu)$ with the highest weight module $L(w_1\cdot\mu)\otimes L(w_1\cdot\nu)$. It is more or less obvious that $q(\mu,\nu)=c_d(L(w_1\cdot\mu)\otimes L(w_1\cdot\nu))$ has degree 6 (compare the proof of Corollary 3.5 of [9]). The proof of Lemma 3.4 of [13] is easily sharpened to show that $\deg p\leqq\deg q$; so $p(\mu,\nu)=c_0\prod_{i=1}^{3}(\mu,\alpha_i)(\nu,\alpha_i)$ as claimed.

Now we proceed as in the proof of Proposition 4.8 of [13] to compute the Bernstein degree of the "coherent continuations" of $V(w_1\cdot\lambda, -\lambda)$. The results of [14] (Theorems 4.12 and 4.14) allow one to express these coherent continuations in terms of irreducible representations. After a calculation, one finds the following: set $w_3=w_1s_{\alpha_0}s_{\alpha_2}s_{\alpha_3}$, and let δ denote the root $(\alpha_1,0)$. Then $V(w_1\cdot\lambda, -\lambda)$ occurs say x times in $\varphi_\delta\psi_\delta(V(w_3\cdot\lambda, -\lambda))$; and the dth Bernstein degree $c_d(V(w_1'\cdot\mu, -\nu))=r(\mu,\nu)$ is given by $r(\mu,\nu)=(2-x)p(\mu,\nu)$. So we must show that x is less than 2. (In fact $x=1$.) Such a result follows rather easily from Conjecture 3.15 of [14]; but since that conjecture remains unproved, another approach is necessary.

The multiplicities of irreducibles in coherent continuations can be used to compute composition series of principal series (cf. [14], Sect. 5). Put $w_4'=s_{\alpha_0}w_1's_{\alpha_0}$, $w_4=s_{\alpha_0}w_1s_{\alpha_0}$. Then another computation shows that $V(w_4\cdot\lambda, -\lambda)$ occurs $2+x$ times in $L(w_4'\cdot\lambda, -\lambda)$.

Suppose first that $\mathfrak{g}\cong\mathfrak{so}(8)$. Let μ be the fundamental weight for the root α_0. By a translation principle ([12], Theorem 6.18), $V(w_4\cdot\mu, -\mu)$ occurs $2+x$ times in $L(w_4'\cdot\mu, -\mu)$. (This is not quite trivial, since μ is singular.) By Zelobenko's theorem (cf. [4]) $V(w_4\cdot\mu, -\mu)|_{\mathfrak{k}}$ contains the adjoint representation of $\mathfrak{k}$ (its lowest $\mathfrak{k}$-type) exactly once. Since $V(w_4'\cdot\mu, -\mu)|_{\mathfrak{k}}$ contains but is not equal to the trivial representation, it must contain the adjoint representation at least once. So $L(w_4'\cdot\mu, -\mu)$ contains the adjoint representation at least $(2+x)+1=3+x$ times. But by Frobenius reciprocity, $L(w_4'\cdot\mu, -\mu)$ contains the adjoint representation 4 times; so $x\leqq 1$ as desired.

Next take $\mathfrak{g}\cong\mathfrak{sp}(4)$. Let μ be the shortest dominant weight such that $\langle\alpha_0,\mu\rangle\neq 0$ and $\mu-\lambda$ is integral for R. Let F be the fundamental representation of $\mathfrak{g}$ corresponding to the middle short simple root. Using the ideas of [14], one can partially compute $V(w_4'\cdot\mu, -\mu)|_{\mathfrak{k}}$; one finds for example that F occurs $3-x$ times. Write $\bar{F}$ for the representation $(F\otimes 1)$ of $\bar{\mathfrak{g}}$. The multiplicity of F in $(V(w_4'\cdot\mu, -\mu)\otimes\bar{F})|_{\mathfrak{k}}$ can then be computed; it is $7-x$. On the other hand, Lemma 5.4 of [12] expresses this tensor product as a combination of various coherent continuations of $V(w_4'\cdot\mu, -\mu)$. These continuations can be expressed in terms of irreducibles by the methods of [14], and the various restrictions to $\mathfrak{k}$ computed. In this way one finds that F occurs 6 times in the tensor product. Hence $x=1$ as desired. Q.E.D.

Some remarks about this proof are in order. The calculations described are not very deep, but they are very long; and I know of no conceptual reason for them to succeed. Various approaches which do *not* work include: proving the non-existence of nilpotent orbits in $\mathfrak{g}$ of appropriate dimensions, estimating Gelfand-Kirillov dimensions using the Duflo ordering principle (although this works when the set B' of the theorem has only two elements), and using Theorem 3.9 of [14] to study $\varphi_\delta\psi_\delta(V(w_3\cdot\lambda, -\lambda))$. It is extremely likely that all composition series of principal series (or Verma modules) with R_λ of type D_4 can be computed in terms of x by the methods of [14]. If for example the multiplicity of some composition factor is of the form $1-x$, then of course we could deduce $x\leqq 1$. Checking this would, however, require more long computations.

Definition 4.2. Suppose $-\lambda\in\mathfrak{h}^*$ is dominant and regular. Let $\sim$ be the equivalence relation on W_λ generated by

a) if w and s_α satisfy the hypotheses of Theorem 2.5, then $w\sim s_\alpha w$; and

b) if w and $s_{B'}$ satisfy the hypotheses of Theorem 4.1, then $w\sim s_B w$.

The equivalence classes of $\sim$ are called the *strong cells* of W_λ; the equivalence class of w is written $\mathcal{SC}_w$, and the set of equivalence classes is written $\mathcal{SC}(W_\lambda)$; in analogy with [8], Sect. 6, this is sometimes called the *strongly contracted graph* of W_λ.

Conjecture 4.3. *Suppose $-\lambda\in\mathfrak{h}^*$ is dominant and regular, and $\mathfrak{g}$ is classical. If w_1, $w_2\in W_\lambda$, then w_1 and w_2 lie in the same strong cell of W_λ if and only if they have the same generalized τ invariant.*

This is consistent with the popular conjecture that the structure of X_λ depends only on W_λ as an abstract reflection group.

Proposition 4.4. *Suppose $-\lambda\in\mathfrak{h}^*$ is dominant and regular. Then the Duflo surjection $W_\lambda\to X_{\tilde\lambda}$ factors through $W_\lambda\to\mathcal{SC}(W_\lambda)$. If $\mathfrak{g}$ is classical, then Conjecture 4.3 implies that $\mathcal{SC}(W_\lambda)\to X_{\tilde\lambda}$ is a bijection, and that Conjecture 3.11 holds.*

This is obvious. Note that Conjecture 4.3 is a purely combinatorial question about W_λ. Furthermore it is enough to verify it in case R_λ is simple. I have done this for R_λ classical of rank $\leqq 4$; and the case R_λ of type A_n is treated in Sect. 6. Accordingly Proposition 4.4 classifies $X_{\tilde\lambda}$ when R_λ is a product of such systems. (This classification was previously obtained by Joseph and Borho and Jantzen.) For R_λ of type F_4, calculations by L. Corman show that Conjecture 4.3 fails. The calculations indicate, but do not quite prove, that Conjecture 3.11 holds.

5. Gelfand-Kirillov Dimension

We collect here some results useful for computing Gelfand-Kirillov dimensions. We use the standard notation Dim for Gelfand-Kirillov dimension. The first is a reduction technique.

Proposition 5.1. *Let $B^0\subseteq B$ be a subset of the simple roots of $\mathfrak{g}$. Suppose $-\lambda\in\mathfrak{h}^*$ is dominant. Let $B_\lambda^0=\mathbb{Z}B^0\cap B_\lambda$, and suppose $w\in W_\lambda$ is a product of reflections in B_λ^0. Let $\mathfrak{g}^0\supseteq\mathfrak{h}$ be the reductive subalgebra of $\mathfrak{g}$ generated by B^0, and $I_{w\cdot\lambda}(\mathfrak{g}^0)\subseteq U(\mathfrak{g}^0)$ the* *primitive ideal corresponding to w under the Duflo map for $\mathfrak{g}^0$. Let $R^0 = \mathbb{Z}B^0 \cap R$. Then*

$$\operatorname{Dim}(U(\mathfrak{g})/I_{w\cdot\lambda}) = |R - R^0| + \operatorname{Dim}(U(\mathfrak{g}^0)/I_{w\cdot\lambda}(\mathfrak{g}^0)).$$

Proof. Let $\mathfrak{p} \supseteq \mathfrak{b}$ be the parabolic subalgebra with Levi factor $\mathfrak{g}^0$. The hypotheses on w immediately imply that

$$I^{\mathfrak{g}}_{\mathfrak{p}}(L^0(w\cdot\lambda)) = U(\mathfrak{g}) \bigotimes_{\mathfrak{p}} [L^0(w\cdot\lambda) \otimes \mathbb{C}_{-\varrho+\varrho^0}]$$

[here $L^0(w\cdot\lambda)$ is the highest weight module for $\mathfrak{g}^0$, etc.] is irreducible. Because it obviously has highest weight $\lambda - \varrho$, we have $L(w\cdot\lambda) \cong I^{\mathfrak{g}}_{\mathfrak{p}}(L^0(w\cdot\lambda))$. By say [13], Lemma 2.3, $\operatorname{Dim}(L(w\cdot\lambda)) = \operatorname{Dim}(L^0(w\cdot\lambda)) + \dim \mathfrak{g}/\mathfrak{p}$. The proposition now follows from Lemma 3.8. Q.E.D.

Proposition 5.2. *The maps $T_{\alpha\beta}$ preserve Gelfand-Kirillov dimension. (In particular, if α and β have different lengths and $T_{\alpha\beta}(I) = (J, J')$, then*

$$\operatorname{Dim}(U(\mathfrak{g})/I) = \operatorname{Dim}(U(\mathfrak{g})/J) = \operatorname{Dim}(U(\mathfrak{g})/J').)$$

Proof. This follows from Corollary 3.3 and Lemma 3.8. Q.E.D.

If B' is a subset of B_λ, we write $w_{B'}$ for the unique product of reflections in B' which takes B' to $-B'$.

Proposition 5.3 ([9], Corollary 3.5). *Suppose $-\lambda \in \mathfrak{h}^*$ is dominant and regular. For each $B' \subsetneq B_\lambda$,*

$$\operatorname{Dim}(U(\mathfrak{g})/I_{w_{B'}\cdot\lambda}) = |R - R'|;$$

here $R' = \mathbb{Z}B' \cap R$.

Write $w_\lambda = w_{B_\lambda}$. Recall from [2], Corollar 2.19, the almost maximal ideals of $X_{\hat{\lambda}}$. These are parametrized by B_λ, via $\alpha \to P_\alpha = I_{(w_\lambda s_\alpha)\cdot\lambda}$; P_α is the smallest primitive ideal I with $\alpha \notin \tau(I)$. The following result was communicated to me by Jantzen.

Proposition 5.4. *Suppose $-\lambda \in \mathfrak{h}^*$ is dominant and regular. If α and β lie in the same connected component of B_λ, then $\operatorname{Dim}(U(\mathfrak{g})/P_\alpha) = \operatorname{Dim}(U(\mathfrak{g})/P_\beta)$.*

Proof. We may assume $\mathfrak{g}$ is simple. If $\dim \mathfrak{h} = 2$, then [2], Satz 2.20 and Proposition 5.3 above imply that both dimensions are $|R| - 2$. So we may assume $\mathfrak{g}$ has no simple factors of type G_2. Clearly it is enough to show that if α and β are adjacent, then

$$\operatorname{Dim}(U(\mathfrak{g})/P_\alpha) \geqq \operatorname{Dim}(U(\mathfrak{g})/P_\beta).$$

Let I be either $T_{\alpha\beta}(P_\alpha)$ or one of the two components of $T_{\alpha\beta}(P_\alpha)$ (according as α and β have the same or different lengths). By Theorem 3.2 and Definition 3.4, $\beta \notin \tau(I)$; so $I \subsetneq P_\beta$. By Proposition 5.2,

$$\operatorname{Dim}(U(\mathfrak{g})/P_\alpha) = \operatorname{Dim}(U(\mathfrak{g})/I) \geqq \operatorname{Dim}(U(\mathfrak{g})/P_\beta). \quad \text{Q.E.D.}$$

Corollary 5.5. *Suppose $-\lambda \in \mathfrak{h}^*$ is dominant and regular, and $\mathfrak{g}$ has no simple factors of type G_2. If α and β are adjacent elements of B_λ, then $T_{\alpha\beta}(P_\alpha) = P_\beta$ or (P_β, P_β).*

Proof. By Proposition 5.4, equality must hold in the last inequality of its proof. Since $I \subsetneq P_\beta$ (in that notation), this implies $I = P_\beta$. Q.E.D.

Proposition 5.6 ([3], Korollar 3.6 and Satz 7.1). *If $I \in \operatorname{Prim} U(\mathfrak{g})$, then $\operatorname{Dim}(U(\mathfrak{g})/I)$ is the dimension of a nilpotent orbit of the adjoint group in $\mathfrak{g}$. If $I \subsetneq J$, with $J \in \operatorname{Prim} U(\mathfrak{g})$, then*

$$\operatorname{Dim}(U(\mathfrak{g})/I) > \operatorname{Dim}(U(\mathfrak{g})/J).$$

Although these techniques are not quite decisive [for example if $\mathfrak{g} \cong \mathfrak{sp}(4)$ and R_λ is of type D_4, they do not establish whether $\operatorname{Dim}(U(\mathfrak{g})P_\alpha)$ is 18 or 14; in fact it is 18] they are extremely effective in determining which of two ideals has a larger Gelfand-Kirillov dimension. This is all that matters for the study of the ordering in $X_{\hat{\lambda}}$. In combination with Corollary 3.6 and Proposition 2.2, these results determine, for example, the order structure of $X_{\hat{\lambda}}$ when R_λ is of type D_4. Unfortunately one finds six inclusions not accounted for by Proposition 2.2; it is not at present clear how to formulate a reasonable conjecture for the order structure of $X_{\hat{\lambda}}$. (Such extra inclusions exist also for R_λ of type B_3, so the problem involves more than the branch in the Dynkin diagram.)

6. Joseph's Theorem for R_λ of Type A_n

In this section we use Joseph's combinatorial results in [8] to establish Conjecture 4.3 when R_λ is of type A_n. In particular we recover the following part of the Jantzen conjecture, first proved by Joseph. As remarked in the introduction, the proof was discovered independently (and first) by Jantzen.

Theorem 6.1 ([10], Corollary 5.3). *Suppose $\lambda \in \mathfrak{h}^*$ is regular, and all simple factors of R_λ are of type A_n. Then the Duflo map $\Sigma_\lambda \to X_{\hat{\lambda}}$ is bijective.*

(Recall that Σ_λ is the set of involutions in W_λ.)

Proof. Joseph has shown ([8], Corollary 7.10) that each strong cell of W_λ contains exactly one involution. (Since the Dynkin diagram of R_λ is unbranched, strong cells are just cells in the sense of [8].) So by Proposition 4.4, it suffices to establish Conjecture 4.3 in this case. This we now do. First we recall some notation from [8].

Fix a positive integer n. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_t)$, $\xi_i \geqq \xi_{i+1}$, $\xi_t \geqq 1$, ξ_i a positive integer, be a partition of n; thus $\Sigma\xi_i = n = |\xi|$. A *standard tableau* $T(\xi)$ of type ξ and order n is an array t_i^j, $1 \leqq i \leqq t$, $1 \leqq j \leqq \xi_i$, of positive integers satisfying

a) $\{t_i^j\} = \{1, 2, \ldots, n\}$,
b) $t_i^j \leqq t_i^k$ when $j \leqq k$,
c) $t_i^j \leqq t_k^j$ when $i \leqq k$.

The position occupied by t_i^j is denoted T_i^j. Let η_j be the largest integer with

$\xi_{\eta_j} \geq j$. Put

$$T_i = \bigcup_{1 \leq j \leq \xi_i} T_i^j$$

$$T^j = \bigcup_{1 \leq i \leq \eta_j} T_i^j$$

the *rows* and *columns* of T respectively. $\bigcup T_i$ is the *skeleton* of $T(\xi)$, and is also written T. The set of standard tableaux of type ξ is written $St(\xi)$. As usual, S_n denotes the permutation group on $\{1, \ldots, n\}$.

Proposition 6.2 (Robinson, Schensted, and Schützenberger – cf. [1], 2.3). *There is a bijection Φ from S_n to $\bigcup_{|\xi|=n} St(\xi) \times St(\xi)$. Write $\Phi(w) = (A(w), B(w))$. Then $B(w) = A(w^{-1})$.*

For $w \in S_n$, let $\xi(w)$ be the partition defined by the rows of $A(w)$. Viewing S_n as the Weyl group of type A_{n-1}, we have

Theorem 6.3 (Joseph [8], Theorem 7.9). *For each $w \in S_n$,*

$$\mathscr{C}_w = \Phi^{-1}\{(A(w), S) : S \in St(\xi(w))\}.$$

The maps $T_{\alpha\beta}$ of Sect. 3 are defined on the level of W. Let $\approx$ denote the equivalence relation generated by $w \approx T_{\alpha\beta}(w)$ [whenever $w \in D(T_{\alpha\beta})$]. The maps $T_{\alpha\beta}$ are formally identical to the equivalence defining the cells of W, except that the reflections appear on the right instead of the left (compare Theorems 2.5 and 3.2). So the proof of Theorem 6.3 also shows

Proposition 6.4. *If W is of type A_{n-1}, then the equivalence class of w under the relation $\approx$ just defined is*

$$\Phi^{-1}\{(S, B(w)) : S \in St(\xi(w))\}.$$

It is now easy to prove

Theorem 6.5. *Conjecture 4.3 holds for W_λ of type A_{n-1}.*

Proof. Suppose w_1 and w_2 in S_n have the same generalized τ invariant. For $w \in S_n$ [considered as $W(A_{n-1})$] define R_w to be the $\mathbb{Z}$ span of $\tau(w)$, and $|w| = |R_w|$. For a partition ξ of n, let $T_0(\xi)$ be the unique standard tableau with $t_i^j = t_{i-1}^j + 1$ for $2 \leq i \leq \eta_j$. By [8], proof of Corollary 7.12, $|w| \leq |\Phi^{-1}(T_0(\xi(w), B(w))|$ for all $w' \approx w_j$ and furthermore this maximum is attained [since $\Phi^{-1}(T_0(\xi(w)), B(w))$ is equivalent to w by Proposition 6.4]. Put $\|w\| = \max_{w' \approx w} |w'|$. Clearly $\|w\|$ depends only on the generalized τ invariant of w, so $\|w_1\| = \|w_2\|$. Choose a sequence of $T_{\alpha\beta}$'s taking w_1 to $w_1' = \Phi^{-1}(T_0(\xi(w_1)), B(w_1))$, and let w_2' denote the result of applying this same sequence of $T_{\alpha\beta}$'s to w_2. Since w_1 and w_2 have the same generalized τ invariant, $\tau(w_1') = \tau(w_2')$. Hence $|w_2'| = |w_1'| = \|w_1\| = \|w_2\|$. Write $w_2' = \Phi^{-1}(A, B)$, $w_3 = \Phi^{-1}(T_0(\xi(w_2)), B)$. We claim that $\tau(w_1) = \tau(w_3)$. Assume this for a moment. Then Lemma 7.6 of [8] shows that $\xi(w_2) = \xi(w_1)$. So $w_1' = \Phi^{-1}(A, B')$, so w_1' and w_2' lie in the same cell of S_n. By Proposition ... [...], ... cells of S_n; so w_1 and w_2 also lie in the same cell, proving Conjecture 4.3.

It remains to show that $\tau(w_1) = \tau(w_3)$. This is proved using the fact that $|w_2'| = \|w_2\|$, which was established above. Write $A = (a_i^j)$. We define a sequence $k_1', \ldots, k_n'$ inductively as follows: $k_1' = 0$; and $k_i' = k_{i-1}' + 2$ if the $(i-1)$ simple root lies in $\tau(w_2')$, and 0 otherwise. Clearly

$$\sum_{i=1}^n k_i' = |w_2'|.$$

Now think of the k_i as arrayed in the skeleton of A, with $k_i'^j = k_{a_i^j}'$ in the A_i^j box. Recall Lemma 7.6 of [8]:

$$\tau(w_2') = \{i : i \in A_j,\ i+1 \in A_k \text{ with } k > j\}.$$

(Here we identify the simple roots with $\{1, \ldots, n-1\}$.) Put $w_3 = \Phi^{-1}(T_0(\xi), B)$, and define a sequence k_i attached to w_3 exactly as above. Since $|w_3| = \|w_2\| = |w_2'|$, we have

$$\sum_{i=1}^n k_i = \sum_{i=1}^n k_i'.$$

It is easy to see that if the k_i are arrayed in A as the k_i' were, then $k_i^j = 2i - 2$. Furthermore an induction on i shows that $k_i'^j \leq 2i - 2$. So we must have $k_i^j = k_i'^j$ for all i, j. Let l_r (resp. l_r') be the number of connected components of order r in $\tau(w_3)$ [resp. $\tau(w_2')$]; then it follows that $l_r = l_r'$ for all r. Since $\tau(w_1) = \tau(w_2')$, it follows that $\tau(w_1)$ and $\tau(w_3)$ have the same number of connected components of each length. In both cases these components sit inside $\{1, \ldots, n-1\}$ in decreasing order of length, separated by single roots not in the τ invariant; so in fact $\tau(w_1) = \tau(w_3)$ as claimed. Q.E.D.

As remarked in Sect. 4, it is enough to prove Conjecture 4.3 for the simple factors of W_λ; so the case needed for Theorem 6.1 follows from Theorem 6.5. This completes the proof of Theorem 6.1. Q.E.D.

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Multiplier Criteria of Hörmander Type for Fourier Series and Applications to Jacobi Series and Hankel Transforms

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1. Introduction

Marcinkiewicz's famous multiplier criterion for Fourier series, which was published [25] in 1939, has led to many analogues, e.g. in the case of Fourier transforms [23, 24], weighted Fourier series [18], Hankel transforms [17, 22], ultraspherical series [3, 5, 27], Jacobi series [1, 14], and spherical harmonics [5, 13]. Another type of multiplier criterion for the n-dimensional Fourier transform was given in 1960 by Hörmander [20] and it also has various successors, e.g. in the case of spherical harmonics [5, 7, 29], ultraspherical series [8, 10], Jacobi series [9, 16] and Hankel transforms [10].

In this paper we derive multiplier criteria of Hörmander type for Fourier series in weighted L^p-spaces, which are a counterpart to Hirschman's Marcinkiewicz type criterion [18, p. 60] and are new even in the unweighted case. As an application, in Sect. 5 we use this result and Askey's transplantation theorem [1] to obtain "lower end point" multiplier criteria for Jacobi series, which are then interpolated with the "upper end point" multiplier criteria in Gasper and Trebels [16] to obtain more general multiplier criteria.

A transplantation argument due to Igari [22] is used in Sect. 6 to give sufficient conditions for Hankel multipliers and these are then compared with necessary conditions. In addition, the techniques developed in [15] are employed to obtain extensions to M_p^r multipliers for both Jacobi series and Hankel transforms.

For the Fourier series case, we consider the space $L_{2\pi}^{p,\tau}$ of 2π-periodic functions $f(\theta)$ with finite norm

$$\|f\|_{p,\tau}=\left(\int_{-\pi}^{\pi}|f(\theta)|^p|\theta|^{\tau p}d\theta\right)^{1/p}, \quad 1\leqq p<\infty .$$

Note that $L_{2\pi}^{p,\tau}\subset L_{2\pi}^{1,0}$, $-1<\tau p<p-1$, so that we can consider Fourier series expansions in these Lebesgue spaces. For simplicity, as in Zygmund [33, Chap. XV, (4.14)], we shall first work with Fourier series of power series type. A

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