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Gelfand-Kirillov Dimension for Harish-Chandra Modules

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1. Introduction

Let G be a connected semisimple matrix group. Denote by $\hat{G}$ the set of infinitesimal equivalence classes of irreducible admissible representations of G. Our goal is to describe and relate several notions of the size of infinite dimensional elements of $\hat{G}$. To explain the results, let $\mathfrak{g}$ be the complexified Lie algebra of G, $U(\mathfrak{g})$ its enveloping algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Fix a positive Weyl chamber $C \subseteq \mathfrak{h}^*$, and some translate Λ of the lattice of weights of finite dimensional representations. Choose a maximal compact subgroup $K \subseteq G$. Suppose we are given a family $\{\pi(\lambda) | \lambda \in \Lambda \cap C\} \subseteq \hat{G}$, which depends coherently on λ ([5]). Let $X(\lambda)$ be the space of K-finite vectors for $\pi(\lambda)$, which is an irreducible $U(\mathfrak{g})$ module. Fix any vector $0 \neq x_\lambda \in X(\lambda)$. Let $U_n(\mathfrak{g})$ be the subspace of $U(\mathfrak{g})$ spanned by products of at most n elements of $\mathfrak{g}$. Put $X_n(\lambda) = U_n(\mathfrak{g})\, x_\lambda$. Recall that a polynomial on $\mathfrak{h}^*$ is called *harmonic* if it is annihilated by all the Weyl group invariant constant coefficient differential operators without constant term.

Theorem 1.1. *With hypotheses as above,* $\dim X_n(\lambda)$ *is asymptotic to* $\dfrac{c(\lambda)}{d!} n^d$ *as* n *goes to infinity. Both* $c(\lambda)$ *and* d *are positive integers independent of the choice of* x_λ. *The function* $c(\lambda)$ *is a harmonic polynomial on* $\mathfrak{h}^*$, *and* $2d$ *is the Gelfand-Kirillov dimension* ([3]) *of* $U(\mathfrak{g})/\mathrm{Ann}\, X(\lambda)$.

We call d the Gelfand-Kirillov dimension of $X(\lambda)$, and $c(\lambda)$ its Bernstein degree.

Let Ω be the Casimir operator of K. If X is a semisimple representation of K, we write $N_X(t)$ for the sum of the dimensions of the eigenspaces of Ω with eigenvalue less than or equal to t^2.

Theorem 1.2. *Suppose* $\pi \in \hat{G}$; *let* X *be the space of* K*-finite vectors in* π, *and* d *the Gelfand-Kirillov dimension of* X. *Then there is a constant* $A > 0$, *depending on* X, *such that for* $t \geqq 1$,

$$A^{-1} t^d \leqq N_X(t) \leqq A t^d.$$

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Theorem 1.1 is proved in Section 4, and Theorem 1.2 in Section 5. We conclude the paper with a rather complicated but complete characterization of those representations with maximal Gelfand-Kirillov dimension (Theorem 6.2). Some possible improvements of these results are stated as Conjectures 4.10 and 5.1.

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2. Preliminaries on Gelfand-Kirillov Dimension

We begin by recalling some facts from commutative algebra. Details may be found in [14], Chapter VII, especially Theorem 41. Let R be the polynomial ring $\mathbb{C}[x_1, \ldots, x_N]$, graded by degree. Let $M = \sum_{q=0}^{\infty} M_q$ be a graded R module: if $p \in R$ is homogeneous of degree t, and $m \in M_q$, then $p \cdot m \in M_{q+t}$. Suppose now that M is finitely generated. Then

$$\varphi_M(n) = \sum_{q \leq n} \dim_{\mathbb{C}} M_q$$

is finite. By a theorem of Hilbert and Serre, there is a polynomial $\bar{\varphi}_M(n)$ in n, of degree at most N, such that $\bar{\varphi}_M(n) = \varphi_M(n)$ for all sufficiently large n. The degree of $\bar{\varphi}_M$ is written $\mathrm{Dim}(M)$, and its leading coefficient $\dfrac{c(M)}{(\mathrm{Dim}(M))!}$; $c(M)$ is an integer. If $d = \mathrm{Dim}(M)$, we may write $c_d(M)$ for emphasis. For $d' > \mathrm{Dim}(M)$, we define $c_{d'}(M) = 0$. For reasons which will become clear, we are not interested in lower order terms of $\bar{\varphi}_M$.

Now suppose $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$. Let $U_n(\mathfrak{g}) \subseteq U(\mathfrak{g})$ be the subspace of $U(\mathfrak{g})$ generated by monomials of the form $g_1 \ldots g_m$, with $m \leq n$, and $g_i \in \mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem, the associated graded algebra $\mathrm{gr}(U(\mathfrak{g}))$ is canonically isomorphic to $S(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$ – a polynomial ring in $\dim \mathfrak{g}$ variables. Suppose V is a finitely generated $U(\mathfrak{g})$ module. Choose a generating subspace V_0, and set $V_n = U_n(\mathfrak{g}) V_0$. Define $d_{V,V_0}(n) = \dim V_n$. Set $M_n = V_n/V_{n-1}$, $M = \sum_{n=0}^{\infty} M_n$. An element $g \in \mathfrak{g}$ acts on M in the following way: if $v \in V_n$, and $\bar{v} = v + V_{n-1} \in M_n$, then

$$g \cdot \bar{v} = g \cdot v + V_n \in M_{n+1}.$$

In this way M becomes a module for $S(\mathfrak{g})$. It is trivial to verify that M_0 generates M, and that

$$d_{V,V_0}(n) = \varphi_M(n).$$

Lemma 2.1. *With notation as above, there is a polynomial $\bar{d}_{V,V_0}(n)$ of degree at most $\dim \mathfrak{g}$, so that $d_{V,V_0}(n) = \bar{d}_{V,V_0}(n)$ for large n. The leading coefficient of $\bar{d}_{V,V_0}$, which we write as $\dfrac{c(V)}{(\mathrm{Dim}\, V)!}$, is independent of the choice of V_0.*

Proof. The existence of $\bar{d}_{V,V_0}$ follows from the discussion above. Suppose then that V_0' also generates V. Then for some large integer k, we have $V_0' \subseteq V_k$, and $V_0 \subseteq V_k'$. (Here of course $V_k' = U_k(\mathfrak{g}) V_0'$.) It is immediate that

$$V_{n-k}' \subseteq V_n \subseteq V_{n+k}'.$$

Hence $\bar{d}_{V,V_0'}(n-k) \leq \bar{d}_{V,V_0}(n) \leq \bar{d}_{V,V_0'}(n+k)$, for all sufficiently large n. Therefore, the polynomials $\bar{d}_{V,V_0'}$ and $\bar{d}_{V,V_0}$ have the same leading term. Q.E.D.

The preceding proof is due to Bernstein [1], who considered only the Heisenberg algebra. The integer $\mathrm{Dim}\, V$ is defined by Joseph [7] in this situation; we will call it the Gelfand-Kirillov dimension of V. The integer $c(V)$ is called the *Bernstein degree* of V. If $d = \mathrm{Dim}\, V$, we sometimes write $c(V) = c_d(V)$ for emphasis; and if $d' > d$, we define $c_{d'}(V) = 0$.

Example. Suppose $I \subseteq U(\mathfrak{g})$ is a two-sided ideal. If we consider the algebra $U(\mathfrak{g})/I$ as a left $U(\mathfrak{g})$ module, then $\mathrm{Dim}(U(\mathfrak{g})/I)$ is by definition its Gelfand-Kirillov dimension in the usual sense ([3]).

Lemma 2.2. *Suppose V is a finitely generated $U(\mathfrak{g})$ module, and F is a finite dimensional $U(\mathfrak{g})$ module of dimension f. Then $\mathrm{Dim}(V \otimes F) = \mathrm{Dim}\, V$, and $c(V \otimes F) = f \cdot c(V)$.*

Proof. Fix a generating subspace V_0 of V. Then $V_0 \otimes F$ generates $V \otimes F$; and in the associated filtration,

$$(V \otimes F)_n = V_n \otimes F.$$

Hence $d_{V \otimes F, V_0 \otimes F}(n) = f \cdot d_{V,V_0}(n)$. The result follows. Q.E.D.

Lemma 2.3. *Suppose $\mathfrak{b} \subseteq \mathfrak{g}$ is a Lie subalgebra, and V is a finitely generated $U(\mathfrak{b})$ module. Define $X = U(\mathfrak{g}) \bigotimes_{\mathfrak{b}} V$. Then X is a finitely generated $U(\mathfrak{g})$ module. Furthermore, $\mathrm{Dim}\, X = \mathrm{Dim}\, V + \dim(\mathfrak{g}/\mathfrak{b})$, and $c(X) = c(V)$.*

Proof. Fix a generating subspace V_0 of V. Then one verifies easily that $1 \otimes V_0 = X_0 \subseteq X$ generates X. Choose a complementary subspace $\mathfrak{r} \subseteq \mathfrak{g}$ to $\mathfrak{b}$. Then $U(\mathfrak{g}) \cong S(\mathfrak{r}) \bigotimes_{\mathbb{C}} U(\mathfrak{b})$, so $X \cong S(\mathfrak{r}) \bigotimes_{\mathbb{C}} V$. If M is the graded module associated to V and V_0, and N is the graded module associated to X and X_0, then $N \cong S(\mathfrak{r}) \bigotimes_{\mathbb{C}} M$ (graded tensor product). Therefore

$$\begin{aligned}
\varphi_N(n) &= \sum_{q \leq n} \dim N_q \\
&= \sum_{q \leq n} \sum_{k=0}^{q} (\dim S_k(\mathfrak{r}))(\dim M_{q-k}) \\
&= \sum_{k=0}^{n} (\dim S_k(\mathfrak{r})) \left(\sum_{q=0}^{n-k} \dim M_q \right) \\
&= \sum_{k=0}^{n} \dim S_k(\mathfrak{r})\, \varphi_M(n-k).
\end{aligned}$$

Put $r=\dim\mathfrak{r}=\dim\mathfrak{g}/\mathfrak{b}$. Then $\dim(S_k(\mathfrak{r}))=\binom{r+k-1}{r-1}$ (the binomial coefficient). So if we replace $\varphi_M(n-k)$ by $\bar{\varphi}_M(n-k)$ (which differs from it for only finitely many values of $n-k$), we make an error which is $O(m^{r-1})$. If we change $\bar{\varphi}_M$ by a polynomial of degree less than $\operatorname{Dim} V$, we make an error which is $O(n^{r-1+(\operatorname{Dim} V-1)})$ in each term of the sum, and hence of order $n^{r+\operatorname{Dim} V-1}$ altogether. So to compute the leading term of $\varphi_N(n)$, we may replace $\bar{\varphi}_M(n-k)$ by $\binom{n-k+\operatorname{Dim} V}{\operatorname{Dim} V}\cdot c(V)$. This amounts to replacing M by $c(V)$ copies of a symmetric algebra on $\operatorname{Dim} V$ variables, so that N is replaced by $c(V)$ copies of a symmetric algebra on $r+\operatorname{Dim} V$ variables. The lemma follows. Q.E.D.

Lemma 2.4. *Suppose $0\to A\to B\to C\to 0$ is an exact sequence of $U(\mathfrak{g})$ modules, and that A and C finitely generated. Then B is finitely generated also;* $\operatorname{Dim} B=\max(\operatorname{Dim} A, \operatorname{Dim} C)=d$, *and* $c_d(B)=c_d(A)+c_d(C)$.

Proof. Fix finite dimensional generating subspaces A_0 and C_0 of A and C, and choose a subspace $B_0\subseteq B$ so that

$$0\to A_0\to B_0\to C_0\to 0$$

is exact. Then B_0 is finite dimensional. Obviously $U(\mathfrak{g})\cdot B_0$ contains $U(\mathfrak{g})\cdot A_0=A$, and the image of $U(\mathfrak{g})\cdot B_0$ in C is $U(\mathfrak{g})\cdot C_0=C$. So B_0 generates B. Put $B_n=U_n(\mathfrak{g})\cdot B_0$, $C_n=U_n(\mathfrak{g})\cdot C_0$, and $A'_n=B_n\cap A$. We have exact sequences

$$0\to A'_n\to B_n\to C_n\to 0;$$

so

$$d_{B,B_0}(n)=\dim A'_n+d_{C,C_0}(n). \qquad (*)$$

Let $\bar{B}=\bigotimes_{n=0}^{\infty} B_n/B_{n-1}$ be the graded $S(\mathfrak{g})$ module associated to B. Since

$$A'_n\cap B_{n-1}=B_{n-1}\cap B_n\cap A=B_{n-1}\cap A=A'_{n-1},$$

the graded $S(\mathfrak{g})$ module $\bar{A}'=\bigotimes_{n=0}^{\infty} A'_n/A'_{n-1}$ may be regarded as a submodule of $\bar{B}$. Now $\bar{B}$ is finitely generated, and $S(\mathfrak{g})$ is Noetherian; so $\bar{A}'$ is finitely generated, say by elements of degree at most m. It follows that $\mathfrak{g}\cdot\bar{A}'_n=\bar{A}'_{n+1}$ for $n\geqq m$. Therefore $\mathfrak{g}\cdot A'_n+A'_n=A'_{n+1}$ for $n\geqq m$. Put $A''_0=A'_m$. By induction,

$$A''_n=U_n(\mathfrak{g})\cdot A''_0=A'_{m+n}$$

for $n\geqq 0$. By $(*)$, we have for $n\geqq m$,

$$d_{B,B_0}(n)=d_{A,A''_0}(n-m)+d_{C,C_0}(n).$$

The result now follows from Lemma 2.1. Q.E.D.

Corollary 2.5. *Finitely generated $U(\mathfrak{g})$ modules of Gelfand-Kirillov dimension at most d form an abelian category; and the Bernstein degree c_d defines a homomorphism of the associated Grothendieck group into the integers.*

3. Preliminaries on Verma Modules

We assume in this section that $\mathfrak{g}$ is a complex semisimple Lie algebra, with Borel subalgebra $\mathfrak{b}=\mathfrak{h}+\mathfrak{n}$. Here $\mathfrak{h}$ is a Cartan subalgebra, and $\mathfrak{n}$ is the nil radical of $\mathfrak{b}$. Then $\mathfrak{b}$ determines a set of positive roots $\Delta^+\subseteq\Delta(\mathfrak{g},\mathfrak{h})$; we write $\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. The Weyl group $W=W(\mathfrak{g},\mathfrak{h})$ acts on $\mathfrak{h}^*$. Fix a nonsingular invariant bilinear form $\langle\,,\rangle$ on $\mathfrak{g}$, which is positive definite on the real span of the roots in $\mathfrak{h}^*$. Define a positive Weyl chamber

$$C=\{\lambda\in\mathfrak{h}^*|\operatorname{Re}\langle\lambda,\alpha\rangle>0 \quad \text{or } \operatorname{Re}\langle\lambda,\alpha\rangle=0 \quad \text{and} \quad \operatorname{Im}\langle\lambda,\alpha\rangle>0 \quad \text{for all } \alpha\in\Delta^+\};$$

we write $\bar{C}$ for its closure, which is a fundamental domain for W. A weight λ is called *regular* or *nonsingular* if $\langle\lambda,\alpha\rangle\neq 0$ whenever $\alpha\in\Delta^+$. Elements of $\bar{C}$ are called *dominant.*

Write $\mathfrak{Z}(\mathfrak{g})$ for the center of $U(\mathfrak{g})$. The (translated) Harish-Chandra homomorphism defines for each $\lambda\in\mathfrak{h}^*$ a map $\chi_\lambda:\mathfrak{Z}(\mathfrak{g})\to\mathbb{C}$; this identifies $\operatorname{Spec}\mathfrak{Z}(\mathfrak{g})$ with W orbits in $\mathfrak{h}^*$. If $\mathfrak{Z}(\mathfrak{g})$ acts in a $U(\mathfrak{g})$ module by the scalars χ_λ, we will say that the module has *infinitesimal character* λ; the infinitesimal character is called nonsingular if λ is. Let $M(\lambda)$ denote the Verma module with highest weight $\lambda-\rho$ (with respect to $\mathfrak{b}$), and $L(\lambda)$ its unique irreducible quotient. Then $M(\lambda)$ and $L(\lambda)$ are finitely generated $U(\mathfrak{g})$ modules, with infinitesimal character λ. For our purposes the main result about Verma modules is

Theorem 3.1 (Joseph [6], Proposition 6.5). $\operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann} L(\lambda))=2\operatorname{Dim} L(\lambda)$.

It would be very nice to relate $c(L(\lambda))$ to $c(U(\mathfrak{g})/\operatorname{Ann} L(\lambda))$.

We will also use a technical result of Joseph's. Let $\mathfrak{a}$ be a finite dimensional Lie algebra, and $\mathfrak{b}$ a solvable ad-algebraic subalgebra. Write $\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n}$, with $\mathfrak{h}$ commutative ad-semisimple, and $\mathfrak{n}$ ad-nilpotent (on $\mathfrak{a}$). Let L be a $U(\mathfrak{a})$ module generated by a finite dimensional $\mathfrak{b}$ submodule L_0, on which $\mathfrak{n}$ acts nilpotently. Put $L_k=U_k(\mathfrak{a})\,L_0$, a finite dimensional $\mathfrak{b}$ submodule of L.

Lemma 3.2. (Joseph [7], Lemma 2.5). *Under the above hypotheses, there is a positive integer t such that for each positive integer k and each $x\in L^k$,*

$$U(\mathfrak{b})\,x=U_{kt}(\mathfrak{b})\,x.$$

Now suppose $\mathfrak{k}\subseteq\mathfrak{a}$ is a reductive subalgebra, and choose a triangular decomposition $\mathfrak{k}=\mathfrak{n}_1^-+\mathfrak{t}_1+\mathfrak{n}_1$. Let L be a $U(\mathfrak{a})$ module generated by a finite dimensional $\mathfrak{k}$ submodule L_0. Put $L_k=U_k(\mathfrak{a})\,L_0$.

Corollary 3.3. *Under the above hypotheses, there is a positive integer s such that for each positive integer k and each $x\in L_k$,*

$$U(\mathfrak{k})\,x=U_{ks}(\mathfrak{k})\,x.$$

Proof. Apply Lemma 3.2 to $\mathfrak{b}_1=\mathfrak{h}_1+\mathfrak{n}_1$ and $\mathfrak{n}_1^-$ respectively, getting positive integers t and t'. Set $s=t'+t$. If $x\in L_k$ and $u\in U(\mathfrak{k})$, we want to show that $u\cdot x\in U_{ks}(\mathfrak{k})\,x$. Write $u=\sum u'_i u''_i$, with $u'_i\in U(\mathfrak{n}_1^-)$ and $u''_i\in U(\mathfrak{b}_1)$; clearly we may as well assume $u=u'u''$. Now $u''\cdot x\in U_{kt}(\mathfrak{b})\,x$ by Lemma 3.2; and $u''\cdot x\in L_k$, since L_k

is a $\mathfrak{k}$ submodule of L. Hence $u'\cdot(u''x)\in U_{kt'}(\mathfrak{n}_1^-)\,U_{kt}(\mathfrak{b}_1)\,x\subseteq U_{k(t+t')}(\mathfrak{k})\,x$ by Lemma 3.2 again. Q.E.D.

We return now to the assumption that $\mathfrak{g}$ is reductive, and that $\mathfrak{b}$ is a Borel subalgebra. Let $\mathfrak{k}$ be a reductive subalgebra of $\mathfrak{g}$ such that $\mathfrak{k}+\mathfrak{b}=\mathfrak{g}$ (not necessarily a direct sum). Let V and L be two $U(\mathfrak{g})$ modules with a nonsingular pairing $\langle,\rangle$. We suppose that V and L are generated by finite dimensional subspaces V_0 and L_0, $\mathfrak{k}$-stable and $\mathfrak{b}$-stable respectively.

Lemma 3.4. *Under the preceding hypotheses,* $\operatorname{Dim} V=\operatorname{Dim} L$, *and* $\operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann} V)=\operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann} L)$.

Proof. Recall the transpose map of $U(\mathfrak{g})$, defined by ${}^t g=-g$ for $g\in\mathfrak{g}$, and extended to an antiautomorphism. Then for $v\in V$, $l\in L$, $u\in U(\mathfrak{g})$, we have $\langle u\cdot v, l\rangle=\langle v, {}^t u\, l\rangle$. Hence $\operatorname{Ann} V={}^t(\operatorname{Ann} L)$, so clearly $\operatorname{Dim} U(\mathfrak{g})/\operatorname{Ann} V=\operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann} L)$. For the first statement, define V_k and L_k as in Section 2; choose t as in Lemma 3.2, and s for V as in Lemma 3.3. We claim that $\dim L_k\leqq\dim V_{kt}$, and that $\dim V_k\leqq\dim L_{ks}$; this will prove the lemma. Notice first that $L=U(\mathfrak{k})\,L_0$, and $V=U(\mathfrak{b})V_0$. We claim that the natural map $V_{kt}\to L_k^*$ defined by $\langle,\rangle$ is surjective; this will suffice for the first claim, and the second is similar. So take $x\in L_k$, and suppose $\langle x, V_{kt}\rangle=0$. By transposition, $\langle U_{kt}(\mathfrak{b})\,x, V_0\rangle=0$. By Lemma 3.2, $\langle U(\mathfrak{b})\,x, V_0\rangle=0$, and then $\langle x, U(\mathfrak{b})\,V_0\rangle=\langle x, V\rangle=0$ by transposition. Thus $x=0$ by the nonsingularity of $\langle,\rangle$. Q.E.D. This argument is due to Joseph (cf. [7], Lemma 3.2).

4. Coherent Continuation

In the present section we assume that G is a linear reductive Lie group, with abelian Cartan subgroups. Infinitesimal equivalence classes of admissible representations of G are identified with compatible $(\mathfrak{g},K)$ modules ([9]). (As in the introduction, K is a maximal compact subgroup of G, and $\mathfrak{g}$ is the complexified Lie algbra of G. We write $\mathfrak{g}_0$ for the real Lie algebra, and use analogous notation for other groups.) Let θ denote the Cartan involution of $\mathfrak{g}$; $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, with $\mathfrak{p}$ the -1 eigenspace of θ. Fix a θ-invariant maximally split Cartan subgroup $H=T^+A$ of G, with $T^+\subseteq K$, and $A\subseteq\exp\mathfrak{p}_0$. We choose an Iwasawa decomposition $G=KAN$. Let M be the centralizer of A in K, so that MAN is a minimal parabolic subgroup of G. If we choose a Borel subalgebra $\mathfrak{b}_m=\mathfrak{t}^++\mathfrak{n}_m$, then $\mathfrak{b}=\mathfrak{h}+(\mathfrak{n}_m+\mathfrak{n})=\mathfrak{h}+\mathfrak{n}'$ is a Borel subalgebra of $\mathfrak{g}$; thus we have a positive Weyl chamber $C\subseteq\mathfrak{h}^*$ as defined in Section 3, etc.

We recall now the theory of coherent continuation developed in [11]. Suppose X is a compatible $(\mathfrak{g},K)$ module with infinitesimal character $\lambda\in C$. (In particular λ is nonsingular.) Write $\Theta(\lambda)=\Theta(X)$ for the global character of X. Whenever $\tilde{\mu}\in\hat{H}$ is a weight of a finite dimensional representation of G, a new virtual character $S_{\tilde{\mu}}(\Theta(\lambda))=\Theta(\lambda+\tilde{\mu})$ was defined in [11]; here $\lambda+\tilde{\mu}$ is a formal symbol, not a sum. Roughly speaking, $\Theta(\lambda+\tilde{\mu})$ is defined as follows. On a small connected open subset U of the regular set in a Cartan subgroup B of G, $\Theta(\lambda)$ may be written in the form

$$\Delta^{-1}(b)\cdot\sum_{w\in W(\mathfrak{g},\mathfrak{b})} a_w\exp(w\lambda'(\log b));$$

here Δ is the Weyl denominator, "log" is continuous and stisfies

$$\exp(\log(b_1 b_2^{-1}))\,b_2=b_1$$

for b_i in U, and $\lambda'\in\mathfrak{b}^*$. Furthermore there is an inner automorphism c of $\mathfrak{g}$ which takes $\mathfrak{h}$ to $\mathfrak{b}$ and λ to λ'. Let μ' be c applied to the differential of $\tilde{\mu}$. Then $\Theta(\lambda+\tilde{\mu})$ is given on U by the same formula, with λ' replaced by $\lambda'+\mu'$. (This is not precisely correct; we refer to [11] for details.) Suppose $\operatorname{Dim} X=d$. Write $\mathscr{A}_d$ for the category of $\mathfrak{Z}(\mathfrak{g})$-finite compatible $(\mathfrak{g},K)$ modules Y with $\operatorname{Dim} Y\leqq d$. Then $\mathscr{A}_d$ is closed under tensor products with finite dimensional modules (Lemma 2.2) and under passage to submodules. It follows from the construction of $\Theta(\lambda+\tilde{\mu})$ that every constituent of $\Theta(\lambda+\tilde{\mu})$ belongs to $\mathscr{A}_d$. By Corollary 2.5, we can therefore define $c(\lambda+\tilde{\mu})=c_d(\Theta(\lambda+\tilde{\mu}))$ by considering $\Theta(\lambda+\tilde{\mu})$ as an element of the Grothendieck group. Let $\mu\in\mathfrak{h}^*$ be the differential of $\tilde{\mu}$. If $\tilde{\gamma}\in\hat{H}$ is another weight of a finite dimensional representation, and $\langle\mu,\alpha\rangle=\langle\gamma,\alpha\rangle$ for all $\alpha\in\Delta(\mathfrak{g},\mathfrak{h})$, then $\tilde{\mu}-\tilde{\gamma}$ is the weight of a one dimensional representation F_1 of G; and $\Theta(\lambda+\tilde{\mu})=\Theta(\lambda+\tilde{\gamma})\cdot\Theta(F_1)$. Hence $c(\lambda+\tilde{\mu})=c(\lambda+\tilde{\gamma})$. So c may be regarded as a function of $\lambda+\mu$, independent of the restriction of $\lambda+\mu$ to the center of $\mathfrak{g}$. By passing to a covering group of G, we may assume that (up to the action of the center) any integral weight is a weight of a finite dimensional representation ([11], Lemma 5.19); so we may assume $c(\lambda+\mu)$ is defined whenever μ is integral.

The following results were suggested by conversations with G. Zuckerman.

Lemma 4.1. *With notation as above, suppose $\mu,\gamma\in\mathfrak{h}^*$ are integral. Then if $W=W(\mathfrak{g},\mathfrak{h})$,*

$$c(\lambda+\mu)=|W|^{-1}\sum_{\sigma\in W} c(\lambda+\mu+\sigma\cdot\gamma). \tag{4.2}$$

Proof. Whenever F is a finite dimensional representation, write $\Delta(F)$ for the set of weights of F with multiplicities. Then

$$\Theta(\lambda+\tilde{\mu})\cdot\Theta(F)=\sum_{\tilde{\gamma}\in\Delta(F)}\Theta(\lambda+\tilde{\mu}+\tilde{\gamma})$$

(see [11], Lemma 5.3). Put $f=\dim F$. Using Lemmas 2.2 and 2.4, we deduce that

$$f\cdot c(\lambda+\mu)=\sum_{\gamma\in\Delta(F)} c(\lambda+\mu+\gamma).$$

By a standard argument (using induction on the length of γ, for example), (4.2) is a linear combination of such equations for various F. Q.E.D.

One should be able to replace W in Lemma 4.1 by the Weyl group W_λ generated by reflections about roots with respect to which λ is integral. It is not at all clear how to prove this, however, Because of the following lemma, such a result would give an enormous amount of information about c. Let $\Lambda = \{\lambda + \mu | \mu$ is integral$\}$.

Lemma 4.3. *Let $\mathscr{H}$ denote the space of functions on Λ satisfying* (4.2). *Then $\mathscr{H}$ is the restriction to Λ of the space of harmonic polynomials on $\mathfrak{h}^*$.*

Proof. Recall that the harmonic polynomials are those satisfying every W-invariant differential operator without constant term. (Elementary properties of these functions may be found for example in [16].) We denote this space by $\mathscr{H}_0$. Recall that $\dim \mathscr{H}_0 = |W|$. It is easy to see that $\mathscr{H}_0 \subseteq \mathscr{H}$; for if $c \in \mathscr{H}_0$, then

$$h(\gamma) = c(\lambda + \mu) - \frac{1}{|W|} \sum_{\sigma \in W} c(\lambda + \mu + \sigma \cdot \gamma)$$

is a W-invariant harmonic polynomial vanishing at zero, and therefore identically. So it is enough to show $\dim \mathscr{H} \leqq |W|$. The referee has pointed out that the desired result can be deduced fairly easily from Theorem 2.2 of [17]. Since that result is itself not entirely trivial, we will give a direct argument. For this we may as well assume $\lambda = 0$ and that $\mathfrak{g}$ is semisimple. For each $\sigma \in W$, we will define an integral weight p_σ. Suppose $c \in \mathscr{H}$, and $c(p_\sigma) = 0$ for all $\sigma \in W$. Then we will show that $c = 0$. This will prove the lemma. Let $B \subset \Delta^+$ be the set of simple roots. If $A \subseteq B$, write χ_A for the characteristic function of A. Define an integral weight δ_A by

$$\frac{2\langle \delta_A, \beta \rangle}{\langle \beta, \beta \rangle} = \chi_A(\beta), \qquad \text{for all } \beta \in B.$$

If $\sigma \in W$, put $S_\sigma = \{\alpha \in \Delta^+ | \sigma\alpha \notin \Delta^+\}$, $\tau(\sigma) = S_\sigma \cap B$. Define $p_\sigma = \sigma \cdot \delta_{\tau(\sigma)}$. Suppose then that $c \in \mathscr{H}$, and $c(p_\sigma) = 0$ for all σ. We want to show that $c(\mu) = 0$ for every integral weight μ. We proceed by induction on the length of μ; suppose then that $c(\mu') = 0$ whenever $|\mu'| < |\mu|$. Choose $\sigma_1 \in W$ so that $\sigma_1 \cdot \mu$ is dominant, and set

$$A = \{\alpha \in B | \langle \alpha, \sigma_1 \cdot \mu \rangle \neq 0\}.$$

Suppose first that there is a root $\alpha \in A$ such that $\dfrac{2\langle \alpha, \sigma_1 \cdot \mu \rangle}{\langle \alpha, \alpha \rangle} \geqq 2$. Then by (4.2),

$$c(\mu - \sigma_1^{-1} \delta_\alpha) = \frac{1}{|W|} \sum_{\tau \in W} c(\mu - \sigma_1^{-1} \delta_\alpha + \tau \delta_\alpha). \tag{$*$}$$

Since $\sigma_1 \mu - \delta_\alpha$ and δ_α are dominant, $|\mu - \sigma_1^{-1} \delta_\alpha| < |\mu|$. Suppose $|\mu - \sigma_1^{-1} \delta_\alpha + \tau \delta_\alpha| \geqq |\mu|$. Then $\sigma_1 \mu - \delta_\alpha$ and $\sigma_1 \tau \cdot \delta_\alpha$ lie in a common Weyl chamber, by an easy argument. Now $\sigma_1 \mu - \delta_\alpha$ annihilates only those roots not in A. Write $W(B-A)$ for the Weyl group generated by those roots. Then there is some $\sigma_2 \in W(B-A)$ such that $\sigma_2(\sigma_1 \tau \cdot \delta_\alpha)$ is dominant; so

$$\sigma_1 \tau \cdot \delta_\alpha = \sigma_2 \cdot \delta_\alpha = \delta_\alpha,$$

since δ_α also annihilates those roots not in A. So $\tau \cdot \delta_\alpha = \sigma_1^{-1} \delta_\alpha$, and $\mu - \sigma_1^{-1} \delta_\alpha + \tau \cdot \delta_\alpha = \mu$. Thus ($*$) may be written as

$$c(\mu) = \sum_{|\mu_i'| < |\mu|} a_i c(\mu_i'),$$

which implies that $c(\mu) = 0$ by induction.

So we may assume that $\dfrac{2\langle \alpha, \sigma_1 \mu \rangle}{\langle \alpha, \alpha \rangle} = \chi_A(\alpha)$, i.e. that $\sigma_1 \mu = \delta_A$. Set $\sigma = \sigma_1^{-1}$, so that $\mu = \sigma \cdot \delta_A$. Modifying σ on the right by an element of $W(B-A)$ changes nothing; so we may assume that $S_\sigma \cap (B-A)$ is empty, i.e. that $\tau(\sigma) \subseteq A$. If $\tau(\sigma) = A$, then $\mu = p_\sigma$, and $c(\mu) = 0$ by hypothesis. Otherwise set $C = A - \tau(\sigma)$. We may assume by induction that $c(\sigma' \delta_A) = 0$ whenever $\langle \sigma' \delta_A, \rho \rangle < \langle \sigma \delta_A, \rho \rangle$. We have

$$c(\sigma \cdot \delta_{\tau(\sigma)}) = \frac{1}{|W|} \sum_{s \in W} c(\sigma \cdot \delta_{\tau(\sigma)} + (\sigma \cdot s) \cdot \delta_C). \tag{$**$}$$

Just as before, $|\sigma \cdot \delta_{\tau(\sigma)}| < |\mu|$. Suppose

$$|\sigma \cdot \delta_{\tau(\sigma)} + (\sigma \cdot s) \cdot \delta_C| \geqq |\mu| = |\delta_{\tau(\sigma)} + \delta_C|.$$

Then $\delta_{\tau(\sigma)}$ and $s \cdot \delta_C$ lie in a common Weyl chamber as before. Hence there is an element $s_1 \in W(B - \tau(\sigma))$ such that $s \cdot \delta_C = s_1 \cdot \delta_C$. Modifying s_1 on the right by an element of $W(B-A)$, we may assume that $\tau(s_1) \subseteq C$. We have

$$\begin{aligned} \sigma \cdot \delta_{\tau(\sigma)} + (\sigma \cdot s)\, \delta_C &= (\sigma s_1)(\delta_{\tau(\sigma)} + \delta_C) \\ &= (\sigma s_1)(\delta_A). \end{aligned}$$

To complete the proof, it is enough to show that if $\sigma s_1(\delta_A) \neq \sigma \delta_A$, then $\langle \sigma s_1(\delta_A), \rho \rangle < \langle \sigma \delta_A, \rho \rangle$; for then we can argue by rearranging ($**$) as in the first case. If $\sigma s_1(\delta_A) \neq \sigma \delta_A$, then since $s_1(\delta_{\tau(\sigma)}) = \delta_{\tau(\sigma)}$, we must have $s_1 \delta_C \neq \delta_C$. Since $s_1 \in W(B - \tau(\sigma))$, it follows that $s_1 \delta_C = \delta_C - Q$, with Q a nonempty sum of roots in $B - \tau(\sigma)$. By definition of $\tau(\sigma)$, $\sigma \cdot Q$ is then a nonempty sum of positive roots. So

$$\langle \sigma s_1 \delta_A - \sigma \delta_A, \rho \rangle = \langle -\sigma Q, \rho \rangle < 0. \quad \text{Q.E.D.}$$

If Θ is a character, write $\operatorname{Dim} \Theta$ for the maximum of the dimensions of the irreducible constituents of Θ.

Corollary 4.4. *Let Θ be a character with nonsingular (dominant) infinitesimal character λ; write $\Theta = \Theta(\lambda)$. Let $\{\Theta(\lambda + \tilde{\mu}) | \tilde{\mu}$ a weight of a finite dimensional representation$\}$ be the associated coherent family. Put $d = \operatorname{Dim} \Theta$. Then there is a harmonic polynomial c on $\mathfrak{h}^*$ such that if μ is the differential of $\tilde{\mu}$,*

$$c(\lambda + \mu) = C_d(\Theta(\lambda + \tilde{\mu})).$$

To complete the proof of Theorem 1.1, we need the following result of Casselman (which is apparently unpublished, but fairly well known).

Lemma 4.5 (Casselman). *Let X be a $\mathfrak{Z}(\mathfrak{g})$-finite compatible $(\mathfrak{g}, K)$ module, and let $\mathfrak{b} \subseteq \mathfrak{g}$ be the Borel subalgebra defined at the beginning of this section (using an Iwasawa decomposition of G). Let $X_{\mathfrak{b}}^*$ denote the $\mathfrak{b}$-finite vectors in the dual of X. Then $X_{\mathfrak{b}}^*$ is finitely generated, and the canonical pairing between X and $X_{\mathfrak{b}}^*$ is nonsingular.*

Corollary 4.6. *In the setting of 4.5, suppose X is irreducible, and G is connected. Then there is a nonsingular pairing between X and some irreducible* $\mathfrak{b}$-*highest weight module $L(\lambda)$.*

Proof. Obviously $X_{\mathfrak{b}}^*$ has some irreducible submodule $L(\lambda)$. The pairing between X and $L(\lambda)$ is non-zero, and hence nonsingular since X is irreducible for $\mathfrak{g}$. Q.E.D.

Corollary 4.7. *If X is an irreducible $(\mathfrak{g}, K)$ module, then* $\operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann} X) = 2 \operatorname{Dim} X$.

Proof. If G is connected, this follows from 3.1, 3.4, and 4.6. In general, let K_0 denote the identity component of K. Obviously

$$X = X_1 \oplus \cdots \oplus X_r$$

as a $(\mathfrak{g}, K_0)$ module, with X_i irreducible, and all X_i conjugate by elements of $\operatorname{Ad}(\mathfrak{g})$ normalizing $\mathfrak{k}$. (Any element of $\operatorname{Ad}(\mathfrak{g})$ normalizing $\mathfrak{k}$ obviously acts on the set of $(\mathfrak{g}, \mathfrak{k})$ modules. The elements we need are those in $\operatorname{Ad}(K)$.) Thus $\operatorname{Dim}(X_i) = \operatorname{Dim}(X_j)$ for all i and j, so $\operatorname{Dim}(X_i) = \operatorname{Dim}(X)$. Now $\operatorname{Ann}(X_i)$ is a two sided ideal in $U(\mathfrak{g})$, and is therefore stable under $\operatorname{Ad}(\mathfrak{g})$; so $\operatorname{Ann}(X_i) = \operatorname{Ann}(X)$, for any i. Hence $\operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann} X) = \operatorname{Dim}(U(\mathfrak{g})/\operatorname{Ann} X_i) = 2 \operatorname{Dim} X_i = 2 \operatorname{Dim} X$. Q.E.D.

Theorem 1.1 now follows from 4.4 and 4.7. We close this section with some related results. Let X be an irreducible compatible $(\mathfrak{g}, K)$ module, with infinitesimal character $\lambda \in \bar{C}$. By [11], 6.18, there is a *unique* coherent family $\Theta(\lambda + \tilde{\mu})$ ($\tilde{\mu}$ a weight of a finite dimensional representation) such that $\Theta(\lambda) = \Theta(X)$, and $\Theta(\lambda + \tilde{\mu})$ is the character of an irreducible representation whenever $\lambda + \mu \in \bar{C}$. The constituents of $\Theta(\lambda + \tilde{\mu})$ are obtained from X by tensoring with finite dimensional representations and taking subrepresentations, so $\operatorname{Dim}(\Theta(\lambda + \tilde{\mu})) \leq \operatorname{Dim} X = d$. So we obtain a canonical harmonic polynomial c associated to X, with $c_d(X) = c(\lambda)$. Let

$$\Delta_\lambda = \left\{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \,\middle|\, \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \right\},$$

and let C_λ be the associated Weyl chamber. By [11], Theorem 5.20, $\Theta(\lambda + \tilde{\mu})$ is the character of an irreducible representation $(X(\lambda + \tilde{\mu})$ whenever $\lambda + \mu \in \bar{C}_\lambda$. If $X(\lambda + \tilde{\mu}) \neq 0$, the remarks above imply that $\operatorname{Dim}(X(\lambda + \tilde{\mu})) = d$, so $c(\lambda + \mu) > 0$. Let B_λ be the set of simple roots for the positive system $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. The Borho-Jantzen-Duflo τ-invariant of X is a subset of B_λ defined by

$$\tau(X) = \{\alpha \in B_\lambda | X(\lambda + \tilde{\mu}) = 0 \quad \text{for all } \lambda + \mu \in \bar{C}_\lambda \text{ such that } \langle \alpha, \lambda + \mu \rangle = 0\}$$

(cf. [2]; the definition in [4] is different). Then $\tau(X)$ depends only on $\operatorname{Ann} X$. Because of the remarks above, this definition can be reformulated as

$$\tau(X) = \{\alpha \in B_\lambda | c(\lambda + \mu) = 0 \quad \text{for all } \lambda + \mu \in \bar{C}_\lambda \text{ such that } \langle \alpha, \lambda + \mu \rangle = 0\}.$$

Set $p_\alpha(\lambda) = \langle \alpha, \lambda \rangle$; then since c is a polynomial, we have

$$\tau(X) = \{\alpha \in B_\lambda | p_\alpha \text{ divides } c\}.$$

Suppose $\alpha \in \tau(X)$. By [11], Theorem 5.20, we have for $\lambda + \mu \in C_\lambda$,

$$\Theta(\lambda + \tilde{\mu}) = -\Theta(s_\alpha(\lambda + \tilde{\mu})).$$

(If $\dfrac{2\langle \alpha, \lambda + \mu \rangle}{\langle \alpha, \alpha \rangle} = n$, $s_\alpha(\lambda + \tilde{\mu})$ simply means $\lambda + (\tilde{\mu} - n\alpha)$, which makes sense since $\tilde{\mu} - n\alpha$ is a weight of finite dimensional representation.) Thus

$$c(\lambda + \mu) = -c(s_\alpha(\lambda + \mu)).$$

Now W acts on polynomials on $\mathfrak{h}^*$ by

$$(\sigma \cdot p)(\gamma) = p(\sigma^{-1} \gamma),$$

so this condition may be written as $s_\alpha \cdot c = -c$. Suppose on the other hand that $\alpha \notin \tau(X)$. By [11], Corollary 6.17, if $\lambda + \mu \in C_\lambda$,

$$\Theta(s_\alpha(\lambda + \tilde{\mu})) = \Theta(\lambda + \tilde{\mu}) + \Theta_0(\lambda + \tilde{\mu})$$

with $\Theta_0(\lambda + \tilde{\mu})$ the character of a representation. We call $\Theta_0(\lambda + \tilde{\mu})$ the *α-trash term*; if Y is any constituent of $\Theta_0(\lambda + \tilde{\mu})$, then $\alpha \in \tau(Y)$ (cf. the remarks after Theorem 5.20 in [11]). In particular, $c(s_\alpha(\lambda + \mu)) \geq c(\lambda + \mu)$ for $\lambda + \mu \in C_\lambda$, so $s_\alpha \cdot c \neq -c$. Thus

$$\tau(X) = \{\alpha \in B_\lambda | s_\alpha \cdot c = -c\}.$$

These ideas have several amusing consequences. For example,

Proposition 4.8. *Suppose α and β are adjacent simple roots of the same length in B_λ, $\alpha \in \tau(X)$, and $\beta \notin \tau(X)$. Then the β-trash term has Gelfand-Kirillov dimension $d = \operatorname{Dim} X$.*

Proof. As constituents of the coherent continuation of X, the constituents of the β-trash term have Gelfand-Kirillov dimension $\leq d$. Suppose strict inequality holds. Then if we write

$$\Theta(s_\beta(\lambda + \tilde{\mu})) = \Theta(\lambda + \tilde{\mu}) + \Theta_0(\lambda + \tilde{\mu}),$$

with $\Theta_0(\lambda + \tilde{\mu})$ the β-trash term, we have $c_d(\Theta_0(\lambda + \tilde{\mu})) = 0$, so that $c(s_\beta(\lambda + \mu)) = c(\lambda + \mu)$. Hence $s_\beta c = c$. Since $\alpha \in \tau(X)$, $s_\alpha c = -c$. Let W_1 be the Weyl group generated by s_α and s_β. Then W_1 acts on c by scalars. But $W_1 \cong S_3$, the permutation group on three elements; so its only one dimensional representations are the trivial representation and the sign representation, and clearly the action on c is neither of these. This contradiction proves the result. Q.E.D.

Among the annihilators of irreducible representations of $\mathfrak{g}$ with infinitesimal character λ, there are a largest and a smallest, I_0 and I_1; representations with these annihilators are characterized by the property $\tau(X) = B_\lambda$ or $\tau(X) = \emptyset$

respectively. Put $r=|\Delta^+|$, $r_\lambda=|\Delta_\lambda^+|$; then

$$\mathrm{Dim}(U(\mathfrak{g})/I_0)=2r-2r_\lambda$$

$$\mathrm{Dim}(U(\mathfrak{g})/I_1)=2r$$

(see [7], Corollary 3.5).

Proposition 4.9. *With notation as above, suppose X is an irreducible $(\mathfrak{g}, K)$ module with infinitesimal character $\lambda\in\bar{C}$; define the harmonic polynomial c as above.*

a) *If $\tau(X)=\emptyset$, then c is constant.*

b) *If $\tau(X)=B_\lambda$, then c is divisible by the dimension function* $\prod_{\alpha\in\Delta_\lambda^+} p_\alpha=\delta_\lambda$ of Δ_λ.

Proof. For (a), suppose $\lambda+\mu\in C$. By the proof of Theorem 6.18 of [11], for any $w\in W_\lambda$,

$$\Theta(w(\lambda+\bar{\mu}))=\Theta(\lambda+\bar{\mu})+\Theta_0;$$

and each constituent Y of Θ_0 satisfies $\tau(Y)\neq B_\lambda$. Hence $\mathrm{Ann}(Y)\neq I_1$. The proof of Lemma 6.6 of [6] shows that $\mathrm{Dim}\,Y<r=\mathrm{Dim}\,X$. Hence $\mathrm{Dim}\,\Theta_0<r$, so $c_r(\Theta_0)=0$, and

$$c(w(\lambda+\mu))=c(\lambda+\mu).$$

It follows that $w\cdot c=c$ for all $w\in W_\lambda$. We have seen that $c(\lambda+\mu)\geqq 0$ whenever $\lambda+\mu\in\bar{C}_\lambda$ (with μ a weight of a finite dimensional representation). Now let $\tilde{c}$ be the highest degree term of c. Then it follows that $\tilde{c}$ is a non-negative harmonic polynomial, and hence a constant.

For (b), we have seen that $s_\alpha\cdot c=-c$ whenever $\alpha\in B_\lambda$; it follows that c transforms according to the sign representation of W_λ. Hence $c=c_1\,\delta_\lambda$, with c_1 a W_λ-invariant polynomial. Q.E.D.

Conjecture 4.10. *If X is an irreducible $(\mathfrak{g}, K)$ module of Gelfand-Kirillov dimension d, and c is the harmonic polynomial associated to X as above, then c is homogeneous of degree $|\Delta^+|-d$, and harmonic with respect to W_λ.*

All of the results of this section can be formulated and proved for highest weight modules without essential change.

5. Asymptotics of $N_X(t)$

We turn now to Theorem 1.2 and related matters; we continue to assume that G is reductive, as in Section 4. Recall that Ω is the Casimir operator of $\mathfrak{k}$. If X is a semisimple representation of K and $t>0$, we define $X_{(t)}$ to be the span of the eigenspaces of Ω in X with eigenvalue at most t^2, and put $N_X(t)=\dim X_{(t)}$. Our goal is to study the asymptotics of N_X. Clearly N_X can be defined in the Grothendieck group. To motivate the results below, we begin with a conjecture.

Conjecture 5.1. *Suppose X is an irreducible $(\mathfrak{g}, K)$ module with infinitesimal character $\lambda\in\bar{C}$. Embed $\Theta(X)=\Theta(\lambda)$ in a coherent family of characters as in the last part of Section 4. Then there is a homogeneous harmonic polynomial $\tilde{c}$ on $\mathfrak{h}^*$, of degree $|\Delta^+|-\mathrm{Dim}\,X$, such that $\tilde{c}(\lambda)\neq 0$ and* $\lim_{t\to\infty}(t^{-\mathrm{Dim}\,X}N_{\Theta(\lambda+\mu)}(t))=\tilde{c}(\lambda+\mu)$ *whenever $\bar{\mu}$ is a weight of a finite dimensional representation.*

Our results fall far short of this, unfortunately. All of the sharp asymptotic results rely on

Proposition 5.2. (Wallach [13]). *Let $K^0\subseteq K$ be a closed subgroup, and F a finite dimensional representation of K^0, of dimension f. Put $d=\dim K/K^0$, $X=\mathrm{Ind}_{K^0}^K F$. Then $N_X(t)\sim C(K,K^0)\cdot f\cdot t^d$.*

(We have replaced Wallach's (G,K,Γ) by $(K,K^0,\{1\})$. Although Wallach's hypotheses no longer hold, his proof still applies, and in fact simplifies substantially. The referee has pointed out that such results have been known for a very long time – see for example [15]. I would like to thank J. Rosenberg for bringing these ideas to my attention.) The proof is by heat equation methods – a representation-theoretic proof might be very enlightening.

Corollary 5.3. *Let $P_1=M_1A_1N_1$ be a parabolic subgroup of G, $\delta\in\hat{M}_1$ a finite dimensional representation, and $v\in\hat{A}_1$ a (non-unitary), character. Put $X=\mathrm{Ind}_{P_1}^G\delta\otimes v\otimes 1$. (Here and henceforth, Ind is used to mean the K-finite vectors in the induced representation.)*

$$N_X(t)\sim c(P_1)(\dim\delta)\,t^{\dim N_1}.$$

Proof. We only need to observe that

$$X|_K=\mathrm{Ind}_{M_1\cap K}^K(\delta|_{M_1\cap K}),$$

and that $\dim K/M_1\cap K=\dim N_1$. Q.E.D.

Proposition 5.4 (*G reductive*). *Let X be a finitely generated $U(\mathfrak{g})$ module, semisimple for $\mathfrak{k}$; put $d=\mathrm{Dim}\,X$. Then there is a constant $A>0$ depending only on $\mathfrak{g}$, such that for all sufficiently large t,*

$$N_X(t)\geqq A\,c(X)\,t^d.$$

Proof. Fix a finite dimensional $\mathfrak{k}$-invariant generating subspace $X_0\subseteq X$, and set $X_n=U_n(\mathfrak{g})\,X_0$. Recall that $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in X; let $S_{(n)}(\mathfrak{p})$ be the space of polynomials of degree at most n in $\mathfrak{p}$. Since X_0 is $\mathfrak{k}$-invariant, X_n is isomorphic to a quotient of $S_{(n)}(\mathfrak{p})\otimes_{\mathbb{C}} X_0$ as a $\mathfrak{k}$ module. Fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Let c_1 be the length of the longest weight of $\mathfrak{t}$ in $\mathfrak{p}$, and c_2 the length of the longest weight of $\mathfrak{t}$ in X_0. Then the longest weight of $\mathfrak{t}$ in $S_{(n)}(\mathfrak{p})\otimes X_0$ has length at most $c_2+n\,c_1$. So the greatest eigenvalue of Ω in $S_{(n)}(\mathfrak{p})\otimes X_0$ is at most $c_3^2(c_2+n\,c_1)^2+c_4(c_2+n\,c_1)$, for some positive constants c_3 and c_4 depending only on $\mathfrak{k}$. So if $B>c_1c_3$, we have

$$X_n\subseteq X_{(Bn)}$$

for all sufficiently large n. The proposition follows. Q.E.D.

Proposition 5.5 (*G reductive*). *Let X be a finitely generated, $\mathfrak{Z}(\mathfrak{g})$-finite $U(\mathfrak{g})$ module, semisimple for $\mathfrak{k}$; put $d=\mathrm{Dim}\,X$. Then there is a constant $B>0$, depending* *only on* $\mathfrak{g}$, *such that for all sufficiently large* t,

$$N_X(t) \leqq B\,c(X)\,t^d.$$

Proof. We would like to reverse the argument given for Proposition 5.4; this would amount to showing that small representations of $\mathfrak{k}$ cannot occur in $S_n(\mathfrak{p})$ for large n. Such a statement is false; but it can be modified along the following lines: let J be the space of $\mathfrak{k}$-invariant polynomials on $\mathfrak{p}$, and let H be the space of harmonic polynomials on $\mathfrak{p}$. (These are the polynomials annihilated by all $\mathfrak{k}$-invariant differential operators on $\mathfrak{p}$ without constant term – for details see [8].) Then by a theorem of Kostant and Rallis, $S(\mathfrak{p}) \cong H \otimes J$. Choose a finite dimensional $\mathfrak{k}$-invariant, $U(\mathfrak{g})^{\mathfrak{k}}$ invariant generating subspace X_0 of X. Let H_n be the space of harmonic polynomials of degree n. Under the identification $U(\mathfrak{g}) \cong S(\mathfrak{p}) \otimes U(\mathfrak{k})$, J is identified with a subspace of $U(\mathfrak{g})^{\mathfrak{k}}$. It follows that, as a $\mathfrak{k}$ module, X_n/X_{n-1} may be identified with a submodule of $H_n \underset{\mathbb{C}}{\otimes} X_0$. We claim that for some constant a_1 depending only on $\mathfrak{g}$, and all large t,

$$X_{(t)} \subseteq X_{a_1 t}.$$

We will specify a_1 in a moment; this will prove the proposition. Suppose not. Then there is an integer $m > a_1 t$, and a $\mathfrak{k}$-type π_1 occurring in $X_{(t)}$, such that π_1 also occurs in X_m/X_{m-1}, and hence in $H_m \otimes X_0$. Let γ_1 be an extremal weight of π_1. Since π_1 occurs in $X_{(t)}$, $|\gamma_1| \leqq a_2 t$, for some constant a_2 depending only on $\mathfrak{k}$. Let a_3 be the length of the longest weight of $\mathfrak{t}$ in X_0. Since π_1 occurs in $H_m \otimes X_0$, H_m has a $\mathfrak{k}$-type π_2 in common with $\pi_1 \otimes X_0^*$; for

$$0 \neq \mathrm{Hom}_{\mathfrak{k}}(\pi_1, H_m \otimes X_0) \cong \mathrm{Hom}_{\mathfrak{k}}(\pi_1 \otimes X_0^*, H_m).$$

By a standard fact about tensor products, π_2 has an extremal weight of the form $\gamma_2 = \gamma_1 + \gamma_0$, with γ_0 a weight of X_0; so $|\gamma_2| \leqq a_2 t + a_3$.

Lemma 5.6. *There is a constant* $a_4 > 0$ *such that if a representation* π_2 *of* $\mathfrak{k}$ *with extremal weight* γ_2 *occurs in* H_m, *then* $|\gamma_2| \geqq a_4 m$.

Assume this lemma for a moment. Then we have

$$a_2 t + a_3 \geqq |\gamma_2| \geqq a_4 m > a_1 a_4 t.$$

If we choose $a_1 > a_2 a_4^{-1}$, this is impossible for large t, proving the proposition.

Proof of Lemma 5.6. Let $\{\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r\}$ be the set of θ-invariant Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{t}$; let $\mathfrak{n}_i$ be the nil radical of $\mathfrak{q}_i$. Choose elements $\{t_1, \ldots, t_r\}$ in $\mathfrak{t}$, and a constant $b_1 > 0$ so that if α is a weight of $\mathfrak{t}$ in $\mathfrak{n}_i$, then $\alpha(t_i) \geqq b_1$; this is certainly possible. By the Schwarz inequality, there is a constant $b_2 > 0$ so that if $\gamma \in \mathfrak{t}^*$, $|\gamma| \geqq b_2 |\gamma(t_i)|$ for all i.

Now H_m is generated as a $\mathfrak{k}$ module by the various $S_m(\mathfrak{n}_i \cap \mathfrak{p})$ (see [8]). So the representation π_2 of the lemma contains some weight γ_3 occurring in $S_m(\mathfrak{n}_i \cap \mathfrak{p})$. By the preceding paragraph, $\gamma_3(t_i) \geqq m b_1$, so $|\gamma_3| \geqq m b_1 b_2$. The longest weights of any representation of $\mathfrak{k}$ are the extremal weights, so $|\gamma_3| \leqq |\gamma_2|$. The lemma follows, with $a_4 = b_1 b_2$. Q.E.D.

Theorem 1.2 follows from 5.4 and 5.5.

Proposition 5.7. *Let* X *be an irreducible* $(\mathfrak{g}, K)$ *module. Then* $\mathrm{Dim}\, X \leqq \dim N$ *(here* $G = KAN$ *is an Iwasawa decomposition of* G*).*

Proof. Recall the minimal parabolic subgroup $P = MAN$ of G described at the beginning of Section 4. By Harish-Chandra's subquotient theorem, there is an irreducible representation $\delta \in \hat{M}$ and a character $\nu \in \hat{A}$ such that X is a constituent of $Y = \mathrm{Ind}_P^G\, \delta \otimes \nu \otimes 1$. By Corollary 5.3,

$$N_X(t) \leqq N_Y(t) \sim c\,t^{\dim N}.$$

The proposition now follows from Theorem 1.2 Q.E.D.

Corollary 5.8. *Let* X *be an irreducible* $(\mathfrak{g}, K)$ *module. If* $\mathrm{Ann}\, X$ *is the minimal ideal described before Proposition* 4.9, *then* G *is quasisplit* (i.e. P *is a Borel subgroup of* G, *and* MA *is abelian*).

6. Large Representations

Definition 6.1. The irreducible $(\mathfrak{g}, K)$ module X is called *large* if its annihilator is the minimal primitive ideal, i.e. $\tau(X) = \emptyset$.

By Corollary 5.8, large representations exist only for quasisplit groups. Recall that for these groups, $MA = H$ is a maximally split Cartan subgroup. The following theorem was suggested in part by B. Speh; related results for complex groups have been obtained by M. Duflo in [4].

Theorem 6.2. *Suppose* G *is quasisplit. Let* X *be an irreducible* $(\mathfrak{g}, K)$ *module. (As always, we assume that* G *is linear and reductive, with abelian Cartan subgroups; it need not be connected.) Set* $r = \dim N = |\Delta^+(\mathfrak{g}, \mathfrak{h})|$. *The following conditions are equivalent:*

a) X *is large.*

b) $\mathrm{Dim}\, X = r$.

c) $c_r(X) = 2^{-s} \cdot a_G$; *here* s *is an integer depending on* X, *and* a_G *depends only on* G.

d) $N_X(t) \sim 2^{-s} \cdot b_G t^r$ *as* $t \to \infty$.

e) *There is a representation* $\delta \in \tilde{M}$ *and a character* $\nu \in \hat{A} \cong \mathfrak{a}^*$ *such that if* $\gamma \in \mathfrak{h}^*$ *is the differential of* $\delta \otimes \nu$, *then*

i) $\mathrm{Re}\, \nu$ *is dominant with respect to the restricted roots of* $\mathfrak{a}$ *in* $\mathfrak{n}$.

ii) *If* $\alpha \in \Delta^+$ *and* $\alpha - \theta\alpha$ *is a root, then* $\dfrac{2\langle\alpha, \gamma\rangle}{\langle\alpha, \alpha\rangle}$ *is not a negative integer.*

iii) X *is infinitesimally equivalent to a subrepresentation of* $\mathrm{Inf}_P^G\, \delta \otimes \nu \otimes 1$.

f) *There is a cuspidal parabolic subgroup* $P' = M'A'N' \supseteq P$, *a representation* $\delta' \in \hat{M}'$ *in the limits of the discrete series, and a character* $\nu' \in A'$, *such that the Weyl chamber in the compact Cartan subalgebra of* $\mathfrak{m}'$ *defined by the Harish-Chandra parameter of* δ' *has all simple roots noncompact; and* X *is infinitesimally equivalent to* $\mathrm{Ind}_{P'}^G\, \delta' \otimes \nu' \otimes 1$. *In this case the integer* s *of* (c) *and* (d) *depends only on* M'.

Proof. The equivalence of (a) and (b) has been noted in the proof of 4.9, and clearly (d) and (c) imply (a). We will show (e) ⇒ (b), (b) ⇒ (e), (a) ⇒ (f), (f) ⇒ (d), and (f) ⇒ (c); this will prove the theorem. (The study of (e) is thus unrelated to the rest of the theorem; the reader may skip it without loss.) For $\gamma \in \mathfrak{h}^*$, define

$$\Delta_\gamma = \left\{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \,\middle|\, \frac{2\langle \alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z} \right\}.$$

If β is a restricted root of $\mathfrak{a}$ in $\mathfrak{n}$, we write s_β for the corresponding simple reflection; s_β acts on $\mathfrak{h}^*$ and $\hat{H}$. Suppose β is a restricted root such that 2β is not a restricted root. Let $\mathfrak{m}^\beta$ be the span of $\mathfrak{h}$ and the root vectors for roots which restrict to $\pm\beta$ and $\pm 2\beta$. Then $\mathfrak{m}^\beta = (\mathfrak{m}_0^\beta)_{\mathbb{C}}$; $\mathfrak{m}_0^\beta$ is a rank one reductive Lie algebra. More precisely, there are three possibilities.

1) β is real, and $\mathfrak{m}_0^\beta \cong \mathfrak{sl}(2, \mathbb{R}) + \text{center}$.

2) β is the restriction of a complex root α orthogonal to $\theta\alpha$; and $\mathfrak{m}_0^\beta \cong \mathfrak{sl}(2, \mathbb{C}) + \text{center}$.

3) β is the restriction of a complex root α not orthogonal to $\theta\alpha$; $\alpha - \theta\alpha$ is a real root (namely 2β), and $\mathfrak{m}_0^\beta \cong \mathfrak{su}(2,1) + \text{center}$.

In cases (2) and (3), β is simple as a restricted root iff α and $-\theta\alpha$ are simple for Δ^+. If β is simple, there is a parabolic subgroup $P^\beta = M^\beta A^\beta N^\beta \supseteq P$, *with* $\mathfrak{m}_0^\beta$ the Lie algebra of M^β, and $\mathfrak{a}^\beta = \ker \beta \subseteq \mathfrak{a}$.

To study condition (e), we need some facts about principal series representations. Call the restricted root β *totally non-integral* with respect to $\gamma \in \mathfrak{h}^*$ if (in case (1)) $\frac{2\langle \beta, \gamma\rangle}{\langle \beta, \beta\rangle} \notin \mathbb{Z}$, or (in cases (2) and (3)) $\frac{2\langle \alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle} \notin \mathbb{Z}$ for α a complex root of $\mathfrak{h}$ in $\mathfrak{m}^\beta$. It is easy to see that if $\gamma \in \mathfrak{h}^*$ is the differential of a character $\delta \otimes v \in \hat{H}$, then Δ_γ is θ-invariant.

Lemma 6.3. *Fix $\delta \otimes v \in \hat{H}$, with differential γ; write $\pi(\delta \otimes v) = \mathrm{Ind}_P^G \delta \otimes v \otimes 1$. Let β be a simple restricted root, and $\alpha \in \Delta^+$ a simple root with $\alpha|_{\mathfrak{a}} = \beta$.*

a) *If β is totally non-integral with respect to γ, then $\pi(\delta \otimes v) \cong \pi(s_\beta(\delta \otimes v))$ (infinitesimal equivalence).*

b) *If $\mathfrak{m}^\beta \cong \mathfrak{sl}(2, \mathbb{C}) + \text{center}$, and $\langle \alpha, \gamma\rangle$ and $\langle -\theta\alpha, \gamma\rangle$ have opposite sign (or one vanishes), then $\pi(\delta \otimes v) \cong \pi(s_\beta(\delta \otimes v))$.*

c) *If $\mathfrak{m}^\beta \cong \mathfrak{sl}(2, \mathbb{C})$, and $\frac{2\langle \alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle} = n$ and $\frac{2\langle \theta\alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle}$ are negative integers, set $\delta_\alpha \otimes v_\alpha = \delta \otimes v - n\alpha$; here we consider α as an element of $\hat{H}$, and write the group operation in $\hat{H}$ additively. Then there is an embedding*

$$\pi(\delta \otimes v) \hookrightarrow \pi(\delta_\alpha \otimes v_\alpha).$$

Proof. If $\mathfrak{g}_0 \cong \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{su}(2,1)$, or $\mathfrak{sl}(2, \mathbb{C})$, the lemma is known. The general case follows using induction by stages (through the parabolic subgroup P^β). Q.E.D.

Proof of (e) ⇒ (b). Recall the module $X_{\mathfrak{b}}^*$ of $\mathfrak{b}$-finite vectors in the dual of X, introduced in Lemma 4.5. Given a representation $\delta' \otimes v' \in \hat{H}$, we can define a compatible $(\mathfrak{g}, M)$ module (in the obvious sense) as follows: let $\mathbb{C}_{\delta' \otimes (v' - \rho)}$ be the one dimensional module for M and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ in which M and $\mathfrak{h}$ act by the

indicated weight, and $\mathfrak{n}$ acts trivially. Define

$$V(\delta' \otimes v') = U(\mathfrak{g}) \bigotimes_{\mathfrak{b}} \mathbb{C}_{\delta' \otimes (v' - \rho)}.$$

By Frobenius reciprocity,

(6.4) $\quad \mathrm{Hom}_{(\mathfrak{g}, K)}(X, \pi(\delta \otimes v)) \cong \mathrm{Hom}_{(\mathfrak{g}, M)}(V(\delta^* \otimes (-v)), X_{\mathfrak{b}}^*).$

Consider the set L of all $\delta' \otimes v' \in \hat{H}$ such that X is infinitesimally equivalent to a subrepresentation of $\pi(\delta \otimes v)$, and such that the differential γ' of $\delta' \otimes v'$ satisfies the following conditions with respect to each restricted root β of $\mathfrak{a}$ in $\mathfrak{n}$

L1 If $\mathfrak{m}_0^\beta \cong \mathfrak{sl}(2, \mathbb{R}) + \text{center}$, then $\frac{2\langle \gamma', \beta\rangle}{\langle \beta, \beta\rangle}$ is not a negative integer.

L2 If $\mathfrak{m}_0^\beta \cong \mathfrak{sl}(2, \mathbb{C}) + \text{center}$, and $\alpha|_{\mathfrak{a}} = \beta$ for some $\alpha \in \Delta^+$, and $\frac{2\langle \alpha, \gamma'\rangle}{\langle \alpha, \alpha\rangle}$ is a negative integer, then $\langle -\theta\alpha, \gamma'\rangle > 0$.

L3 If $\mathfrak{m}_0^\beta \cong \mathfrak{su}(2,1) + \text{center}$, and $\alpha|_{\mathfrak{a}} = \beta$ for some $\alpha \in \Delta^+$, then $\frac{2\langle \alpha, \gamma'\rangle}{\langle \alpha, \alpha\rangle}$ is not a negative integer.

It is trivial to check that the element $\delta \otimes v$ given by (e) lies in L; so L is non-empty. Now choose an element $\delta \otimes v \in L$ so that $\operatorname{Re} v$ is as long as possible. We claim that in this case $\frac{2\langle \alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle}$ is never a negative integer, for any root $\alpha \in \Delta^+$. For suppose not. Let $R \subseteq \Delta_\gamma$ be the positive root system

$$R = \{\alpha \in \Delta_\gamma \mid \langle \alpha, \gamma\rangle > 0 \text{ or } \langle \alpha, \gamma\rangle = 0 \text{ and } \alpha \in \Delta^+\}.$$

Let $\alpha \in \Delta^+$ be a root with $\frac{2\langle \alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle}$ a negative integer. By L1–L3, $\langle -\theta\alpha, \gamma\rangle > 0$. Replacing α by $-\theta\alpha$, we have a root α with $\frac{2\langle \alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle}$ a positive integer, and $\frac{2\langle -\theta\alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle}$ a negative integer. Write $\alpha = \delta_1 + \cdots + \delta_r$, a sum of simple roots in R. For some i, we must have $\frac{2\langle -\theta\delta_i, \gamma\rangle}{\langle \delta_i, \delta_i\rangle}$ a negative integer. If $\langle \delta_i, \gamma\rangle = 0$, then $-\delta_i \in R$ by L2; so $\frac{2\langle \delta_i, \gamma\rangle}{\langle \delta_i, \delta_i\rangle}$ is a positive integer. Suppose for definiteness that $\delta_i \in \Delta^+$; the other case is very similar, and is left to the reader. Assume that all choices have been made so that δ_i involves as few simple roots of Δ^+ as possible. We claim that then δ_i is simple. Suppose not; choose a simple restricted root β so that $\langle \beta, \delta_i|_{\mathfrak{a}}\rangle > 0$, and write $\delta_i|_{\mathfrak{a}} = d \cdot \beta + \beta'$, with $\beta' = s_\beta(\delta_i|_{\mathfrak{a}})$. We want to replace γ by $s_\beta \gamma$; this will replace δ_i by $s_\beta \delta_i$, which involves fewer simple roots of Δ^+ than δ_i. This will contradict the choice of γ and δ_i originally made, and prove that δ_i is simple. If β is totally nonintegral with respect to γ, then X embeds in $\pi(s_\beta(\delta \otimes v))$ by Lemma 6.3 (a); and it is easy to check that $s_\beta \gamma$ satisfies L1–L3. So in this case we may in fact replace γ by $s_\beta \gamma$. If $\mathfrak{m}_0^\beta \cong \mathfrak{sl}(2, \mathbb{C}) + \text{center}$, and $\alpha|_{\mathfrak{a}} = \beta$ for some $\alpha \in \Delta^+$, then we may as well assume $\frac{2\langle \alpha, \gamma\rangle}{\langle \alpha, \alpha\rangle} = n \in \mathbb{Z}$ (the totally non-integral case having been treated already). If $\langle \alpha, \gamma\rangle$ and $\langle -\theta\alpha, \gamma\rangle$

have opposite sign, then X embeds in $\pi(s_\beta(\delta\otimes\nu))$ by Lemma 6.3 (b); and $s_\beta\gamma$ satisfies L1–L3. By L2, we may assume $\langle\alpha,\gamma\rangle$ and $\langle-\theta\alpha,\gamma\rangle$ are both non-negative. Either $\langle\alpha,\delta_i\rangle$ or $\langle-\theta\alpha,\delta_i\rangle$ is positive; assume for definiteness that the first holds. Then $\delta_i=t\alpha+\alpha'$, for $\alpha'=s_\alpha\delta_i$. Since $\alpha\in\Delta_\gamma$ and $\delta_i\in\Delta_\gamma$, $\alpha'\in\Delta_\gamma$. But $\langle\alpha,\gamma\rangle\geqq 0$, and $\alpha\in\Delta^+$, so $\alpha\in R$. By the simplicity of δ_i in R, $\langle\alpha',\gamma\rangle<0$. On the other hand, $-\theta\delta_i=-t\theta\alpha-\theta\alpha'$. By hypothesis, $\langle-\theta\delta_i,\gamma\rangle<0$ and $\langle-\theta\alpha,\gamma\rangle\geqq 0$; so $\langle-\theta\alpha',\gamma\rangle<0$. Thus α' and $-\theta\alpha'$ are positive, and $\frac{2\langle\alpha',\gamma\rangle}{\langle\alpha',\alpha'\rangle}$ and $\frac{2\langle-\theta\alpha',\gamma\rangle}{\langle\alpha',\alpha'\rangle}$ are negative integers; this contradicts L2 and L3. This contradiction proves that $\mathfrak{m}_0^\beta\neq\mathfrak{sl}(2,\mathbb{C})+$center. If $\mathfrak{m}_0^\beta\cong\mathfrak{su}(2,1)+$center, an identical argument produces a contradiction; in this case $\langle\alpha,\gamma\rangle$ and $\langle-\theta\alpha,\gamma\rangle$ are non-negative by L3. If $\mathfrak{m}_0^\beta\cong\mathfrak{sl}(2,\mathbb{R})+$center, we may assume as before that $\frac{2\langle\beta,\gamma\rangle}{\langle\beta,\beta\rangle}$ is an integer, which is non-negative by L1. Then $\delta_i=d\cdot\beta+\alpha'$, with $\alpha'=s_\beta\delta_i$; as before $\alpha'\in\Delta_\gamma$. Now $\beta\in R$, so $\alpha'\notin R$ by the simplicity of δ_i in R; so $\langle\alpha',\gamma\rangle<0$. On the other hand, $-\theta\delta_i=d\cdot\beta-\theta\alpha'$; so $\langle-\theta\alpha',\gamma\rangle<0$ as before, contradicting L2 and L3. Hence δ_i is in fact simple in Δ^+. Set $\beta=\delta_i|_{\mathfrak{a}}$. By Lemma 6.3 (b) and a simple argument like those already given, we may replace γ by $s_\beta\gamma$; this makes $-\delta_i$ a simple root for Δ^+. (If we had considered the case $-\delta_i\in\Delta^+$ earlier, we would have ended up with this same situation.) We now take $-\delta_i=\alpha$ in Lemma 6.3 (c). Put $n=\frac{2\langle\gamma,\delta_i\rangle}{\langle\delta_i,\delta_i\rangle}$, $\delta'\otimes\nu'=\delta\otimes\nu-n\delta_i$, and let γ' be the differential of $\delta'\otimes\nu'$. We would like to say that $\delta'\otimes\nu'\in L$. This is not quite true. By Lemma 6.3 (c), X embeds in $\pi(\delta'\otimes\nu')$. We claim that γ' satisfies L1 and L3, and

L2′: If $\mathfrak{m}_0^\beta\cong\mathfrak{sl}(2,\mathbb{C})+$center, and $\alpha|_{\mathfrak{a}}=\beta$ for some $\alpha\in\Delta^+$, and $\frac{2\langle\alpha,\gamma'\rangle}{\langle\alpha,\alpha\rangle}$ is a negative integer, then $\langle-\theta\alpha,\gamma'\rangle\geqq 0$.

We will verify only L2′. Now $\gamma'=s_{\delta_i}\gamma$; so $s_{\delta_i}\alpha\in\Delta_\gamma$, and $\langle s_{\delta_i}\alpha,\gamma\rangle<0$. Hence $s_{\delta_i}\alpha\in-R$. Since $\alpha\neq\delta_i$ (for $\delta_i\notin\Delta^+$), and δ_i is simple for R, $\alpha\in-R$. Since $\alpha\in\Delta^+$, $\langle\alpha,\gamma\rangle<0$. By L2, $\langle-\theta\alpha,\gamma\rangle>0$, so $-\theta\alpha\in R$. Since $-\theta\alpha\neq\delta_i$, $s_{\delta_i}(-\theta\alpha)\in R$, so $\langle-\theta\alpha,\gamma'\rangle\geqq 0$; this proves L2′.

Let $\delta''\otimes\nu''\in\hat{H}$ be chosen so that $\mathrm{Re}\langle\nu'',\rho\rangle$ is as large as possible with respect to the conditions

1) X embeds in $\pi(\delta''\otimes\nu'')$;
2) The differential γ'' of $\delta''\otimes\nu''$ satisfies L1, L3, and L2′; and
3) $|\mathrm{Re}\,\nu''|=|\mathrm{Re}\,\nu'|$.

We claim that $\delta''\otimes\nu''\in L$. It is enough to show that $\mathrm{Re}\,\nu''$ is dominant; for then L2′ easily implies L2. Suppose not; let β be a simple restricted root with $\mathrm{Re}\langle\beta,\nu''\rangle<0$. By L1, L2′, and L3, either β is totally nonintegral with respect to γ, or (with $\alpha\in\Delta^+$, $\alpha|_{\mathfrak{a}}=\beta$) $\mathfrak{m}_0^\beta\cong\mathfrak{sl}(2,\mathbb{C})+$center, and $\langle\alpha,\gamma''\rangle$ and $\langle-\theta\alpha,\gamma''\rangle$ have opposite sign (or one vanishes). By Lemma 6.3 (a) and (b), X embeds in $\pi(s_\beta(\delta''\otimes\nu''))$; it is easy to verify that $s_\beta\gamma''$ satisfies L1, L2′, and L3; and $|\mathrm{Re}\,\nu''|=|\mathrm{Re}\,s_\beta\nu''|$. But $\mathrm{Re}\langle s_\beta\nu'',\rho\rangle>\mathrm{Re}\langle\nu'',\rho\rangle$, a contradiction. So $\delta''\otimes\nu''\in L$. But $|\mathrm{Re}\,\nu''|=|\mathrm{Re}\,\nu'|$; and it is easy to compute that $|\mathrm{Re}\,\nu'|>|\mathrm{Re}\,\nu|$ (cf. [11], Corollary 6.3). This contradicts the choice of γ (with $|\mathrm{Re}\,\nu|$ maximal), proving our claim that $\frac{2\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}$ is never a negative integer. It follows that the Verma module $V(\delta^*\otimes(-\nu))$ is irreducible. Since we have a non-zero map

$$X\hookrightarrow\pi(\delta\otimes\nu),$$

(6.4) provides an embedding of $V(\delta^*\otimes(-\nu))$ in $X_{\mathfrak{b}}^*$. By Lemma 3.4, $\mathrm{Dim}\,X=\mathrm{Dim}\,V(\delta^*\otimes(-\nu))$, which is r by Lemma 2.3. This proves (e) $\Rightarrow$ (b).

Proof of (b) $\Rightarrow$ (e). By the Casselman-Jacquet subrepresentation theorem, we can find some $\delta\otimes\nu\in\hat{H}$ so that X embeds in $\pi(\delta\otimes\nu)$. Choose such $\delta\otimes\nu$ with differential γ so that $\mathrm{Re}\langle\gamma,\rho\rangle$ is as large as possible. To simplify the notation we will write $V(-\gamma)$ for $V(\delta^*\otimes(-\nu))$. By (6.4), we can find a non-zero map $V(-\gamma)\to X_{\mathfrak{b}}^*$; write $\bar{V}(-\gamma)$ for the image. We claim $\bar{V}(-\gamma)$ is irreducible. Suppose not; then there is some $\gamma'\neq\gamma$ and a non-zero map $V(-\gamma')\to V(-\gamma)\subseteq X_{\mathfrak{b}}^*$. By (6.4), X embeds in some $\pi(\delta'\otimes\nu')$ with γ' the differential of $\delta'\otimes\nu'$. But clearly $\gamma'=\gamma+\Sigma n_i\alpha_i$, with $\alpha_i\in\Delta^+$ and $n_i\geqq 0$; so $\mathrm{Re}\langle\gamma',\rho\rangle>\mathrm{Re}\langle\gamma,\rho\rangle$, a contradiction. We have a non-zero pairing between X and $\bar{V}(-\gamma)$; since both of these are irreducible, it is nonsingular. By Lemma 3.6, $\mathrm{Dim}\,X=\mathrm{Dim}\,\bar{V}(-\gamma)$. Now the irreducible quotient of a reducible Verma module has Gelfand-Kirillov dimension strictly less than r (see for example [6], Proposition 6.5); so since $\mathrm{Dim}\,X=r$, $\bar{V}(-\gamma)\cong V(-\gamma)$ is irreducible. Thus $\frac{2\langle\alpha,\gamma\rangle}{\langle\alpha,\alpha\rangle}$ is not a negative integer for any $\alpha\in\Delta^+$. To show that $\delta\otimes\nu$ satisfies the conditions of (e), it remains only to verify that $\mathrm{Re}\,\nu$ is dominant. A similar statement was proved above at the end of the proof that (e) $\Rightarrow$ (b); the same proof works here. This proves (b) $\Rightarrow$ (e).

Proof of (a) $\Rightarrow$ (f). Let X be an irreducible $(\mathfrak{g},K)$ module with $\tau(X)=\emptyset$. We want to show that X is of the form specified by (f). A simple argument using Zuckerman's "periodicity" and the Borho-Jantzen "translation principle" (cf. [2], Theorem 2.12, and [11], Corollary 5.12 and Theorem 5.15) shows that we may assume the infinitesimal character $\lambda\in\mathfrak{h}^*$ of X to be nonsingular; we may as well take $\lambda\in C$. By the Langlands classification theorem, there is a parabolic subgroup $P'=M'A'N'$ containing P, a discrete series representation $\delta\in\hat{M}'$, and a character $\nu\in\hat{A}'$, so that X is infinitesimally equivalent to the Langlands quotient $\bar{\pi}(\delta\otimes\nu)$ of $\pi(\delta\otimes\nu)=\mathrm{Ind}_{P'}^{G}\,\delta\otimes\nu\otimes 1$. Suppose $\pi(\delta\otimes\nu)$ is reducible. By [11], Theorems 6.15 and 6.16, it is easy to see that $\tau(X)$ is non-empty. (In the terminology of [11], X is α-singular for some α.) This contradicts $\tau(X)=\emptyset$; so $\pi(\delta\otimes\nu)$ is irreducible, and hence

$$X\cong\bar{\pi}(\delta\otimes\nu)\cong\pi(\delta\otimes\nu).$$

Let $\Theta(\lambda+\tilde{\mu})$ be the coherent family of characters with $\Theta(\lambda)=\Theta(X)$ (defined in Section 4). By Proposition 4.9 (a), $\Theta(\lambda+\tilde{\mu})$ never vanishes. Suppose that in the Weyl chamber defined by the Harish-Chandra parameter for δ, there is a simple compact root. By [11], Corollary 5.12, and the first Hecht-Schmid character identity ([5], Proposition 3.6) we can choose $\tilde{\mu}$ (at least for some covering group of G) so that $\Theta(\lambda+\tilde{\mu})=0$; we need only make $\lambda+\mu$ lie on an appropriate root hyperplane. This contradiction completes the proof that (a) $\Rightarrow$ (f).

Proof of (f)$\Rightarrow$(d). We proceed by downward induction on $\dim A'$. If $\dim A' = \dim A$, then $A = A'$, $P = P'$. Since M is abelian, δ is one dimensional; so the result follows from Corollary 5.3. Suppose then that $A' \neq A$, and that the result is proved for all parabolic subgroups of parabolic rank greater than $\dim A'$. We will need the following lemma, which is of some interest in its own right.

Lemma 6.5. *Let G be a reductive Lie group (not necessarily quasisplit), and $P = MAN$ a parabolic subgroup (not necessarily minimal). Let Y be a finitely generated compatible $(\mathfrak{m}+\mathfrak{a}, M \cap K)$ module; K is a maximal compact subgroup of G, and the Langlands decomposition of P is chosen so that $M \cap K = P \cap K$. Suppose there is a nonsingular pairing of Y with Y_1. Put $X = \mathrm{Ind}_P^G(Y \otimes 1)$, $X_1 = U(\mathfrak{g}) \bigotimes_{\mathfrak{p}} Y_1$. (Here $\mathfrak{p}$ is the Lie algebra of P.) Then there is a nonsingular pairing of X with X_1. If Y is $\mathfrak{Z}(\mathfrak{m}+\mathfrak{a})$-finite, then* $\mathrm{Dim}\, X = \mathrm{Dim}\, Y + \dim N$.

Proof. If $x \in X$, we may consider x as an analytic function on G/MAN with values in Y; recall that we consider only the K-finite vectors in the induced representation. Thus $x(1) \in Y$; so if $y \in Y_1$, $\langle y, x(1)\rangle$ is well defined. This gives a mapping $Y_1 \hookrightarrow X^*$. The image of Y_1 is annihilated by $\mathfrak{n}$, so this induces a map $X_1 \to X^*$. We may identify X_1 with $S(\mathfrak{n}^-) \bigotimes_{\mathbb{C}} Y_1$; here $\mathfrak{p}^- = \mathfrak{m}+\mathfrak{a}+\mathfrak{n}^-$ is the parabolic opposite to $\mathfrak{p}$. Now X may be identified with the K-finite sections of the vector bundle on $K/M \cap K$ defined by $Y|_{M \cap K}$. In this way X_1 is identified with a space of differential operators on this bundle, supported at the identity. From this point of view it is clear that X_1 actually embeds in X^*; and that any element of X annihilated by X_1 vanishes to infinite order at the identity. Since X consists of analytic functions, such an element is zero. Thus the pairing between X and X_1 is nonsingular. If Y is $\mathfrak{Z}(\mathfrak{m}+\mathfrak{a})$ finite, we may take Y_1 to consist of $\mathfrak{b}$-finite vectors (cf. Lemma 4.5), in which case $\mathrm{Dim}\, X = \mathrm{Dim}\, X_1 = \mathrm{Dim}\, Y_1 + \dim N = \mathrm{Dim}\, Y + \dim N$ by Lemma 3.4 and Lemma 2.3. Q.E.D.

Corollary 6.6. *In the setting of 6.5, suppose Y is irreducible but not large. Then no constituent of X is large.*

We return now to our quasisplit group G. Let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots of $\Delta^+(\mathfrak{g}, \mathfrak{h})$; choose corresponding root vectors $X_{\pm\alpha_i}$. If α_i is real, we may assume $X_{\pm\alpha_i} \in \mathfrak{g}_0$; if α_i is complex, then (writing bar for the complex structure of $\mathfrak{g}$) we have $\bar{\alpha}_i = -\theta\alpha_i$, and we may assume $\bar{X}_{\pm\alpha_i} = X_{\mp\theta\alpha_i}$. Let $\{\beta_1 \ldots \beta_s\}$ be the simple restricted roots. For each subset $B \subseteq \{\beta_1 \ldots \beta_s\}$, we define an automorphism σ_B of G as follows. We first define σ_B on $\mathfrak{g}$: σ_B centralizes $\mathfrak{h}$, and $\sigma_B \cdot X_{\pm\alpha_i} = \varepsilon \cdot X_{\pm\alpha_i}$; here $\varepsilon = 1$ if $\alpha_i|_{\mathfrak{a}} \notin B$, and $\varepsilon = -1$ if $\alpha_i|_{\mathfrak{a}} \in B$. Then it is easy to check that σ_B preserves $\mathfrak{g}_0$. Thus σ_B is the differential of an automorphism σ_B of some linear covering group $\tilde{G}_0$ of G_0, and preserves the center of $\tilde{G}_0$ pointwise. Hence σ_B actually defines an automorphism of G_0. Now σ_B centralizes $M \cap G_0$; defining σ_B to centralize M, we get an extension of σ_B to $MG_0 = G$, centralizing MA. It is easy to see that any automorphism of $\mathfrak{g}_0$ coming from $\mathrm{Ad}(\mathfrak{g})$ differs from some σ_B by an inner automorphism from G_0. We write R for the group generated by the σ_B.

Recall now our parabolic subgroup $P' = M'A'N' \supseteq P$. We construct the compact Cartan subgroup T' of M' as follows: choose a maximal strongly orthogonal set $\{\gamma_1, \ldots, \gamma_l\}$ of real roots of $\mathfrak{h}$, with root vectors X_{γ_i}; then $\mathfrak{m}_0$ and the various $Z_i = X_{\gamma_i} + \theta X_{\gamma_i}$ span a compact Cartan subalgebra $\mathfrak{t}'_0$ of $\mathfrak{m}'_0$. If $B \subseteq \{\beta_j\}$, then σ_B normalizes M', and $\sigma_B \cdot Z_i = \sigma_B \cdot Z_i = \pm Z_i$; so σ_B normalizes $\mathfrak{t}'_0$, and hence defines an element $w_B \in W(\mathfrak{m}'/\mathfrak{t}')$. Suppose $w_B \in W(M'/T')$. Then we can modify σ_B by an inner automorphism (from M') so that the resulting automorphism σ'_B centralizes $\mathfrak{t}$. But any automorphism of a connected reductive group centralizing a compact Cartan subgroup is inner; so σ_B is inner for M'. Let $R_{M'}$ be the preimage of $W(M'/T')$ in R under the map $\sigma_B \to w_B$; then we have shown that $R_{M'}$ is the subgroup of M'-inner automorphisms. Define the integer $s = s(M')$ by $2^s = |R/R_{M'}|$.

Let Ψ be a system of positive roots for $\mathfrak{t}'$ in $\mathfrak{m}'$. We call Ψ *good* if every simple root of Ψ is noncompact. (Since $\mathfrak{t}'$ preserves the Cartan decomposition $\mathfrak{m}' = (\mathfrak{m}' \cap \mathfrak{k}) \oplus (\mathfrak{m}' \cap \mathfrak{p})$, the roots of $\mathfrak{t}'$ in $\mathfrak{m}'$ fall into two classes, called compact and noncompact.) Two systems of positive roots are called *equivalent* if they are conjugate under $W(M'/T')$; we write (Ψ) for the equivalence class of Ψ. To each representation $\delta' \in \hat{M}'$ in the limits of the discrete series, Harish-Chandra has associated an equivalence class (Ψ) of positive root systems for $\mathfrak{t}'$ in $\mathfrak{m}$ (the extension to limits of discrete series may be found in [5]). Any element $w \in W(\mathfrak{m}'/\mathfrak{t}')$ which preserves the sets of compact and noncompact roots can be realized by an automorphism of $\mathfrak{m}'_0$ coming from $\mathrm{Ad}(\mathfrak{m}')$, and hence (up to inner automorphism) by an element of R. It follows that any two equivalence classes of good positive root systems are conjugate by R; the stabilizer of such a class is clearly $R_{M'}$. Thus there are 2^s equivalence classes of good positive root systems.

By the hypotheses of (f), we are given a representation $\delta' \in \hat{M}'$ in the limits of the discrete series, associated to a good equivalence class (Ψ). Recall that M' is noncompact, and that (f)$\Rightarrow$(c) is known for all smaller parabolic subgroups. By the second Hecht-Schmid character identity (cf. [10], Theorem 9.4, and [11], Proposition 5.22), we can arrange the following situation. There is a cuspidal parabolic subgroup $P'' = M''A''N''$ contained in P', with $\dim A'' = \dim A' + 1$; a representation $\delta'' \in \hat{M}''$ in the limits of the discrete series; and a character $\nu'' \in \hat{A}''$. Choose a compact Cartan subalgebra $\mathfrak{t}''_0 \subseteq \mathfrak{m}''_0$, and extend it to a compact Cartan subalgebra $\mathfrak{t}'_0$ of $\mathfrak{m}'_0$. (This choice of $\mathfrak{t}'$ may differ from our previous one, but they are conjugate by $M'_0 \cap K$.) Choose a representative Ψ of the equivalence class of positive root systems associated to δ'. Then we may take δ'' associated to the restriction of Ψ to $\mathfrak{t}''_0$. The annihilator of $\mathfrak{t}''$ in $(\mathfrak{t}')^*$ is spanned by a single noncompact root β. Let $s_\beta \in W(\mathfrak{m}'/\mathfrak{t}')$ be the corresponding reflection. Set $Y = \mathrm{Ind}_{P''}^{G} \delta'' \otimes \nu'' \otimes 1$. There are three cases to consider. If $s_\beta \in W(M'/T')$, then Y has two (possibly reducible) composition factors X and Y_0; and any constituent Y'_0 of Y_0 satisfies $\tau(Y'_0) \neq \emptyset$. (For this and similar assertions below, see [11], Proposition 5.22.) Hence $N_{Y_0}(t) \leq A t^{r-1}$ by Theorem 1.2. If δ'' is not large, then by Corollary 6.6, X is not large, a contradiction. So δ'' is large. As indicated in the proof that (a)$\Rightarrow$(f), it follows that the associated positive root system is good. If we replace Y by $Y_\nu = \mathrm{Ind}_{P''}^{G} \delta'' \otimes \nu \otimes 1$, then $N_Y(t)$ does not change; but for almost all ν, Y_ν is irreducible. By inductive hypothesis, $N_Y(t) \sim 2^{-s(M'')} b_G t^r$; so the same holds for $N_X(t) = N_Y(t) - N_{Y_0}(t)$. So we only have to show that $s(M'') = s(M')$. Clearly $R_{M''} \supseteq R_{M'}$. Fix $\sigma_B \in R_{M''}$; we must show $\sigma_B \in R_{M'}$. We may assume that $\mathfrak{t}''$ is constructed as indicated earlier for M', so that σ_B preserves $\mathfrak{t}''$.

Then σ_B preserves the centralizer of $\mathfrak{t}''$ in $\mathfrak{m}'\cap\mathfrak{k}$, which is just $\mathfrak{t}'$. Write $\mathfrak{t}_\beta$ for the orthogonal complement of $\mathfrak{t}''$ in $\mathfrak{t}'$; then σ_B preserves $\mathfrak{t}''$ and $\mathfrak{t}_\beta$. Since $\sigma_B\in R_{M''}$, we can find $\tau_B\in W(M''/T'')$ so that $\tau_B\sigma_B$ centralizes $\mathfrak{t}''$. Hence $\tau_B\sigma_B$ acts by ± 1 on $\mathfrak{t}_\beta$. Possibly modifying τ_B by s_β, we find $\tau'_B\in W(M'/T')$ so that $\tau'_B\sigma_B$ centralizes $\mathfrak{t}'$; so $\sigma_B\in R_{M'}$. This proves (d) when $s_\beta\in W(M'/T')$.

Suppose next that $s_\beta\notin W(M'/T')$; we will divide this into two cases. Then Y has three composition factors, Y_0, X, and X_1. Any constituent Y'_0 of Y_0 satisfies $\tau(Y'_0)\neq\emptyset$, so we can disregard Y_0 as before. X_1 is of the form $\operatorname{Ind}_{P'}^G\delta'_1\otimes\nu'\otimes 1$; and δ'_1 is a representation in the limits of the discrete series, associated to the root system $s_\beta\cdot\Psi$. Suppose first that $s_\beta\cdot\Psi$ is good. Then we can find an element $\sigma_B\in R$ so that $\sigma_B\cdot\Psi$ is equivalent to $s_\beta\cdot\Psi$. In this case (as follows from the detailed form of the Hecht-Schmid character identity) $\sigma_B\cdot\delta'\cong\delta'_1$; here we make the automorphism σ_B of M' act on $\hat{M}'$. It follows that $N_X(t)=N_{X_1}(t)$; since $N_Y(t)=N_{Y_0}(t)+N_X(t)+N_{X_1}(t)$, an argument like that given in the first case shows that

$$N_X(t)\sim 2^{-[s(M'')+1]}b_G t^r.$$

So we must show that $|R_{M''}/R_{M'}|=2$. Since $(\sigma_B\Psi)=(s_\beta\Psi)$, we can find an element $\tau_B\in W(M'/T')$ so that $\tau_B\sigma_B$ acts on $\mathfrak{t}'$ by s_β. Since $s_\beta\notin W(M'/T')$, it follows that $\sigma_B\notin R_{M'}$. We want to show that $\sigma_B\in R_{M''}$. By the argument given in the first case, σ_B preserves $\mathfrak{t}''$ and $\mathfrak{t}_\beta$; since s_β does as well, τ_B preserves $\mathfrak{t}''$ and $\mathfrak{t}_\beta$. Let X_β and $X_{-\beta}$ be root vectors for the root β, chosen so that $(\mathfrak{a}_\beta)_0=\mathfrak{a}''_0\cap\mathfrak{m}'_0$ is spanned by $X_\beta+X_{-\beta}$. Choose a representative m_β for τ_β, with $m_\beta\in M'\cap K$; then $m_\beta\cdot X_\beta=cX_{\pm\beta}$. Modifying m_β by an element of T', we may take $c=1$, so that m_β centralizes $\mathfrak{a}_\beta$. Then $m_\beta\in M''$, so $\tau_B|_{\mathfrak{t}''}\in W(M''/T'')$. Since $\tau_B\sigma_B|_{\mathfrak{t}''}$ is the identity, it follows that $\sigma_B\in R_{M''}$. The argument given in case 1 shows that σ_B generates $R_{M''}/R_{M'}$, which therefore has order two.

Finally suppose $s_\beta\cdot\Psi$ is not good (with $s_\beta\notin W(M'/T')$). Then $s_\beta\cdot\Psi$ has a compact simple root; and as noted before, it follows that δ'_1 is not large. So no constituent of X_1 is large. Hence $N_X(t)\sim 2^{-s(M'')}b_G t^r$, and we need only show that $R_{M''}\subseteq R_{M'}$. Suppose $\sigma_B\in R_{M''}$; choose $\tau_B\in W(M''/T'')$ so that $\tau_B\sigma_B$ centralizes $\mathfrak{t}''$. Everything extends to $\mathfrak{t}'$; it follows that $\tau_B\sigma_B$ is 1 or s_β on $\mathfrak{t}'$. If $\tau_B\sigma_B=s_\beta$, then $(s_\beta\cdot\Psi)=(\tau_B\sigma_B\Psi)=(\sigma_B\Psi)$, which is good; a contradiction. So $\tau_B\sigma_B=1$, and $\sigma_B\in R_{M'}$. This completes the proof that (f) $\Rightarrow$ (d).

Proof of (f) $\Rightarrow$ (c). One can give an identical argument here; the only difficulty is starting the induction. So suppose $P=MAN$ is the minimal parabolic; put $\pi(\delta\otimes\nu)=\operatorname{Ind}_P^G\delta\otimes\nu\otimes 1$. To start the induction, we need only show that $c_r(\pi(\delta\otimes\nu))$ is independent of δ and ν. We first fix δ, and show that it is independent of ν. All of the $\pi(\delta\otimes\nu)$ are realized on the fixed space $X(\delta\otimes\nu)=X(\delta)=\operatorname{Ind}_M^K\delta$ (K-finite vectors in the induced representation, as usual). Fix a bounded set $\Omega\subseteq\hat{A}$; we will show $c_r(\pi(\delta\otimes\nu))$ is constant as a function of ν on Ω. There is a finite set $F\subseteq\hat{K}$ such that for $\nu\in\Omega$, any irreducible constituent of $\pi(\delta\otimes\nu)$ contains a K-type $\gamma\in F$ (this follows for example from [12], Theorem 1). It follows that $X(\delta)$ has a finite dimensional K-invariant subspace X_0 which is $U(\mathfrak{g})$-cyclic in $X(\delta\otimes\nu)$ for all $\nu\in\Omega$. Now symmetrization provides a map $\beta: S(\mathfrak{p})\to U(\mathfrak{g})$ which commutes with adjoint action of $\mathfrak{k}$. Since $U(\mathfrak{g})\cong S(\mathfrak{p})\otimes U(\mathfrak{k})$,

we may identify X_n/X_{n-1} with

$$\bar{X}_n(\delta\otimes\nu)=S_n(\mathfrak{p})\cdot X_0(\delta\otimes\nu).$$

If $\gamma\in\hat{K}$, write $d(n,\gamma,\nu)$ for the multiplicity of γ in $\bar{X}_n$. Then $d(n,\gamma,\nu)$ is the rank of some matrix whose entries are analytic functions of ν. Thus there is an integer $\bar{d}(n,\gamma)$ such that

$$d(n,\gamma,\nu)\leqq\bar{d}(n,\gamma),$$

and equality holds for almost all ν. So we can choose ν such that $d(n,\gamma,\nu)=\bar{d}(n,\gamma)$ for all n; it follows that $\sum_{n=0}^{\infty}\bar{d}(n,\gamma)$ is the multiplicity of γ in $X(\delta)$, which is finite. On the other hand, if $\nu\in\Omega$, then $\sum_{n=0}^{\infty}d(n,\gamma,\nu)$ is also the multiplicity of γ in $X(\delta)$; so for $\nu\in\Omega$, $d(n,\gamma,\nu)=\bar{d}(n,\gamma)$. It follows that

$$\dim\bar{X}_n(\delta\otimes\nu)=\sum_{\gamma\in\hat{K}}d(n,\gamma,\nu)(\dim\gamma)$$

is independent of ν for $\nu\in\Omega$. Therefore $c_r(\pi(\delta\otimes\nu))$ is independent of ν for $\nu\in\Omega$. Since Ω was an arbitrary bounded subset of $\hat{A}$, $c_r(\pi(\delta\otimes\nu))$ is independent of ν; we denote it by $c(\delta)$.

Now choose δ and ν so that the differential γ of $\delta\otimes\nu$ is dominant and nonsingular, and $\pi(\delta\otimes\nu)$ is irreducible. Let $\tilde{\mu}\in\hat{H}$ be a weight of a finite dimensional representation, and let $\Theta(\gamma+\tilde{\mu})$ be the corresponding coherent continuation of $\pi(\delta\otimes\nu)$ (defined in Sect. 4). By [11], Corollary 5.12, $\Theta(\gamma+\tilde{\mu})=\Theta(\pi(\delta\otimes\nu+\tilde{\mu}))$. By Proposition 4.9,

$$c_r(\gamma+\tilde{\mu})=c(\delta+\tilde{\mu}|_M)$$

is constant. But $\tilde{\mu}|_M$ can be arbitrary; so $c(\delta)=a_G$ is independent of δ. The proof that (f) $\Rightarrow$ (c) is now identical to the one that (f) $\Rightarrow$ (d). This proves Theorem 6.2. Q.E.D.

We could add to Theorem 6.2 two supplementary results: necessary and sufficient conditions for the reducibility of the induced representations considered in 6.2 (f), and a determination of all large composition factors of principal series representations. These results are immediate consequences of the theorems of [11]; but since their statement require some technical ideas from [11], we omit them. The following result is relatively easy, however.

Corollary 6.7. *In the setting of Theorem 6.2, suppose $\delta\otimes\nu\in\hat{H}$. Then the principal series representation $\operatorname{Ind}_P^G\delta\otimes\nu\otimes 1=Y$ has a large composition factor X, which occurs with multiplicity one. X is unique up to conjugation by outer automorphisms of G coming from $\operatorname{Ad}(\mathfrak{g})$, and fixing Y.*

Proof. Y has finitely many irreducible composition factors, say $X_1,\dots,X_l$; so $N_Y(t)=\sum_{i=1}^{l}N_{X_i}(t)$. Since $N_Y(t)\sim b_G t^r$ by Corollary 5.3, at least one of the X_i is large; say X_1 is. Choose a cuspidal parabolic subgroup $P'=M'A'N'\supseteq P$, a representation $\delta' \in \hat{M}'$ in the limits of the discrete series, and a character $v' \in \hat{A}'$ as in 6.2 (f), so that $X_1 \cong \mathrm{Ind}_{P'}^{G} \otimes v' \otimes 1$. Recall now the proof of (f) $\Rightarrow$ (d). We had a group $R = \{\sigma_B\}$ of automorphisms of G, centralizing MA, and a subgroup $R_{M'}$. The various σ_B preserve M'; and, in the induced action on representations, $\sigma_B \cdot \delta'$ is equivalent to δ' iff $\sigma_B \in R_{M'}$. By computing the global character of X_1, one can deduce that $\sigma_B \cdot X_1$ is equivalent to X_1 iff $\sigma_B \in R_{M'}$. On the other hand, σ_B centralizes MA, so $\sigma_B \cdot Y \cong Y$. So all of the various $\sigma_B \cdot X_1$ occur as composition factors of Y; and there are $|R/R_{M'}| = 2^s$ such representations. Furthermore, by Theorem 6.2 (d), $N_{X_1}(t) \sim 2^{-s} b_G t^r$. So the representations $\sigma_B \cdot X_1$ account completely for the asymptotic behaviour of $N_Y(t)$; so Y can have no other large composition factors. Q.E.D.

Using Corollary 6.7, Kostant has recently shown that large representations are precisely those which admit a so-called Whittaker model. Similar results for $G = GL(n, \mathbb{R})$ have been obtained by Casselman and Zuckerman.

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Erratum

On Certain Problems of Chebyshev, Zolotarev, Bernstein and Akhieser

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We correct here a few misprints appeared in [1].

Page	↓ *Lines*	*Should be*	*For*
87	5	$\lvert P^k(0)\rvert \leqq M n^k$	$\max\limits_{-1 \leqq x \leqq 1} \lvert P^k(x)\rvert \leqq \frac{M n^k}{(k)!}$
87	16	$2m = 2n + 4$	$m = 2n + 4$
88	12	$n 4^{2n}$	$n 4^n$
98	12	$\frac{k}{2n}$	$\frac{k}{n}$
99	3	$(n-1)$	$(2n-1)$
99	5	$\frac{p}{q}$	p_n/q_n
99	5, 7, 17	x^n	x^{2n}

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