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IRREDUCIBLE CHARACTERS OF SEMISIMPLE LIE GROUPS IV. CHARACTER-MULTIPLICITY DUALITY

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1. Introduction. Let G be a real linear reductive Lie group. (It is important to allow G to be disconnected; precise hypotheses on G are formulated in section 2.) To each irreducible admissible representation of G, Langlands in [17] has associated a natural induced representation, of a kind we will call *standard*. Roughly speaking, the standard representations are non-unitarily induced from discrete series representations. This association sets up a bijection between the irreducible representations and the standard ones. Write $\bar{\pi}$ for the irreducible representation corresponding to the standard representation π. The standard representations are fairly well understood—much better, at least, than the irreducible representations. One way to describe irreducible representations is to write them as (finite) integer combinations of standard representations, in an appropriate Grothendieck group. That is, we look for an expression

$$\bar{\pi} = \sum_{\rho \text{ standard}} M(\rho, \bar{\pi})\rho \qquad (M(\rho, \bar{\pi}) \in \mathbf{Z}). \tag{1.1}$$

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The expression on the right is called the *formal character* of $\bar{\pi}$. As was first observed by Zuckerman, the formal character of any irreducible admissible representation exists and is unique—see [25], Proposition 6.6.7. (The difficult problem of computing the integers $M(\rho, \bar{\pi})$ explicitly will not directly concern us here. For the moment it suffices to know that there is a conjecture on how to compute them ([23]), which has been proved by Lusztig, Beilinson, and Bernstein in some cases—see [24], and section 12 below.)

The inverse problem is more elementary, and is just as important in representation theory. It is the problem of decomposing a standard representation into its irreducible constituents. That is, we seek an expression

$$\pi = \sum_{\bar{\rho} \text{ irreducible}} m(\bar{\rho}, \pi)\bar{\rho} \qquad (m(\bar{\rho}, \pi) \in \mathbf{N} = \{0, 1, 2, \dots\}). \tag{1.2}$$

Here $m(\bar{\rho}, \pi)$ is the multiplicity of $\bar{\rho}$ as a composition factor of π. Since π is not a direct sum of irreducible representations, (1.2) must also be understood in a Grothendieck group. Combining (1.1) and (1.2) gives

$$\begin{aligned} \pi &= \sum_{\bar{\rho}} m(\bar{\rho}, \pi)\bar{\rho} \\ &= \sum_{\bar{\rho}, \mu} m(\bar{\rho}, \pi) M(\mu, \bar{\rho})\mu \\ &= \sum_{\mu} \left[\sum_{\bar{\rho}} M(\mu, \bar{\rho}) m(\bar{\rho}, \pi) \right] \mu. \end{aligned}$$

By the uniqueness of formal characters,

$$\sum_{\bar{\rho}} M(\mu, \bar{\rho}) m(\bar{\rho}, \pi) = \delta_{\mu, \pi} \tag{1.3}$$

(the Kronecker δ); that is, the matrices m and M are inverses of each other. So the two problems just described—computing formal characters and computing multiplicities—are equivalent. In an appropriate ordering of the standard representations, M and m are upper triangular with one's on the diagonal ([25], Proposition 6.6.7). This equivalence is therefore even quite accessible computationally. Our goal in this paper is to show that sometimes the problems are not only similar, but actually "the same". To understand what this should mean, let us consider first the simpler case of Verma modules for a complex semisimple Lie algebra $\mathfrak{g}$. Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ of $\mathfrak{g}$, with $\mathfrak{h}$ a Cartan subalgebra and the nil radical. Write ρ for half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{n}$, and W for the Weyl group. If $w \in W$, define

$$Z(w) = U(\mathfrak{g}) \underset{\mathfrak{b}}{\otimes} \mathbf{C}_{w\rho - \rho}$$

to be the Verma module of highest weight $w\rho - \rho$, and $L(w)$ to be its unique irreducible quotient. Then there are expressions

$$L(w) = \sum_{y \in W} M(y,w) Z(y) \qquad (M(y,w) \in \mathbf{Z}) \tag{1.4}$$

$$Z(w) = \sum_{y \in W} m(y,w) L(y) \qquad (m(y,w) \in \mathbf{N}). \tag{1.5}$$

Let w_0 be the longest element of W. Then $Z(w_0)$ is irreducible; that is, $L(w_0) = Z(w_0)$. The module $L(1)$ is the trivial representation, and the Weyl character formula (see [10]) may be interpreted to mean

$$L(1) = \sum_{y \in W} (-1)^{l(y)} Z(y), \tag{1.6}$$

or

$$M(y,1) = (-1)^{(y)}; \tag{1.6$'$}$$

here l is the length function on W. For $\mathfrak{g} = \mathfrak{sl}(2)$, these facts tell us everything. W is the two element group $\{1, w_0\}$, and we compute

$$\begin{aligned} L(w_0) &= Z(w_0) & Z(w_0) &= L(w_0) \\ L(1) &= Z(1) - Z(w_0) & Z(1) &= L(1) + L(w_0). \end{aligned} \tag{1.7}$$

In terms of the matrices M and m this is

$$M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad m = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{1.7$'$}$$

These formulas suggest the guess that, for general $\mathfrak{g}$,

$$M(y,w) = (-1)^{l(y)-l(w)} m(y,w).$$

Applied to (1.6′), this would give

$$m(y,1) = 1, \qquad \text{all } y;$$

that is, it would suggest that $Z(1)$ should contain each $L(y)$ exactly once. Unfortunately, this is known to be false—for example, in $\mathfrak{sl}(4)$ (see [4]). The easiest result saying that something occurs with multiplicity one is Verma's theorem that $Z(w_0)$ (which is equal to $L(w_0)$) occurs exactly once in every $Z(y)$; that is,

$$m(w_0, y) = 1, \qquad \text{all } y. \tag{1.8}$$

In conjunction with (1.6′) and (1.7′), this suggests

$$M(y,w) = (-1)^{l(y)-l(w)} m(w_0 w, w_0 y). \tag{1.9}$$

Formula (1.9) was conjectured by Jantzen on the basis of extensive calculations. Kazhdan and Lusztig deduced it from their conjectured formulas for $M(y,w)$ in [15]. As these formulas have now been proved ([3], [6]), (1.9) is true. It makes precise the idea that the problems of computing m and M are exactly the same.

Theorem 1.15 below extends (1.9) to real groups. To see what is involved in that, we begin with $G = \mathrm{SL}(2,\mathbf{R})$. There are exactly three irreducible representations of G on which the center $Z(G)$ of G and the center $\mathfrak{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$ both act as in the trivial representation. They are the trivial representation $\bar{\pi}_0$, and two discrete series representations $\bar{\pi}_1$ and $\bar{\pi}_2$. Write π_i for the corresponding standard representations; π_0 is a spherical principal series representation, and $\pi_1 = \bar{\pi}_1$, $\pi_2 = \bar{\pi}_2$. The decomposition of π_0 into irreducibles is well known; we find

$$\begin{aligned}
&\bar{\pi}_1 = \pi_1 &\qquad& \pi_1 = \bar{\pi}_1\\
&\bar{\pi}_2 = \pi_2 && \pi_2 = \bar{\pi}_2\\
&\bar{\pi}_0 = \pi_0 - \pi_1 - \pi_2 && \pi_0 = \bar{\pi}_0 + \bar{\pi}_1 + \bar{\pi}_2 .
\end{aligned} \tag{1.10}$$

Here again there is an obvious conjecture like the one formulated after (1.7); but here again that conjecture is false for other groups (for example $SU(2,1)$). To get an analogue of (1.9), we must find three irreducible representations $\bar{\rho}_0$, $\bar{\rho}_1$, $\bar{\rho}_2$ satisfying

$$M(\pi_j, \bar{\pi}_i) = \epsilon_{ij} m(\bar{\rho}_i, \rho_j) \qquad (\epsilon_{ij} = \pm 1). \tag{1.11}$$

This says that the multiplicity with which $\bar{\rho}_i$ occurs in ρ_j should equal the coefficient of π_j in the formal character of $\bar{\pi}_i$, up to sign. By inspection of (1.10), this gives

$$\begin{aligned}
&\bar{\rho}_1 = \rho_1 - \rho_0 &\qquad& \rho_1 = \bar{\rho}_1 + \bar{\rho}_0\\
&\bar{\rho}_2 = \rho_2 - \rho_0 && \rho_2 = \bar{\rho}_2 + \rho_0\\
&\bar{\rho}_0 = \rho_0 && \rho_0 = \bar{\rho}_0
\end{aligned} \tag{1.12}$$

Now $\mathrm{SL}(2,\mathbf{R})$ has only one standard representation having exactly two composition factors (the reducible unitary principal series); so these formulas cannot hold for any three representations of $\mathrm{SL}(2,\mathbf{R})$. What one has to do is look at a different group. The group needed here is sometimes called $\mathrm{SL}^{\pm}(2,\mathbf{R})$:

$$\begin{aligned}
\mathrm{SL}^{\pm}(2,\mathbf{R}) &= \{ g \in \mathrm{GL}(2,\mathbf{R}) \mid \det g = \pm 1 \}\\
&= \mathrm{SL}(2,\mathbf{R}) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathrm{SL}(2,\mathbf{R}).
\end{aligned} \tag{1.13}$$

(The group $\mathrm{PGL}(2,\mathbf{R})$ could also be used, but $\mathrm{SL}^{\pm}(2,\mathbf{R})$ is closer to the one we will actually construct in the proof of Theorem 1.15.) The Weyl group of a compact Cartan subgroup of $\mathrm{SL}^{\pm}(2,\mathbf{R})$ has two elements; so by Harish-Chandra's general theory (or trivial calculation), there is exactly one discrete series representation $\bar{\rho}_0$ in which $\mathfrak{Z}(\mathfrak{g})$ acts as in a one dimensional representation.

On the other hand, the group has two components, so there are two one dimensional representations $\bar{\rho}_1$, $\bar{\rho}_2$. Writing ρ_i for the corresponding standard representations, one easily verifies (1.12) (see [25], Chapter 1).

There is another subtlety here, however. The group $SU(2,1)$ has six representations $\bar{\pi}_1, \ldots, \bar{\pi}_6$ having the same infinitesimal and central characters as the trivial representation. It is an entertaining exercise to match them with six representations $\bar{\rho}_1 \ldots \bar{\rho}_6$ of $SL(3, \mathbb{R})$ so that (1.11) holds. (The reader more interested in enlightenment than in entertainment may consult (16.3a) below.) However, $SL(3,\mathbb{R})$ has a seventh irreducible representation $\bar{\rho}_0$ having the same infinitesimal and central character as the other $\bar{\rho}_i$; $\bar{\rho}_0$ is an irreducible principal series representation. So the matching does not seem to work perfectly. The point is that $\bar{\rho}_0$ must be matched with the trivial representation of $SU(3)$: the second group involved depends not only on the first, but also on which representations of the first are considered. To keep track of this in general, one needs the notion of blocks.

Definition 1.14. *Block equivalence* of irreducible (or standard) representations of G is the equivalence relation $\overset{\approx}{B}$ generated by

$$\bar{\pi}_1 \overset{\approx}{B} \bar{\pi}_2 \qquad \text{if} \quad m(\bar{\pi}_1, \pi_2) \neq 0.$$

The equivalence classes are called *blocks*.

There is a more natural definition in terms of Ext groups, but this is equivalent (see [25], Chapter 9).

THEOREM 1.15. (Theorem 13.13 below) *Let $G^{\mathbb{C}}$ be a complex connected reductive Lie group, and G a real form of $G^{\mathbb{C}}$ (in the weak sense of section 2). Fix a block $\{\bar{\pi}_1, \ldots, \bar{\pi}_r\}$ of irreducible representations of G, having the same infinitesimal character as some finite dimensional representation of G. Write $\check{G}^{\mathbb{C}}$ for the complex simply connected semisimple group whose root system is dual to that of $G^{\mathbb{C}}$. Then there is a real form $\check{G}$ of $\check{G}^{\mathbb{C}}$, and a block $\{\bar{\rho}_1, \ldots, \bar{\rho}_r\}$ of irreducible representations of $\check{G}$, such that*

$$M(\pi_j, \bar{\pi}_i) = \epsilon_{ij} m(\bar{\rho}_i, \rho_j)$$

$$M(\rho_j, \bar{\rho}_i) = \epsilon_{ij} m(\bar{\pi}_i, \pi_j)$$

for all i, j; here $\epsilon_{ij} = \pm 1$. In the correspondence $\bar{\rho}_i \leftrightarrow \bar{\pi}_i$, discrete series corresponds to Langlands quotients of principal series for split groups; and finite dimensional representations correspond to representations whose annihilator is a minimal primitive ideal.

In this correspondence, $\check{G}$ will be specified essentially by the block $\{\bar{\pi}_i\}$, and the block $\{\bar{\rho}_i\}$ will be specified by the real form G. The signs ϵ_{ij} will be specified precisely. The proof is entirely constructive, in the sense that $\check{G}$ and $\{\bar{\rho}_i\}$ are easily computable in examples. Despite the symmetry of the conclusion in G and

$\check{G}$, the proof does not necessarily produce G and $\{\overline{\pi}_i\}$ when applied to $\check{G}$ and $\{\overline{\rho}_i\}$. The hypothesis that $\overline{\pi}_i$ have the same infinitesimal character as some finite dimensional representation is needed only because a proof of the conjecture of [23] has been published only in that case. J. Bernstein has informed me that the proof can be carried out in general, however, (see [24]). We will state a generalization (Theorem 13.13) depending on the conjecture. In it, $\check{G}^{\mathbf{C}}$ has the root system dual to the *integral* root systems defined by $\overline{\pi}_i$; otherwise the formulation is unchanged.

The idea of the proof is to associate to the pair $(G, \overline{\pi}_i)$ (for each i) a collection of additional structures on the root system R of G (Proposition 4.11; among other things, an involutive automorphism θ of R). These structures pass in a natural way to the same kind of structures on the dual root system $\check{R}$—for example, θ is replaced by $-\theta$ on $\check{R}$. From the structures, one can reconstruct a group $\check{G}$ and a representation $\overline{\rho}_i$ (Theorem 11.1). All of the difficulties stem from the fact that $\check{G}$ and $\overline{\rho}_i$ are not uniquely defined. For example, if $G = \mathrm{SL}^{\pm}(2, \mathbf{R})$ and $\overline{\pi}_i$ is the trivial representation, then the construction says that $\check{G}$ should be $\mathrm{SL}(2, \mathbf{R})$, and $\overline{\rho}_i$ should be a discrete series; but it does not say *which* discrete series. Even if we fix the infinitesimal character, the two candidates differ by an automorphism of $\mathrm{SL}(2, \mathbf{R})$, so they cannot possibly be distinguished in any intrinsic way. Most of our effort, therefore, is devoted to showing that all choices can be made in a compatible way as $\overline{\pi}_i$ varies. We will choose two special (but not quite canonical) elements $\overline{\pi}_1$ and $\overline{\pi}_2$ of the block, and use them to construct $\check{G}$, $\overline{\rho}_1$, and $\overline{\rho}_2$. All the other $\overline{\pi}_i$ will be obtained from $\overline{\pi}_1$ and $\overline{\pi}_2$ by various canonical operations (mostly related to Cayley transforms—see section 7). These can be duplicated in $\check{G}$ to give the corresponding $\overline{\rho}_i$ in terms of ρ_1 and ρ_2 (Theorem 10.1). All of the indeterminacy in the construction is therefore concentrated in the choice of π_1, π_2, ρ_1, and ρ_2. Once the $\overline{\rho}_i$ are specified, one only needs to verify some very simple formal properties of the correspondence; then Theorem 1.15 follows from [24] just as (1.9) follows from the Kazhdan–Lusztig conjectures (Proposition 13.12).

Section 16 is devoted to a fairly detailed description of several examples of various aspects of the definitions and results of the main body of the paper. The paper might almost be read by beginning with these and referring back to the general results only as they are needed to understand the examples. (It was certainly written in very much that way, with proofs being modified until their conclusions agreed with the examples.) Experienced readers may prefer to consider their own favorite examples, but in any case one should avoid trying to make sense of the results in some abstract sense.

The form of Theorem 1.15 suggests that we should try to invoke the theory of dual groups in some form; in particular, that we should use ${}^{L}(G^{\mathbf{C}})^0$ instead of $\check{G}^{\mathbf{C}}$ (see [5]). This is possible, at least if we require G to be *precisely* the set of real points of $G^{\mathbf{C}}$. A reformulation of Theorem 1.15 along these lines is sketched in section 15. Lusztig has suggested that the reformulation ought to be much easier to prove, but I have not been able to see that. There are many technical simplifications, however. (In this context it is also interesting that Langlands' L-packets play an important part in the paper (Definition 8.1).)

It is a pleasure to thank George Lusztig for many very helpful discussions, and particularly for explaining how to deduce Theorem 1.15 from a formal duality theory.

2. Notation and assumptions on G. Unexplained notation will in general follow [25]; we recall here some of the main points. Fix once and for all a connected reductive algebraic group $G^{\mathbf{C}}$, and a connected real form $G_0 \subseteq G^{\mathbf{C}}$. Write $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, and $\mathfrak{g} = \mathrm{Lie}(G^{\mathbf{C}})$ for the complexification of $\mathfrak{g}_0$. Choose a Cartan involution θ of G_0, and write

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

for the Cartan decomposition. Fix a non-degenerate symmetric invariant bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}_0$, preserved by θ, which is negative on $\mathfrak{k}_0$ and positive on $\mathfrak{p}_0$. Fix once and for all a real Lie group G, with identity component G_0, satisfying

$$G/G_0 \text{ is finite} \tag{2.1a}$$

$$\mathrm{Ad}(G) \subseteq \mathrm{Ad}(G^{\mathbf{C}}) \text{ as automorphism groups of } \mathfrak{g} \tag{2.1b}$$

$$\text{the Cartan subgroups of } G \text{ are abelian.} \tag{2.1c}$$

(By a Cartan subgroup, we mean the centralizer in G of a Cartan subalgebra of $\mathfrak{g}_0$.) We summarize these conditions by saying that G is a *real form* of $G^{\mathbf{C}}$; note however that G need not be isomorphic to a subgroup of $G^{\mathbf{C}}$. The unique maximal compact subgroup of G containing K_0 is called K. Put

$$Z(G) = \text{center of } G$$

$$U(\mathfrak{g}) = \text{universal enveloping algebra of } \mathfrak{g}$$

$$\mathfrak{Z}(\mathfrak{g}) = \text{center of } U(\mathfrak{g}).$$

Fix a Cartan subgroup H of G. Put

$$\Delta(\mathfrak{g}, \mathfrak{h}) = \{\text{roots of } \mathfrak{h} \text{ in } \mathfrak{g}\};$$

we may regard these roots either as elements of $\mathfrak{h}^*$, or as elements of the character group $\hat{H}$. Put

$$W(\mathfrak{g}, \mathfrak{h}) = W(\Delta(\mathfrak{g}, \mathfrak{h})) = \text{complex Weyl group}$$

$$\begin{aligned} W(G, H) &= \text{Weyl group of } H \text{ in } G \text{ (the real Weyl group)} \\ &= (\text{normalizer of } H \text{ in } G)/H \end{aligned}$$

$W(\mathfrak{g},\mathfrak{h})$ acts on $\mathfrak{h}$ or $\mathfrak{h}^*$. $W(G,H)$ may be regarded as a subgroup of $W(\mathfrak{g},\mathfrak{h})$, but it also acts on H and $\hat{H}$. If $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$, the *coroot* $\check{\alpha}$ is

$$\check{\alpha} = 2\alpha/\langle\alpha,\alpha\rangle.$$

Define $s_\alpha \in W(\mathfrak{g},\mathfrak{h})$ to be the reflection through α: if $\lambda \in \mathfrak{h}^*$, then

$$s_\alpha(\lambda) = \lambda - \langle\check{\alpha},\lambda\rangle\alpha.$$

The roots are classified as *real*, *complex*, or *imaginary*, according to their values on $\mathfrak{h}_0$. If α is imaginary, then the root vectors $X_{\pm\alpha}$ generate a subalgebra of $\mathfrak{g}$ whose intersection with $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or to $\mathfrak{su}(2)$; we call α *noncompact* (*singular* in Harish-Chandra's terminology) or *compact* accordingly. We will say that H is *split* if all roots are real.

Suppose H is a Cartan subgroup, and $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is either real or noncompact imaginary. Then the root vectors for α generate a subalgebra whose intersection with $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$; so we get a map

$$\phi_\alpha : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_0. \tag{2.2}$$

If α is real, we choose ϕ_α so that

$$\phi_\alpha\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \in \mathfrak{h}_0, \qquad \phi_\alpha\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \in \{\alpha \text{ root space}\};$$

and if α is imaginary, so that

$$\phi_\alpha\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \in \mathfrak{h}_0.$$

If H is θ-stable, we can and do also arrange

$$\phi_\alpha(-{}^tX) = \theta_\alpha(X).$$

These conditions do not specify ϕ_α uniquely, but this is not a problem. Since G is linear, ϕ_α exponentiates to

$$\Phi_\alpha : \mathrm{SL}(2,\mathbb{R}) \to G. \tag{2.3}$$

These maps are fundamental to the theory of Cayley transforms; they make it possible to reduce many questions to $\mathrm{SL}(2,\mathbb{R})$. Put

$$\begin{aligned} \sigma_\alpha &= \Phi_\alpha\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\ m_\alpha &= \sigma_\alpha^2 = \Phi_\alpha\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \in H. \end{aligned} \tag{2.4}$$

If H is θ-stable, σ_α and m_α both belong to K. Clearly they have order 4 and 2 (or

2 and 1, if Φ_α has a kernel) respectively. The element m_α does not depend on the choice of ϕ_α, but changing ϕ_α may replace σ_α by $\sigma_\alpha m_\alpha$. If α is real and H is θ-stable, then σ_α normalizes H, and represents $s_\alpha \in W(G,H)$.

If $\mathfrak{h} \subseteq \mathfrak{g}$ is any Cartan subalgebra, we have a Harish-Chandra isomorphism

$$\mathfrak{Z}(\mathfrak{g}) \to S(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})}.$$

If $\lambda \in \mathfrak{h}^*$, write

$$\chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \to \mathbf{C}$$

for the composition of this isomorphism with evaluation at λ. This identifies the maximal ideals in $\mathfrak{Z}(\mathfrak{g})$ with Weyl group orbits in $\mathfrak{h}^*$. We call λ and χ_λ *nonsingular* if $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$. If a positive root system $\Delta^+(\mathfrak{g},\mathfrak{h})$ is specified, we call λ *dominant* if

$$-\langle \check{\alpha}, \lambda \rangle \notin \mathbf{N} \qquad (\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})).$$

If $\langle \check{\alpha}, \lambda \rangle \in \mathbf{Z}$, then λ is said to be *integral with respect to* α. Put

$$\begin{aligned} R(\lambda) &= \{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) \mid \langle \check{\alpha}, \lambda \rangle \in \mathbf{Z}\}, \\ W(\lambda) &= W(R(\lambda)) \end{aligned} \tag{2.5a}$$

the *system of integral roots* and the *integral Weyl group* for λ. If λ is nonsingular, put

$$\begin{aligned} R^+(\lambda) &= \{\alpha \in R(\lambda) \mid \langle \alpha, \lambda \rangle > 0\} \\ \Pi(\lambda) &= \{\text{simple roots of } R^+(\lambda)\} \\ S(\lambda) &= \{s_\alpha \mid \alpha \in B(\lambda)\} \subseteq W(\lambda), \end{aligned} \tag{2.5b}$$

the *positive integral roots*, the *simple integral roots*, and the *simple integral reflections*.

It is convenient to be able to identify roots and Weyl groups on different Cartan subalgebras. There are natural ways to do this, but for our purposes it is enough to fix, once and for all, a Cartan subalgebra $\mathfrak{h}^a \subseteq \mathfrak{g}$, and a positive root system

$$(\Delta^a)^+ \subseteq \Delta(\mathfrak{g},\mathfrak{h}^a) = \Delta^a.$$

We call these the *abstract Cartan subalgebra* and the *abstract positive root system*. Fix once and for all a $(\Delta^+)^a$-dominant nonsingular weight

$$\lambda^a \in (\mathfrak{h}^a)^*, \tag{2.6a}$$

and define

$$\chi = \chi_{\lambda^a} : \mathfrak{Z}(\mathfrak{g}) \to \mathbf{C}$$

$$\begin{aligned}
R^a &= R(\lambda^a) \subseteq \Delta^a && \text{(the \textit{abstract integral roots})} \\
(R^a)^+ &= R^+(\lambda^a) \subseteq (\Delta^a)^+ && \text{(the \textit{abstract positive integral roots})} \\
\Pi^a &= \Pi(\lambda^a) && \text{(the \textit{abstract simple integral roots})} \\
W^a &= W(\lambda^a) \subseteq W(\mathfrak{g}, \mathfrak{h}^a) && \text{(the \textit{abstract integral Weyl group})} \\
S^a &= S(\lambda^a) \subseteq W^a && \text{(the \textit{abstract integral simple reflections})}
\end{aligned} \tag{2.6b}$$

(notation as in (2.5)). Suppose now that $\mathfrak{h} \subseteq \mathfrak{g}$ is any Cartan subalgebra, $\lambda \in \mathfrak{h}^*$, and that χ_λ is equal to the fixed χ defined above. Then there is a unique isomorphism

$$i_\lambda : ((\mathfrak{h}^a)^*, \lambda^a) \to (\mathfrak{h}^*, \lambda) \tag{2.7a}$$

which is inner for $G^{\mathbf{C}}$. This isomorphism satisfies

$$\begin{aligned}
i_\lambda &: R^a \to R(\lambda) \\
i_\lambda &: W^a \to W(\lambda),
\end{aligned} \tag{2.7b}$$

and so forth. If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)$ and $w \in W(\mathfrak{g}, \mathfrak{h}^a)$, write

$$\begin{aligned}
\alpha_\lambda &= i_\lambda(\alpha) \in \Delta(\mathfrak{g}, \mathfrak{h}) \\
w_\lambda &= i_\lambda(w) \in W(\mathfrak{g}, \mathfrak{h})
\end{aligned} \tag{2.7c}$$

these are called the *root corresponding to the abstract root* α, etc. It is often important to understand how i_λ changes when λ is modified by $W(\mathfrak{g}, \mathfrak{h})$. This is trivial, but the result is worth recording; if $w \in W(\mathfrak{g}, \mathfrak{h})$, then

$$\begin{aligned}
i_{w\lambda}(\chi) &= w[i_\lambda(\chi)] && (\chi \in (\mathfrak{h}^a)^*) \\
\alpha_{w\lambda} &= w\alpha_\lambda && (\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^a)) \\
i_{w\lambda}(\sigma) &= w i_\lambda(\sigma) w^{-1} && (\sigma \in W(\mathfrak{g}, \mathfrak{h}^a)) \\
\sigma_{w\lambda} &= w\sigma_\lambda w^{-1} && (\sigma \in W(\mathfrak{g}, \mathfrak{h}^a)).
\end{aligned} \tag{2.8}$$

We are going to consider only representations having the infinitesimal character χ defined in (2.6). Thus we are excluding representations with singular infinitesimal character. In the context of results like Theorem 1.15, this is a natural thing to do. A representation of G with singular infinitesimal character should (formally) correspond to one of $\check{G}$ whose infinitesimal character is "at infinity" in some sense. Such objects may very well exist, but there is at present no theory of them.

Let $H = TA$ be a θ-stable Cartan subgroup of G; such notation will always be interpreted to mean

$$\begin{aligned} T &= H \cap K \\ A &= \exp(\mathfrak{h}_0 \cap \mathfrak{p}_0). \end{aligned} \tag{2.9}$$

Then T is compact abelian (but may be disconnected), and A is a vector group. Recall from [21] or [25], Definition 6.6.1, the notion of a *regular character* γ of H; this is an ordered pair

$$\gamma = (\Gamma, \bar{\gamma}), \qquad \Gamma \in \hat{H}, \quad \gamma \in \mathfrak{h}^* \tag{2.10}$$

satisfying the following conditions. Write $G^{\mathfrak{a}} = MA$ for the Langlands decomposition of the centralizer of A in G; thus $T \subseteq M$ is a compact Cartan subgroup. The first assumption is

$$\bar{\gamma}|_{\mathfrak{t}} \in i\mathfrak{t}_0^* \quad \text{is nonsingular for } \Delta(\mathfrak{m}, \mathfrak{t}). \tag{2.10a}$$

Write

$$\Delta^+(\mathfrak{m}, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{m}, \mathfrak{t}) \mid \langle \alpha, \bar{\gamma} \rangle > 0\},$$

and $\rho_{\mathfrak{m}}$ (respectively $\rho_{\mathfrak{m} \cap \mathfrak{k}}$) for half the sum of the roots in $\Delta^+(\mathfrak{m}, \mathfrak{t})$ (respectively $\Delta^+(\mathfrak{m} \cap \mathfrak{k}, \mathfrak{t})$). The second assumption is

$$d\Gamma = \bar{\gamma} + \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{k}}. \tag{2.10b}$$

If H is split, these conditions mean that Γ may be arbitrary, and $\bar{\gamma}$ is required to be its differential. If H is compact, then Γ must be a highest weight of the lowest K-type of a discrete series representation; and γ is required to be the Harish-Chandra parameter of that discrete series.

Attached to γ there is a standard representation $\pi(\gamma)$ (see [21] or [25]); it may be defined by parabolic induction from a discrete series representation on a certain cuspidal parabolic subgroup of the form $P = MAN$, with MA as above. The group N is chosen so that the Langlands subquotients appear as subrepresentions of $\pi(\gamma)$; and we set

$$\bar{\pi}(\gamma) = \text{maximal completely reducible subrepresentation of } \pi(\gamma), \tag{2.11}$$

the *Langlands subrepresentation* of $\pi(\gamma)$. When $\bar{\gamma}$ is nonsingular (the only case we will consider), $\bar{\pi}(\gamma)$ is irreducible. Define

$$\begin{aligned} \hat{H}' &= \text{set of regular characters of } H \\ \hat{H}'_\chi &= \{\gamma = (\Gamma, \bar{\gamma}) \in \hat{H}' \mid \chi_{\bar{\gamma}} = \chi\} \\ &= \{\gamma = (\Gamma, \bar{\gamma}) \mid ((\mathfrak{h}^a)^*, \lambda^a) \text{ is } G^{\mathbb{C}}\text{-conjugate to } (\mathfrak{h}^*, \bar{\gamma})\} \\ &= \{\gamma \in \hat{H}' \mid \pi(\gamma) \text{ has infinitesimal character } \chi\} \end{aligned} \tag{2.12}$$

(notation (2.6)). Occasionally we may write $(\hat{H}')^G$ and $\pi_G(\gamma)$ to emphasize the dependence on the group G.

THEOREM 2.13. (Langlands—see [25]). *Suppose $\pi \in \hat{G}$ has infinitesimal character χ. Then there is a θ-stable Cartan subgroup $H^1 \subseteq G$, and $\gamma^1 \in (\hat{H}^1)'_\chi$, such that*

$$\pi \cong \bar{\pi}(\gamma^1).$$

If (H^2, γ^2) is another pair of the same kind, then $\bar{\pi}(\gamma^1) \cong \bar{\pi}(\gamma^2)$ if and only if (H^1, γ^1) is conjugate to (H^2, γ^2) under K.

This theorem says that performing formal operations on representations is the same as performing formal operations on conjugacy classes of regular characters; and in fact this paper is full of Cartan subgroups and roots, and empty of Hilbert spaces and operators. All the work of relating the formalism to the behavior of representations is already done, and summarized in [24].

Suppose $\gamma \in \hat{H}'_\chi$. Write

$$cl(H, \gamma) = \{(H^1, \gamma^1) \mid (H^1, \gamma^1) \text{ is conjugate to } (H, \gamma) \text{ by } K\}, \tag{2.14}$$

the *equivalence class of* (H, γ). Often we will write simply $cl(\gamma)$. If $\gamma^i \in (\hat{H}^i)'_\chi$, $i = 1, 2$, then we may identify $(\mathfrak{h}^1)^*$ and $(\mathfrak{h}^2)^*$ by the isomorphism

$$\begin{gathered} i_{\bar{\gamma}^1, \bar{\gamma}^2} : ((\mathfrak{h}^1)^*, \bar{\gamma}^1) \to ((\mathfrak{h}^2)^*, \bar{\gamma}^2) \\ i_{\bar{\gamma}^1, \bar{\gamma}^2} = i_{\bar{\gamma}^2} \circ i_{\bar{\gamma}^1}^{-1}. \end{gathered} \tag{2.15}$$

In this way we can speak of (say) the root for $\mathfrak{h}^2$ "corresponding" to some root for $\mathfrak{h}^1$. We also define

$$m(\gamma^1, \gamma^2) = m(\bar{\pi}(\gamma^1), \pi(\gamma^2)), \tag{2.16}$$

and similarly for M (see (1.1), (1.2)). Similarly, write

$$\gamma^1 \underset{B}{\approx} \gamma^2 \Leftrightarrow \bar{\pi}(\gamma^1) \underset{B}{\approx} \bar{\pi}(\gamma^2). \tag{2.17}$$

(Definition 1.14) This is called *block equivalence* of regular characters.

3. Root systems with involutions. One of the ingredients of the proof of Theorem 1.15 is a detailed knowledge of the Weyl group of an arbitrary Cartan subgroup of G. Much of what we need is known, the most sophisticated results being those of [16]. Published proofs of these results are rarely in quite the form we need, however, and can often be simplified substantially. Therefore, we will prove what we need more or less from scratch, assuming mainly some familiarity with the case of compact or complex groups. In this section we deal with formal preliminaries; the result we are aiming at is Proposition 4.14.

Definition 3.1. A (*reduced*) *root system* is a triple $(V, \langle\ ,\ \rangle, R)$, satisfying the following conditions.

(a) V is a finite dimensional real or complex vector space, $\langle\ ,\ \rangle$ is a nondegenerate symmetric bilinear form on V, and $R \subseteq V - \{0\}$ is a finite subset.

(b) $\langle\ ,\ \rangle$ is positive definite on the real subspace spanned by R.

(c) If α, $\beta \in R$, then $2\langle\alpha, \beta\rangle/\langle\alpha,\alpha\rangle \in \mathbf{Z}$.

(d) If $\alpha \in R$, write $s_\alpha : V \to V$,

$$s_\alpha(\chi) = \chi - (2\langle\alpha,\chi\rangle/\langle\alpha,\alpha\rangle)\alpha$$

for the simple reflection through α. Then s_α preserves R.

(e) If $\alpha \in R$, then $2\alpha \notin R$.

A *possibly non-reduced root system* is a triple $(V, \langle\ ,\ \rangle, R)$ satisfying (a)–(d). The unqualified term "root system" will always mean a reduced root system. We allow V to be complex (which is not the usual convention) so that $(\mathfrak{h}^*, \langle\ ,\ \rangle, \Delta(\mathfrak{g},\mathfrak{h}))$ will be a root system if $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra; of course this does not change anything serious. When V and $\langle\ ,\ \rangle$ are understood or not particularly important, we will call R itself a root system.

Definition 3.2. Suppose $(V, \langle\ ,\ \rangle, R)$ is a root system. The *dual root system* is $(V, \langle\ ,\ \rangle, \check{R})$, with

$$\check{R} = \{\check{\alpha} \mid \alpha \in R\}$$

$$\check{\alpha} = 2\alpha/\langle\alpha,\alpha\rangle;$$

we call $\check{\alpha}$ the *coroot* corresponding to α. Put

$$\begin{aligned} W(R) &= \text{group generated by } \{s_\alpha \mid \alpha \in R\} \\ &= W(\check{R}), \end{aligned}$$

the *Weyl group* of R (or $\check{R}$).

That $\check{R}$ is a root system and $W(R) = W(\check{R})$ follows from the easily verified formulas

$$2\langle\check{\alpha}, \check{\beta}\rangle/\langle\check{\alpha}, \check{\alpha}\rangle = 2\langle\alpha, \beta\rangle/\langle\beta, \beta\rangle$$
$$s_{\check{\alpha}} = s_\alpha$$
$$s_\alpha(\check{\beta}) = [s_\alpha(\beta)]^{\vee},$$

with $\alpha, \beta \in R$. The root system of type B_n and C_n are dual to each other; all other simple roots systems are equivalent to their dual systems (but not necessarily by the map $\alpha \to \check{\alpha}$). The map $R \to \check{R}$ is at the heart of Theorem 1.15; as we add more structure to R, we will always have to transfer it to $\check{R}$ in some way.

Definition 3.3. Suppose R is a root system. Put

$$L = L(R) = \mathbf{Z}\text{-module generated by } R \subseteq V,$$

the *root lattice* in V;

$$P = P(R) = \{\lambda \in V \mid \langle\check{\alpha}, \lambda\rangle \in \mathbf{Z}, \text{ all } \alpha \in R\},$$

the *integral weights* in V,

$$P_1 = P_1(R) = P \cap \mathbf{Q}L,$$

the *small weight lattice* in V. Clearly $L \subseteq P_1 \subseteq P$. Write $\check{L}$, $\check{P}$, etc. for the same objects defined using R, and define

$$Z_0 = Z_0(R) = \{\lambda \in V \mid \langle\alpha, \lambda\rangle = 0, \text{ all } \alpha \in R\}$$
$$Z = Z(R) = \check{P}/\check{L}$$
$$Z_1 = Z_1(R) = \check{P}_1/\check{L}.$$

It is easy to see that

$$\check{P} = \check{P}_1 + Z_0, \tag{3.4a}$$

and therefore that

$$Z = Z_1 + Z_0. \tag{3.4b}$$

Suppose for definiteness that V is complex. Let $G(R)$ be a complex simply connected reductive group (not algebraic) with a Cartan subalgebra $\mathfrak{h}(R)$ in its Lie algebra $\mathfrak{g}(R)$, so that

$$(V, R) \cong (\mathfrak{h}(R)^*, \Delta(\mathfrak{g}(R), \mathfrak{h}(R))). \tag{3.5a}$$

Then

$$Z(G(R)) \cong Z(R); \tag{3.5b}$$

this group has identity component $Z_0(R)$, and component group $Z_1(R)$. We will not use these facts, but they motivate the notation. The group $Z_1(R)$ is finite. It is the product of the corresponding groups for the simple factors of R; and these in turn are well known (see for example [10], p. 68).

Definition 3.6. Suppose R is a root system. We will call a positive root system $R^+ \subseteq R$ an *ordering* of R, or say that R is *ordered* if R^+ is specified. In this case $\check{R}$ is also ordered, by

$$\check{R}^+ = \{\check{\alpha} \mid \alpha \in R^+\}$$

Define

$$\rho = \rho(R^+) = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

$$\check{\rho} = \rho(\check{R}^+).$$

A positive root α is *simple* for R^+ if it is not the sum of two positive roots; or, equivalently if $\check{\alpha}$ is simple for $\check{R}^+$. Set

$$\Pi = \Pi(R^+) = \text{set of simple roots of } R^+.$$

LEMMA 3.7. *In the setting of Definition* 3.6, $\rho \in P_1(R)$, *and* $\check{\rho} \in \check{P}_1(R) = P_1(\check{R})$. *We have for* $\alpha \in R$

$$\alpha \in R^+ \Leftrightarrow \langle \check{\alpha}, \rho \rangle > 0 \Leftrightarrow \langle \alpha, \check{\rho} \rangle > 0$$

$$\alpha \text{ simple} \Leftrightarrow \langle \check{\alpha}, \rho \rangle = 1 \Leftrightarrow \langle \alpha, \check{\rho} \rangle = 1.$$

The statements about ρ are well known, and those for $\check{\rho}$ are the same ones applied to $\check{R}$. It is *not* true in general that $\langle \alpha, \check{\rho} \rangle = \langle \check{\alpha}, \rho \rangle$, for example in B_2.

One final general fact about root systems will be used often.

PROPOSITION 3.8 (Chevalley). *Suppose R is a root system in V, and $X \subseteq V$. Define*

$$R^X = \{\alpha \in R \mid \langle \check{\alpha}, \lambda \rangle = 0, \textit{all } \lambda \in X\}$$

$$W^X = \{w \in W(R) \mid w\lambda = \lambda, \textit{all } \lambda \in X\}$$

$$R(X) = \{\alpha \in R \mid \langle \check{\alpha}, \lambda \rangle \in \mathbf{Z}, \textit{all } \lambda \in X\}$$

$$W(X) = \{w \in W(R) \mid w\lambda - \lambda \in L(R), \textit{all } \lambda \in X\}.$$

(*Definition* 3.3) *Then* R^X *and* $R(X)$ *are root systems, and*

$$W^X = W(R^X)$$

$$W(X) = W(R(X)).$$

For a proof, see [7]. When $X = \{\lambda\}$, we write $R^X = R^\lambda$, etc.

COROLLARY 3.9. *In the setting of Definition* 3.6, *suppose* $w \in W(R)$. *Then* $w\rho = \rho$ *if and only if* $w = 1$.

For R^ρ is empty by Lemma 3.7.

Definition 3.10. A *root system with involution* is a pair (R, θ), with R a root system on V, and $\theta : V \to V$ an orthogonal linear transformation of order two, such that

$$\theta R = R.$$

The *dual root system with involution* is the pair $(\check{R}, -\theta)$. Write $W(R)^\theta$ for the subgroup of $W(R)$ commuting with θ; then $W(R)^\theta = W(\check{R})^{(-\theta)}$. Define

$$R^{\mathbb{R}} = R^{\mathbb{R}}(\theta) = \{\alpha \in R \mid \theta\alpha = -\alpha\}$$

$$W^{\mathbb{R}} = W^{\mathbb{R}}(\theta) = W(R^{\mathbb{R}}),$$

the *real roots* and *real Weyl group* for (R, θ); and

$$R^{i\mathbb{R}} = R^{i\mathbb{R}}(\theta) = \{\alpha \in R \mid \theta\alpha = \alpha\}$$

$$W^{i\mathbb{R}} = W^{i\mathbb{R}}(\theta) = W(R^{i\mathbb{R}}),$$

the *imaginary roots* and *imaginary Weyl group* for (R, θ). The roots which are neither real nor imaginary are called *complex*. We call (R, θ) *quasisplit* if $R^{i\mathbb{R}} = \emptyset$, *fundamental* if $R^{\mathbb{R}} = \emptyset$, and *complex* if it is both fundamental and quasisplit. The *compact part* of V is

$$V_c = V_c(\theta) = \{x \in V \mid \theta x = x\};$$

the *split part* is

$$V_s = V_s(\theta) = \{x \in V \mid \theta x = -x\}.$$

The example of a graded root system of interest to us is the action of the Cartan involution on the roots of a θ-stable Cartan subalgebra. This accounts for some of the terminology. Suppose (R, θ) is a root system with involution. We want to describe $W(R)^\theta$. We begin with a special case.

LEMMA 3.1. *Let* (R, θ) *be a root system with involution*; *and suppose it is complex* (*Definition* 3.10). *Then we can write*

$$R = R_1 \cup R_2,$$

a disjoint orthogonal union, in such a way that

$$\theta : R_i \to R_{3-i} \qquad (i = 1,2)$$

is an isomorphism. By this isomorphism,

$$W(R) = W(R_1) \times W(R_2)$$

$$W(R)^\theta = \{(w, \theta w) \mid w \in W(R_1)\}$$

$$\cong W(R_1).$$

In particular, $W(R)^\theta$ *is generated by the elements* $s_\alpha s_{\theta\alpha}$ *with* $\alpha \in R$.

Proof. Since there are no real roots, we can find $x \in V_c$ such that $\langle \alpha, x \rangle$ is a non-zero real number for all $\alpha \in R$. Define

$$R^+ = \{\alpha \in R \mid \langle \alpha, x \rangle > 0\}.$$

Since $\theta x = x$, $\theta R^+ = R^+$. Write

$$R = R^1 \cup \cdots \cup R^l,$$

an orthogonal union of simple subsystems. Of course θ permutes these. The lemma will follow if we show that no R^i is fixed by θ. Suppose to the contrary that $\theta R^i = R^i$. Let $\beta \in (R^i)^+$ be the highest root of R^i with respect to the ordering R^+ ([10, p. 54). By the uniqueness of the highest root, $\theta\beta = \beta$. This contradicts the hypothesis that there are no imaginary roots, and proves the claim. Q.E.D.

PROPOSITION 3.12. *Suppose* (R, θ) *is a root with involution; use the notation of Definition* 3.10. *Fix positive systems* $(R^{\mathbb{R}})^+$, $(R^{i\mathbb{R}})^+$ *for the real and imaginary roots. Set*

$$\check{\rho}^{\mathbb{R}} = \check{\rho}\left((R^{\mathbb{R}})^+\right)$$

$$\rho^{i\mathbb{R}} = \rho\left((R^{i\mathbb{R}})^+\right)$$

as in Definition 3.6; *and define*

$$R^q = \{\alpha \in R \mid \langle \alpha, \rho^{i\mathbb{R}} \rangle = 0\}$$

$$R^f = \{\alpha \in R \mid \langle \alpha, \check{\rho}^{\mathbb{R}} \rangle = 0\}$$

$$R^{\mathbb{C}} = R^q \cap R^f$$

$$W^q = W(R^q), \qquad W^f = W(R^f), \qquad W^{\mathbb{C}} = W(R^{\mathbb{C}})$$

(a) (R^q, θ), (R^f, θ), *and* $(R^{\mathbb{C}}, \theta)$ *are root systems with involution*; *they are quasisplit, fundamental, and complex, respectively.*

(b) *W^{R} and $W^{i\mathsf{R}}$ are normal subgroups of $W(R)^\theta$.*

(c)
$$W(R)^\theta = (W^{\mathsf{C}})^\theta \ltimes (W^{\mathsf{R}} \times W^{i\mathsf{R}})$$
$$= (W^q)^\theta \ltimes W^{i\mathsf{R}}$$
$$= (W^f)^\theta \ltimes W^{\mathsf{R}}$$

All of these are semidirect products, with the second factor normal.

(We use imaginary roots and real coroots in the definition of R^q and R^f only to preserve symmetry in the duality $(R,\theta) \to (\check{R}, -\theta)$. This result remains true of $\check{\rho}^{\mathsf{R}}$ is replaced by ρ^{R}.)

Proof. We have

$$\theta\rho^{i\mathsf{R}} = \rho^{i\mathsf{R}}, \qquad \theta\check{\rho}^{\mathsf{R}} = -\check{\rho}^{\mathsf{R}}:$$

so θ preserves R^q, R^f, and R^{C}. By Lemma 3.7, every imaginary root α satisfies $\langle \alpha, \rho^{i\mathsf{R}} \rangle \neq 0$; so R^q does not meet $R^{i\mathsf{R}}$, and R^q is quasisplit. Similarly (or by considering $(\check{R}, -\theta)$) R^f is fundamental, and (a) follows. For (b), suppose $\alpha \in R^{\mathsf{R}}$, and $w \in W(R)^\theta$. Then $\theta\alpha = -\alpha$, and

$$\theta(w\alpha) = w(\theta\alpha) = w(-\alpha) = -w\alpha,$$

so $w\alpha \in R^{\mathsf{R}}$. Since $ws_\alpha w^{-1} = s_{w\alpha}$, it follows that W^{R} is normal in $W(R)^\theta$. The claim for $W^{i\mathsf{R}}$ is identical. For the first formula of (c), suppose $w \in W(R)^\theta$. Then $w^{-1}(R^{\mathsf{R}})^+$ and $w^{-1}(R^{i\mathsf{R}})^+$ are positive systems for the real and imaginary roots respectively; so we can find $\sigma \in W^{\mathsf{R}}$, $\tau \in W^{i\mathsf{R}}$ such that

$$w^{-1}\left((R^{\mathsf{R}})^+, (R^{i\mathsf{R}})^+\right) = \left(\sigma^{-1}(R^{\mathsf{R}})^+, \tau^{-1}(R^{i\mathsf{R}})^+\right),$$

or

$$\left((R^{\mathsf{R}})^+, (R^{i\mathsf{R}})^+\right) = w\sigma^{-1}\tau^{-1}\left((R^{\mathsf{R}})^+, (R^{i\mathsf{R}})^+\right).$$

In particular, $w_1 = w\sigma^{-1}\tau^{-1}$ satisfies

$$w_1(\check{\rho}^{\mathsf{R}}, \rho^{i\mathsf{R}}) = (\check{\rho}^{\mathsf{R}}, \rho^{i\mathsf{R}}).$$

In the notation of Proposition 3.8,

$$R^{\mathsf{C}} = R^{\{\check{\rho}^{\mathsf{R}}, \rho^{i\mathsf{R}}\}};$$

so that proposition implies that $w_1 \in W(R^{\mathsf{C}}) = W^{\mathsf{C}}$. Since, w, σ, and τ commute with θ, w_1 does as well; so

$$w = w_1\sigma\tau \in (W^{\mathsf{C}})^\theta W^{\mathsf{R}} W^{i\mathsf{R}}.$$

The first formula will follow when we show that

$$(W^{\mathbf{C}})^{\theta} \cap W^{\mathbf{R}} W^{i\mathbf{R}} = \{1\}.$$

But every element of $(W^{\mathbf{C}})^{\theta}$ fixes $(\check{\rho}^{\mathbf{R}}, \rho^{i\mathbf{R}})$; so this follows from Corollary 3.9.

Applying the first formula to R^q gives

$$(W^q)^{\theta} = (W^{\mathbf{C}})^{\theta} \ltimes W^{\mathbf{R}},$$

so the second follows; and the third is identical. Q.E.D.

The next structures to be introduced are $\mathbf{Z}/2\mathbf{Z}$ gradings. The prototype here is the separation of imaginary roots into compact and noncompact ones (discussed in section 2).

Definition 3.13. Let R be a root system on a vector space V. A $\mathbf{Z}/2\mathbf{Z}$ *grading* of R is a map

$$\epsilon : R \to \mathbf{Z}/2\mathbf{Z} = \{0, 1\}$$

such that

$$\epsilon(\alpha + \beta) = \epsilon(\alpha) + \epsilon(\beta)$$
$$\epsilon(\alpha) = \epsilon(-\alpha)$$

whenever α, β, and $\alpha + \beta$ are in R. Put

$$R_0 = R_0(\epsilon) = \epsilon^{-1}(0),$$

the *even roots*, and

$$R_1 = R_1(\epsilon) = \epsilon^{-1}(1),$$

the *odd roots*. The pair (R, ϵ)—or, equivalently, the triple (R, R_0, R_1)—is called a $\mathbf{Z}/2\mathbf{Z}$-*graded root system*. Define

$$W_0 = W_0(\epsilon) = W(R_0)$$
$$W_2 = W_2(\epsilon) = \{w \in W(R) \mid wR_0 = R_0\}$$
$$E(R) = \text{set all } \mathbf{Z}/2\mathbf{Z} \text{ gradings of } R.$$

Since we will discuss only $\mathbf{Z}/2\mathbf{Z}$ gradings, we may call them simply gradings without danger of confusion.

LEMMA 3.14. *Suppose R is a root system; use the notation of Definitions* 3.3 *and* 3.13.

(a) *There are natural bijections*

$$E(R) \cong \operatorname{Hom}_{\mathbf{Z}}(L(R), \mathbf{Z}/2\mathbf{Z})$$
$$\cong \check{P}_1(R)/2\check{P}_1(R),$$

defined as follows. If $\epsilon \in E(R)$, *extend* ϵ *by* $\mathbf{Z}$*-linearity to a map from* $L(R)$ *to* $\mathbf{Z}/2\mathbf{Z}$. *If* $\lambda \in \check{P}_1(R)$, *define* $\epsilon(\lambda) \in E(R)$ *by*

$$\epsilon(\alpha) = \langle \alpha, \lambda \rangle \pmod{2\mathbf{Z}}.$$

This depends only on $\lambda \bmod 2\check{P}_1$, *and defines the last isomorphism.*

(b) *Suppose* R^+ *is a positive system for* R, *with simple roots* Π. *Then the first bijection of* (a) *defines by restriction to* Π *a bijection*

$$E(R) \cong (\mathbf{Z}/2\mathbf{Z})^{\Pi},$$

the functions from Π *to* $\mathbf{Z}/2\mathbf{Z}$.

This is very easy, and we leave it to the reader. Most of the subtleties we must face center on the difference between W_2 and W_0 in Definition 3.13. Fortunately, this is very well understood, mostly by the work of Knapp.

Definition 3.15. Suppose (R, ϵ) is a $\mathbf{Z}/2\mathbf{Z}$ graded root system. The *universal* $\mathfrak{R}$*-group for* ϵ is

$$\mathfrak{U}(\epsilon) = W_2(\epsilon)/W_0(\epsilon)$$

(notation as in Definition 3.13). An $\mathfrak{R}$*-group for* ϵ is a subgroup

$$\mathfrak{R} \subseteq W_2(\epsilon)/W_0(\epsilon);$$

or, equivalently, a subgroup W_1 satisfying

$$W_0(\epsilon) \subseteq W_1 \subseteq W_2(\epsilon).$$

The phrasing of this definition assumes that $W_0(\epsilon)$ is normal in $W_2(\epsilon)$. This is very easy to check directly; but it is also a consequence of the following result.

PROPOSITION 3.16. *Suppose* (R, ϵ) *is a* $\mathbf{Z}/2\mathbf{Z}$*-graded root system. Then there is a natural homomorphism*

$$\xi(\epsilon) : W_2(\epsilon) \to Z_1(R),$$

(*Definitions* 3.3 *and* 3.13) *with kernel exactly* $W_0(\epsilon)$.

Proof. By Lemma 3.14, there is a $\lambda \in \check{P}_1(R)$ such that

$$\epsilon(\alpha) = \langle \alpha, \lambda \rangle \qquad \pmod{2\mathbf{Z}}.$$

Suppose $w \in W(R)$, and $\alpha \in R$. Then

$$\begin{aligned} \epsilon(w^{-1}\alpha) &= \langle w^{-1}\alpha, \lambda \rangle \qquad &\pmod 2 \\ &= (\alpha, w\lambda \rangle \qquad &\pmod 2). \end{aligned}$$

So w preserves R_0 if and only if

$$\langle \alpha, w\lambda - \lambda \rangle \equiv 0 \qquad (\mathrm{mod}\, 2)$$

for all $\alpha \in R$; that is, if and only if

$$\tfrac{1}{2}(w\lambda - \lambda) \in \check{P}_1(R).$$

So we get a map

$$\xi_\lambda : W_2(\epsilon) \to \check{P}_1(R)$$
$$\xi_\lambda(w) = \tfrac{1}{2}(w\lambda - \lambda)$$

Suppose now that $\mu \in \check{P}_1(R)$. By Proposition 3.8 applied to $\check{R}$,

$$w\mu - \mu \in L(\check{R}) = L(\check{R}) \tag{3.17}$$

for all $w \in W$. In particular, for $w \in W_2(\epsilon)$

$$\begin{aligned} \xi_{\lambda+2\mu}(w) &= \tfrac{1}{2}(w\lambda - \lambda) + (w\mu - \mu) \\ &\equiv \xi_\lambda(w) \qquad (\mathrm{mod}\, \check{L}(R)). \end{aligned}$$

Since ϵ defines λ mod $2\check{P}_1$, we get a well defined map

$$\xi(\epsilon) = W_2(\epsilon) \to \check{P}_1(R)/\check{L}(R) = Z_1(R)$$
$$\xi(\epsilon)(w) = \xi_\lambda(w) \qquad (\mathrm{mod}\, \hat{L}(R)).$$

To see that $\xi(\epsilon)$ is a group homomorphism, suppose $\sigma, \tau \in W_2(\epsilon)$. Then

$$\tfrac{1}{2}(\tau\lambda - \lambda) \in \check{P}_1(R);$$

so by (3.17),

$$\sigma\left[\tfrac{1}{2}(\tau\lambda - \lambda)\right] \equiv \tfrac{1}{2}(\tau\lambda - \lambda) \qquad (\mathrm{mod}\, \check{L}(R)).$$

We now compute

$$\begin{aligned} \xi(\epsilon)(\sigma\tau) &= \tfrac{1}{2}(\sigma\tau\lambda - \lambda) && (\mathrm{mod}\, \check{L}(R)) \\ &= \sigma\left(\tfrac{1}{2}(\tau\lambda - \lambda)\right) + \tfrac{1}{2}(\sigma\lambda - \lambda) && (\mathrm{mod}\, \check{L}(R)) \\ &= \tfrac{1}{2}(\tau\lambda - \lambda) + \tfrac{1}{2}(\sigma\lambda - \lambda) && (\mathrm{mod}\, \check{L}(R)) \\ &= \xi(\epsilon)(\tau) + \xi(\epsilon)(\sigma). \end{aligned}$$

So $\xi(\epsilon)$ is a group homomorphism. Its kernel is

$$\{w \in W(R) \mid \tfrac{1}{2}(w\lambda - \lambda) \in \check{L}(R)\}.$$

By Proposition 3.8 (applied to $\check{R}$), this is exactly the Weyl group of the root system

$$\begin{aligned}\check{R}(\tfrac{1}{2}\lambda) &= \{\check{\alpha} \in \check{R} \mid \tfrac{1}{2}\langle\alpha,\lambda\rangle \in \mathbf{Z}\}\\ &= \{\alpha \in \check{R} \mid \langle\alpha,\lambda\rangle \equiv 0 \pmod 2\}\\ &= \{\check{\alpha} \in \check{R} \mid \epsilon(\alpha) = 0\}\\ &= \check{R}_0.\end{aligned}$$

Its Weyl group is by definition $W_0(\epsilon)$. Q.E.D.

COROLLARY 3.17. *In the setting of Definition* 3.15, *$W_0(\epsilon)$ is a normal subgroup of $W_2(\epsilon)$. The quotient $\mathcal{U}(\epsilon)$ is canonically isomorphic to a subgroup of $Z_1(R)$, and is therefore abelian.*

This is in some ways the nicest result about $\mathcal{U}(\epsilon)$, and it will be important technically. However, we sometimes need the more concrete information provided by Knapp's cross section for W_2/W_0; it will allow us to reduce some problems to $SL(2,\mathbf{R})$ and closely related groups.

Definition 3.18. A set $\{\alpha_1, \cdots, \alpha_l\}$ of roots in a root system R is called *strongly orthogonal* if $\alpha_i \pm \alpha_j$ is never a root or zero for $i \neq j$. It is called *superorthogonal* if the only roots in the linear span of $\{\alpha_i\}$ are $\{\pm\alpha_i\}$, and these are all distinct.

Suppose $\alpha, \beta \in R$, $\alpha \neq \pm\beta$, and $\langle\alpha, \beta\rangle \neq 0$. Then either $\alpha - \beta$ is a root, or $\alpha + \beta$ is a root (or both). It follows that a strongly orthogonal set of roots is orthogonal. A superorthogonal set is obviously strongly orthogonal. As one sees in B_2, however, neither of the converse implications is true in general.

LEMMA 3.19. *Suppose R is a root system. A set $\{\alpha_1, \cdots, \alpha_l\} \subseteq R$ is superorthogonal if and only if there is a positive system R^+ in which the α_i are non-adjacent simple roots. Suppose $\{\alpha_i\}$ is superorthogonal. If $R_0 \subseteq R$ is a subsystem not meeting $\{\alpha_i\}$, then*

$$W(\{\pm\alpha_i\}) \cap W(R_0) = \{1\},$$

Proof. Suppose that $\{\alpha_i\}$ is superorthogonal. Write V for the underlying vector space of R, and $V^\perp$ for the orthogonal complement of $\{\alpha_i\}$. Choose $v \in V^\perp$ so that if $\beta \in R$, then $\langle v, \beta\rangle = 0$ if and only if β is the span of $\{\alpha_i\}$. Thus

$$R = \{\beta \in R \mid \langle v, \beta\rangle \neq 0\} \cup \{\pm\alpha_i\}.$$

Put

$$R^+ = \{\beta \in R \mid \langle v, \beta \rangle > 0\} \cup \{\alpha_i\}.$$

(If V is complex, we need to choose $v \in \mathbf{Q}R$, say, for this to make sense.) Obviously s_{α_j} preserves $R^+ - \{\alpha_j\}$ for each j, since $s_{\alpha_j} v = v$. So each α_j is simple in R^+. Conversely, a set of nonadjacent simple roots is clearly superorthogonal, as the simple roots are a basis of the root lattice.

If $w \in W(\{\pm\alpha_i\})$, then w fixes the element v defined above. By Proposition 3.8 (applied to v and R_0), if $w \in W(R_0)$, then w is a product of reflections in $R_0 \cap \{\pm\alpha_i\}$. As this set is empty, $w = 1$. Q.E.D.

PROPOSITION 3.20 (Knapp). *Suppose (R, ϵ) is a $\mathbf{Z}/2\mathbf{Z}$-graded root system. In the notation of Definition 3.13 and 3.15, there is a superorthogonal set $\{\alpha_1, \cdots, \alpha_l\} \subseteq R_1(\epsilon)$ such that, if we set*

$$Q = W_2(\epsilon) \cap W(\{\pm\alpha_1, \cdots, \alpha_l\}),$$

then

$$W_2(\epsilon) = Q \ltimes W_0(\epsilon),$$

a semidirect product with the second factor normal.

Proof. Fix a positive system $R_0^+(\epsilon)$ for the even roots, and write ρ_0 for half the sum of the roots in it. Set

$$Q = \{w \in W_2 \mid wR_0^+ = R_0^+\}.$$

The proof of Proposition 3.12(a) shows that

$$W_2 = Q \propto W_0(\epsilon);$$

it remains only to describe Q. Obviously

$$Q = \{w \in W_2 \mid w\rho_0 = \rho_0\}.$$

Choose a positive system R^+ making ρ_0 dominant, and put

$$R^{\rho_0} = \{\alpha \in R \mid \langle \alpha, \rho_0 \rangle = 0\}.$$

By Lemma 3.7 (applied to R_0), $R^{\rho_0} \subseteq R_1$. Since ρ_0 is dominant, R^{ρ_0} is spanned by the R^+-simple roots $\{\alpha_1, \cdots, \alpha_l\}$ contained in it. Suppose α_i is adjacent to α_j. Then $\alpha_i + \alpha_j \in R^{\rho_0} \subseteq R_1$. But α_i and α_j also belong to R_1, so $\alpha_i + \alpha_j \in R_0$, a contradiction. So the α_i are all non-adjacent. By Lemma 3.19, they are superorthogonal; so

$$R^{\rho_0} = \{\pm\alpha_i\}.$$

By Proposition 3.8,

$$W(R^{\rho_0}) = \{w \in W \mid w\rho_0 = \rho_0\}.$$

So

$$Q = W_2 \cap W(R^{\rho_0}). \quad \text{Q.E.D.}$$

Definition 3.21. Suppose R is a root system. A *cograding* of R is a map

$$\check{\delta} : R \to \mathbf{Z}/2\mathbf{Z}$$

such that the map

$$\delta : \check{R} \to \mathbf{Z}/2\mathbf{Z}, \qquad \delta(\check{\alpha}) = \check{\delta}(\alpha) \qquad (\alpha \in R)$$

is a grading of $\check{R}$. Define

$$R_i = R_i(\check{\delta}) = (\check{R}_i(\delta))^{\vee} \qquad (i = 0, 1)$$

$$W_i(\check{\delta}) = W_i(\delta) \qquad (i = 0, 2).$$

(Definition 3.13), and

$$\mathfrak{U}(\check{\delta}) = W_2(\check{\delta})/W_0(\check{\delta}).$$

(Definition 3.15). An $\mathfrak{R}$-*group for* $\check{\delta}$ is a subgroup

$$\mathfrak{R} \subseteq \mathfrak{U}(\check{\delta}),$$

or, equivalently, a group W_1 satisfying

$$W_0(\check{\delta}) \subseteq W_1 \subseteq W_2(\check{\delta}).$$

All the definitions and results about gradings will be applied to cogradings (with appropriate minor modifications) without comment.

Definition 3.22. Suppose R is a root system on V. A *weak bigrading* of R is a triple $g = (\theta, \epsilon, \check{\delta})$ such that

(a) *θ is an involution of R (Definition 3.10; use that notation).*

(b) $\epsilon \in E(R^{i\mathbb{R}})$, $\check{\delta} \in \check{E}(R^{\mathbb{R}})$ (*Definitions* 3.13 *and* 3.21).

We write $\theta = \theta(g)$, etc. The set of all bigradings of R is written $\mathcal{G}(R)$. The notation of Definitions 3.10, 3.13, 3.15, and 3.21 will be extended to bigradings in fairly obvious ways; for example, $W_2^{i\mathbb{R}}(g)$ denotes the subgroup $W_2(\epsilon)$ of $W(R^{i\mathbb{R}})$. A *strong bigrading* of R is a five-tuple $\bar{g} = (\theta, \epsilon, W_1^{i\mathbb{R}}, \check{\delta}, W_1^{\mathbb{R}})$, such that if we write $g = (\theta, \epsilon, \check{\delta})$, then $g \in \mathcal{G}(R)$; and

$$W_0^{i\mathbb{R}}(g) \subseteq W_1^{i\mathbb{R}} \subseteq W_2^{i\mathbb{R}}(g)$$

$$W_0^{\mathbb{R}}(g) \subseteq W_1^{\mathbb{R}} \subseteq W_2^{\mathbb{R}}(g).$$

The set of strong bigradings of R is written $\bar{\mathcal{G}}(R)$. Again we may write $W_1^{i\mathbb{R}}(\bar{g})$, $\theta(\bar{g})$, etc.

To every representation of G with infinitesimal character χ (cf. (2.6)), we are going to attach a strong bigrading of the abstract ordered integral root system $(R^a,(R^a)^+)$ (Definition 4.12). The duality of Theorem 1.15 will correspond to the next definition.

Definition 3.23. Suppose R is a root system, and

$$\bar{g} = (\theta, \epsilon, W_1^{i\mathbb{R}}, \check{\delta}, W_1^{\mathbb{R}}) \in \bar{\mathcal{G}}(R).$$

The *dual strong bigrading* is

$$\check{\bar{g}} = (-\theta, \delta, W_1^{\mathbb{R}}, \check{\epsilon}, W_1^{i\mathbb{R}}) \in \bar{\mathcal{G}}(\check{R}).$$

Here

$$\delta(\check{\alpha}) = \check{\delta}(\alpha)$$

for $\alpha \in R^{\mathbb{R}}(\theta) = [\check{R}^{i\mathbb{R}}(-\theta)]^{\vee}$; and similarly for $\check{\epsilon}$.

The main point is still that of Definition 3.10—replacing θ on R by $-\theta$ on $\check{R}$. The other structure (which will not appear at all in complex groups, for example) can to some extent be regarded as a technicality.

As was mentioned in the introduction, we are going to make use of several ways of producing new representations from old. The first of these involves only the Weyl group, and we can describe the formal part of it now.

Definition 3.24. Suppose R is a root system. The *cross action* of the Weyl group $W(R)$ on strong bigradings is defined as follows. Suppose

$$\bar{g} = (\theta, \epsilon, W_1^{i\mathbb{R}}, \check{\delta}, W_1^{\mathbb{R}}) \in \bar{\mathcal{G}}(R),$$

and $w \in W(R)$. Put

$$w \times \bar{g} = (w\theta w^{-1}, w\cdot\epsilon, wW_1^{i\mathbb{R}}w^{-1}, w\cdot\check{\delta}, wW_1^{\mathbb{R}}w^{-1}).$$

More explicitly, $w\cdot\epsilon$ is defined as a function on the $w\theta w^{-1}$-imaginary roots by

$$(w\cdot\epsilon)(\alpha) = \epsilon(w^{-1}\alpha).$$

For this to make sense, we need to know that α is $w\theta w^{-1}$-imaginary if and only if $w^{-1}\alpha$ is θ-imaginary; that is,

$$(w\theta w^{-1})(\alpha) = \alpha \Leftrightarrow \theta(w^{-1}\alpha) = w^{-1}\alpha.$$

This is clear, and the rest of the verification that $w\times\bar{g} \in \bar{\mathcal{G}}(R)$ is similar. Obviously the cross action is defined on $\mathcal{G}(R)$ as well. We can regard $E(R)$ and

$\check{E}(R)$ as subsets of $\mathcal{G}(R)$, by

$$\begin{aligned}&\epsilon \to (\mathrm{id}, \epsilon, \phi)\\&\check{\delta} \to (-\mathrm{id}, \phi, \check{\delta}\,),\end{aligned} \tag{3.25}$$

where $\epsilon \in E(R)$, $\check{\delta} \in \check{E}(R)$, and ϕ is the unique function from the emtpy set to $\mathbf{Z}/2\mathbf{Z}$. In this identification, $E(R)$ and $\check{E}(R)$ are stable under the cross action; so we have cross actions on $E(R)$ and $\check{E}(R)$.

We will give a representation—theoretic interpretation of the cross action in the next section. The cross action in $E(R)$ may be slightly more familiar. Suppose R is isomorphic to the root system of a compact Cartan subalgebra $\mathfrak{t}_0$ of our real Lie algebra $\mathfrak{g}_0$. Then $\Delta(\mathfrak{g}, \mathfrak{t})$ is graded, with the even roots the compact ones (defined before (2.2)). So the isomorphism $R \xrightarrow{\phi} \Delta(\mathfrak{g}, \mathfrak{t})$ defines a grading $\epsilon \in E(R)$. But ϕ is not canonical; and if we modify it by an element $w \in W(R)$ (replacing ϕ by $\phi \circ w^{-1}$), then ϵ is replaced by $w \times \epsilon$. It is not very hard to check that this construction sets up a bijection

$$\begin{aligned}&\{\text{real forms of } \mathfrak{g} \text{ with a compact Cartan}\} \leftrightarrow\\&\{\text{orbits of the cross action of } W(\mathfrak{g}, \mathfrak{h}^a) \text{ on } E(\Delta^a)\};\end{aligned} \tag{3.26}$$

here $\Delta^a = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ is the (abstract) root system defined before (2.6). This is related to Kac's proof of the Cartan classification of all real forms (see [9], Chapter 10). (There is an important technicality to bear in mind in interpreting (3.26); see the remarks after Table 16.4(h).)

4. The cross action on regular characters.

Definition 4.1 ([25], Definition 8.3.1). Suppose $\gamma = (\Gamma, \bar{\gamma}) \in \hat{H}'_\chi$ (notation (2.12)), and $w \in W(\bar{\gamma})$ (notation 2.5(a)). Write

$$w\bar{\gamma} = \bar{\gamma} - \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} n_\alpha \alpha,$$

for some $n_\alpha \in \mathbf{Z}$; this is possible by Proposition 3.8. Define $\rho^w_{\mathfrak{m}}, \rho^w_{\mathfrak{m} \cap \mathfrak{k}}$ as in (2.10), using $w\bar{\gamma}$ instead of γ. Write

$$\rho^w_{\mathfrak{m}} - 2\rho^w_{\mathfrak{m} \cap \mathfrak{k}} = \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m} \cap \mathfrak{k}} - \sum_{\alpha \in \Delta(\mathfrak{m}, \mathfrak{h})} m_\alpha \alpha,$$

for some $m_\alpha \in \mathbf{Z}$. Define $w \times \gamma$, *w cross γ*, by

$$\begin{aligned}&w \times \gamma = (w \times \Gamma, w\bar{\gamma}) \in \hat{H}'_\chi\\&w \times \Gamma = \Gamma - \sum (n_\alpha + m_\alpha)\alpha \in \hat{H};\end{aligned}$$

here the roots α are interpreted as characters of H.

The most serious checking required in this defintion is of the fact that any sum of roots $\lambda \in \mathfrak{h}^*$ is the differential of a *unique* sum of roots in $\hat{H}$. This is not obvious, since the particular expression $\lambda = \sum n_\alpha \alpha$ is certainly not unique. A detailed discussion of the definition may be found in [25]. It arises naturally in the study of tensor products of finite dimensional and standard representations; but for us, only formal properties matter.

Definition 4.2. Suppose $\gamma \in \hat{H}'_\chi$, and $w \in W^a$ (the abstract integral Weyl group; see (2.6)). Recall the element $w_{\bar{\gamma}} \in W(\gamma)$ of (2.7)(c); and define $w \times \gamma$, *w cross* γ by

$$w \times \gamma = w_{\bar{\gamma}}^{-1} \times \gamma,$$

the action on the right being that of Definition 4.1. (This is an action by (2.8), which also explains the inverse appearing in the definition.)

Because we used such a naive definition of the "abstract" Cartan $\mathfrak{h}^a$, there is technically a danger of ambiguity in this notation. It could happen that $\mathfrak{h}^a$ coincides with $\mathfrak{h}$, and so w might itself belong to $W(\bar{\gamma})$. In this case the definitions of $w \times \gamma$ given by (4.1) and (4.2) differ. We could avoid this by calling the second "$\times^a$", for example; but since it will always be clear from context which action is meant, this is unnecessary.

PROPOSITION 4.3 ([25], Theorem 9.2.11). *If* $\gamma \in \hat{H}'_\chi$ *and* $w \in W^a$, *then* $\bar{\pi}(\gamma)$ *and* $\bar{\pi}(w \times \gamma)$ *belong to the same block* (*Definition* 1.14).

Of course it suffices to prove this when w is a simple reflection, and that is the case treated in [25].

We turn now to the construction of a bigrading of the integral roots attached to the regular character $\gamma = (\Gamma, \bar{\gamma}) \in \hat{H}'_\chi$. The most delicate part is defining a cograding of the real integral roots. Recall from (2.4) the element $m_\alpha \in H$ attached to the real root α. Since m_α has order two,

$$\Gamma(m_\alpha) = \pm 1. \tag{4.4a}$$

If α, β, and γ are real roots, and $\check{\alpha} + \check{\beta} = \check{\gamma}$, then

$$m_\alpha m_\beta = m_\gamma \tag{4.4b}$$

([25], Corollary 4.3.20). Thus $(-1)^{\check{\phi}(\alpha)} = \Gamma(m_\alpha)$ is a cograding of the real roots. However, the Jantzen–Zuckerman translation principle (see [25]) suggests that any construction which has representation—theoretic meaning ought to be invariant when γ is changed by a weight $\Lambda \in \hat{H}$ of a finite dimensional representation of G. Now for such Λ,

$$\Lambda(m_\alpha) = (-1)^{\langle \check{\alpha}, d\Lambda \rangle}, \tag{4.4c}$$

by calculation in SL(2, R). This indicates that we should consider the cograding

$$(-1)^{\check{\phi}(\alpha)} = \Gamma(m_\alpha)(-1)^{\langle\check{\alpha},\bar{\gamma}\rangle},$$

now defined only on the integral real roots. The objection to this is more subtle: it is not well behaved under Langlands functoriality—that is, when G is changed in certain natural ways. (This problem manifests itself in more obvious ways—for example, in the formulation of the result in [21] on reducibility of the standard representation $\pi(\gamma)$. There what enters is not the sign of $\Gamma(m_\alpha)(-1)^{\langle\check{\alpha},\gamma\rangle}$, but the more complicated invariant of Definition 4.5. However, the explanation of this fact, as can be seen by inspecting the proof, is what happens when G is changed.)

Definition 4.5 ([25], Definition 8.3.11). Suppose $H = TA$ is a θ-stable Cartan subgroup of G, and $\gamma = (\Gamma, \bar{\gamma}) \in \hat{H}'$ is a regular character. For each real root α, define $\epsilon_\alpha^G = \epsilon_\alpha = \pm 1$ by [25], Definition 8.3.11. If α is integral for $\bar{\gamma}$, define $\check{\delta}(\gamma)(\alpha) = \check{\delta}(\alpha) \in \mathbf{Z}/2\mathbf{Z}$ by

$$(-1)^{\check{\delta}(\alpha)} = \epsilon_\alpha \Gamma(m_\alpha)(-1)^{\langle\check{\alpha},\bar{\gamma}\rangle}$$

this is the *cograding of the real integral roots defined by* γ.

It is not obvious from the definition that $\check{\delta}(\gamma)$ is a cograding. As is shown in [25], we can find a subgroup $L \subseteq G$ (the centralizer of T_0), containing H, and a regular character for L

$$\gamma_{\mathfrak{q}} = (\Gamma_{\mathfrak{q}}, \bar{\gamma}_{\mathfrak{q}}) \in (\hat{H}')^L,$$

so that

$$\begin{aligned}
&\text{(a) If } \alpha \text{ is real, } \langle\check{\alpha}, \bar{\gamma} - \bar{\gamma}_{\mathfrak{q}}\rangle \in 2\mathbf{Z}\\
&\text{(b) } \epsilon_\alpha^G \Gamma(m_\alpha) = \epsilon_\alpha^L \Gamma_{\mathfrak{q}}(m_\alpha) \qquad\qquad (4.6)\\
&\text{(c) } \epsilon_\alpha^L = 1.
\end{aligned}$$

(The first two conditions hold for some larger class of subgroups $L \supseteq H$, and are related to the functoriality discussed earlier. The third depends on the assumption $L = G^{T_0}$.) By (4.6)(a)–(b), the maps $\check{\delta}(\gamma)$ (defined for G) and $\check{\delta}(\gamma_{\mathfrak{q}})$ (defined for L) coincide. By (4.6)(c), and (4.4)(a)–(b), $\check{\delta}(\gamma_{\mathfrak{q}})$ is a cograding; so $\check{\delta}(\gamma)$ is as well.

Definition 4.7. Suppose $\gamma \in \hat{H}'$. Define

$$R_0^{\mathbf{R}}(\gamma) = \check{\delta}(\gamma)^{-1}(0),$$

the *real integral roots not satisfying the parity condition* for γ, and

$$R_1^{\mathbf{R}}(\gamma) = \check{\delta}(\gamma)^{-1}(1),$$

the *real integral roots satisfying the parity condition for* γ.

Thus the "parity condition" on a real root relates the parity of $\langle\check{\alpha},\bar{\gamma}\rangle$ and the sign of $\Gamma(m_\alpha)$.

LEMMA 4.8 ([25], Lemma 8.3.17). *Suppose $\gamma \in \hat{H}'$, and α is a real integral root not satisfying the parity condition. Then*

$$s_\alpha \times \gamma = s_\alpha \gamma.$$

Here the action on the left is the cross action of Definition 4.1; *and that on the right is the obvious action of the real Weyl group $W(G, H)$ by conjugation.*

LEMMA 4.9 ([25], Lemma 8.3.3). *Suppose $H = TA$ is a θ-stable Cartan subgroup of G, and $\gamma \in \hat{H}'$. Write G^A for the centralizer of A in G. If $w \in W(G^A, H)$, then*

$$w \times \gamma = w \cdot \gamma.$$

LEMMA 4.10. *Suppose $\gamma \in \hat{H}'$, and $w \in W(\gamma)$ (notation* 2.5(a)). *Then*

$$R_i^{\mathbf{R}}(\gamma) = R_i^{\mathbf{R}}(w \times \gamma) \qquad (i = 0, 1)$$

(*Definition* 4.7); *that is, the cross action does not change the set of roots satisfying the parity condition.*

Proof. This is immediate from Definitions 4.1 and 4.5, and (4.4c). (We have only to notice that if β is real and α is imaginary, then $\langle \alpha, \beta \rangle = 0$. Therefore the term $\sum_{\alpha \in \Delta(\mathfrak{m}, \mathfrak{h})} m_\alpha \alpha$ in the definition of $w \times \Gamma$ does not affect $(w \times \Gamma)(m_\beta)$.) Q.E.D.

PROPOSITION 4.11. *Suppose $H = TA$ is a θ-stable Cartan subgroup of G, and $\gamma \in \hat{H}'_\chi$ (notation (2.12)). Then γ defines in $\mathfrak{h}^*$ an ordered root system $(R(\gamma), R^+(\gamma))$ with strong bigrading $\bar{g}(\gamma) = (\theta(\gamma), \epsilon(\gamma), W_1^{i\mathbf{R}}(\gamma), \check{\delta}(\gamma), W_1^{\mathbf{R}}(\gamma))$ (Definition* 3.22) *as follows.*

$$R(\gamma) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \langle \check{\alpha}, \gamma \rangle \in \mathbf{Z}\}$$

$$R^+(\gamma) = \{\alpha \in R(\gamma) \mid \langle \alpha, \bar{\gamma} \rangle > 0\}$$

$$\theta(\gamma) = \theta = \textit{Cartan involution on } \mathfrak{h}^*$$

$$R_0^{i\mathbf{R}}(\gamma) = \textit{compact imaginary roots}$$

$$R_1^{i\mathbf{R}}(\gamma) = \textit{noncompact imaginary roots}$$

$$\epsilon(\gamma) = \epsilon \in E(R^{i\mathbf{R}}) \textit{ corresponding grading (Definition 3.13)}$$

$$R_0^{\mathbf{R}}(\gamma), R_1^{\mathbf{R}}(\gamma), \check{\delta}(\gamma) = \check{\delta} \textit{ as in Definition 4.7}$$

$$W_1^{i\mathbf{R}}(\gamma) = W(R^{i\mathbf{R}}) \cap W(G, H)$$

$$= W(G^A, H)$$

$$W_1^{\mathbf{R}}(\gamma) = \{w \in W(R^{\mathbf{R}}) \mid w \times \gamma = w \cdot \gamma\}.$$

Proof. That ϵ is a grading of the imaginary roots is an elementary fact, discussed before (3.26). Since the compact simple reflections lie in $W(G,H)$ and generate (by definition) $W_0^{i\mathbb{R}}(\gamma)$, we have $W_1^{i\mathbb{R}}(\gamma) \supseteq W_0^{i\mathbb{R}}(\gamma)$. Since $W(G,H)$ preserves the set of compact roots, $W_1^{i\mathbb{R}}(\gamma) \subseteq W_2^{i\mathbb{R}}(\gamma)$. That $\check{\delta}$ is a cograding was established after Definition 4.5. The containments $W_0^{\mathbb{R}}(\gamma) \subseteq W_1^{\mathbb{R}}(\gamma) \subseteq W_2^{R}(\gamma)$ follow from Lemmas 4.8 and 4.10, respectively. Q.E.D.

Definition 4.12. Suppose $\gamma \in \hat{H}'_\chi$. Use the isomorphism $i_{\bar{\gamma}}$ of (2.7) to define

$$\bar{g}^a(\gamma) = \left(\theta^a(\gamma), \epsilon^a(\gamma), W_1^{i\mathbb{R}}(\gamma)^a, \check{\delta}^a(\gamma), W_1^{\mathbb{R}}(\gamma)^a\right) \in \bar{\mathcal{G}}(R^a),$$

the strong bigrading of the abstract root system defined by γ. Clearly this depends only on the conjugacy class of γ; so if π is an irreducible admissible representation of G with infinitesimal character χ, then we can define $\bar{g}^a(\pi)$ as well.

When we construct the dual objects $(\check{G}, \check{\gamma})$ of Theorem 1.15, we will have

$$R^a(\check{G}) = \left[R^a(G)\right]^\vee$$

$$\bar{g}^a(\check{\gamma}) = \left[\bar{g}^a(\gamma)\right]^\vee$$

(the duality of Definition 3.23). Part of the definition of $\check{\gamma}$ will be the requirement

$$(w \times \gamma)^\vee = w \times \check{\gamma}.$$

In order for this to be well-defined on conjugacy classes, we need to know that $w \times \gamma$ is conjugate to γ if and only if $w \times \check{\gamma}$ is conjugate to $\check{\gamma}$. This suggests the following definition.

Definition 4.13. Suppose $\gamma \in \hat{H}'_\chi$. The *cross stabilizer* of γ is the group

$$W_1(\gamma) = \left\{ w \in W(G,H) \cap W(\gamma) \mid w \times \gamma = w \cdot \gamma \right\}.$$

Using $i_{\bar{\gamma}}$, we can define the corresponding group

$$W_1(\gamma)^a \subseteq W^a.$$

PROPOSITION 4.14. *In the setting of Definition* 4.13 *and Proposition* 4.11, *define* $R^{\mathbb{C}} \subseteq R$ *as in Proposition* 3.12 (*using* $R^+ \cap R^{i\mathbb{R}}$, $R^+ \cap R^{\mathbb{R}}$ *for the positive imaginary and real roots*). *Then*

$$W_1(\gamma) = W(R^{\mathbb{C}})^\theta \ltimes \left(W_1^{i\mathbb{R}}(\gamma) \times W_1^{\mathbb{R}}(\gamma)\right),$$

a semidirect product with the second factor normal.

Proof. By Proposition 3.12

$$W(\gamma)^\theta = W(R^{\mathbb{C}})^\theta \ltimes \left(W^{i\mathbb{R}}(\gamma) \times W^{\mathbb{R}}(\gamma)\right).$$

Now every element of $W(G,H)$ has a representative in K, and so commutes with θ. So by the definition of $W_1^{i\mathsf{R}}(\gamma)$,

$$W(\gamma) \cap W(G,H) = W(\gamma)^\theta \cap W(G,H)$$
$$\subseteq W\big(R^{\mathsf{C}\theta} \ltimes (W^{i\mathsf{R}})(\gamma) \times W^{\mathsf{R}}(\gamma)\big).$$

We will show that $W_1(\gamma) \supseteq W(R^{\mathsf{C}})^\theta$. Assuming this, the conclusion of the proposition is immediate from Lemma 4.9, and the definitions of $W_1^{i\mathsf{R}}(\gamma)$ and $W_1^{\mathsf{R}}(\gamma)$. To prove the claim, recall the elements $\check{\rho}^{\mathsf{R}}$ and $\check{\rho}^{i\mathsf{R}}$ of $\mathfrak{h}^*$ which appear in the definition of R^{C} (Lemma 3.4). Choose elements $\check{r}^{\mathsf{R}} \in \mathfrak{a}_0$, $r^{i\mathsf{R}} \in i\mathfrak{t}_0$ dual to these; and set

$$\tilde{G}^{\mathsf{C}} = \text{centralizer of } \check{r}^{\mathsf{R}} \text{ and } r^{i\mathsf{R}} \text{ in } G.$$

(This need not be a complex group, because of the presence of nonintegral real roots.) We have

$$\Delta(\tilde{\mathfrak{g}}^{\mathsf{C}}, \mathfrak{h}) = \big\{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) \,|\, \langle \alpha, \check{\rho}^{\mathsf{R}} \rangle = \langle \alpha, \rho^{i\mathsf{R}} \rangle = 0\big\}.$$

Therefore

$$\Delta(\tilde{\mathfrak{g}}^{\mathsf{C}}, \mathfrak{h}) \cap R(\gamma) = R^{\mathsf{C}}.$$

We define a $\tilde{G}^{\mathsf{C}}$-regular character γ^{C} of H by

$$\gamma^{\mathsf{C}} = (\Gamma^{\mathsf{C}}, \bar{\gamma}^{\mathsf{C}})$$
$$\Gamma^{\mathsf{C}} = \Gamma + (\text{sum of positive compact imaginary roots})$$
$$\bar{\gamma}^{\mathsf{C}} = \bar{\gamma} + \rho^{i\mathsf{R}}.$$

Since there are no imaginary roots of H in $\tilde{G}^{\mathsf{C}}$, condition 2.10(b) shows that γ^{C} is really a $\tilde{G}^{\mathsf{C}}$-regular character. Also

$$R(\gamma^{\mathsf{C}}) = \big\{\alpha \in \Delta(\tilde{\mathfrak{g}}^{\mathsf{C}}, \mathfrak{h}) \,|\, \langle \check{\alpha}, \gamma^{\mathsf{C}} \rangle \in \mathbf{Z}\big\}$$
$$= \Delta(\tilde{\mathfrak{g}}^{\mathsf{C}}, \mathfrak{h}) \cap R(\gamma) = R^{\mathsf{C}}$$

since $\rho^{i\mathsf{R}} = \bar{\gamma}^{\mathsf{C}} - \bar{\gamma}$ kills the roots of $\tilde{\mathfrak{g}}^{\mathsf{C}}$. Finally, if $w \in W(R^{\mathsf{C}})^\theta$, then

$$w \times \gamma^{\mathsf{C}} = (w \times \gamma)^{\mathsf{C}}$$
$$w \cdot \gamma^{\mathsf{C}} = (w \cdot \gamma)^{\mathsf{C}}$$

by inspection of Definition 4.1. So we are reduced to proving the claim for $\tilde{G}^{\mathsf{C}}$; that is, we may assume that H is maximally split in the quasisplit group G, and that there are no real integral roots. Fix a root $\alpha \in R^{\mathrm{C}}$, and put

$$\mathfrak{h}^{\alpha} = \{x \in \mathfrak{h} \mid \alpha(x) = \theta\alpha(x) = 0\}$$

$$G^{\alpha} = \text{centralizer of } \mathfrak{h}^{\alpha} \text{ in } G.$$

The element $w_\alpha = s_\alpha s_{\theta\alpha}$ of $W(R^{\mathrm{C}})$ has a representative in G^α; and since these elements generate $W(R^{\mathrm{C}})^\theta$ (Lemma 3.11), it is enough to show that $w_\alpha \cdot \gamma = w_\alpha \times \gamma$. So we may assume $G = G^\alpha$. In that case, $\Delta(\mathfrak{g}, \mathfrak{h})$ is spanned by orthogonal roots α and $\theta\alpha$ of the same length. Since G is quasisplit, no rational multiple of $\alpha + \theta\alpha$ is a root; so $\Delta(\mathfrak{g}, \mathfrak{h})$ is not of type B_2. So it is of type $A_1 \times A_1$, and G is complex. It follows that

$$H = Z(G) \cdot H_0;$$

for every $\mathrm{Ad}(\mathfrak{g})$—inner automorphism of a complex reductive Lie algebra $\mathfrak{g}_0$ comes from $\mathrm{Ad}(G_0)$. (Of course we only need this for $SL(2, \mathbf{C})$.) Obviously $w_\alpha \cdot \gamma$ and $w_\alpha \times \gamma$ agree on H_0 and on $Z(G)$, so they coincide. Q.E.D.

LEMMA 4.15. *Suppose $\gamma \in \hat{H}'_x$, $w \in W^a$, and $y \in W(\gamma)$. Then*

$$\bar{g}^a(w \times \gamma) = w \times \bar{g}^a(\gamma)$$

$$\bar{g}(y \times \gamma) = \bar{g}(\gamma)$$

(*notation* (3.24), (4.1), (4.11), *and* (4.12)).

Proof. The second assertion follows from Lemma 4.10 and easy checking. The first is a consequence of (2.8) and the second. Q.E.D.

Although we will make no use of it, we may as well record here how Knapp's result on $W(G, H)$ follows from Proposition 4.14.

PROPOSITION 4.16 (Knapp [16]). *Let $H = TA$ be a θ-stable Cartan subgroup of G. Write $R = \Delta(\mathfrak{g}, \mathfrak{h})$, and use the notation of Definition* 3.10 *and Proposition* 3.12. *Then*

(a)
$$W(G, H) \cong (W^{\mathrm{C}})^{\theta} \ltimes \left(W^{\mathrm{R}} \times W(G^A, H)\right)$$
$$\cong (W^q)^{\theta} \ltimes \left(W(G^A, H)\right)$$

Here G^A denotes the centralizer of A in H. The set

(b)
$$\bar{R}^q = \{\alpha|_{\mathfrak{a}} \mid \alpha \in R^q\} \subseteq \mathfrak{a}^*$$

is a (possibly non-reduced) root system in $\mathfrak{a}^*$, and

(c)
$$W(G, H)|_{\mathfrak{a}} = (W^q)^{\theta}|_{\mathfrak{a}} = W(\bar{R}^q).$$

Furthermore, in the notation of Definition 3.13,

(d) $$W_0^{i\mathsf{R}}(\epsilon) \subseteq W(G^A, H) \subseteq W_2^{i\mathsf{R}}(\epsilon).$$

Here ϵ is the grading by compact and noncompact roots of the imaginary roots.

The group $W(G^A, H)$ depends on G/G_0, and so cannot be specified in terms of Lie algebra data alone. In Proposition 10.20(a), we will describe exactly which possibilities can occur. A method for computing $W(G^A, H)$ is also given in [20].

Part (c) is Knapp's result that $W(G, A)$ is the Weyl group of a root system. The root system $\overline{R}^q$ is in general smaller than Knapp's "useful roots" ([16]), but the two can easily be shown to have the same Weyl group.

Proof of Proposition 4.16. Choose a regular character $\gamma \in \hat{H}'$ with regular integral infinitesimal character. Then $R(\gamma) = R$, so Proposition 4.14 implies that $(W^{\mathsf{C}})^\theta \subseteq W(G, H)$. (It would be more reasonable to say, "By the proof of Proposition 4.14, $(W^{\mathsf{C}})^\theta \subseteq W(G, H)$.") By the remarks after (2.4), $W^{\mathsf{R}} \subseteq W(G, H)$. Now (a) follows from Proposition 3.12. For (b), suppose α, $\beta \in R^q$; write $\overline{\alpha}$, $\overline{\beta} \in \overline{R}^q$ for their restrictions to $\mathfrak{a}$. We must show that $\langle \check{\overline{\alpha}}, \overline{\beta} \rangle \in \mathbf{Z}$, and $s_{\overline{\alpha}} \overline{\beta} \in \overline{R}^q$. If α is real, then $\overline{\alpha} = \alpha$, and these follow from the fact that R^q is a root system. So suppose $\alpha \neq \theta\alpha$. Since R^q contains no imaginary roots, $\alpha + \theta\alpha$ is not a root. If $\alpha - \theta\alpha = \gamma$ is a root, then it is real; and

$$\overline{\alpha} = \tfrac{1}{2}\gamma \qquad \check{\overline{\alpha}} = 2\check{\gamma}$$

Therefore

$$\langle \check{\overline{\alpha}}, \beta \rangle = 2\langle \check{\gamma}, \beta \rangle \in \mathbf{Z}$$

$$s_{\overline{\alpha}} \overline{\beta} = \overline{s_\gamma \beta} \in \overline{R}^q.$$

So we may as well suppose $\alpha \pm \theta\alpha$ are not roots. Then they generate an $\mathrm{SL}(2, \mathbf{C})$. Put

$$w_\alpha = s_\alpha s_{\theta\alpha} \in W(R^q)^\theta.$$

By computation in $\mathrm{SL}(2, \mathbf{C})$,

$$w_\alpha|_{\mathfrak{a}} = s_{\overline{\alpha}}$$

$$\langle \overline{\alpha}, \overline{\alpha} \rangle = \tfrac{1}{2}\langle \alpha, \alpha \rangle.$$

By the first of these,

$$s_{\overline{\alpha}} \overline{\beta} = w_\alpha \overline{\beta} \in \overline{R}^q.$$

By the second,

$$\begin{aligned}\langle \check{\bar{\alpha}}, \bar{\beta} \rangle &= 2\langle \bar{\alpha}, \bar{\beta} \rangle / \langle \bar{\alpha}, \bar{\alpha} \rangle \\ &= 4\langle \bar{\alpha}, \beta \rangle / \langle \alpha, \alpha \rangle \\ &= 2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle - 2\langle \theta\alpha, \beta \rangle / \langle \theta\alpha, \theta\alpha \rangle \\ &= \langle \check{\alpha}, \beta \rangle - \langle \theta\check{\alpha}, \beta \rangle \in \mathbf{Z}.\end{aligned}$$

For (c), the argument just given also showed that

$$(W^q)^\theta|_\mathfrak{a} \supseteq W(\bar{R}^q) \supseteq W(R^\mathbf{C})^\theta|_\mathfrak{a}, W(R^\mathbf{R})|_\mathfrak{a}.$$

By Proposition 3.12(c), $(W^q)^\theta$ is generated by $W(R^\mathbf{C})^\theta$ and $W(R^\mathbf{R})$; so (c) follows. Part (d) is obvious (proof of Proposition 4.11). Q.E.D.

5. Cayley transforms: formal theory. In this section we begin the study of operations on graded (and, eventually, bigraded) root systems, which correspond to changing Cartan subgroups in G. On the level of root systems with involution, we will simply be modifying the involution θ by an involution in W which commutes with θ. The effect of this on bigradings is complicated, and will occupy the next two sections.

LEMMA 5.1 (D. Garfinkle). *Let R be a root system, and $\alpha \in R$. Put*

$$R^\alpha = \{ \beta \in R \mid \langle \alpha, \beta \rangle = 0 \}$$

Define $\tilde{\epsilon} : R^\alpha \to \mathbf{Z}/2\mathbf{Z}$ by

$$\begin{aligned}\tilde{\epsilon}(\beta) &= 0 \qquad \text{if } \alpha \pm \beta \text{ are not roots} \\ \tilde{\epsilon}(\beta) &= 1 \qquad \text{if } \alpha \pm \beta \text{ are roots.}\end{aligned}$$

Then $\tilde{\epsilon} \in E(R^\alpha)$ (Definition 3.13).

Proof. Suppose β, γ and $\beta + \gamma$ are all roots orthogonal to α. First, assume that $\tilde{\epsilon}(\beta) = \tilde{\epsilon}(\gamma) = 0$; we must show that $\tilde{\epsilon}(\beta + \gamma) = 0$. Suppose not; that is, that $\alpha + \beta + \gamma$ is a root. Since $\langle \beta + \gamma, \beta + \gamma \rangle$ is positive, either $\langle \beta, \beta + \gamma \rangle > 0$ or $\langle \gamma, \beta + \gamma \rangle > 0$. Suppose for definiteness that the first holds. Since $\beta \in R^\alpha$,

$$\langle \beta, \alpha + \beta + \gamma \rangle = \langle \beta, \beta + \gamma \rangle > 0;$$

so

$$\alpha + \gamma = (\alpha + \beta + \gamma) - \beta \in R,$$

contradicting $\tilde{\epsilon}(\gamma) = 0$.

Next, suppose that $\tilde{\epsilon}(\beta) = 0$, $\tilde{\epsilon}(\gamma) = 1$. We want to show that $\tilde{\epsilon}(\beta + \gamma) = 1$. Suppose not; that is, that $\tilde{\epsilon}(\beta + \gamma) = 0$. Applying the first case with $(\beta, \gamma, \beta + \gamma)$ replaced by $(-\beta, \beta + \gamma, \gamma)$, we deduce that $\tilde{\epsilon}(\gamma) = 0$; a contradiction.

Finally, suppose that $\tilde{\epsilon}(\beta) = \tilde{\epsilon}(\gamma) = 1$, but that $\tilde{\epsilon}(\beta + \gamma) = 1$. Then $\alpha \pm \beta$, $\alpha \pm \gamma$, and $\alpha \pm (\beta + \gamma)$ are all roots. It follows that β, γ, and $\beta + \gamma$ all have the same length as α, and that $\alpha + \beta$ has twice that length. So β and γ generate an A_2, and $\langle \check{\beta}, \gamma \rangle = -1$. Since $\gamma \in R^\alpha$, we compute

$$\begin{aligned}\langle (\alpha + \beta)^\vee, \gamma \rangle &= 2\langle \alpha + \beta, \gamma \rangle / \langle \alpha + \beta, \alpha + \beta \rangle \\ &= 2\langle \beta, \gamma \rangle / 2\langle \beta, \beta \rangle \\ &= \tfrac{1}{2}\langle \check{\beta}, \gamma \rangle = -\tfrac{1}{2}.\end{aligned}$$

This contradicts the axioms for a root system. Q.E.D.

Definition 5.2 (Schmid [20]). Suppose (R, ϵ) is a graded root system, $\alpha \in R$, and $\epsilon(\alpha) = 1$. Set $R^\alpha = \{\beta \in R \mid \langle \alpha, \beta \rangle = 0\}$. The *Cayley transform of* (R, ϵ) *through* α is the pair $c^\alpha(R, \epsilon) = (R^\alpha, c^\alpha(\epsilon))$, with

$$c^\alpha(\epsilon) = (\epsilon|_{R^\alpha}) \cdot \tilde{\epsilon};$$

here $\tilde{\epsilon}$ is the grading of Lemma 5.1. More explicitly,

$$\begin{aligned}R_0^\alpha &= \{\beta \in R_0 \mid \alpha \pm \beta \text{ are not roots}\} \\ &\quad \cup \{\beta \in R_1 \mid \langle \alpha, \beta \rangle = 0, \text{ and } \alpha \pm \beta \text{ are roots}\} \\ R_1^\alpha &= \{\beta \in R_1 \mid \alpha \pm \beta \text{ are not roots}\} \\ &\quad \cup \{\beta \in R_0 \mid \langle \alpha, \beta \rangle = 0, \text{ and } \alpha \pm \beta \text{ are roots}\}.\end{aligned}$$

An "explanation" of this definition may be found in the proof of Lemma 10.9.

Definition 5.3. Suppose (R, ϵ) is a graded root system. An ordered set $S = \{\alpha_1, \ldots, \alpha_l\} \subseteq R$ is called *admissible* if the iterated Cayley transform

$$c^S = c^{\alpha_l} \ldots c^{\alpha_1}$$

is defined. That is, we assume that $\epsilon(\alpha_1) = 1$, and that $\{\alpha_2, \ldots, \alpha_l\}$ is an admissible sequence in R^{α_1}.

In particular, an admissible sequence must be orthogonal. A strongly orthogonal sequence (Definition 3.18) contained in $R_1 = \epsilon^{-1}(1)$ is automatically admissible; this is clear from Definition 5.2. These are the most important examples, as we will show that any iterated Cayley transform is equivalent to one given by such a sequence (Corollary 5.7).

LEMMA 5.4. *Suppose α, β and γ are orthogonal roots in R, and that $\alpha \pm \beta$ and $\alpha \pm \gamma$ are roots. Then $\beta \pm \gamma$ are roots as well.*

Proof. Since α, β, and γ are orthogonal,

$$\langle \alpha + \beta, \alpha \mp \gamma \rangle = \langle \alpha, \alpha \rangle > 0.$$

So

$$(\alpha + \beta) - (\alpha \mp \gamma) = \beta \pm \gamma$$

is a root. Q.E.D.

LEMMA 5.5. *Suppose $S = \{\alpha_1, \ldots, \alpha_l\}$ is an orthogonal sequence of roots in a graded root system (R, ϵ). Set*

$$R^S = \{\beta \in R \mid \langle \beta, \alpha_i \rangle = 0, \text{ all } i\}$$

$$n_i = |\{j < i \mid \alpha_j \pm \alpha_i \text{ are roots}\}|$$

$(i = 1, \ldots, l)$; and

$$n_\beta = |\{j \mid \alpha_j \pm \beta \text{ are roots}\}| \qquad (\beta \in R^S).$$

Then S is admissible (Definition 5.3) if and only if $\epsilon(\alpha_i) \equiv n_i + 1 \pmod 2$ for $i = 1, \ldots, l$. In that case,

$$c^S(\epsilon)(\beta) \equiv \epsilon(\beta) + n_\beta \qquad (\mathrm{mod}\ 2).$$

This is clear from Definition 5.2.

LEMMA 5.6. *Suppose (R, ϵ) is a graded root system, and $S = \{\alpha_1 \ldots \alpha_l\}$ is an admissible sequence which is not strongly orthogonal. Choose $i < j$ as small as possible so that $\alpha_i' = \alpha_i + \alpha_j$ and $\alpha_j' = \alpha_i - \alpha_j$ are both roots. Then*

$$S' = \{\alpha_1, \ldots, \alpha_i', \ldots, \alpha_j', \ldots, \alpha_l\}$$

is an admissible sequence; and

$$\left(R^S, c^S(\epsilon)\right) = \left(R^{S'}, c^{S'}(\epsilon)\right).$$

Proof. Since S and S' have the same span, $R^S = R^{S'}$. Write n_k, n_β as in Lemma 5.5, and n_k', n_β' for S'. What must be shown is that

$$n_k + \epsilon(\alpha_k) \equiv n_k' + \epsilon(\alpha_k') \qquad (\mathrm{mod}\ 2) \quad (k = 1, \ldots, l)$$

$$n_\beta + \epsilon(\beta) \equiv n_\beta' + \epsilon(\beta) \qquad (\mathrm{mod}\ 2), \quad \text{all} \quad \beta \in R^S.$$

We consider only the first of these; the second is formally identical. If $k < i$, the assertion is obvious. Since α_i' and α_j' are long, they are strongly orthogonal to the other α_k; so $n_i' = n_j' = 0$. By the minimality of (i, j) and Lemma 5.5,

$$n_i = 0, \qquad \epsilon(\alpha_i) = 1$$

$$n_j = 1, \qquad \epsilon(\alpha_j) = 0.$$

So $\epsilon(\alpha_i') = \epsilon(\alpha_j') = 1$, and the claim follows for $k = i, j$. If $i < k < j$, the minimality of (i, j) guarantees $n_k = n_{k'} = 0$. Suppose $k > j$. If $\alpha_k \pm \alpha_j$ and $\alpha_k \pm \alpha_i$ are not roots, then $n_k = n_k'$, and the claim holds. Otherwise, Lemma 5.4 guarantees that α_k is strongly othogonal to neither α_i nor α_j. Therefore $n_k' = n_k - 2$, and the claim still holds. Q.E.D.

COROLLARY 5.7. *Suppose (R, ϵ) is a graded root system, and $S \subseteq R$ is an admissible sequence. Then there is a strongly orthogonal subset S' of $R_1 = \epsilon^{-1}(1)$, having the same span as S, such that $c^S(\epsilon) = c^{S'}(\epsilon)$.*

Proof. Suppose S is not strongly orthogonal. Lemma 5.6 provides a new admissible sequence with the same span and same Cayley transform. The new sequence consists of longer roots than the first. If it is not strongly orthogonal we repeat the process. Since the roots keep getting longer, this ends in a finite number (at most $[l/2]$) of steps. Q.E.D.

We want to show that R^S depends only on the span of the admissible sequence S. The following proposition is the key to that fact, and we will have many other uses for it as well.

PROPOSITION 5.8. *Let (R, ϵ) be a graded root system, and suppose $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ are maximal admissible sequences (Definition 5.3) in R. Then $m = n$, and there is an element $w \in W(R_0)$ such that*

$$w\langle \alpha_1, \ldots, \alpha_m \rangle = \langle \beta_1, \ldots, \beta_n \rangle.$$

(We write $\langle X \rangle$ for the span of a set X.) If both $\{\alpha_i\}$ and $\{\beta_j\}$ are strongly orthogonal, then we can arrange

$$w\{\pm\alpha_i\} = \{\pm\beta_j\},$$

an equality of unordered sets.

Proof. Write S, S' for the two sequences. To say that they are maximal amounts to saying that R^S and $R^{S'}$ have the trivial grading; that, $c^S(\epsilon)$ and $c^{S'}(\epsilon)$ are identically zero. By Corollary 5.7, we can replace $\{\alpha_i\}$ and $\{\beta_j\}$ by strongly orthogonal sequences having the same span and the same Cayley transform. By the description of maximality just given, the new sequences will also be maximal. So we may as well assume that $\{\alpha_i\}$ and $\{\beta_j\}$ are strongly orthogonal, and try to prove the last claim; for everything else will follow from it. Clearly we may assume that R is simple and spans V. Write $R_i = \epsilon^{-1}(i)$, $W_0 = W(R_0)$ as usual. We proceed by induction on the dimension of V. If $\dim V = 2$ and $R_1 \neq 0$, then it is easily checked that (R, R_0) is one of the pairs (A_2, A_1), (B_2, A_1) (with A_1 consisting of short roots), $(B_2, A_1 \times A_1)$ (with $A_1 \times A_1$ the long roots), or $(G_2, A_1 \times A_1)$. For G_2 the proposition is easily checked; so we assume R is not of type G_2 for simplicity. The difficulties are caused mostly by roots α having the following special properties (corresponding to rank two subsystems of the second type mentioned above):

$$\epsilon(\alpha) = 1, \alpha \text{ is short, and } \alpha = \tfrac{1}{2}(\beta_1 + \beta_2), \text{ with } \beta_i \text{ orthogonal long roots satisfying } \epsilon(\beta_i) = 1. \tag{$*$}$$

Step I. *Suppose* $\alpha, \beta \in R$, $\epsilon(\alpha) = 1$, *and* $\langle \alpha, \beta \rangle \neq 0$. *Then either* $\beta \in \langle W_0\alpha \rangle$ (*the span of* $W_0\alpha$), *or* α *satisfies* $(*)$. *If also* $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ *and* $\epsilon(\beta) = 1$, *then* $\pm\beta \in W_0\alpha$. Obviously this can be reduced to the case $\dim V = 2$, and checked case by case.

Step II. *If* $\epsilon(\alpha) = 1$, *then either* $\langle W_0\alpha \rangle = V$, *or* α *satisfies* $(*)$. Suppose $(*)$ fails. Set

$$\Phi = \langle W_0\alpha \rangle \cap R$$

$$\Phi^\perp = \{ \beta \in R \mid \langle \beta, W_0\alpha \rangle = 0 \}.$$

By Step I, $R = \Phi \cup \Phi^\perp$, an orthogonal union of root systems. Since R is simple, $\Phi^\perp$ is empty.

Step III. *If* $\epsilon(\alpha) = \epsilon(\beta) = 1$, *and* $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$, *then* $\pm\beta \in W_0\alpha$. Suppose first that $(*)$ fails. By Step II, there is a $w \in W_0$ such that $\langle \beta, w\alpha \rangle \neq 0$. Now apply Step I.

Next, suppose $(*)$ holds; choose β_i accordingly. By Step II applied to β_1 instead of α, we can find a $w \in W_0$ so that $\langle w\beta_1, \beta \rangle \neq 0$. Replacing α by $\pm w\alpha$, we may assume that $\langle \beta_1, \beta \rangle > 0$. As the β_i are long and β (which has the same length as α) is short, this gives

$$\langle \check{\beta}_1, \beta \rangle = 1$$

$$\langle \check{\beta}_2, \beta \rangle = 1, 0, \text{ or } -1.$$

If $\langle \check{\beta}_2, \beta \rangle$ is 1 or 0, then

$$\langle \alpha, \beta \rangle = \tfrac{1}{2}(\langle \beta_1, \beta \rangle + \langle \beta_2, \beta \rangle) \neq 0;$$

so $\beta \in W_0\alpha$ by Step I. So suppose $\langle \check{\beta}_2, \beta \rangle = -1$. Then

$$\begin{aligned} \langle \beta, \tfrac{1}{2}(\beta_1 - \beta_2) \rangle &= \langle \beta_1, \beta_1 \rangle \left(\tfrac{1}{4}\langle \beta, \check{\beta}_1 \rangle - \tfrac{1}{4}\langle \beta, \check{\beta}_2 \rangle \right) \\ &= \tfrac{1}{2}\langle \beta_1, \beta_1 \rangle \\ &= \langle \tfrac{1}{2}(\beta_1 \pm \beta_2), \tfrac{1}{2}(\beta_1 \pm \beta_2) \rangle \\ &= \langle \alpha, \alpha \rangle. \end{aligned}$$

By the Cauchy–Schwarz inequality, $\beta = \frac{1}{2}(\beta_1 - \beta_2)$; so $\alpha = \beta + \beta_2$. Since $\epsilon(\alpha) = \epsilon(\beta) = \epsilon(\beta_2) = 1$, this is impossible.

Step IV. *If there are two root lengths in R_1, then one of the α_i is long.* Suppose not. Fix a long root $\alpha \in R_1$. By Step II, we may assume (after replacing α by $\pm w\alpha$, for some $w \in W_0$) that $\langle \alpha, \alpha_1 \rangle > 0$. Since $\langle \alpha, \alpha \rangle > \langle \alpha_1, \alpha_1 \rangle$, $\alpha - \alpha_1$ is a short root. Since $\epsilon(\alpha) = \epsilon(\alpha_1) = 1$, $\epsilon(\alpha - \alpha_1) = 0$. Suppose α is also adjacent to some other α_i; say (possibly after replacing α_i by $-\alpha_i$) $\langle \alpha, \alpha_i \rangle > 0$. Then $\alpha - \alpha_i$ is a root. Since α_1 is orthogonal to α_i,

$$\langle \alpha - \alpha_1, \alpha - \alpha_i \rangle = \langle \alpha, \alpha \rangle > 0;$$

so

$$(\alpha - \alpha_1) - (\alpha - \alpha_i) = \alpha_i - \alpha_1 \in R.$$

This contradicts the assumption that $\{\alpha_1, \ldots, \alpha_l\}$ is strongly orthogonal. So $\{\alpha_1, \ldots, \alpha_l, \alpha - \alpha_1\}$ is orthogonal, and $\alpha - \alpha_1 \in R_0$. We claim that this sequence is admissible. This will contradict the maximality of S, and prove the claim. Since $\alpha_1 \pm (\alpha - \alpha_1)$ are roots, it suffices (by Lemma 5.5) to show that $\alpha - \alpha_1$ is strongly orthogonal to all the other α_i. Since $\{\alpha_1, \ldots, \alpha_l\}$ is strongly orthogonal, and $\{\alpha_1, \alpha - \alpha_1\}$ is not, this follows from Lemma 5.4. (The alert reader will have noticed that this last argument is essentially the standard one on equine pigmentation.)

We now complete the proof of the proposition. By Steps III and IV, we may assume (after renumbering and acting by W_0 on S) that $\alpha_1 = \pm \beta_1$. Define $\tilde{\epsilon}$ using α_1 as in Lemma 5.1, and set

$$\Psi = \tilde{\epsilon}^{-1}(0) \subseteq R^{\alpha_1}.$$

Then

$$c^{\alpha_1}(\epsilon)|_{\Psi} = \epsilon|_{\Psi}.$$

The sequences $\{\alpha_2, \ldots, \alpha_m\}$, $\{\beta_2, \ldots, \beta_n\}$ therefore satisfy the hypotheses of the proposition for $(\Psi, \epsilon|_{\Psi})$. By induction, there is a $w \in W_0(\Psi) \subseteq W_0$ such that

$$w\{\pm \alpha_2, \ldots, \pm \alpha_m\} = \{\pm \beta_2, \ldots, \pm \beta_n\}$$

up to order. Since $w \in W(\Psi)$, $w\alpha_1 = \alpha_1 = \pm \beta_1$. Q.E.D.

COROLLARY 5.9. *Suppose S, S' are admissible sequences in the graded root system (R, ϵ), and that S and S' have the same span. Then $c^S(\epsilon) = c^{S'}(\epsilon)$.*

Proof. By Corollary 5.7, we may assume that $S = \{\alpha_1, \ldots, \alpha_l\}$ and $S' = \{\beta_1, \ldots, \beta_l\}$ are both strongly orthogonal. Write $\mathfrak{s} \subseteq V$ for their mutual span, and $\Phi = R \cap \mathfrak{s}$, a graded root system on $\mathfrak{s}$. By Proposition 5.8 applied to Φ, there is a $w \in W(\Phi_0) \subseteq W(R_0)$ such that $w\{\pm \alpha_i\} = \{\pm \beta_j\}$. Suppose $\beta \in R^S = \{\gamma \in R \mid \langle \gamma, \mathfrak{s} \rangle = 0\}$. Then $w\beta = \beta$, since w is a product of reflections fixing β.

In the notation of Lemma 5.5, therefore,

$$\begin{aligned} n_\beta &= |\{i \mid \beta \pm \alpha_i \text{ are roots}\}| \\ &= |\{i \mid w(\beta \pm \alpha_i) \text{ are roots}\}| \\ &= |\{i \mid \beta \pm w\alpha_i \text{ are roots}\}| \\ &= |\{j \mid \beta \pm \beta_j \text{ are roots}\}| \\ &= n'_\beta . \end{aligned}$$

By Lemma 5.5, $c^S(\epsilon) = c^{S'}(\epsilon)$. Q.E.D.

Definition 5.10. Suppose (R, ϵ) is a graded roots system on V. A subspace $\mathfrak{S} \subseteq V$ is called *ϵ-admissible* if it is the span of some admissible sequence $S = \{\alpha_1, \ldots, \alpha_l\}$ (Definition 5.3); or, equivalently, if it is the span of a strongly orthogonal subset $\{\beta_1, \ldots, \beta_l\}$ of $R_1(\epsilon)$. (Lemma 5.6). *The Cayley transform of (R, ϵ) through $\mathfrak{S}$* is the graded root system $c^{\mathfrak{S}}(R, \epsilon) = (R^{\mathfrak{S}}, c^{\mathfrak{S}}(\epsilon))$ defined by

$$c^{\mathfrak{S}}(R) = R^{\mathfrak{S}} = \{\gamma \in R \mid \langle \gamma, \mathfrak{S} \rangle = 0\},$$

$$c^{\mathfrak{S}}(\epsilon) = c^S(\epsilon).$$

(Definition 5.3) with $S \subseteq \mathfrak{S}$ any admissible sequence. (This is well defined by Corollary 5.9.) *Cayley transforms of cogradings* are defined similarly, and written $c_{\mathfrak{S}}$.

We turn now to some technical results about the case when the span of all the roots is an admissible subspace.

Proposition 5.11. *Suppose (R, ϵ) is a graded root system such that the span $\mathfrak{S}$ of all roots is an admissible subspace* (*Definition* 5.10). *Fix an admissible sequence $S = \{\alpha_1, \ldots, \alpha_l\}$ whose span is $\mathfrak{S}$. Call a positive root system R^+ S-compatible if for each i, one of the roots $\pm\alpha_i$ is dominant for $R^+ \cap (\text{span of } \alpha_1, \ldots, \alpha_i)$.*

(a) *There exists an S-compatible positive root system R^+; any other S-compatible positive system is of the form wR^+, with w a product of reflections in S.*

(b) *There exists a positive root system, all of whose simple roots β satisfy $\epsilon(\beta) = 1$.*

(c) *If $\tilde{R}^+$ is as in* (b), *then there is a $w \in W_0(\epsilon)$ such that $w\tilde{R}^+$ is S-compatible.*

The proof uses the following lemma. A much quicker proof of the lemma can be given with the machinery of group representations, by comparing Gelfand–Kirillov dimensions in the Hecht–Schmid character identity. The proof given here is elementary, but not very illuminating.

Lemma 5.12. *Suppose (R, ϵ) is a graded root system, R^+ is a positive system, α is R^+-simple, and $\epsilon(\alpha) = 1$. Set*

$$(R^\alpha)^+ = R^+ \cap R^\alpha,$$

and assume that every simple root β of $(R^{\alpha})^{+}$ satisfies $c^{\alpha}(\epsilon)(\beta) = 1$. Then either R^{+} or $s_{\alpha}R^{+}$ has this same property for ϵ.

Proof. Every simple root β of R^{+} which is orthogonal to α is simple in $(R^{\alpha})^{+}$, and so satisfies $c^{\alpha}(\epsilon)(\beta) = 1$. Since $\beta - \alpha$ is not a root (as α and β are R^{+}-simple), $\epsilon(\beta) = 1$. So we need only consider simple roots of R^{+} adjacent to α. Passing to an appropriate subsystem of R, we may assume that every simple root of R^{+} is adjacent to α. This gives rise to about ten cases, which are all treated in much the same way; so we will discuss only two of the more complicated ones. Suppose first that R is of type D_4, and that γ_1, γ_2, and γ_3 are the other R^{+}-simple roots. The $(R^{\alpha})^{+}$-simple roots are $\alpha + \gamma_i + \gamma_j$, with $\{i, j\} \subseteq \{1,2,3\}$. By hypothesis,

$$\epsilon(\alpha + \gamma_i + \gamma_j) = 1, \qquad \text{all} \quad \{i, j\} \subseteq \{1,2,3\}.$$

Since $\epsilon(\alpha) = 1$, it follows that

$$\epsilon(\gamma_i) = \epsilon(\gamma_j), \qquad \text{all} \quad \{i, j\} \subseteq \{1,2,3\}.$$

So all $\epsilon(\gamma_i)$ are 1, or all are 0. In the first case we are done. In the second, replace R^{+} by $s_{\alpha}R^{+}$. Then γ_i is replaced by $s_{\alpha}\gamma_i = \gamma_i + \alpha$; and

$$\begin{aligned} \epsilon(s_{\alpha}\gamma_i) &= \epsilon(\gamma_i) + \epsilon(\alpha) \\ &= 0 + 1 = 1, \end{aligned}$$

as desired.

Next, suppose R is of type C_3, with α the middle root, γ_1 the short end, and γ_2 the long end. Then $\alpha + \gamma_2$ is simple in $(R^{\alpha})^{+}$, and not strongly orthogonal to α; so $\epsilon(\alpha + \gamma_2) = 0$. It follows that $\epsilon(\gamma_2) = 1$. If $\epsilon(\gamma_1) = 1$, we are done. Otherwise, replace R^{+} by $s_{\alpha}(R^{+})$. This does not affect the first part of the argument, so $\epsilon(s_{\alpha}\gamma_2) = 1$. But γ_1 is replaced by $s_{\alpha}(\gamma_1) = \gamma_1 + \alpha$; so

$$\begin{aligned} \epsilon(s_{\alpha}(\gamma_1)) &= \epsilon(\gamma_1) + \epsilon(\alpha) \\ &= 0 + 1 = 1, \end{aligned}$$

as desired. Q.E.D.

LEMMA 5.13. *Let R be a root system, and $\{\alpha_1, \ldots, \alpha_l\} = S$ an orthogonal set of roots spanning R. Set*

$$\begin{aligned} R^{j} &= \{\alpha \in R \mid \langle \alpha, \alpha_i \rangle = 0, i = j+1, \ldots, l\} \\ &= \langle \alpha_1, \ldots, \alpha_j \rangle \cap R \\ {}^{j}R &= \{\alpha \in R \mid \langle \alpha, \alpha_i \rangle = 0, i = 1, \ldots, j\} \\ &= \langle \alpha_{j+1}, \ldots, \alpha_l \rangle \cap R \end{aligned}$$

(a) *Let R^+ be an S-compatible positive root system (defined as in Proposition* 5.11). *Define a sequence $\phi = \{\phi_1, \ldots, \phi_l\}$ of ± 1's by $\phi_j \alpha_j \in R^+$. Then*

(1) *$\phi_j \alpha_j$ is dominant for $(R^j)^+$*

(2) *$R^+ = \{\alpha \in R \mid$ there is a j with $\langle \alpha, \alpha_i \rangle = 0$ for $i > j$, and $\langle \alpha, \phi_j \alpha_j \rangle > 0\}$*

(3) *$\phi_j \alpha_j$ is simple in ${}^{(j-1)}R^+$*

(b) *Suppose $\phi = \{\phi_1, \ldots, \phi_l\}$ is a sequence of ± 1's. Define R^+ by* (a2) *above. Then R^+ is S-compatible.*

Proof. Part (a1) is the definition of an S-compatible positive root system. For (a2), suppose $\alpha \in R$. Clearly there is a unique j so that $\alpha \in R^j - R^{j-1}$. This means that

$$\langle \alpha, \alpha_i \rangle = 0, \qquad i > j$$

$$\langle \alpha, \alpha_j \rangle \neq 0.$$

By (a1), $\alpha \in R^+$ if and only if $\langle \alpha, \phi_j \alpha_j \rangle$ is positive. For (a3), suppose $\phi_j \alpha_j = \beta + \gamma$, with β and γ both in ${}^{j-1}R^+$. Write

$$\beta = \sum_{i \geqslant j} m_i(\phi_i \alpha_i), \qquad \gamma = \sum_{i \geqslant j} n_i(\phi_i \alpha_i),$$

with $m_i, n_i \in \mathbb{Q}$. Since $\phi_j \alpha_j = \beta + \gamma$, $m_i = -n_i$ for $i > j$. Since β and γ are both in R^+, (a2) now guarantees that $m_i = n_i = 0$ for $i > j$; that is, that β and γ are both multiples of $\phi_j \alpha_j$. Since R is a reduced root system, this is a contradiction. Assertion (b) is obvious. Q.E.D.

Proof of Proposition 5.11. Part (a) is immediate from Lemma 5.13(a2), (b). To prove (b), we are going to find a sequence $\phi = \{\phi_1, \ldots, \phi_l\}$ of ± 1's, with the following property. Define R^+, ${}^jR$ as in Lemma 5.13, and give ${}^jR$ the grading

$${}^j\epsilon = c^{\langle \alpha_1, \ldots, \alpha_j \rangle}(\epsilon).$$

The property we want is:

$$\text{Every simple root of } {}^{(j-1)}R^+ \text{ belongs to } {}^{(j-1)}R_1({}^{j-1}\epsilon). \tag{$*$}$$

We will choose ϕ_j and verify $(*)$ by downward induction on j. Put $\phi_l = 1$; then $(*)$ holds for $j = l$ since $\{\alpha_1, \ldots, \alpha_l\}$ is an ϵ-admissible sequence. For the inductive step, suppose $(*)$ holds for some $j + 1$. We have

$${}^j\epsilon = c^{\alpha_j}({}^{j-1}\epsilon). \tag{$**$}$$

By Lemma 5.13(a3), $\phi_j \alpha_j$ is simple in ${}^{j-1}R^+$, with either choice of ϕ_j. By Lemma 5.12, $(**)$, and the inductive hypothesis, we can make this choice so as to satisfy $(*)$. This completes the induction; and for $j = 1$, $(*)$ is Proposition 5.11(b).

For (c), we proceed by induction on l. Choose an S-compatible positive system R^+ as in (b). Define $w_2 \in W(R)$ by

$$w_2 \tilde{R}^+ = R^+.$$

Since all simple roots in R^+ and $\tilde{R}^+$ are in $R_1(\epsilon)$, w_2 must preserve ϵ; that is, $w_2 \in W_2(\epsilon)$ (Definition 3.13). Define

$$\beta_i = w_2^{-1}\alpha_i \qquad (i = 1, \ldots, l)$$

$$S' = \{\beta_i\}.$$

Since $w_2 \in W_2$, S' is an admissible sequence. What we will actually do is produce an element $w \in W(R_0)$ such that $w\beta_i = \pm\alpha_i$ for all i; clearly this suffices. Since S' is admissible, $\beta_1 \in R_1$; and $\alpha_1 \in W(R)\beta_1$. By Step III in the proof of Proposition 5.8, we can find a $w_0 \in W(R_0)$ so that

$$\alpha_1 = \pm w_0\beta_1.$$

After replacing $\tilde{R}^+$ by $w_0\tilde{R}^+$ (which replaces w_2 by $w_2w_0^{-1}$, and β_i by $w_0\beta_i$) we may therefore assume that $\alpha_1 = \pm\beta_1$. Therefore $R^{\alpha_1} = R^{\beta_1}$. Since $w_2\beta_1 = \pm\alpha_1$, and $w_2 \in W_2$, w_2 preserves R^{α_1} and its grading. This is also true for $s_{\alpha_1}w_2$. One of these elements fixes α_1, and so lies in $W(R^{\alpha_1})$; calling it w_2', we have $w_2' \in W_2(R^{\alpha_1})$, and $\alpha_i = w_2'\beta_i$ for $i \geqslant 2$. By inductive hypothesis, we can find $w_0' \in W(R_0^{\alpha_1})$ such that $w_0'\beta_i = \pm\alpha_i$ for $i \geqslant 2$. We claim that w_0' (acting on R^{α_1}) is the restriction to R^{α_1} of an element w of $W(R_0)$ satisfying $w\alpha_1 = \pm\alpha_1$. This element w will then satisfy the requirements of the lemma.

The claim will be proved for any element of $W(R_0^{\alpha_1})$; so it is enough to consider a reflection s_γ, with $\gamma \in R_0^{\alpha_1}$. If γ and α_1 are strongly orthogonal, then $\gamma \in R_0$, and $s_\gamma \in W(R_0)$, as desired. Otherwise $\gamma \in R_1$, and $\gamma \pm \alpha_1$ are roots; as $\alpha_1 \in R_1$, $\gamma \pm \alpha_1 \in R_0$. Set $w_\gamma = s_{\gamma+\alpha_1}s_{\gamma-\alpha_1} \in W(R_0)$. Clearly

$$w_\gamma\alpha_1 = -\alpha_1$$

$$w_\gamma|_{R_1^\alpha} = s_\gamma|_{R_1^\alpha},$$

as we wished to show. Q.E.D.

COROLLARY 5.14. *Suppose (R, ϵ) is a graded root system, and $S = \{\alpha_1, \ldots, \alpha_m\}$ is a maximal admissible sequence in R (Definition 5.3). Then every element of $W_2(\epsilon)/W_0(\epsilon)$ (Definition 3.13) has a representative which is a product of reflections in S.*

Proof. Choose a superorthogonal set $\{\beta_1, \ldots, \beta_l\} \subseteq R_1(\epsilon)$ as in Proposition 3.20, and extend it to a maximal admissible sequence $\{\beta_1, \ldots, \beta_n\}$. By Proposition 5.8, the spans of $\{\alpha_i\}$ and $\{\beta_j\}$ are conjugate by $W_0(\epsilon)$; so we may as well suppose that they coincide. By Proposition 3.20, every element of $W_2(\epsilon)/W_0(\epsilon)$ has a representative which is a product of certain s_{β_j}. Therefore we may as well replace R by the common span of $\{\alpha_i\}$ and $\{\beta_j\}$. By Proposition 5.11, we can choose an S-admissible positive system R^+ so that all simple roots belong to R_1. Fix $w \in W_2(\epsilon)$, and set

$$\tilde{R}^+ = w^{-1}R^+;$$

then $\tilde{R}^+$ has the same property, by the definition of $W_2(R)$. Proposition 5.12

now guarantees that w is of the form $w_S w_0$, with $w_0 \in W(R_0)$, and w_S a product of reflections in S, as required. Q.E.D.

Next, we want to be able to recover a graded root system from a Cayley transform of it. This is not quite possible, but we can determine precisely the extent to which it is.

Definition 5.15. Suppose R is a root system on V, and $\mathfrak{s} \subseteq V$. Set

$$c^{\mathfrak{s}}(R) = R^{\mathfrak{s}} = \{\alpha \in R \mid \langle \alpha, \mathfrak{s} \rangle = 0\}.$$

Fix a grading $\epsilon^{\mathfrak{s}} \in E(R^{\mathfrak{s}})$, and define

$$\begin{aligned} d_{\mathfrak{s}}(\epsilon^{\mathfrak{s}}) &= \{\epsilon \in E(R) \mid \mathfrak{s} \text{ is } \epsilon\text{-admissible, and } \epsilon^{\mathfrak{s}} = c^{\mathfrak{s}}(\epsilon)\} \\ &= (c^{\mathfrak{s}})^{-1}(\epsilon^{\mathfrak{s}}) \subseteq E(R), \end{aligned}$$

the inverse Cayley transform of $\epsilon^{\mathfrak{s}}$.

PROPOSITION 5.16. *In the setting of Definition* 5.15, *set*

$$\Phi = \Phi(\mathfrak{s}) = R \cap \mathfrak{s}$$

$$W(\mathfrak{s}) = W(\Phi).$$

Then $d_{\mathfrak{s}}(\epsilon^{\mathfrak{s}})$ *either is empty, or consists of exactly one orbit of* $W(\mathfrak{s})$ *in its cross action on* $E(R)$ (*Definition* 3.24). *More explicitly, suppose* $\epsilon = (R, R_0, R_1)$ *and* $\tilde{\epsilon} = (R, \tilde{R}_0, \tilde{R}_1)$ *are two gradings of R such that* $\mathfrak{s}$ *is admissible for both gradings, and*

$$c^{\mathfrak{s}}(\epsilon) = c^{\mathfrak{s}}(\epsilon) = \epsilon^{\mathfrak{s}}.$$

Then there is an element $w \in W(\mathfrak{s})$ *such that*

$$wR_0 = \tilde{R}_0, \qquad wR_1 = \tilde{R}_1.$$

The last assertion really is a more explicit version of the first; for if such a w exists then $\mathfrak{s}$ is ϵ-admissible if and only if it is $\tilde{\epsilon}$-admissible, and in that case the Cayley transforms coincide. This is obvious.

Proof. We proceed by induction on $\dim \mathfrak{s}$. Since the span of the roots is admissible in either grading of Φ, Proposition 5.11 shows that we can find positive systems $\Phi^+, \tilde{\Phi}^+$ for which every simple root belongs to Φ_1 or to $\tilde{\Phi}_1$ respectively. Choose $w_1 \in W(\mathfrak{s})$ taking Φ^+ to $\tilde{\Phi}^+$; then clearly

$$w_1 \times (\epsilon|_{\Phi}) = \tilde{\epsilon}|_{\Phi}.$$

So, replacing ϵ by $w_1 \times \epsilon$, we may assume that

$$\epsilon|_{\Phi} = \tilde{\epsilon}|_{\Phi}.$$

Now choose a root $\alpha \in \Phi_1 \subseteq R_1 \cap \tilde{R}_1$, and let $\mathfrak{s}^\alpha$ be the orthogonal complement of α in $\mathfrak{s}$. Then

$$c^{\mathfrak{s}^\alpha}\left[c^\alpha(\epsilon)\right] = c^{\mathfrak{s}^\alpha}\left[c^\alpha(\tilde{\epsilon})\right] = \epsilon^{\mathfrak{s}}$$

by Definition 5.10. By inductive hypothesis, we can find $w^\alpha \in W(\mathfrak{s}^\alpha) \subseteq W(\mathfrak{s})$ such that

$$w^\alpha \times c^\alpha(\epsilon) = c^\alpha(\tilde{\epsilon}).$$

Since w^α fixes α, it is clear that

$$w^\alpha \times c^\alpha(\epsilon) = c^\alpha(w^\alpha \times \epsilon);$$

so, replacing ϵ by $w^\alpha \times \epsilon$, we may assume that

$$c^\alpha(\epsilon) = c^\alpha(\tilde{\epsilon}). \tag{5.17}$$

Now choose a positive root system R^+ making α dominant. We will show that for some choice of $w \in \{1, s_\alpha\}$, we have

$$(w \times \epsilon)(\beta) = \tilde{\epsilon}(\beta) \tag{5.18}$$

for every simple root β for R^+. This will complete the proof. If β is orthogonal to α, then (5.18) follows from (5.17), with either choice of w. Therefore we only need to consider the simple roots β adjacent to α; so we may as well assume R is simple. If R is of type A_1, then $\alpha = \beta$, and $\epsilon(\alpha) = \tilde{\epsilon}(\alpha) = 1$ by the choice of α. Otherwise there are exactly one or two simple roots adjacent to (but distinct from) α; we consider these cases separately.

First suppose there are two such roots, β_1 and β_2. Then R is of type A_n ($n \geqslant 2$), and

$$\alpha = \beta_1 + \beta_2 + \sum \gamma_i,$$

with the γ_i simple roots orthogonal to α. We know that

$$\epsilon(\alpha) = \tilde{\epsilon}(\alpha) = 1,$$

and we have already seen that

$$\epsilon(\gamma_i) = \tilde{\epsilon}(\gamma_i).$$

It follows that

$$\epsilon(\beta_1) + \epsilon(\beta_2) = \tilde{\epsilon}(\beta_1) + \tilde{\epsilon}(\beta_2).$$

If $\epsilon(\beta_1) = \tilde{\epsilon}(\beta_1)$, this implies that $\epsilon(\beta_2) = \tilde{\epsilon}(\beta_2)$, and (5.18) holds with $w = 1$.

Otherwise, choose $w = s_\alpha$. Then

$$\begin{aligned}(s_\alpha \times \epsilon)(\beta_i) &= \epsilon(s_\alpha\beta_i)\\ &= \epsilon(\beta_i - \alpha)\\ &= \epsilon(\beta_i) - 1\\ &= \tilde{\epsilon}(\beta_i);\end{aligned}$$

so (5.18) holds.

Next, suppose α is adjacent to a unique simple root β. If $\epsilon(\beta) = \tilde{\epsilon}(\beta)$, we are done; so suppose $\epsilon(\beta) = 1 - \tilde{\epsilon}(\beta)$. If $\langle\check{\alpha}, \beta\rangle$ is odd, then

$$\begin{aligned}s_\alpha\beta &= \beta - \langle\check{\alpha}, \beta\rangle\alpha\\ \epsilon(s_\alpha\beta) &= \epsilon(\beta) = 1;\end{aligned}$$

and (5.18) holds with $w = s_\alpha$ as in the first case. So we may assume that $\langle\check{\alpha}, \beta\rangle = 2$. Write

$$\alpha = m\beta + \sum a_j\gamma_j,$$

with the sum over simple roots γ_j orthogonal to α. We know that $\epsilon(\gamma) = \tilde{\epsilon}(\gamma)$; so

$$\begin{aligned}\epsilon(\alpha) - \tilde{\epsilon}(\alpha) &\equiv m(\epsilon(\beta) - \tilde{\epsilon}(\beta)) \qquad (\mathrm{mod}\ 2)\\ &\equiv m\end{aligned}$$

since we are assuming $\epsilon(\beta) = 1 - \tilde{\epsilon}(\beta)$. But we also have $\epsilon(\alpha) = \tilde{\epsilon}(\alpha) = 1$; so m is necessarily even. On the other hand,

$$2 = \langle\check{\alpha},\alpha\rangle = m\langle\check{\alpha}, \beta\rangle + \sum a_j\langle\check{\alpha},\gamma_j\rangle = m\langle\check{\alpha}, \beta\rangle = 2m,$$

since α is orthogonal to all the γ_j, and $\langle\check{\alpha}, \beta\rangle = 2$. So $m = 1$, a contradiction. Q.E.D.

Definition 5.19. Suppose R is a root system, and $g = (\theta, \epsilon, \check{\delta}) \in \mathcal{G}(R)$ is a weak bigrading of R (Definition 3.22). We use the notation of Definitions 3.10, 3.13, 3.15, and 3.21. Suppose $\mathfrak{s} \subseteq V_s(\theta)$ is $\check{\delta}$-admissible (Definition 5.10). Since $\mathfrak{s}$ is spanned by an orthogonal set of roots, the orthogonal involution $w_\mathfrak{s}$ with -1 eigenspace $\mathfrak{s}$ belongs to W^{R}. Define

$$c_\mathfrak{s}(\theta) = \theta_\mathfrak{s} = \theta \circ w_\mathfrak{s}.$$

Since $\mathfrak{s} \subseteq V_s(\theta)$, θ and $w_\mathfrak{s}$ commute; so $\theta_\mathfrak{s}$ is an involutive automorphism of R. Furthermore,

$$\begin{aligned}V_c(\theta_\mathfrak{s}) &= V_c(\theta) \oplus \mathfrak{s}\\ V_s(\theta_\mathfrak{s}) &= \{x \in V_s(\theta) \mid \langle x, \mathfrak{s}\rangle = 0\}.\end{aligned}$$

Define

$$\begin{aligned} c_{\mathfrak{s}}\big(R^{\mathbb{R}}(\theta)\big) &= R^{\mathbb{R}}(\theta_{\mathfrak{s}}) \\ &= \big\{\alpha \in R^{\mathbb{R}}(\theta) \mid \langle \alpha, \mathfrak{s}\rangle = 0\big\} \\ c_{\mathfrak{s}}(\check{\delta}\,) &= \check{\delta}_{\mathfrak{s}} \in \check{E}\big(R^{\mathbb{R}}(\theta_{\mathfrak{s}})\big). \end{aligned}$$

(Definition 5.10). Recall from Definition 5.15 the inverse Cayley transform

$$d_{\mathfrak{s}}(\epsilon) \subseteq E^{i\mathbb{R}}\big(R^{i\mathbb{R}}(\theta_s)\big).$$

Define the *Cayley transform of the bigrading g through* $\mathfrak{s}$,

$$c_{\mathfrak{s}}(g) \subseteq \mathcal{G}(R),$$

to be the set of weak bigradings of R defined by

$$\begin{aligned} c_{\mathfrak{s}}(g) &= \big(\theta_{\mathfrak{s}}, d_{\mathfrak{s}}(\epsilon), \check{\delta}_{\mathfrak{s}}\big) \\ &= \big\{\, g_{\mathfrak{s}} = \big(\theta_{\mathfrak{s}}, \sigma, \check{\delta}_{\mathfrak{s}}\big) \mid \sigma \in d_{\mathfrak{s}}(\epsilon)\big\} \\ &= \big\{\big(\theta_{\mathfrak{s}}, \sigma, \check{\delta}_{\mathfrak{s}}\big) \mid c^{\mathfrak{s}}(\sigma) = \epsilon\big\} \end{aligned}$$

Similarly, if $\mathfrak{u} \subseteq V_c$ is ϵ-admissible, we can define $c^{\mathfrak{u}}(g)$.

PROPOSITION 5.20. *Suppose R is a root system, $g = (\theta, \epsilon, \check{\delta}) \in \mathcal{G}(R)$, and $\mathfrak{s} \subseteq V_s(\theta)$ is $\check{\delta}$-admissible. Write*

$$\begin{aligned} \Phi &= R \cap \mathfrak{s} \\ W(\mathfrak{s}) &= W(\Phi). \end{aligned}$$

(a) *The Cayley transform $c_{\mathfrak{s}}(g) \subseteq \mathcal{G}(R)$ (Definition 5.19) either is empty, or consists of a single orbit under the cross action of $W(\mathfrak{s})$ (Definition 3.24).*

(b) *If $w \in W(R)$ is arbitrary, then*

$$w \times c_{\mathfrak{s}}(g) = c_{w\mathfrak{s}}(w \times g).$$

(c) *Suppose $\check{g}$ is the dual weak bigrading of $\check{R}$ (Defintion 3.23). Then*

$$\begin{aligned} \big[c_{\mathfrak{s}}(g)\big]^{\vee} &= c^{\mathfrak{s}}(\check{g}) \\ \big[w \times g\big]^{\vee} &= w \times \check{g} \qquad (w \in W(R)). \end{aligned}$$

Part (a) is immediate from Proposition 5.16. The other assertions are clear, and are included mainly to set the mood for increasingly complicated versions of them which will appear later. From a formal point of view, the main shortcoming of this formulation is that there is no discussion of $\mathfrak{R}$ groups. They present enough difficulties to warrant a section of their own.

6. Formal Cayley transforms of $\mathfrak{R}$ groups.

LEMMA 6.1. *Suppose (R,ϵ) is a graded root system, and $\mathfrak{S}\subseteq V$ is an ϵ-admissible subspace (Definition 5.10). Set*

$$\mathfrak{S}^{\perp}=\{x\in V\mid\langle x,\mathfrak{S}\rangle=0\}$$
$$W_2(\epsilon,\mathfrak{S})=\{w\in W_2(\epsilon)\mid w\mathfrak{S}=\mathfrak{S}\}$$
$$W_0(\epsilon,\mathfrak{S})=W_2(\epsilon,\mathfrak{S})\cap W_0$$
$$c^{\mathfrak{S}}(W_2(\epsilon))=\{w\in W(R)|w|_{\mathfrak{S}}=1,w|_{\mathfrak{S}^{\perp}}\in W_2(\epsilon,\mathfrak{S})|_{\mathfrak{S}^{\perp}}\}$$
$$c^{\mathfrak{S}}(W_0(\epsilon))=\{w\in W(R)|w|_{\mathfrak{S}}=1,w|_{\mathfrak{S}^{\perp}}\in W_0(\epsilon,\mathfrak{S})|_{\mathfrak{S}^{\perp}}\}.$$

Then

$$W_2(c^{\mathfrak{S}}(\epsilon))\supseteq c^{\mathfrak{S}}(W_2(\epsilon))\supseteq c^{\mathfrak{S}}(W_0(\epsilon))\supseteq W_0(c^{\mathfrak{S}}(\epsilon))$$

Proof. Suppose $w\in W_2(\epsilon,\mathfrak{S})$. Choose a strongly orthogonal sequence $S=\{\alpha_1,\ldots,\alpha_l\}$ spanning $\mathfrak{S}$. Then wS is another admissible sequence spanning $\mathfrak{S}$; so by Corollary 5.9,

$$\begin{aligned}c^{\mathfrak{S}}(\epsilon)&=c^{S}(\epsilon)\\&=c^{wS}(\epsilon)\\&=c^{S}(w^{-1}\times\epsilon)\\&=c^{\mathfrak{S}}(w^{-1}\times\epsilon)\\&=w^{-1}\times c^{\mathfrak{S}}(\epsilon).\end{aligned}$$

It follows that the automorphism of $R^{\mathfrak{S}}$ defined by w preserves $c^{\mathfrak{S}}(\epsilon)$. Therefore

$$W_2(c^{\mathfrak{S}}(\epsilon))\supseteq c^{\mathfrak{S}}(W_2(\epsilon)).$$

The second inclusion is obvious. For the last, we simply elaborate on the end of the proof of Proposition 5.11. Suppose $\alpha\in R_0^{\mathfrak{S}}$; it is enough to show that $s_\alpha\in c^{\mathfrak{S}}(W_0(\epsilon))$. If $\alpha\in R_0$, then $s_\alpha\in W_0(\epsilon)$, and $s_\alpha=1$ on $\mathfrak{S}$; so the claim is obvious. If $\alpha\in R_1$, then Lemma 5.5 shows that there is a root $\beta\in\mathfrak{S}\cap R_1$ such that α is orthogonal to β, and $\alpha\pm\beta$ are roots. Since α and β are in R_1, $\alpha\pm\beta\in R_0$. The element $w_\alpha=s_{\alpha+\beta}s_{\alpha-\beta}$ is the orthogonal involution with -1 eigenspace spanned by α and β; so $w_\alpha\in W_0(\epsilon,\mathfrak{S})$, and $w_\alpha|_{\mathfrak{S}^{\perp}}=s_\alpha|_{\mathfrak{S}^{\perp}}$. It follows that $s_\alpha\in c^{\mathfrak{S}}(W_0(\epsilon))$, as required. Q.E.D.

Definition 6.2. Suppose (R,ϵ,W_1) is a graded root system with $\mathfrak{R}$ group (Definition 3.15), and $\mathfrak{S}$ is an ϵ-admissible subspace (Definition 5.10). Define $W_1(\epsilon,\mathfrak{S})$ and $c^{\mathfrak{S}}(W_1)$ in analogy with the notation for W_0 and W_2 in Lemma 6.1.

The *Cayley transform of* (R, ϵ, W_1) *through* $\mathfrak{S}$ is $(\mathfrak{R}^{\mathfrak{S}}, c^{\mathfrak{S}}(\epsilon), c^{\mathfrak{S}}W_1))$. Similarly we define *Cayley transformations of cogradings with* $\mathfrak{R}$ *groups*. By Lemma 6.1, $(R^{\mathfrak{S}}, c^{\mathfrak{S}}(\epsilon), c^{\mathfrak{S}}(W_1))$ is a graded root system with $\mathfrak{R}$ group.

The main subtlety we must face in extending this to bigradings is in the definition of some sort of inverse Cayley transform. We have a natural Cayley transform defined on $\mathfrak{R}$ groups, but it is not injective: if W_1 and W_1' are distinct $\mathfrak{R}$ groups for (R, ϵ), then it can happen that $c^{\mathfrak{S}}(W_1) = c^{\mathfrak{S}}(W_1')$. Of course we had the same situation for gradings; but the various gradings in the inverse image of one could all correspond to the same group G. Changing $\mathfrak{R}$ groups corresponds roughly to changing the component group G/G_0; so a multi-valued inverse image is not natural. We begin with the machinery needed to remedy this.

Definition 6.3. A graded root system (R, ϵ) is called *principal* if there is a positive root system R^+ such that for every simple root α, $\epsilon(\alpha) = 1$. It is called *admissible* if the span of the roots is an admissible subspace (Definition 5.11).

Obviously principal gradings of any root system exist, and constitute a single orbit of $W(R)$ in the cross action of Definition 3.24. By Definition 5.11, a root system can have an admissible grading only if it is spanned by an orthogonal set of roots; and not every root system has this property. (An example is the system of type A_2.) With these comments in mind, we can state the basic result about principal gradings.

PROPOSITION 6.4. *Let R be a root system. Suppose $\mathfrak{S}, \mathfrak{S}' \subseteq V$ are subspaces spanned by orthogonal sets of roots, and $\mathfrak{S}$ is maximal with respect to this property.*

(a) *$\mathfrak{S}'$ is conjugate by $W(R)$ to a subspace of $\mathfrak{S}$.*

(b) *$\mathfrak{S}$ is equal to the span of R if and only if $Z_1(R)$ (Definition 3.3) is a product of copies of $\mathbf{Z}/2\mathbf{Z}$.*

(c) *A grading of R is admissible if and only if it is principal, and $\mathfrak{S}$ is the span of R.*

(d) *If ϵ is a principal grading, then the image of the injection $W_2(\epsilon)/W_0(\epsilon) \to Z_1(R)$ (Corollary 3.17) is exactly the subgroup of elements of order 2. In particular, if ϵ is admissible, the map is an isomorphism.*

(e) *A grading ϵ of R is principal if and only if every sequence of orthogonal roots is $W(R)$-conjugate to an ϵ-admissible sequence.*

Before proving this, we will show how to use it to define Cayley transforms of bigradings with $\mathfrak{R}$ groups.

LEMMA 6.5. *Let R be a root system. In the notation of Definition* 3.3, $Z_1(R) \cong [Z_1(\check{R})]^\wedge$.

Here $[Z_1(\check{R})]^\wedge$ denotes the character group of the finite abelian group $Z_1(\check{R})$. This is an elementary fact about lattices, which is well known.

COROLLARY 6.6. *Suppose* (ϵ, R) *is an admissible graded root system, and* $\check{\delta}$ *is an admissible cograding of* R. *There are natural isomorphisms*

$$W_2(\epsilon)/W_0(\epsilon) \cong Z_1(R) \cong [Z_1(\check{R})]^\wedge \cong [W_2(\check{\delta})/W_0(\check{\delta})]^\wedge.$$

Therefore, to every subgroup $\mathcal{R} \subseteq W_2(\epsilon)/W_0(\epsilon)$, *there is naturally associated a subgroup*

$$\operatorname{Ann}\mathcal{R} \subseteq W_2(\check{\delta})/W_0(\check{\delta})$$

defined by

$$\operatorname{Ann}\mathcal{R} = \{\chi \in W_2(\check{\delta})/W_0(\check{\delta}) \mid \chi(r) = 1 \textit{ for all } r \in W_2(\epsilon)/W_0(\epsilon)\};$$

here $\chi(r) \in \mathbf{C}$ *is defined by the isomorphisms above. The correspondence* $\mathcal{R} \to \operatorname{Ann}\mathcal{R}$ *is bijective and order-reversing on the set of subgroups of the groups in question*; *and*

$$\operatorname{Ann}(\operatorname{Ann}\mathcal{R}) = \mathcal{R}.$$

This is immediate from Proposition 6.4(d) and Lemma 6.5.

We can now define Cayley transforms of bigradings with $\mathcal{R}$ groups. Probably this could be done in some generality; but we will confine ourselves to a simple case adequate for our purposes (the assumption that ϵ is trivial below).

Definition 6.7. Suppose $\bar{g} = (\theta, \epsilon, W_1^{i\mathbf{R}}, \check{\delta}, W_1^{\mathbf{R}})$ is a strong bigrading of the roots sytem R (Definition 3.22), $\mathcal{S} \subseteq V_s(\theta)$ is $\check{\delta}$ admissible, and the grading ϵ is trivial. Define the *Cayley transform of the bigrading* $\bar{g}$ *through* $\mathcal{S}$,

$$c_{\mathcal{S}}(\bar{g}) \subseteq \bar{\mathcal{G}}(R)$$

to be the set of strong bigradings

$$\bar{g}_{\mathcal{S}} = (\theta_{\mathcal{S}}, \sigma, W_1^{i\mathbf{R}}(g_{\mathcal{S}}), \check{\delta}_{\mathcal{S}}, c_{\mathcal{S}}(W_1^{\mathbf{R}}))$$

such that, if $g = (\theta, \epsilon, \check{\delta}) \in \mathcal{G}(\mathrm{R})$, then

(a) $g_{\mathcal{S}} = (\theta_{\mathcal{S}}, \sigma, \check{\delta}_{\mathcal{S}}) \in c_{\mathcal{S}}(g)$ (Definition 5.19)

(b) $c_{\mathcal{S}}(W_1^{\mathbf{R}})$ is given by Definition 6.2.

(c) Put $\Phi = R \cap \mathcal{S}$, and $W(\mathcal{S}) = W(\Phi)$; thus $(\Phi, \check{\delta}|_\Phi)$ is an admissible cograding, and $(\Phi, \sigma|_\Phi)$ is an admissible grading. Put

$$\mathcal{R}(\mathcal{S}, \check{\delta}) = (W_1^{\mathbf{R}} \cap W(\mathcal{S}))/(W_0^{\mathbf{R}} \cap W(\mathcal{S}))$$

$$\subseteq W_2(\check{\delta}|_\Phi)/W_0(\check{\delta}|_\Phi)$$

and define

$$\operatorname{Ann}\mathcal{R} \subseteq W_2(\sigma|_\Phi)/W_0(\sigma|_\Phi)$$

by Corollary 6.6. Define $W_1(\sigma|_\Phi)$ to be the preimage of $\operatorname{Ann}\mathfrak{R}$ in $W_2(\sigma|_\Phi)$. Then our last condition on $\bar{g}_{\mathfrak{s}}$ is that

$$W_1^{i\mathbb{R}}(\bar{g}_{\mathfrak{s}}) \cap W(\mathfrak{s}) = W_1(\sigma|_\Phi).$$

If $\mathfrak{u} \subset V_c(\theta)$ is ϵ-admissible and $\check{\delta}$ is trivial then $c^{\mathfrak{u}}(\bar{g})$ is defined analogously.

LEMMA 6.8. *In the setting of Definition* 6.7, *fix* $g_{\mathfrak{s}} = (\theta_{\mathfrak{s}}, \sigma, \check{\delta}_{\mathfrak{s}}) \in c_{\mathfrak{s}}(g)$. *Then*

(a) $$W_0^{\mathbb{R}} \cap W(\mathfrak{s}) = W_0(\check{\delta}) \cap W(\mathfrak{s}) = W_0(\check{\delta}|_\Phi)$$

$$W_0(\sigma) \cap W(\mathfrak{s}) = W_0(\sigma|_\Phi)$$

(b) *There is a natural inclusion*

$$W_2(\sigma)/W_0(\sigma) \to W_2(\sigma|_\Phi)/W_0(\sigma|_\Phi).$$

(c) *The condition* (c) *of Definition* 6.7 *is satisfied by at most one* $\mathfrak{R}$ *group* $W_1^{i\mathbb{R}}(\bar{g}_{\mathfrak{s}})$ *for* σ.

(d) *The conclusions of Proposition* 5.20 *hold also for Cayley transforms of strong bigradings*.

Proof. For (a), write

$$\mathfrak{s}^\perp = \{x \in V \mid \langle x, \mathfrak{s}\rangle = 0\}.$$

Because of Proposition 3.8, we have

$$W(\Delta \cap \mathfrak{s}) = \{w \in W(\Delta) \mid w|_{\mathfrak{s}^\perp} = 1\}$$

for any root system $\Delta \subseteq R$. Thus

$$W(\Delta \cap \mathfrak{s}) = W(\Delta) \cap W(\mathfrak{s}).$$

Taking $\Delta = R_0^{\mathbb{R}}(\check{\delta})$ and $\Delta = R_0^{i\mathbb{R}}(\sigma)$, (a) follows. For (b), there is a natural inclusion

$$W_2(\sigma) \cap W(\mathfrak{s})/W_0(\sigma) \cap W(\mathfrak{s}) \to W_2(\sigma)/W_0(\sigma). \tag{$*$}$$

By definition, $c^{\mathfrak{s}}(\sigma) = \epsilon$ is trivial; so $\mathfrak{s}$ is maximal admissible for σ. By Corollary 5.14, $(*)$ is an isomorphism. By (a), there is a natural inclusion

$$W_2(\sigma) \cap W(\mathfrak{s})/W_0(\sigma) \cap W(\mathfrak{s}) \to W_2(\sigma|_\Phi)/W_0(\sigma|_\Phi).$$

Now $(*)$ gives (b). Part (c) follows. Part (d) is a consequence of (c) and Proposition 5.20. Q.E.D.

Since the inclusion of Lemma 6.8(b) is not onto, it can easily happen that $c_{\mathfrak{s}}(\bar{g})$ is empty even when $c_{\mathfrak{s}}(g)$ is not.

We turn now to the proof of Proposition 6.4.

LEMMA 6.9. *Suppose (R, ϵ) is a principal graded root system (Definition 6.3), and R^+ is a positive system for which each simple root α satisfies $\epsilon(\alpha) = 1$. Let β be the highest root of a simple factor of R, not of type A_{2n+1}.*

(a) $\epsilon(\beta) = 1$

(b) *$c^\beta(\epsilon)$ is principal. More precisely, every simple root α of $(R^\beta)^+ = R^\beta \cap R^+$ (notation 5.1) is simple (in R^+), and strongly orthogonal to β; so*

$$c^\beta(\epsilon)(\alpha) = 1.$$

Proof. Part (a) says that if we write

$$\beta = \sum_{\alpha \text{ simple}} n_\alpha \alpha,$$

then $\sum n_\alpha$ is odd. This is easily checked (see [10], p. 66); $\sum n_\alpha + 1$ is the Coxeter number, which is well known to be even except in type A_{2n+1}. (In the setting of McKay's beautiful ideas in [19], this can be deduced from the fact the only finite subgroups of SU(2) of odd order are cyclic.) In (b) the second formulation is obvious, and gives the first statement. Q.E.D.

We need an analogous fact for A_{2n+1}.

LEMMA 6.10. *Suppose (R, ϵ) is a principal graded root system, and R^+ is a positive system for which each simple root α satisfies $\epsilon(\alpha) = 1$. Let β be an extremal simple root in a simple factor of R of type A_n.*

(a) $\epsilon(\beta) = 1$.

(b) *$c^\beta(\epsilon)$ is principal. More precisely, every simple root α of $(R^\beta)^+$ is simple in R^+, and strongly orthogonal to β; so*

$$c^\beta(\epsilon)(\alpha) = 1.$$

Proof. Part (a) is trivial, and is included only to preserve the analogy with Lemma 6.9. For (b), we may assume R is simple. Then one knows (or can check) that R^β is of type A_{n-2}. Since there are clearly $n - 2$ roots α as described (those not adjacent to β), they must be all the simple roots of $(R^\beta)^+$. Q.E.D.

LEMMA 6.11. *Suppose (R, ϵ) is a graded root system, $\alpha \in R$, and $\epsilon(\alpha) = 1$. If $c^\alpha(\epsilon)$ is principal, then so is ϵ.*

Proof. Choose $(R^\alpha)^+$ making $c^\alpha(\epsilon)(\beta) = 1$ for each $(R^\alpha)^+$-simple root β. Regard $(R^\alpha)^+$ as defined by an ordering of

$$V^\alpha = \{x \in V \mid \langle \alpha, x \rangle = 0\}.$$

Every root but $\pm\alpha$ has a non-zero restriction to V^α; so this ordering defines a unique positive root system R^+ containing α. Clearly α is simple for R^+. By Lemma 5.12, ϵ is principal. Q.E.D.

PROPOSITION 6.12. *Suppose (R,ϵ) is a principal graded root system, $\alpha \in R$, and $\epsilon(\alpha) = 1$. Then $c^{\alpha}(\epsilon)$ is principal.*

Proof. We may assume R is simple. By Step III in the proof of Proposition 5.8, we can replace α by any root β of the same length, satisfying $\epsilon(\beta) = 1$. If α is long, Lemmas 6.9 and 6.10 provide a β which works; so assume there are two root lengths, and α is short. Let β_0 be any long root satisfying $\epsilon(\beta_0) = 1$; this exists by the definition of principal, since there must be a long simple root in each positive system. Necessarily β_0 is adjacent to some short root α_0 (that is, $\langle\alpha_0, \beta_0\rangle > 0$); so if we put $\alpha_1 = \beta_0 - \alpha_0$, $\beta_1 = \alpha_0 - \alpha_1$, then α_0 and α_1 are short, and $\beta_0 = \alpha_0 + \alpha_1$. So (perhaps after interchanging α_0 and α_1)

$$\epsilon(\alpha_0) = 0, \qquad \epsilon(\alpha_1) = 1$$

$$\epsilon(\beta_0) = \epsilon(\beta_1) = 1.$$

The sequences (α_1, α_0) and (β_1, β_0) are admissible, with the same span; so

$$c^{\alpha_0}c^{\alpha_1}(\epsilon) = c^{\beta_0}c^{\beta_1}(\epsilon).$$

The right side is principal by the first part of the proof; so the left side is as well. By Lemma 6.11, $c^{\alpha_1}(\epsilon)$ is principal, as we wished to show. Q.E.D.

Proof of Proposition 6.4. Part (a) is well known, and can be proved by a simplified version of the proof of Proposition 5.8. We leave this to the reader. For (b), suppose first that $\mathfrak{S} = \langle\alpha_1, \ldots, \alpha_l\rangle$ (with the α_i orthogonal) spans R. Then, in the notation of Definition 3.3,

$$\begin{aligned}
\check{P}_1(R) &= \left\{\lambda = \sum_{i=1}^{l} q_i\check{\alpha}_i \mid q_i \in \mathbf{Q}, \langle\lambda, \beta\rangle \in \mathbf{Z} \text{ for all } \beta \in R\right\}\\
&\subset \left\{\lambda = \sum q_i\alpha_i \mid \langle\lambda, \alpha_i\rangle \in \mathbf{Z}, \text{ all } i\right\}\\
&= \left\{\lambda = \sum q_i\check{\alpha}_i \mid 2q_i \in \mathbf{Z}, \text{ all } i\right\}\\
&\subseteq \tfrac{1}{2}\check{L}(R).
\end{aligned}$$

So

$$Z_1(R) = \check{P}_1(R)/\check{L}(R) \subseteq \tfrac{1}{2}\check{L}(R)/\check{L}(R),$$

which is the "only if" part of (b). The "if" part (which we will not need in any case) can be checked case by case: the simple root systems not spanned by orthogonal sets of roots are:

$$\begin{array}{lll}
R = A_n & (n > 1); & Z_1(R) = \mathbf{Z}/(n+1)\mathbf{Z}\\
R = D_{2n+1} & (n > 1); & Z_1(R) = \mathbf{Z}/4\mathbf{Z}\\
R = E_6 & ; & Z_1(R) = \mathbf{Z}/3\mathbf{Z}.
\end{array}$$

We prove (e) next, by induction on $|R|$. Suppose ϵ is principal, and $(\alpha_1, \ldots, \alpha_l)$ is an orthogonal sequence of roots. Since every root is $W(R)$-conjugate to a simple one, we may assume $\epsilon(\alpha_1) = 1$. By Proposition 6.12, $c^{\alpha_1}(\epsilon)$ is principal. By induction, we can replace $(\alpha_2, \ldots, \alpha_l)$ by a $W(R^\alpha)$ conjugate which is $c^{\alpha_1}(\epsilon)$-admissible. By definition, this means that $(\alpha_1, \ldots, \alpha_l)$ is ϵ-admissible. For the converse, suppose (R, ϵ) has the property in question. If R is non-empty, then ϵ cannot be trivial; so fix α with $\epsilon(\alpha) = 1$. We claim that $(R^\alpha, c^\alpha(\epsilon))$ has the desired property. Let $(\alpha_2, \ldots, \alpha_l)$ be an orthogonal sequence in R^α. By hypothesis, there is a $w \in W(R)$ with $(w\alpha, w\alpha_2, \ldots, w\alpha_l)$ admissible. In particular, $\epsilon(w\alpha) = 1$. By Step III in the proof of Proposition 5.8, we can find $w_1 \in W_0(\epsilon)$ so that

$$w_1 w\alpha = \pm\alpha.$$

Acting by w_1 does not change the property of being admissible, so $(\pm\alpha, w_1 w\alpha_2, \ldots, w_1 w\alpha_l)$ is admissible. Now $w_1 w\alpha_i$ is orthogonal to α, so s_α fixes $w_1 w\alpha_i$. Taking $y = w_1 w$ or $s_\alpha w_1 w$ as needed, we get

$$y\alpha = \alpha$$

$$y(\alpha, \alpha_2, \ldots, \alpha_l) = (\alpha, w_1 w\alpha_2 \ldots w_1 w\alpha_l).$$

The first property shows that $y \in W(R^\alpha)$, and the second that $(\alpha, y\alpha_2, \ldots, y\alpha_l)$ is ϵ-admissible. By the definition of admissible, it follows that $(y\alpha_2, \ldots, y\alpha_l)$ is $c^\alpha(\epsilon)$-admissible. Thus $c^\alpha(\epsilon)$ has the desired property, and is principal by induction, proving (e).

The only part of (c) which is not immediate from (e) is that an admissible grading is principal. We proceed by induction on $|R|$. Choose an ϵ-admissible sequence $(\alpha_1, \ldots, \alpha_l)$ spanning R. Then $c^{\alpha_1}(\epsilon)$ is obviously admissible—$(\alpha_2, \ldots, \alpha_l)$ is $c^{\alpha_1}(\epsilon)$-admissible and spans R^α—so $c^{\alpha_1}(\epsilon)$ is principal by induction. By Lemma 6.11, ϵ is principal.

Consider now (d). Choose a positive system R^+ for which every simple root α satisfies $\epsilon(\alpha) = 1$. Fix $z \in Z_1(R)$ of order 2, and a representative $\zeta \in \check{P}_1(R)$. Thus $2\zeta \in \check{L}(R)$; so

$$\zeta = \sum_{\alpha \text{ simple}} q_\alpha \check{\alpha}, \qquad 2q_\alpha \in \mathbf{Z}. \tag{6.13a}$$

Choose $\lambda \in \check{P}_1(R)$ as in the proof of Proposition 3.16. By our assumptions on ϵ and R^+,

$$\langle \lambda, \alpha \rangle \equiv 1 \qquad (\text{mod } 2) \quad (\alpha \text{ simple}) \tag{6.13b}$$

We seek an element $w \in W(R)$ such that

$$\tfrac{1}{2}(w\lambda - \lambda) \equiv \zeta \qquad (\text{mod } \check{L}(R)); \tag{6.14}$$

by Proposition 3.16, such an element will automatically lie in $W_2(\epsilon)$, and will satisfy

$$\zeta(\epsilon)(w) = z,$$

as desired. Define

$$F = \{\alpha \text{ simple} \mid q_\alpha \in \mathbf{Z}\}.$$

We claim that F is orthogonal. To see this, order the simple roots as $(\alpha_1, \ldots, \alpha_l)$ in such way that each α_i is adjacent to at most one of its successors, and the lengths are increasing. Suppose F is not orthogonal. Choose $\alpha_i, \alpha_j \in F$, with $i < j$ so that $\langle \alpha_i, \alpha_j \rangle \neq 0$, and i is minimal with these properties. Then, writing $q_m = q_{\alpha_m}$

$$\begin{aligned}
\langle \zeta, \alpha_i \rangle &= \sum q_m \langle \check{\alpha}_m, \alpha_i \rangle \\
&\equiv \sum_{\substack{\alpha_m \text{ adjacent to } \alpha_i \\ \alpha_m \in F}} \frac{1}{2} \langle \check{\alpha}_m, \alpha_i \rangle \qquad (\operatorname{mod} \mathbf{Z}) \\
&= \frac{1}{2} \langle \check{\alpha}_i, \alpha_i \rangle + \frac{1}{2} \langle \check{\alpha}_j, \alpha_i \rangle \\
&\quad + \sum_{\substack{\alpha_m \text{ adjacent to } \alpha_i \\ \alpha_m \in F \\ m \neq i, j}} \frac{1}{2} \langle \check{\alpha}_m, \alpha_i \rangle.
\end{aligned}$$

The first term here is 1, and the second $1/2$ (since $\langle \alpha_j, \alpha_j \rangle \geqslant \langle \alpha_i, \alpha_i \rangle$). Suppose m is in the third summation. If $m < i$, then the pair (α_m, α_i) contradicts the minimality assumption on (i, j). Since $j > i$, α_i can have no other adjacent successors; so $m \not> i$. So the last sum is empty, and

$$\langle \zeta, \alpha_i \rangle \equiv \tfrac{1}{2} \qquad (\operatorname{mod} \mathbf{Z}),$$

contradicting the assumption that $\zeta \in \check{P}_1(R)$. So F is orthogonal. Put

$$w = \prod_{\alpha \in F} s_\alpha;$$

then (6.14) is immediate from (6.13) and the choice of F. Q.E.D.

7. Cayley transforms of regular characters. In [21] or [25], Cayley transforms are defined which take regular characters on one Cartan subgroup H^1 to those on a second Cartan H^2, such that $H^1 \cap H^2$ has codimension one in both H^i. Here we want something more general (and therefore easier to define). Although we will make little explicit use of the special case, a familiarity with it (see Section 8.3 of [25]) would be helpful.

Definition 7.1. Let $H^1 = T^1 A^1$ be a θ-stable Cartan subgroup of G, and $\mathfrak{s} \subseteq \mathfrak{a}^1$ a subspace spanned by an orthogonal set of (real) roots. (To make sense of this, we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ using $\langle\ ,\ \rangle$.) Define $\mathfrak{a}^2$ to be the orthogonal complement of $\mathfrak{s}$ in $\mathfrak{a}^1$. Write

$$M^2A^2 = G^{A^2}$$

for the Langlands decomposition of the centralizer of A^2 in G; then T^1S is a Cartan subgroup of M^2. Because of the assumption on $\mathfrak{s}$, M^2 also contains a θ-stable compact Cartan subgroup T^2 which is unique up to conjugacy by $M^2 \cap K$; we can and do choose T^2 so that

$$T^2 \supseteq T_0^1 .$$

Thus

$$H^2 = T^2A^2 = c_{\mathfrak{s}}(H^1)$$

is a Cartan subgroup of G, the *Cayley transform of* H^1 *through* $\mathfrak{s}$. Put

$$L = L(\mathfrak{s}) = \text{centralizer of } T_0^1 \text{ in } M^2.$$

Then L contains T^1S as a split Cartan subgroup, and T^2 as a compact Cartan subgroup; and H^2 is defined up to conjugation by $L \cap K$.

We want a map $c_{\mathfrak{s}}$ from $(\hat{H}^1)'$ to $(\hat{H}^2)'$. Its characteristic property should be (very roughly speaking) that $\bar{\pi}(c_{\mathfrak{s}}(\gamma))$ is a composition factor of $\pi(\gamma)$ whenever $\gamma \in (\hat{H}^1)'$. Always one should think of the case when H^1 is split and H^2 is compact. Then $\pi(\gamma)$ is a principal series representation, and $\pi(c_{\mathfrak{s}}(\gamma))$ is a discrete series. Now most principal series contain no discrete series at all; and those that do generally contain several. So $c_{\mathfrak{s}}$ can only be defined on a subset of $(\hat{H}^1)'$, and there it will be multivalued. To describe the domain, we need to recall the notation of Definition 4.7.

LEMMA 7.2. *In the setting of Definition* 7.1, *suppose H^2 is compact, and H^1 is split* (*that is, that $G = L$*). *Fix $\gamma \in (\hat{H}^1)'_\chi$. Then there is a $\phi \in (\hat{H}^2)'_\chi$ belonging to the same block as γ* (*Definition* 1.14) *if and only if the real integral roots $R^{\mathbb{R}}(\gamma)$ span* $(\mathfrak{a}^1)^*$, *and the cograding $\check{\delta}(\gamma)$ is admissible* (*Definition* 6.3); *that is, there is a subset $\{\alpha_1, \ldots, \alpha_r\} \subseteq R_1^{\mathbb{R}}(\gamma)$ such that $\{\check{\alpha}_i\}$ is strongly orthogonal and spans* $(\mathfrak{a}^1)^*$.

If such a ϕ exists, then $\phi' \in (\hat{H}^2)'_\chi$ belongs to the same block as γ if and only if

(a) $$\phi|_{Z(G)} = \phi'|_{Z(G)};$$

or, equivalently, if and only if

(b) $$\phi' \in W(\mathfrak{g}, \mathfrak{h}^2) \times \phi.$$

(*Definition* 4.1).

Proof. The conditions for the existence of ϕ follow from the theory of the compactness of a block ([25], Proposition 9.2.12). So suppose ϕ exists. Fix

$\phi' \in (\hat{H}^2)'_\chi$. By Proposition 4.3,

$$(\mathrm{b}) \Rightarrow \phi' \text{ and } \gamma \text{ in same block} \Rightarrow (\mathrm{a}).$$

So assume (a); it is enough to prove (b). Since ϕ and ϕ' define the same infinitesimal character, we can find $w \in W(\mathfrak{g}, \mathfrak{h}^2)$ with $\bar{\phi}' = w\bar{\phi}$; here we write $\phi = (\Phi, \bar{\phi})$ for the regular character as usual. Define $\tilde{\phi} = w \times \phi$. Then $\bar{\tilde{\phi}} = \bar{\phi}'$, so

$$\phi'|_{H_0^2} = \tilde{\phi}|_{H_0^2}.$$

By hypothesis,

$$\tilde{\phi}|_{Z(G)} = \phi|_{Z(G)} = \phi'|_{Z(G)}.$$

Since $H^2 = H_0^2 \cdot Z(G)$ (because H^2 is compact), these give $\tilde{\phi} = \phi'$. Q.E.D.

To extend this result to the general case, we need a rather messy definition from [23] or [25]. We will not give all of the definition here, but a summary follows.

Definition 7.3. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable Levi decomposition of a θ-stable parabolic subalgebra of $\mathfrak{g}$, and assume that $\mathfrak{l} = \bar{\mathfrak{l}}$; here bar denotes the complex structure defined by the real form $\mathfrak{g}_0$. (From now on we may say "let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$," with the remaining assumptions understood.) Let L be the normalizer of $\mathfrak{q}$ in G (a reductive group with complexified Lie algebra $\mathfrak{l}$). Suppose $H = TA \subseteq L$ is a θ-stable Cartan subgroup. Write

$$2\rho(\mathfrak{u}) = \text{sum of the roots of } \mathfrak{h} \text{ in } \mathfrak{u},$$

an element of $\mathfrak{h}^*$. Define $\mathfrak{m}$ as in (2.10), and fix a positive system $\Delta^+(\mathfrak{m}, \mathfrak{t})$. Then there is a naturally defined character

$$\tau = \tau(\mathfrak{q}, \Delta^+(\mathfrak{m}, \mathfrak{t}), \mathfrak{h}) \in \hat{T};$$

this is the character of T on the vector space V^* defined by Lemma 8.1.1 of [25]. (Here τ stands for "twist". The definition of V^*, which is complicated, is what we are omitting.) Now suppose $\gamma = (\Gamma, \bar{\gamma}) \in (\hat{H}')^G$ is a regular character, and $\Delta^+(\mathfrak{m}, \mathfrak{t})$ is the positive system defined by $\bar{\gamma}|_\mathfrak{t}$ (see (2.10)). Put

$$\begin{aligned} \bar{\gamma}_\mathfrak{q} &= \bar{\gamma} - \rho(\mathfrak{u}) \\ \Gamma_\mathfrak{q}|_A &= \Gamma|_A \\ \Gamma_\mathfrak{q}|_T &= \Gamma|_T \otimes \tau \\ \gamma_\mathfrak{q} &= (\Gamma_\mathfrak{q}, \bar{\gamma}_\mathfrak{q}). \end{aligned}$$

LEMMA 7.4. *In the setting of Definition* 7.3, $\gamma_{\mathfrak{q}} \in (\hat{H}')^L$; *that is, the map* $\gamma \to \gamma_{\mathfrak{q}}$ *defines a correspondence between regular characters of H with respect to G, and those with respect to the smaller group L. Suppose now that* $\gamma, \phi \in (\hat{H}')^G_{\check{\chi}}$.

(a) $\check{\delta}(\gamma) = \check{\delta}(\gamma_{\mathfrak{q}})$ (*Definition* 4.5); *that is* γ *and* $\gamma_{\mathfrak{q}}$ *define the same cograding of the real integral roots* (*all of which lie in* $\Delta(\mathfrak{l}, \mathfrak{h})$).

(b) *If* $\gamma_{\mathfrak{q}}$ *and* $\phi_{\mathfrak{q}}$ *lie in the same block for L, then* γ *and* ϕ *lie in the same block for G.*

Proof. That $\gamma_{\mathfrak{q}}$ is a regular character for L is Lemma 8.1.2 of [25]. Part (a) is Lemma 8.3.12 of [25]. Part (b) is not explicitly stated in [25], but is established in the proof of Proposition 9.3.1 there. Q.E.D.

The converse of (b) is false: $\gamma_{\mathfrak{q}}$ and $\phi_{\mathfrak{q}}$ need not even have the same infinitesimal character when γ and ϕ do, since $W(\mathfrak{l}, \mathfrak{h})$ is smaller than $W(\mathfrak{g}, \mathfrak{h})$. The correspondence $\gamma \to \gamma_{\mathfrak{q}}$, on the level of group representations, is a simple (but only partly understood) case of Langlands' functoriality ideas. Since the parabolic $\mathfrak{q}$ is not defined over $\mathbf{R}$, the correspondence is not realized by ordinary induction. See [25] and [21] for some details about it.

Definition 7.5. In the setting of Definition 7.1, choose a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{m}^2$; here we want $\mathfrak{l}$ to be the centralizer of $\mathfrak{t}^1$ in $\mathfrak{m}^2$, as in Definition 7.1. Suppose $\gamma \in (\hat{H}^1)'_{\check{\chi}}$. We say that $\gamma \in \mathfrak{D}(c_{\mathfrak{s}})$ (the domain of $c_{\mathfrak{s}}$) if $\mathfrak{s}$ is admissible (definition 5.11) with respect to the cograding $\check{\delta}(\gamma)$ of Definition 4.5. (To make sense of this, we identify $\mathfrak{s}$ with a subspace of $\mathfrak{h}^*$ using $\langle\ ,\ \rangle$.) Suppose $\gamma \in \mathfrak{D}(c_{\mathfrak{s}})$. The *Cayley transform of* γ *through* $\mathfrak{s}$, $c_{\mathfrak{s}}(\gamma)$, is the set of all ϕ in $(\hat{H}^2)'_{\check{\chi}}$, satisfying the following conditions. Write (notation after (2.12))

$$\phi^1 = \phi|_{T^2} \in \left[(\hat{T}^2)'\right]^{M^2}$$

$$\gamma^1 = \gamma|_{T^1S} \in \left[(T^{\hat{1}}S)'\right]^{M^2}.$$

Then we want

(a) $\phi|_{A^2} = \gamma|_{A^2}$

(b) $\phi^1_{\mathfrak{q}}$ and $\gamma^1_{\mathfrak{q}}$ lie in the same block for L.

LEMMA 7.6. *In the setting of Definition* 7.5, *suppose* $\gamma = (\Gamma, \bar{\gamma}) \in \mathfrak{D}(c_{\mathfrak{s}})$, *and* $\phi = (\Phi, \bar{\phi}) \in (\hat{H}^2)'_{\check{\chi}}$. *Then* $\phi \in c_{\mathfrak{s}}(\gamma)$ *if and only if*

(a) $\bar{\phi}$ *and* $\bar{\gamma}$ *are conjugate by* Ad($\mathfrak{l}$); *and*

(b) $\Phi|_{Z(L)}$ *differs from* $\Gamma|_{Z(L)}$ *by a sum of roots of* $Z(L)$ *in* $\mathfrak{g}$.

Proof. The necessity of (a) and (b) is straightforward, and we leave it to the reader. So assume (a) and (b). The twists τ of Definition 7.3 are sums of roots of T in $\mathfrak{g}$. It follows that conditions (a) and (b) hold with γ and ϕ replaced by $\gamma^1_{\mathfrak{q}}$, $\phi^1_{\mathfrak{q}}$. By Lemma 7.2, it follows that there is a regular character $\psi = (\Psi, \bar{\psi}) \in [(\hat{H}^2)']^L$, lying in the same block for L as $\gamma^1_{\mathfrak{q}}$, and satisfying

$$\bar{\phi}^1_{\mathfrak{q}} = \bar{\psi}. \qquad (*)$$

Since regular characters in the same block agree on the center of the group, $\Gamma^1_{\mathfrak{q}}$ and Ψ agree on $Z(L)$. By (b),

$$\Phi^1_{\mathfrak{q}}|_{Z(L)} = \left[\Psi + \sum_{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}^2)} m_\alpha \alpha\right]\Bigg|_{Z(L)} \tag{**}$$

By (*), the differential of $(\sum m_\alpha \alpha)|_{Z(L)}$ is zero. Choose a positive system $\Delta^+(\mathfrak{g},\mathfrak{h}^2)$ compatible with $\mathfrak{q}$. Then we can write

$$\sum m_\alpha \alpha = \sum_{\beta \text{ simple}} n_\beta \beta.$$

The simple roots in $\mathfrak{l}$ are zero on the Lie algebra of $Z(L)$, and the rest of the simple roots are linearly independent there. We therefore conclude from the differentiated version of (**) that $\sum m_\alpha \alpha$ is a sum of roots in $\mathfrak{l}$. Since these are trivial on $Z(L)$, (**) gives

$$\Phi^1_{\mathfrak{q}}|_{Z(L)} = \Psi|_{Z(L)}\,.$$

Since T^2 is a compact Cartan in L, this condition and (*) give $\phi^1_{\mathfrak{q}} = \psi$. By the choice of ψ, hypothesis (b) of Definition 7.5 follows. Hypothesis 7.5(a) is immediate from condition (a) of the lemma, since $\mathrm{Ad}(\mathfrak{l})$ centralizes $\mathfrak{a}^2$. Q.E.D.

COROLLARY 7.7. *In the setting of Definition* 7.5, *suppose* $\gamma \in \mathfrak{D}(c_{\mathfrak{s}})$ *and* $w \in W^a$. *Then*

$$c_{\mathfrak{s}}(w \times \gamma) = w \times c_{\mathfrak{s}}(\gamma)$$

(*notation* 4.2)).

This follows from Lemma 7.6 and the defintion of $w \times \gamma$.

PROPOSITION 7.8. *In the setting of Definition* 7.5, *suppose* $\gamma \in \mathfrak{D}(c_{\mathfrak{s}})$. *Then* $c_{\mathfrak{s}}(\gamma) \subseteq (H^2)'_\chi$ *consists of exactly one orbit of* $W(\mathfrak{l} + \mathfrak{a}^2, \mathfrak{h}^2)$ *under the cross action. Every element of* $c_{\mathfrak{s}}(\gamma)$ *belongs to the same block as* γ.

Proof. The first claim follows from Lemma 7.2(b) and the definitions. For the second, suppose $\phi \in c_{\mathfrak{s}}(\gamma)$; define ϕ^1, γ^1 as in Definition 7.5. By that definition, $\phi^1_{\mathfrak{q}}$ and $\gamma^1_{\mathfrak{q}}$ belong to the same block for$L$. By Lemma 7.4(b), ϕ^1 and γ^1 belong to the same block for M^2. Since also $\phi|_{A^2} = \gamma|_{A^2}$, the proposition is a consequence of the following simple result.

LEMMA 7.9. *Suppose* $P = MAN$ *is a parabolic subgroup of* G. *Let* $H^i = T^iA^i$ $(i = 1,2)$ *be* θ-*stable Cartan subgroups of* G, *satisfying* $A^i \supseteq A$ (*so that* $H^i \subseteq MA$). *Fix* $\gamma \in (\hat{H}^1)'$, $\delta \in (\hat{H}^2)'$. *Write*

$$\gamma^1 = \gamma|_{H^1 \cap M}, \qquad \delta^1 = \delta|_{H^1 \cap M}\,.$$

Assume that

(a) $\bar{\gamma}$ *and* $\bar{\delta}$ *are nonsingular.*

(b) $\gamma|_A = \delta|_A$, *and*
(c) γ^1 *and* δ^1 *lie in the same block for* M.
Then γ *and* δ *lie in the same block for* G.

Proof. By Lemma 9.2.7 of [25], we may as well assume that $\bar{\pi}_M(\delta^1)$ is a subquotient of $\pi_M(\gamma^1)$ (notation (2.11)). By induction by stages, $\mathrm{Ind}_P^G[\pi_M(\gamma^1) \otimes \gamma|_A \otimes 1]$ has the same composition series as $\pi(\delta)$; and $\bar{\pi}(\delta)$ is a subquotient of $\mathrm{Ind}_P^G[\bar{\pi}_M(\delta^1) \otimes \delta|_A \otimes 1]$. So $\bar{\pi}(\delta)$is a subquotient of $\pi(\gamma)$. By Definition 1.14, γ and δ are in the same block. Q.E.D.

Next, we want to give another perspective on the definition of $c_{\mathfrak{s}}(\gamma)$. Before doing this, we should recall exactly how canonical $c_{\mathfrak{s}}(\gamma)$ is. The only choice involved is in the construction of H^2; and this is unique up to conjugation by$L \cap K$. In some sense, the Cayley transform ought to be defined on conjugacy classes of regular characters. The next definition accomplishes this.

Definition 7.10. Suppose $\gamma \in (\hat{H}^1)'_\chi$, and $\mathfrak{s}^a \subseteq \mathfrak{h}^a$ (the abstract Cartan subalgebra—see (2.6)). Write $\mathfrak{s} = i_{\bar{\gamma}}(\mathfrak{s}^a)$ (notation (2.7)). We say that $\gamma \in \mathfrak{D}(c_{\mathfrak{s}^a})$ if $\mathfrak{s}^a$ is $\check{\delta}^a(\gamma)$-admissible (Definition 4.12); or, equivalently, if $\gamma \in \mathfrak{D}(c_{\mathfrak{s}})$. In this case we define

$$c_{\mathfrak{s}^a}(\gamma) = c_{\mathfrak{s}}(\gamma),$$

the *Cayley transform of* γ *through* $\mathfrak{s}^a$. In the notation of (2.14), it is clear that

$$cl(c_{\mathfrak{s}^a}(\gamma)) = \{ cl(\phi) \mid \phi \in c_{\mathfrak{s}^a}(\gamma) \}$$

depends only on $cl(\gamma)$; we may therefore call it $c_{\mathfrak{s}^a}(cl(\gamma))$.

LEMMA 7.11. *Suppose* $\gamma \in \hat{H}'$, $\mathfrak{s}^a \subseteq \mathfrak{h}^a$, *and* $w \in W^a$ (*notation* 2.6). *Then*

$$w \times \gamma \in \mathfrak{D}(c_{w\mathfrak{s}^a}) \Leftrightarrow \gamma \in \mathfrak{D}(c_{\mathfrak{s}^a});$$

and if these hold, then

$$c_{w\mathfrak{s}^a}(w \times \gamma) = w \times c_{\mathfrak{s}^a}(\gamma).$$

This is immediate from Corollary 7.7.

There are two other basic properties of Cayley transforms which we need. Although they appear more or less unrelated, it is most convenient to prove them simultaneously. The first relates our present definitions to the formal ones given earlier, and the second concerns iterated Cayley transforms.

LEMMA 7.12. *In the setting of Definition* 7.10, *suppose* $\gamma \in \mathfrak{D}(c_{\mathfrak{s}^a})$, *and* $\phi \in c_{\mathfrak{s}^a}(\gamma)$. *Then, in the notation of Definition* 4.12

$$\theta^a(\phi) = c_{\mathfrak{s}^a}(\theta^a(\gamma)) \qquad (\textit{Definition } 5.19)$$

$$(R^a)^{\mathsf{R}}(\phi) = \{ \beta \in (R^a)^{\mathsf{R}}(\gamma) \mid \langle \beta, \mathfrak{s}^a \rangle = 0 \} = (R^a)^{\mathsf{R}}(\gamma)^{\mathfrak{s}^a}$$

$$\check{\delta}(\phi) = c_{\mathfrak{s}^a}(\check{\delta}(\gamma)) \qquad (\textit{Definition } 5.10)$$

LEMMA 7.13. *Suppose* $\mathfrak{s}^a = \mathfrak{u}^a \oplus \mathfrak{v}^a \subseteq \mathfrak{h}^a$ (*orthogonal direct sum*), $\gamma \in (\hat{H}^1)'_\chi$, *and* $\gamma \in \mathfrak{D}(c_{\mathfrak{s}^a}) \cap \mathfrak{D}(c_{\mathfrak{u}^a})$. *Then*

(a) $c_{\mathfrak{u}^a}(\gamma) \subseteq \mathfrak{D}(c_{\mathfrak{v}^a})$

(b) $c_{\mathfrak{v}^a}(c_{\mathfrak{u}^a}(\gamma)) \subseteq c_{\mathfrak{s}^a}(\gamma)$.

We will need an auxiliary formal result.

LEMMA 7.14. *Suppose* (R, ϵ) *is a graded root system on* V, *and* $\mathfrak{s} = \mathfrak{u} \oplus \mathfrak{v} \subseteq V$ (*orthogonal direct sum*). *Assume that* $\mathfrak{s}$ *and* $\mathfrak{u}$ *are* ϵ-*admissible. Then* $\mathfrak{v}$ *is* $c^{\mathfrak{u}}(\epsilon)$-*admissible, and*

$$c^{\mathfrak{s}}(\epsilon) = c^{\mathfrak{v}}(c^{\mathfrak{u}}(\epsilon)).$$

Proof. Choose an ϵ-admissible sequence $U = (\alpha_1, \ldots, \alpha_r)$ spanning $\mathfrak{u}$. By Proposition 5.8 applied to $R \cap \mathfrak{s}$, U can be extended to an ϵ-admissible sequence $S = (\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_s)$ spanning $\mathfrak{s}$. Then $V = (\alpha_{r+1}, \ldots, \alpha_s)$ spans the orthogonal complement $\mathfrak{v}$ of $\mathfrak{u}$ in $\mathfrak{s}$. The lemma is now obvious from Definition 5.10. Q.E.D.

We prove Lemmas 7.12 and 7.13 at the same time, by induction on $\dim \mathfrak{s}$. If this is zero, there is nothing to prove; so suppose $\dim \mathfrak{s} > 0$, and the results are known for lower dimensions.

Proof of Lemma 7.13. If $\mathfrak{u}^a$ or $\mathfrak{v}^a$ is zero, there is nothing to prove; so suppose both are non-zero. Then $\dim \mathfrak{u}^a < \dim \mathfrak{s}^a$; so if $\psi \in c_{\mathfrak{u}^a}(\gamma)$, we have

$$\check{\delta}(\psi) = c_{\mathfrak{u}^a}(\check{\delta}(\gamma))$$

by the inductive hypothesis for Lemma 7.12. Now (a) follows from Lemma 7.14 (or rather its analogue for cogradings, which is equivalent). For (b), we define $\mathfrak{s}, \mathfrak{u}, \mathfrak{v} \subseteq \mathfrak{h}^1$ as in Definition 7.10. Choose the Cartan subgroups $H^2 = c_{\mathfrak{s}}(H^1)$ and $H^3 = c_{\mathfrak{u}}(H^1)$ in such a way that

$$T_0^1 \subseteq T_0^3 \subseteq T_0^2$$

$$A^1 = UA^3 = SA^2 = UVA^2$$

$$A^1 \supseteq A^3 \supseteq A^2.$$

Then $H^2 = c_{\mathfrak{v}}(H^3)$. Since

$$L(\mathfrak{s})A^2 = \text{centralizer of } T_0^1 A^2$$

$$L(\mathfrak{u})A^3 = \text{centralizer of } T_0^1 A^3$$

$$L(\mathfrak{v})A^2 = \text{centralizer of } T_0^3 A^2,$$

we find

$$L(\mathfrak{s})A^2 \supseteq L(\mathfrak{u})A^3, L(\mathfrak{v})A^2. \tag{$*$}$$

As these groups share the Cartan H^3, we deduce that

$$Z(L(\mathfrak{s}))A^2 \subseteq \left[Z(L(\mathfrak{u}))A^3\right] \cap \left[Z(L(\mathfrak{v}))A^2\right]. \tag{**}$$

Using Lemma 7.6, one sees that (b) is a formal consequence of (a), (*), and (**). Q.E.D.

Proof of Lemma 7.12. The first two assertions are immediate. For the third, the case $\dim \mathfrak{s}^a = 1$ is (9.2.15) of [25]. In general, the first conclusion of Proposition 7.8 shows that we may replace ϕ by any other element of $c_{\mathfrak{s}^a}(\gamma)$. Choose an admissible sequence $S = \{\alpha_1, \ldots, \alpha_r\}$ spanning $\mathfrak{s}^a$, and choose

$$\phi \in c_{\alpha^r}(\ldots (c_{\alpha^1}(\gamma))).$$

The iterated Cayley transform is defined at each step by the case $\dim \mathfrak{s}^a = 1$, and

$$\check{\delta}^a(\phi) = c_{\mathfrak{s}^a}(\check{\delta}(\gamma))$$

by the case $\dim \mathfrak{s}^a = 1$ and Definition 5.10. By Lemma 7.13 (which is now available for this dimension of $\mathfrak{s}^a$), ϕ belongs to $c_{\mathfrak{s}^a}(\gamma)$. Q.E.D.

We turn now to the "co" theory. In the formal version of Section 5, results about cogradings were equivalent to results about gradings, because of the duality of Definition 3.23. Now, however, we are trying to *create* such a duality for regular characters. All the results must therefore be proved again, often by somewhat different methods. This is not particularly hard—in fact most of it will be left to the reader—but it is not trivial.

Definition 7.15. Let $H^2 = T^2A^2$ be a θ-stable Cartan subgroup of G, and $\mathfrak{u} \subseteq \mathfrak{t}^2$ a subspace spanned by a strongly orthogonal set of noncompact (imaginary) roots. That is, we assume that $\mathfrak{u}$ is ϵ-admissible, with ϵ the grading of Proposition 4.11. Put

$$\mathfrak{t}^1 = \{x \in \mathfrak{t}^2 \mid \langle x, \mathfrak{u}\rangle = 0\}$$

$$M^1 = \text{centralizer of } \mathfrak{t}^1 \text{ in } G,$$

so that H^2 is a Cartan subgroup of M^1. Let M^2A^2 be the Langlands decomposition of the centralizer of A^2 in G, and put

$$L = L(\mathfrak{u}) = M^1 \cap M^2.$$

By [20], for example, there is a θ-stable Cartan subgroup

$$c^{\mathfrak{u}}(H^2) = H^1 = T^1A^1 \subseteq LA^2;$$

here $\mathfrak{t}^1$ is defined above, so H^1 is split in M^1. H^1 is unique up to conjugacy by $L \cap K$; we call it *the Cayley transform of* H^2 *through* $\mathfrak{u}$. Write

$$S = A^1 \cap M^2, \qquad \mathfrak{s} = \text{Lie}(S).$$

Then

$$\mathfrak{a}^1 = \mathfrak{s} + \mathfrak{a}^2,$$

an orthogonal direct sum, and $H^2 = c_{\mathfrak{s}}(H^1)$.

Notice that now we need to worry about the grading even to define $c^{\mathfrak{u}}(H^2)$; for $c_{\mathfrak{s}}$, the grading entered only in the definition $c_{\mathfrak{s}}(\gamma)$. This is complemented by the fact that $c^{\mathfrak{u}}$ will be defined on all of $(\hat{H}^2)'_X$.

Definition 7.16. In the setting of Definition 7.15, suppose $\phi \in (\hat{H}^2)'_X$. Using the notation of Definition 7.5, we define the *Cayley transform of* ϕ *through* $\mathfrak{u}$, $c^{\mathfrak{u}}(\phi)$, to consist of those $\gamma \in (\hat{H}^1)'_X$ such that

(a) $\gamma|_{A^2} = \phi|_{A^2}$

(b) $\gamma^1_{\mathfrak{q}}$ and $\phi^1_{\mathfrak{q}}$ lie in the same block for L.

In the definition, $\bar{\pi}_L(\phi^1_{\mathfrak{q}})$ is a discrete series representation of L, since $H^2 \cap L = T^2$ is compact. By Harish-Chandra's subquotient theorem for L, $\bar{\pi}_L(\phi^1_{\mathfrak{q}})$ is a constituent of some ordinary principal series representation $\pi(\gamma^1_{\mathfrak{q}})$ of L; the corresponding regular character $\gamma^1_{\mathfrak{q}}$ belongs to $[(T^1\hat{S})']^L$, and lies in the same block as $\phi^1_{\mathfrak{q}}$ by definition. The notation $\gamma^1_{\mathfrak{q}}$ is justified: by Corollary 8.13 of [25], $\gamma^1_{\mathfrak{q}}$ is really constructed for some $\gamma^1 \in [(T^1\hat{S})']^{M^2}$, which can then be extended to $\gamma \in (\hat{H}^1)'$ by the requirement (7.16)(a). Then $\gamma \in c^{\mathfrak{u}}(\phi)$, so the Cayley transform is nonempty. Suppose $\gamma \in c^{\mathfrak{u}}(\phi)$. The theory of compactness of blocks ([25], Proposition 9.2.12) guarantees that $\mathfrak{s}$ must be $\check{\delta}(\gamma^1_{\mathfrak{q}})$-admissible, and therefore $\check{\delta}(\gamma)$-admissible by Lemma 7.4. In light of the formal resemblances between Definitions 7.16 and 7.5, Lemma 7.6 implies the following result.

LEMMA 7.17. *In the setting of Definition* 7.15, *suppose* $\phi \in (\hat{H}^2)'_X$. *Then*

$$\begin{aligned} c^{\mathfrak{u}}(\phi) &= \{\gamma \in \mathfrak{D}(c_{\mathfrak{s}}) \mid \phi \in c_{\mathfrak{s}}(\gamma)\} \\ &= \Big\{\gamma \in \mathfrak{D}(c_{\mathfrak{s}}) \mid \bar{\gamma} \textit{ and } \bar{\phi} \textit{ are conjugate by} \\ &\qquad \mathrm{Ad}(\mathfrak{l}), \textit{ and } \Gamma|_{Z(L)} \textit{ differs from } \Phi|_{Z(L)} \\ &\qquad \textit{by a sum of roots of } Z(L) \textit{ in } \mathfrak{g}\Big\}. \end{aligned}$$

COROLLARY 7.18. *In the setting of Definition* 7.5, *suppose* $\phi \in (\hat{H}^2)'$ *and* $w \in W^a$. *Then*

$$c^{\mathfrak{u}}(w \times \phi) = w \times c^{\mathfrak{u}}(\phi).$$

This is proved in the same way as Corollary 7.7.

PROPOSITION 7.19. *In the setting of Definition* 7.5, *suppose* $\phi \in (\hat{H}^2)'_X$. *Then* $c^{\mathfrak{u}}(\phi)$ *consists of the* $W(LA^2, H^1)$ *conjugates of a single orbit of* $W(\mathfrak{l} + \mathfrak{a}^2, \mathfrak{h}^1)$ *on* $(\hat{H}^1)'$ *in the cross action. Every element of* $c^{\mathfrak{u}}(\phi)$ *belongs to the same block as* ϕ.

Proof. Just as in Proposition 7.8, one reduces to the case $G = L$. Then the second claim is part of the definition of $c^{\mathfrak{u}}(\phi)$, and the first follows from Lemma 9.3.12 of [25]. Q.E.D.

COROLLARY 7.20. *In the setting of Definition* 7.5, *suppose* $\phi \in (\hat{H}^2)'_\chi$, *and* $\gamma \in \mathfrak{D}(c_{\mathfrak{s}})$. *Then*

(a) $c_{\mathfrak{s}}c^{\mathfrak{u}}(\phi) = W(\mathfrak{l} + \mathfrak{a}^2, \mathfrak{h}^2) \times \phi$

(b) $c^{\mathfrak{u}}c_{\mathfrak{s}}(\gamma) = W(LA^2, H^1) \cdot [W(\mathfrak{l} + \mathfrak{a}^2, \mathfrak{h}^2) \times \gamma]$.

Since we are interested only in conjugacy classes of regular characters, the extra conjugation action on the outside in Corollary 7.20(b) does not seriously detract from the symmetry of the result.

Definition 7.21. Suppose $\phi \in (\hat{H}^2)'_\chi$, and $\mathfrak{s}^a \subseteq \mathfrak{h}^a$. Write $\mathfrak{u} = i_{\bar{\phi}}(\mathfrak{s}^a)$ (notation 2.7). We say that $\phi \in \mathfrak{D}(c^{\mathfrak{s}^a})$ if $\mathfrak{s}^a$ is $\epsilon^a(\phi)$-admissible (Definition 4.12); or, equivalently, if $\mathfrak{u} \subseteq \mathfrak{h}$ actually lies in the compact part $\mathfrak{t}^2$, and satisfies the hypotheses of Definition 7.15. In this case we define

$$c^{\mathfrak{s}^a}(\phi) = c^{\mathfrak{u}}(\phi).$$

Just as in Definition 7.10, we write

$$\begin{aligned} c^{\mathfrak{s}^a}(cl(\phi)) &= cl\, c^{\mathfrak{s}^a}(\phi) \\ &= \left\{ cl(\gamma) \mid \gamma \in c^{\mathfrak{s}^a}(\phi) \right\}. \end{aligned}$$

LEMMA 7.22. *Suppose* $\phi \in (\hat{H}^2)'_\chi$, $\mathfrak{s}^a \subseteq \mathfrak{h}^a$, *and* $w \in W^a$. *Then*

$$w \times \phi \in \mathfrak{D}(c^{w\mathfrak{s}^a}) \Leftrightarrow \phi \in \mathfrak{D}(c^{\mathfrak{s}^a});$$

and if these hold,

$$c^{w\mathfrak{s}^a}(w \times \phi) = w \times c^{\mathfrak{s}^a}(\phi).$$

This is a consequence of Corollary 7.18.

PROPOSITION 7.23. *Suppose* $\gamma \in (H^1)'$, $\phi \in (\hat{H}^2)'$ *and* $\mathfrak{s}^a \subseteq \mathfrak{h}^a$. *Then the following conditions are equivalent.*

(a) $\gamma \in \mathfrak{D}(c_{\mathfrak{s}^a})$ *and* $cl(\phi) \in c_{\mathfrak{s}^a}(cl(\gamma))$

(b) $\phi \in \mathfrak{D}(c^{\mathfrak{s}^a})$ *and* $cl(\gamma) \in c^{\mathfrak{s}^a}(cl(\phi))$.

Write $W(\mathfrak{s}^a) = W(R^a \cap \mathfrak{s}^a)$ *as usual. If* (a) *and* (b) *hold, then*

(c) $c^{\mathfrak{s}^a}c_{\mathfrak{s}^a}(cl(\gamma)) = W(\mathfrak{s}^a) \times cl(\gamma)$

(d) $c_{\mathfrak{s}^a}c^{\mathfrak{s}^a}(cl(\phi)) = W(\mathfrak{s}^a) \times cl(\phi)$.

This follows from Lemma 7.17 and Corollary 7.20.

LEMMA 7.24. *In the setting of Definition* 7.21, *suppose* $\phi \in \mathcal{D}(c^{\mathfrak{s}^a})$ *and* $\gamma \in c^{\mathfrak{s}^a}(\phi)$. *Then, in the notation of Definition* 4.12,

$$\theta^a(\gamma) = c^{\mathfrak{s}^a}(\theta)(\phi) \qquad (\textit{Definition } 5.19)$$

$$(R^a)^{i\mathbb{R}}(\gamma) = \left\{ \beta \in (R^a)^{i\mathbb{R}}(\phi) \mid \langle \beta, \mathfrak{s}^a \rangle = 0 \right\} = (R^a)^{i\mathbb{R}}(\phi)^{\mathfrak{s}^a}$$

$$\epsilon(\gamma) = c^{\mathfrak{s}^a}(\epsilon(\phi)) \qquad (\textit{Definition } 5.20)$$

Just as before, it is most convenient to prove this with

LEMMA 7.25. *Suppose* $\mathfrak{s}^a = \mathfrak{u}^a \oplus \mathfrak{v}^a \subseteq \mathfrak{h}^a$ (*orthogonal direct sum*), $\phi \in (\hat{H}^2)'_\chi$, *and* $\phi \in \mathcal{D}(c^{\mathfrak{s}^a}) \cap \mathcal{D}(c^{\mathfrak{v}^a})$. *Then*
(a) $c^{\mathfrak{v}^a}(\phi) \subseteq \mathcal{D}(c^{\mathfrak{u}^a})$
(b) $c^{\mathfrak{u}^a}(c^{\mathfrak{v}^a}(\phi)) \subseteq c^{\mathfrak{s}^a}(\phi)$.

The two lemmas are proved simultaneously, as before. In this setting, the case of Lemma 7.24 when $\dim \mathfrak{s}^a = 1$ is due to Schmid [20]; his proof appears in [25], Proposition 5.3.1.

PROPOSITION 7.26. *Suppose* $\mathfrak{s}^a \subseteq \mathfrak{h}^a$, *and* $\gamma \in \hat{H}'$
(a) *If* $\gamma \in \mathcal{D}(c_{\mathfrak{s}^a})$, *then*

$$g^a(c_{\mathfrak{s}^a}(\gamma)) = c_{\mathfrak{s}^a}(g^a(\gamma)) \qquad (\textit{Definitions } 4.12, 5.19);$$

that is, the set of weak bigradings of R^a *defined by the elements of* $c_{\mathfrak{s}^a}(\gamma)$ *is precisely the* (*formal*) *Cayley transform of the weak bigrading defined by* γ, *through* $\mathfrak{s}^a$.
(b) *If* $\gamma \in \mathcal{D}(c^{\mathfrak{s}^a})$, *then*

$$g^a\left(c^{\mathfrak{s}^a}(\gamma)\right) = c^{\mathfrak{s}^a}(g^a(\gamma)).$$

Proof. By Lemmas 7.12 and 7.24, and (a)$\Leftrightarrow$(b) in Proposition 7.23, we find

$$g^a(c_{\mathfrak{s}^a}(\gamma)) \subseteq c_{\mathfrak{s}^a}(g^a(\gamma)),$$

and similarly for (b). Now equality follows from Proposition 7.23(c) and (d), and Proposition 5.20. Q.E.D.

The problem of extending this result to strong bigradings is most conveniently dealt with later.

8. A standard form for blocks. As was mentioned in the introduction, we intend to describe each block in terms of two special elements of it. The machinery to do this is now in place. Because the proofs are tedious, we will begin this section by formulating all the major results before proving any of them.

Definition 8.1. Suppose $\phi \in \hat{H}'_\chi$. The *c-packet of* ϕ, $cp(\phi)$, consists of all conjugacy classes of regular characters meeting the orbit of ϕ under the cross action of $W^{i\mathbb{R}}(\phi)$, the Weyl group of imaginary roots (notation 4.11). (Here c stands for compact.) That is

$$cp(\phi) = \{\gamma \mid \gamma \in W^{i\mathbb{R}}(\phi)^a \times cl(\phi)\}.$$

If $\mathfrak{s} \subseteq \mathfrak{h}^a$, and $\phi \in \mathfrak{D}(c_{\mathfrak{s}})$ (Definition 7.11), we set

$$cp_{\mathfrak{s}}(\phi) = cp(c_{\mathfrak{s}}(\phi)) \supseteq c_{\mathfrak{s}}(\phi);$$

the last containment is Proposition 7.8. Similarly, we define

$$rp(\phi) = w^{\mathbb{R}}(\phi)^a \times cl(\phi)$$

$$rp^{\mathfrak{s}}(\phi) = rp(c^{\mathfrak{s}}(\phi)) \supseteq c^{\mathfrak{s}}(\phi) \qquad (\phi \in \mathfrak{D}(c^{\mathfrak{s}})).$$

Analogous definitions are made with ϕ replaced by a bigrading of an abstract root system.

If $\pi(\phi)$ is a discrete series, then $\{\pi(\gamma) \mid \gamma \in cp(\phi)\}$ consists of all other discrete series with the same infinitesimal and central character as $\pi(\phi)$; and $rp(\phi) = cl(\phi)$, the conjugacy class of ϕ. If $\pi(\phi)$ is a principal series representation for a split group, then $\{\pi(\gamma) \mid \gamma \in rp(\phi)\}$ consists of all other principal series sharing a composition factor with $\pi(\phi)$. In general, $cp(\phi)$ coincides with the Langlands L-packet of ϕ (on the level of group representations). We have not used his terminology in order to keep clear the duality between c-packets and r-packets.

A number of facts about packets are obvious but worth stating.

LEMMA 8.2. *The c-packets partition each block with infinitesimal character* χ; *that is, having a common c-packet is an equivalence relation. The abstract cograding* $\check{\delta}^a(\phi)$ *is constant on c-packets; so if* $\phi \in \mathfrak{D}(c_{\mathfrak{s}})$, *then* $cp(\phi) \subseteq \mathfrak{D}(c_{\mathfrak{s}})$. *Analogous results hold for r-packets.*

We will leave the verification of this to the reader. For our purposes, the most important property of packets is

PROPOSITION 8.3. *An r-packet and a c-packet can intersect in at most one conjugacy class.*

We will parametrize each block by parametrizing the r-packets and c-packets separately, and describing which have a non-empty intersection.

Definition 8.4. Suppose $\gamma \in \hat{H}'_\chi$. In the notation of Proposition 4.11, we say that γ is *almost minimal* if the cograding $\check{\delta}(\gamma)$ is trivial, and *almost maximal* if the grading $\epsilon(\gamma)$ is trivial.

Thus γ is almost maximal exactly when H is maximally split in G; this means that $\pi(\gamma)$ is an ordinary principal series representation. If $\pi(\gamma)$ is a discrete series, or even a fundamental series representation, then γ is almost minimal; but the converse is not true. This is dual to the fact that a maximally split Cartan need not be split, or even quasisplit.

THEOREM 8.5. *Fix a block B of regular characters with infinitesimal character χ (cf. (2.17)). Then we can find regular characters $\gamma^i \in B$ ($i = 1,2$), such that*

(a) *γ^1 is almost maximal, and γ^2 is almost minimal.*

These elements are uniquely determined up to conjugacy and the cross action of W^a. Fix any other element $\phi \in B$. Then we can find $\mathfrak{s}^i \subseteq \mathfrak{h}^a$, $w^i \in W^a$ ($i = 1,2$), such that

(b) $w^1 \times \gamma^1 \in \mathfrak{D}(c_{\mathfrak{s}^1})$, $w^2 \times \gamma^2 \in \mathfrak{D}(c^{\mathfrak{s}^2})$

(c) $\phi \in c_{\mathfrak{s}^1}(cl(w^1 \times \gamma^1)) \cap c^{\mathfrak{s}^2}(cl(w^2 \times \gamma^2))$ (*Definitions* 7.10 *and* 7.21).

In particular, we can find $\mathfrak{s}^i, w^i$ so that

(d) $cp(\phi) = cp_{\mathfrak{s}^1}(w^1 \times \gamma^1)\ rp(\phi) = rp^{\mathfrak{s}^2}(w^2 \times \gamma^2)$ (*Definition* 8.1).

Now Proposition 8.3 shows that B is parametrized by quadruples $(w^1, \mathfrak{s}^1, w^2, \mathfrak{s}^2)$, via Theorem 8.5(d). To make this more precise (so that it can be used in our proposed duality theory) we need a formal description of which quadruples occur, and which define the same ϕ.

PROPOSITION 8.6. *Suppose $\gamma \in \hat{H}'_\chi$ is almost maximal, $\mathfrak{s}, \mathfrak{u} \subseteq \mathfrak{a}^1$, $w, y \in W^a$, and*

$$w \times \gamma \in \mathfrak{D}(c_{\mathfrak{s}}), \qquad y \times \gamma \in \mathfrak{D}(c_{\mathfrak{u}}).$$

Define

$$\mathfrak{a}(\mathfrak{s}) = \{x \in \mathfrak{h}^a \mid c_{\mathfrak{s}}(\theta^a(w \times \gamma))x = -x\}.$$

(*Definitions* 5.19 *and* 4.12)

$$W^{\mathfrak{a}(\mathfrak{s})} = \{\tau \in W^a | \tau|_{\mathfrak{a}(\mathfrak{s})} = 1\}.$$

Then the following conditions are equivalent

(a) $cp_{\mathfrak{s}}(w \times \gamma) = cp_{\mathfrak{u}}(y \times \gamma)$

(b) *There is a $\tau \in W^{\mathfrak{a}(\mathfrak{s})}$ such that $\tau\mathfrak{u} = \mathfrak{s}$, and $w^{-1}\tau y \in W_1^a(\gamma)$* (*Definition* 4.13).

Analogous results hold for $c^{\mathfrak{s}}$ when γ is almost minimal.

The point of this result is not that it is particularly understandable, but that it is phrased entirely in terms of the strong bigrading $\bar{g}^a(\gamma)$ (in light of the description of $W_1^a(\gamma)$ in Proposition 4.14).

PROPOSITION 8.7. *Fix a block B, and γ^i ($i = 1,2$) as in Theorem 8.5. Suppose $\mathfrak{s}^i \subseteq \mathfrak{h}^a$, $w^i \in W^a$, and*

$$w^1 \times \gamma^1 \in \mathfrak{D}(c_{\mathfrak{s}^1}), \qquad w^2 \times \gamma^2 \in \mathfrak{D}(c^{\mathfrak{s}^2}).$$

Then the following conditions are equivalent.

(a) $cp_{\mathfrak{s}^1}(w^1 \times \gamma^1) \cap rp^{\mathfrak{s}^2}(w^2 \times \gamma^2) \neq \emptyset$

(b) $cp_{\mathfrak{s}^1}(w^1 \times g^a(\gamma^1)) \cap rp^{\mathfrak{s}^2}(w^2 \times g^a(\gamma^2)) \neq \emptyset$.

Here in (b) *we use the notion of packets for* (*weak*) *bigradings mentioned in Definition* 8.1.

We now begin the proofs.

Proof of Proposition 8.3. Suppose $\xi^i \in cp(\phi) \cap rp(\gamma)$, $i = 1,2$. Since ξ^1 and ξ^2 are both in $cp(\phi)$,

$$cl(\xi^2) = cl(w \times \xi^1),$$

with some $w \in W^{i\mathsf{R}}(\xi^1)$. Since they are both in $\underline{rp}(\gamma)$,

$$cl(\xi^2) = cl(y \times \xi^1),$$

with some $y \in W^{\mathsf{R}}(\xi^1)$. Thus

$$cl(w \times \xi^1) = cl(y \times \xi^1),$$

so $y^{-1}w \in W_1(\xi^1)$ (Definition 4.13). By Proposition 4.14, $w \in W_1^{i\mathsf{R}}(\xi^1)$, so

$$cl(\xi^2) = cl(w \times \xi^1) = cl(\xi^1),$$

as we wished to show. Q.E.D.

For Theorem 8.5, we need some *a priori* control on the size of B. This is provided by

THEOREM 8.8 ([25], Theorem 9.2.11). *The blocks of regular characters with infinitesimal character* χ *are the smallest sets closed under conjugacy, the cross action of* W^a, *and the Cayley transformations of Definitions* 7.10 *and* 7.21.

LEMMA 8.9. *Suppose* $\phi \in \hat{H}'_\chi$. *Choose a maximal* $\epsilon^a(\phi)$-*admissible subspace* $\mathfrak{s}^1 \subseteq \mathfrak{h}^a$, *and a maximal* $\check{\delta}^a(\phi)$-*admissible subspace* $\mathfrak{s}^2 \subseteq \mathfrak{h}^a$ (*Definitions* 4.12 *and* 5.10). *Then* $\phi \in \mathfrak{D}(c^{\mathfrak{s}^1}) \cap \mathfrak{D}(c_{\mathfrak{s}^2})$; *choose* $\gamma^1 \in c^{\mathfrak{s}^1}(\phi)$, $\gamma^2 \in c_{\mathfrak{s}^2}(\phi)$ (*Definitions* 7.10 *and* 7.21).

(a) γ^1 *is almost maximal*, γ^2 *is almost minimal*, $\gamma^1 \in \mathfrak{D}(c_{\mathfrak{s}^1})$, $\gamma^2 \in \mathfrak{D}(c^{\mathfrak{s}^2})$ *and*

$$cl(\phi) \in c_{\mathfrak{s}^1}(cl(\gamma^1)) \cap c^{\mathfrak{s}^2}(cl(\gamma^2))$$

(b) *The* W^a *cross orbits* $W^a \times cl(\gamma^1)$ *and* $W^a \times cl(\gamma^2)$ *depend only on* $W^a \times cl(\phi)$ (*and not on any of the other choices in the definition of* γ^i).

Proof. That ϕ belongs to the domains of the Cayley transforms in question follows from the choice of $\mathfrak{s}^1$. That γ^1 is almost maximal and γ^2 almost minimal is a consquence of Proposition 7.26 (compare the remark at the beginning of Proposition 5.8). Now (a) follows from the first part of Proposition 7.23. For (b), suppose $\mathfrak{u}^1$ is maximal admissible for $\epsilon^a(w \times \phi)$, and $\xi^1 \in c^{\mathfrak{u}^1}(w \times \phi)$. We want to show that $\xi^1 \in W^a \times cl(\gamma^1)$. By Lemma 7.22, $w^{-1}\mathfrak{u}^1$ is maximal admissible for

$\epsilon^a(\phi)$, and $\xi^1 \in w \times c^{w^{-1}\mathfrak{u}^1}(\phi)$. By Proposition 5.8, there is a $y \in W_0^{i\mathbb{R}}(\phi)^a$; such that $w^{-1}\mathfrak{u}^1 = y\mathfrak{s}$. By Proposition 4.14, $y \in W_1(\phi)^a$; so $cl(\phi) = cl(y \times \phi)$, and Proposition 7.23 gives

$$\begin{aligned} c^{w^{-1}\mathfrak{u}^1}(cl(\phi)) &= c^{y\mathfrak{s}}(cl(\phi)) \\ &= c^{y\mathfrak{s}}(cl(y \times \phi)) \\ &= y \times c^{\mathfrak{s}}(cl(\phi)). \end{aligned}$$

Therefore $\xi^1 \in (wy) \times c^{\mathfrak{s}}(cl(\phi))$; and by definition, $\gamma^1 \in c^{\mathfrak{s}}(cl(\phi))$. By Proposition 7.19, $\xi^1 \in W^a \times cl(\gamma^1)$, as we wished to show. The proof for γ^2 is identical. Q.E.D.

LEMMA 8.10. *The regular characters γ^i of Lemma* 8.9 *depend only on the block of ϕ, up to conjugacy and the cross action of W^a.*

Proof. By Lemma 8.9(b), $W^a \times cl(\gamma^i)$ depends only on $W^a \times cl(\phi)$. By Theorem 8.8, it therefore suffices to prove the following claim: suppose there is another regular character ψ, a subspace $\mathfrak{u} \subseteq \mathfrak{h}^a$, that $\psi \in \mathfrak{D}(c_{\mathfrak{u}})$, and that $\phi \in c_{\mathfrak{u}}(\psi)$. Then ϕ and ψ define the same elements γ^i, up to conjugacy and W^a. (Once this is proved, Lemma 7.23 will give the analogous result for $c^{\mathfrak{u}}$, by interchanging ϕ and ψ.) Consider first γ^1; say we define it by choosing $\mathfrak{s}^1$ maximal admissible for $\epsilon^a(\psi)$, and choose

$$\gamma^1 \in c^{\mathfrak{s}^1}(\psi).$$

We claim that this same γ^1 works for ϕ. Since $\mathfrak{u}$ is $\check{\delta}^a(\phi)$-admissible, $\mathfrak{u}$ is contained in the -1 eigenspace of $\theta^a(\phi)$. Similarly $\mathfrak{s}^1$ is contained in the $+1$ eigenspace, so $\mathfrak{s}^1$ and $\mathfrak{u}$ are orthogonal. We know that

$$\phi \in c_{\mathfrak{u}}(\psi), \qquad \psi \in c_{\mathfrak{s}^1}(\gamma^1).$$

By Lemma 7.12, $\check{\delta}^a(\psi) = c_{\mathfrak{s}^1}(\check{\delta}^a(\gamma^1))$; and $\mathfrak{u}$ is $\check{\delta}^a(\psi)$-admissible by assumption. The inductive form of the definition of admissible now guarantees that $\mathfrak{s}^1 \oplus \mathfrak{u}$ is $\check{\delta}^a(\gamma^1)$-admissible. By Lemma 7.13, $\phi \in c_{\mathfrak{s}^1 \oplus \mathfrak{u}}(\gamma^1)$. By Lemma 7.23, $\mathfrak{s}^1 \oplus \mathfrak{u}$ is $\epsilon^2(\phi)$-admissible, and $\gamma^1 \in c^{\mathfrak{s}^1 \oplus \mathfrak{u}}(\phi)$. So γ^1 can be obtained from ϕ by the construction of Lemma 8.7, as claimed. The case of γ^2 is identical. Q.E.D.

Theorem 8.5 is a consequence of Lemmas 8.9 and 8.10.

Proof of Proposition 8.6. If $\phi \in cp_{\mathfrak{s}}(w \times \gamma)$, then

$$\begin{aligned} \mathfrak{a}(\mathfrak{s}) &= -1 \text{ eigenspace of } \theta^a(\phi) \\ W^{\mathfrak{a}(\mathfrak{s})} &= W^{i\mathbb{R}}(\phi)^a. \end{aligned} \tag{8.11}$$

Now assume (b). Since $w^{-1}\tau y \in W_1(\gamma)^a$,

$$cl(\gamma) = cl\big((w^{-1}\tau y) \times \gamma\big)$$

$$cl(w \times \gamma) = cl((\tau y) \times \gamma).$$

Therefore

$$\begin{aligned} cp_{\mathfrak{s}}(w \times \gamma) &= cp_{\tau\mathfrak{u}}(\tau \times (y \times \gamma)) \\ &= \tau \times cp_{\mathfrak{u}}(y \times \gamma) \end{aligned}$$

(by Lemma 7.11). Therefore,

$$\tau^{-1} \times cp_{\mathfrak{s}}(w \times \gamma) = cp_{\mathfrak{u}}(y \times \gamma).$$

By (8.11), the hypothesis $\tau \in W^{\mathfrak{a}(\mathfrak{s})}$ gives (a).

Next assume (a). This means that there is an element $\phi \in c_{\mathfrak{s}}(cl(w \times \gamma))$, and a $\sigma \in W^{\mathfrak{a}(\mathfrak{s})}$, such that

$$\sigma^{-1}\phi \in c_{\mathfrak{u}}\big(cl(y \times \gamma)\big).$$

Using Lemma 7.11, we can rewrite this as

$$\begin{aligned} &\phi \in \sigma \times c_{\mathfrak{u}}\big(cl(y \times \gamma)\big) \\ &\phi \in c_{\sigma\mathfrak{u}}\big(cl(\sigma y \times \gamma)\big). \end{aligned} \tag{$*$}$$

Since γ is almost maximal, $w \times \gamma$ and $\sigma y \times \gamma$ are as well; so $\mathfrak{s}$ and $\sigma\mathfrak{u}$ must be maximal $\epsilon(\phi)$-admissible subspaces of $\mathfrak{h}^a$. By Proposition 5.8, we can find $\sigma_0 \in W_0^{i\mathbb{R}}(\phi)^a$ with

$$\mathfrak{s} = \sigma_0\sigma\mathfrak{u}. \tag{$**$}$$

By Proposition 4.14, $cl(\sigma_0^{-1} \times \phi) = cl(\phi)$; so by Lemma 7.11,

$$\begin{aligned} c^{\mathfrak{s}}\big(cl(\phi)\big) &= \sigma_0 \times c^{\sigma_0^{-1}\mathfrak{s}}\big(cl(\sigma_0^{-1} \times \phi)\big) \\ &= \sigma_0 \times c^{\sigma\mathfrak{u}}\big(cl(\phi)\big) \\ &\supseteq cl(\sigma_0\sigma y \times \gamma); \end{aligned}$$

the last containment is $(*)$ and Proposition 7.23. On the other hand, $\phi \in c_{\mathfrak{s}}(cl(w \times \gamma))$; so the left side above is

$$W(\mathfrak{s}) \times cl(w \times \gamma)$$

by Proposition 7.8. So we can find $\sigma_1 \in W(\mathfrak{s})$ so that

$$cl(w \times \gamma) = cl((\sigma_1\sigma_0\sigma)y \times \gamma).$$

Now σ was chosen in $W^{\mathfrak{a}(\mathfrak{s})}$; and σ_0 and σ_1 belong to $W^{\mathfrak{a}(\mathfrak{s})}$ by (8.11). So if we set $\tau = \sigma_1\sigma_0\sigma$, then

$$\tau \in W^{\mathfrak{a}(\mathfrak{s})}$$

$$cl(w \times \gamma) = cl(\tau y \times \gamma).$$

Finally, $\sigma_1\mathfrak{s} = \mathfrak{s}$; so (**) gives

$$\tau\mathfrak{u} = \mathfrak{s}.$$

These three conditions are (8.6)(b). Q.E.D.

Proof of Proposition 8.7. The implication (a)$\Rightarrow$(b) is obvious; so assume (b). Replacing γ^i by $w^i \times \gamma^i$, we may assume $w^i = 1$. By Proposition 7.26, we can choose

$$\tilde{\phi} \in cp_{\mathfrak{s}^1}(\gamma^1) \tag{8.12a}$$

so that

$$g^a(\tilde{\phi}) \in rp^{\mathfrak{s}^2}\big(g^a(\gamma^2)\big). \tag{8.12b}$$

By (8.12a), we can write

$$\tilde{\phi} = \sigma^1 \times \phi, \qquad \sigma^1 \in W^{i\mathsf{R}}(\phi)^a, \qquad \phi \in c_{\mathfrak{s}^1}(\gamma^1). \tag{8.12c}$$

By (8.12b), we can find σ^2 so that

$$g^a(\sigma^2 \times \tilde{\phi}) \in c^{\mathfrak{s}^2}\big(g^a(\gamma^2)\big), \qquad \sigma^2 \in W^{\mathsf{R}}(\phi)^a. \tag{8.12d}$$

Therefore

$$g^a(\sigma \times \phi) \in c^{\mathfrak{s}^2}\big(g^a(\gamma^2)\big), \qquad \sigma = \sigma_1\sigma_2. \tag{8.12e}$$

This last inclusion means that $\mathfrak{s}^2$ is $\check{\delta}^a(\sigma \times \phi)$-admissible; so by Proposition 5.20,

$$c_{\mathfrak{s}^2}\big(g^a(\sigma \times \phi)\big) = W(\mathfrak{s}^2) \times g^a(\gamma^2). \tag{8.12f}$$

We may therefore choose ψ so that

$$\psi \in c_{\mathfrak{s}^2}(\sigma \times \phi) \tag{8.13a}$$

$$g^a(\psi) = g^a(\gamma^2). \tag{8.13b}$$

If ψ were conjugate to γ^2, we could easily get the conclusion of 8.7(a), by reading back through the construction of ψ. It need not be, however. Two things will save us. First, we made several choices in getting ψ, so we can modify it a little without changing its properties. Second, Lemma 8.10 guarantees that there is an element $w \in W^a$ such that

$$cl(\gamma^2) = cl(w \times \psi). \tag{8.13c}$$

We will use (8.13b) to deduce that w has a very particular form. Then $w \times \psi$ will be one of the allowable modifications of ψ mentioned above. By 8.13(b)–(c),

$$w \times g^a(\psi) = g^a(\psi). \tag{8.14a}$$

In particular, w commutes with $\theta^a(\psi)$. By Proposition 3.12,

$$w = \tau^1\tau^2\tilde{w}, \qquad \tilde{w} \in W^{\mathrm{C}}(\psi)^a, \qquad \tau^1 \in W^{i\mathrm{R}}(\psi)^a, \qquad \tau^2 \in W^{\mathrm{R}}(\psi)^a \tag{8.14b}$$

(in the notation of Proposition 3.12). Since ψ is almost minimal, $\check{\delta}^a(\psi)$ is trivial; so Proposition 4.14 gives

$$\tau^2\tilde{w} \in W_1(\psi)^a, \qquad \tau^1 \in W_2^{i\mathrm{R}}(\psi)^a. \tag{8.14c}$$

Therefore we may replace w by τ^1, and get (from (8.13c))

$$cl(\gamma^2) = cl(w \times \psi), \qquad w \in W_2^{i\mathrm{R}}(\psi)^a. \tag{8.15}$$

By (8.12c) and Lemma 7.11,

$$\sigma \times \phi \in c_{\sigma \mathfrak{s}^1}(\sigma \times \gamma^1); \tag{8.16a}$$

so $\sigma\mathfrak{s}^1$ is $\epsilon^a(\sigma \times \phi)$-admissible. By (8.13a) and the inductive nature of the definition of admissible,

$$\mathfrak{s}^2 \text{ and } \sigma\mathfrak{s}^1 \oplus \mathfrak{s}^2 \text{ are } \epsilon^a(\psi)\text{-admissible.} \tag{8.16b}$$

Since γ^1 is almost maximal, $\sigma\mathfrak{s}^1 \oplus \mathfrak{s}^2$ is a maximal $\epsilon^a(\psi)$-admissible subspace. By Corollary 5.14 and (8.15), we can write

$$w = s^1s^2w^0, \qquad s^1 \in W(\sigma\mathfrak{s}^1), \qquad s^2 \in W(\mathfrak{s}^2), \qquad w^0 \in W_0^{i\mathrm{R}}(\psi)^a. \tag{8.16c}$$

By Proposition 4.14, $w^0 \in W_1(\psi)^a$; so (8.15) becomes

$$cl(\gamma^2) = cl(w \times \psi), \qquad w = s^1s^2, \qquad s^1 \in W(\sigma\mathfrak{s}^1), \qquad s^2 \in W(\mathfrak{s}^2). \tag{8.17}$$

By Proposition 7.8 and (8.13a),

$$s^2 \times \psi \in c_{\mathfrak{s}^2}(\sigma \times \phi). \tag{8.18a}$$

Now σs^1 and $\mathfrak{s}^2$ are orthogonal, so s^1 fixes $\mathfrak{s}^2$. By Lemma 7.11,

$$s^1s^2 \times \psi \in c_{\mathfrak{s}^2}(s^1\sigma \times \phi) = c_{\mathfrak{s}^2}\big(\sigma(\sigma^{-1}s^1\sigma) \times \phi\big). \tag{8.18b}$$

Now $s^1 \in W(\sigma\mathfrak{s}^1)$, so

$$\tilde{s}^1 = \sigma^{-1}s^1\sigma \in W(\mathfrak{s}^1). \tag{8.18c}$$

By (8.12c) and Proposition 7.8,

$$\phi' = \tilde{s}^1 \times \phi \in c_{\mathfrak{s}^1}(\gamma^1). \tag{8.18d}$$

By (8.17), (8.18b), and (8.18d), we get

$$cl\gamma^2 \in c_{\mathfrak{s}^2}\big(cl(\sigma \times \phi')\big), \qquad \phi' \in c_{\mathfrak{s}^1}(\gamma^1). \tag{8.19a}$$

By Proposition 7.23, this can be written

$$cl(\sigma \times \phi') \in c_{\mathfrak{s}^2}\big(cl(\gamma^2)\big), \qquad \phi' \in c_{\mathfrak{s}^1}(\gamma^1).$$

But $\sigma = \sigma^1\sigma^2$ (see (8.12)); so

$$\begin{gathered} cl(\sigma^1 \times \phi') \in (\sigma^2)^{-1} \times c^{\mathfrak{s}^2}\big(cl(\gamma^2)\big), \qquad \phi' \in c_{\mathfrak{s}^1}(\gamma^1), \\ \sigma^1 \in W^{i\mathsf{R}}(\phi^a), \qquad \sigma^2 \in W^{\mathsf{R}}(\phi)^a. \end{gathered} \tag{8.19b}$$

But (8.18c)–(8.18d) shows that ϕ and ϕ' define the same abstract real and imaginary roots; so (8.19b) amounts to

$$cl(\sigma^1 \times \phi') \subseteq rp^{\mathfrak{s}^2}(\gamma^2), \qquad \sigma^1 \times \phi' \in cp_{\mathfrak{s}^1}(\gamma^1).$$

Thus

$$cp_{\mathfrak{s}^1}(\gamma^1) \cap rp^{\mathfrak{s}^2}(\gamma^2) \neq \emptyset. \quad \text{Q.E.D.}$$

9. Cayley transforms of $\mathfrak{R}$ groups. Just as in the formal theory, there are an easy case and a hard case.

LEMMA 9.1. *Suppose $\gamma \in \hat{H}'_\chi$, $\mathfrak{s} \subseteq \mathfrak{h}^a$, $\gamma \in \mathfrak{D}(c_{\mathfrak{s}})$, and $\phi \in c_{\mathfrak{s}}(\gamma)$. Then*

$$W_1^{\mathsf{R}}(\phi)^a = c_{\mathfrak{s}}\big(W_1^{\mathsf{R}}(\gamma)\big)$$

(*Definitions* 4.13 *and* 6.2). *That is,*

$$W_1^{\mathsf{R}}(\phi)^a = \big\{ w \in W(\mathfrak{s}) \cdot W_1^{\mathsf{R}}(\gamma) \mid w|_{\mathfrak{s}} = 1 \big\}.$$

The analogous result holds for $c^{\mathfrak{s}}$.

The proof that $W_1^{\mathsf{R}}(\phi)^a$ contains the indicated elements is along the lines of the proof of Proposition 8.3; and the proof that it contains only these is like the proof of Proposition 8.6. We therefore leave this to the reader. (The analogue for $c^{\mathfrak{s}}$, which can be proved in the same way, is due to Schmid [20].)

PROPOSITION 9.2. *In the setting of Lemma* 9.1, *suppose also that γ is almost maximal. Then*

$$\bar{g}^a(c_{\mathfrak{s}}(\gamma)) = c_{\mathfrak{s}}(\bar{g}^a(\gamma))$$

(*Definition* 6.7). *In particular, if* $\phi \in c_{\mathfrak{s}}(\gamma)$, *then*

$$\mathcal{R}^s = \left[W_2^{\mathbf{R}}(\gamma)^a \cap W(\mathfrak{s}) \right] / \left[W_1^{\mathbf{R}}(\gamma)^a \cap W(\mathfrak{s}) \right]^{\wedge}$$

and

$$\mathcal{R}^c = \left[W_1^{i\mathbf{R}}(\phi)^a \cap W(\mathfrak{s}) \right] / \left[W_0^{i\mathbf{R}}(\phi)^a \cap W(\mathfrak{s}) \right],$$

are isomorphic by the natural map of Corollary 6.6. *Analogous results hold for* $c^{\mathfrak{s}}$ *when* γ *is almost minimal.*

By Lemma 9.1 and Definition 6.7, we only have to prove the natural isomorphism $\mathcal{R}^s \cong \mathcal{R}^c$. Recall the group L of Definition 7.1. By the definition of $c_{\mathfrak{s}}(\gamma)$ and the $\mathcal{R}$ groups, we can immediately reduce to the case $G = L$. That is, we make

Assumption 9.3. G contains both a compact Cartan $H^2 = T^2$, and a split Cartan $H^1 = T^1A^1$. We have $\gamma \in (\hat{H}^1)'_\chi$, $\phi \in (\hat{H}^2)'_\chi$. The infinitesimal character χ is integral. Write $\mathfrak{s}$ = span of the abstract roots. The $\epsilon(\phi)$ and $\check{\delta}(\gamma)$ are an admissible grading and cograding (Definition 6.3); and $\phi \in c_{\mathfrak{s}}(\gamma)$, $\gamma \in c^{\mathfrak{s}}(\phi)$. We will define $\mathcal{R}^s$ and $\mathcal{R}^c$ as in Proposition 9.2. Define

$$\mathcal{R} = \mathcal{R}(G) = \mathrm{Ad}(G)/\mathrm{Ad}(G_0).$$

We will prove Proposition 9.2 by proving that $\mathcal{R}^s$ and $\mathcal{R}^c$ are both naturally isomorphic to $\mathcal{R}$. A little notation is helpful.

Definition 9.4. In the general setting (2.1), put

$$G^{\#} = \{x \in G \mid \mathrm{Ad}(x) \in \mathrm{Ad}(G_0)\}$$

$$= \mathrm{Ad}^{-1}(\mathrm{Ad}(G_0))$$

$$G^2 = \{x \in G^{\mathbf{C}} \mid \mathrm{Ad}(x) \text{ normalizes } \mathfrak{g}_0 \subseteq \mathfrak{g}\}$$

$$G^1 = \{x \in G^{\mathbf{C}} \mid \mathrm{Ad}(x) \in \mathrm{Ad}(G)\}$$

$$G^0 = \{x \in G^{\mathbf{C}} \mid \mathrm{Ad}(x) \in \mathrm{Ad}(G_0)\}$$

$$\mathcal{U}(\mathfrak{g}_0) = G^2/G^0, \text{ the } \textit{universal } \mathcal{R} \textit{ group} \text{ of } \mathfrak{g}_0$$

$$\mathcal{R}(G) = G^1/G^0 \cong G/G^{\#} \cong \mathrm{Ad}(G)/\mathrm{Ad}(G_0) \subseteq \mathcal{U}(\mathfrak{g}_0).$$

We say that G is *small* if $\mathcal{R}(G) = \{1\}$, and *large* if $\mathcal{R}(G) = \mathcal{U}(\mathfrak{g}_0)$.

Proposition 9.5. *Suppose G contains a compact Cartan subgroup T^2, and $\phi \in (\hat{T}^2)'_\chi$. There are natural isomorphisms*

(a) $W_2^{i\mathbf{R}}(\phi)/W_0^{i\mathbf{R}}(\phi) \cong \mathcal{U}(\mathfrak{g}_0)$

(b) $W_1^{i\mathbf{R}}(\phi)/W_0^{i\mathbf{R}}(\phi) \cong \mathcal{R}(G)$ (*notation* 4.11).

Proof. Every component of G^2 meets the normalizer of $\mathfrak{t}^2$, since the compact Cartan is unique up to conjugacy by G_0. Write $\tilde{T}^2$ for the centralizer of T^2 in G^2. It is well known (and easy to check) that $\mathrm{Ad}(\tilde{T}^2) = \mathrm{Ad}(\tilde{T}_0^2)$—that is, that $\tilde{T}^2 \subseteq G^0$—and that

$$W(G^0, \tilde{T}^2) = W(\mathfrak{k}, \mathfrak{t}^2) = W_0^{i\mathbb{R}}(\gamma).$$

Therefore

$$G^2/G^0 = W(G^2, \tilde{T}^2)/W(G^0, \tilde{T}^2) = W(G^2, \tilde{T}^2)/W_0^{i\mathbb{R}}(\phi).$$

It is easy to verify that every Weyl group element preserving the notion of compact root has a representative in $G^{\mathbb{C}}$ normalizing $\mathfrak{g}_0$; so

$$W(G^2, \tilde{T}^2) = W_2^{i\mathbb{R}}(\phi).$$

This proves (a); and (b) is similar. Q.E.D.

PROPOSITION 9.6. *Suppose G contains a split Cartan subgroup $H^1 = T^1A^1$. Put*

$$(H^1)^{\#} = G^{\#} \cap H^1$$

$$\tilde{H}^1 = \text{centralizer of } \mathfrak{h}^1 \text{ in } G^2 \qquad (\text{notation } 9.4).$$

Then there are natural isomorphisms

(a) $\tilde{H}^1/[\tilde{H}^1 \cap G^0] \cong \mathfrak{U}(\mathfrak{g}_0)$

(b) $(\tilde{H}^1 \cap G^1)/(\tilde{H}^1 \cap G^0) \cong H^1/(H^1)^{\#} \cong \mathfrak{R}(G)$

(c) $\mathfrak{U}(\mathfrak{g}_0) \cong \check{P}_1(R)/(\check{L}(R) + 2\check{P}_1(R)) = Z_1(R)/2Z_1(R)$

(*notation* (3.3), *using* $R = \Delta(\mathfrak{g}^a, \mathfrak{h}^a)$). *Now suppose $\gamma \in (\hat{H}^1)'_\chi$ is integral. Recall from Proposition* 3.16 *the map*

$$\xi : W_2(\check{\delta}(\gamma)) \to Z_1(\check{R}) \cong [Z_1(R)]^{\wedge}$$

defined by the cograding $\check{\delta}(\gamma)$. Use (c) *to identify $[\mathfrak{U}(\mathfrak{g}_0)]^{\wedge}$ as a subgroup of $[Z_1(R)]^{\wedge}$ (namely the elements of order* 2); *thus we have*

(d) $\xi : W_2(\delta(\gamma)) \to [\mathfrak{U}(\mathfrak{g}_0)]^{\wedge}$.

Then

(e) $W_1(\check{\delta}(\gamma)) = \xi^{-1}(\mathrm{Ann}\,\mathfrak{R}(G))$

and so we have a natural surjections

(f) $\mathfrak{U}(\mathfrak{g}_0) \to [W_2(\check{\delta}(\gamma))/W_0(\check{\delta}(\gamma))]^{\wedge}$

(g) $\mathfrak{R}(G) \to [W_2(\check{\delta}(\gamma))/W_1(\check{\delta}(\gamma))]^{\wedge}$

If $\check{\delta}(\gamma)$ is a principal grading (Definition 6.3), *these maps are isomorphisms.*

Proof. Since the split Cartan is unique up to conjugation by G_0, every component of G^2 has a representative normalizing $\mathfrak{h}_0^1$. As all roots of $\mathfrak{h}_0^1$ are real,

$$W(G_0, H^1 \cap G_0) = W(\mathfrak{g}, \mathfrak{h}^1);$$

so in fact every component of G^2 meets $\tilde{H}^1$. Now (a) and (b) are clear. For (c), list the simple roots of $\mathfrak{h}_0^1$ in $\mathfrak{g}_0$ (for some positive system) as $\{\alpha_1, \ldots, \alpha_l\}$. To specify an automorphism of $\mathfrak{g}_0$ centralizing $\mathfrak{h}_0$, we must specify its eigenvalue on each root vector X_{α_i}; and these may be any real number. The positive real eigenvalues are realized by $\mathrm{Ad}(H_0^1)$, so

$$\mathrm{Ad}(\tilde{H}^1)/\mathrm{Ad}(\tilde{H}_0^1) \cong \check{P}_1(R)/2\check{P}_1(R),$$

the correspondence being

$$\lambda \in \check{P}_1(R) \leftrightarrow h \in \tilde{H}^1 \qquad \text{if} \quad (-1)^{\langle \alpha, \lambda \rangle} = \alpha(h)/|\alpha(h)|.$$

Now $\tilde{H}^1 \cap G^0$ is generated by $\tilde{H}_0^1$, $Z(G^0)$, and the elements m_β of (2.4). Since

$$\alpha(m_\beta) = (-1)^{\langle \alpha, \check{\beta} \rangle}$$

(by (4.4c)), (c) follows.

For (e), suppose $w \in W_2(\check{\delta}(\gamma))$. We want to show that $w \times \gamma = w \cdot \gamma$ if and only if $\xi(w)$ is trivial on $\mathfrak{R}(G)$. So fix $r \in \mathfrak{R}(G)$, and a representative x of r in T^1 (see (b)). We will show that

$$w \times \gamma(x) = (w \cdot \gamma)(x) \Leftrightarrow \xi(w)(r) = 1. \tag{$*$}$$

Clearly w can be replaced by any ww_0, with $w_0 \in W_0(\check{\delta}(\gamma))$. By Proposition 3.20, we may therefore assume that there is a superorthogonal set $\beta_1, \ldots, \beta_s$, such that

$$\check{\delta}(\gamma)(\beta_i) = 1, \qquad \text{all } i$$

$$w = \prod s_{\beta_i}$$

To say that x is a representative of r means that, if $\lambda \in \check{P}_1(R)$ is a representative of r in the isomorphism of (c), then

$$\alpha(x) = (-1)^{\langle \alpha, \lambda \rangle}, \qquad \text{all} \quad \alpha \in R.$$

By the definition of ξ (proof of Proposition 3.16),

$$\xi(w)(r) = \xi\Big(\prod s_{\beta_i}\Big)(r)$$

$$= \prod_i (-1)^{\langle \beta_i, \lambda \rangle}$$

$$\xi(w)(r) = \prod_i \beta_i(x).$$

On the other hand, by [25], proof of Lemma 8.3.17, we have

$$[(w \times \gamma)(x)][(w \cdot \gamma)(x)]^{-1} = \prod_i \beta_i(x),$$

proving (*). Parts (f) and (g) follow; and the last claim is Proposition 6.4(d). Q.E.D.

PROPOSITION 9.7. *Suppose we are in the setting* (9.3). *Consider the diagram* (*of isomorphisms*)

$$\begin{tikzcd}[column sep=small] & \mathfrak{U}(\mathfrak{g}_0) \arrow[dl] \arrow[dr] & \\ \left[W_2^{\mathbb{R}}(\gamma)/W_0^{\mathbb{R}}(\gamma) \right]^{\wedge} \arrow[rr] & & \left[W_2^{i\mathbb{R}}(\phi)/W_0^{i\mathbb{R}}(\phi) \right]. \end{tikzcd}$$

Here the map on the left if Proposition 9.6(f), *that on the right is Proposition* 9.5(a), *and that on the bottom is Corollary* 6.6. *This diagram commutes, and identifies the subgroups* $\mathfrak{R}^s$ *on the left*, $\mathfrak{R}^c$ *on the right, and* $\mathfrak{R}(G)$ *on top.*

Proof. We need only worry about commutativity; then the last assertion is a consequence of Propositions 9.5(b) and 9.6(e). For the commutativity, all maps are really defined inside $G^{\mathbf{C}}$, without direct reference to the group G. So we can reduce to the case when $\mathfrak{g}$ is simple. By Corollary 6.6, all the groups are isomorphic to $Z_1(R)$, which is of course the fundamental group of $\mathrm{Ad}(G^{\mathbf{C}})$. Since it must also be a product of copies of $\mathbf{Z}/2\mathbf{Z}$, we see by inspection of cases that $\mathfrak{U}(\mathfrak{g}_0)$ has order two except when $\mathfrak{g}$ is of type D_{2n}. Since the arrows are all isomorphisms, the diagram commutes except perhaps in this case. So suppose $\mathfrak{g}$ is of type D_{2n}; thus $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$. We may take

$$G^{\mathbf{C}} = \mathrm{SO}(4n, \mathbf{C}) \supseteq \mathrm{SO}_e(2n, 2n) = G.$$

As representatives of generators of $\mathfrak{U}(\mathfrak{g}_0) = G^2/G^0$ we can take the diagonal matrix r with entries

$$[(-1, 1, \ldots, 1), (-1, \ldots, 1)]$$

and the matrix

$$s = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Now the maps in the proposition can be computed explicitly on these elements, and one finds that the diagram commutes. The simple but tedious details are left to the reader. Q.E.D.

One can also prove this proposition by observing that the maps are in an appropriate (very restricted) sense functorial in G: one considers the category of subgroups containing both H^1 and H^2, and satisfying (9.3) for appropriate twists of ϕ and γ. One can show easily that there is such a subgroup locally isomorphic to a product of SL(2, R)'s; so the proposition is reduced to the trivial case of SL(2, R). This is a slightly more satisfactory argument, but the necessary verifications seem to me to be a little worse than for the proof given first.

In light of the discussion after its statement, Proposition 9.2 follows from Proposition 9.7.

10. Characterization of $\bar{g}^a(\gamma)$. We begin by using what we have proved so far about the structure of blocks to formalize the problem of constructing $\check{G}$.

THEOREM 10.1. *Suppose $\check{G}$ is another real reductive group, with abstract Cartan $\check{\mathfrak{h}}^a$. Fix a regular infinitesimal character $\check{\chi}$ for $\check{G}$, and a corresponding weight $\check{\lambda}^a \in (\check{\mathfrak{h}}^a)^*$ as in* (2.6). *Suppose that the integral root system $R(\check{\lambda}^a)$ is isomorphic to $\check{R}^a$, by an isomorphism taking $(\check{R}^a)^+$ to $R^+(\check{\lambda}^a)$. Fix such an isomorphism, and use it to identify W^a and $W(R(\check{\lambda}^a))$, and the span of R^a in $(\mathfrak{h}^a)^*$ with the span of $R(\check{\lambda}^a)$ in $(\check{\mathfrak{h}}^a)^*$. Write $\check{R}^a$ instead of $R(\check{\lambda}^a)$, using the isomorphism.*

Now fix blocks B, $\check{B}$ for G and $\check{G}$, with infinitesimal characters χ and $\check{\chi}$. Suppose there are elements $\gamma^1 \in B$, $\check{\gamma}^1 \in \check{B}$, such that

(a) $[\bar{g}^a(\gamma^1)]^{\vee} = \bar{g}^a(\check{\gamma}^1)$ (*Definitions* 3.23 *and* 4.12)

(b) γ^1 *is almost maximal.* (*Definition* 8.2).

Then there is a bijection ("*duality*")

$$d : \text{conjugacy classes in } B \to \text{conjugacy classes in } \check{B},$$

with the following properties. Suppose $\phi \in B$, $w \in W^a$, and $\mathfrak{s} \subseteq \mathfrak{h}^a$ is in the span of R^a; write $\mathfrak{s}$ also for the corresponding subspace of $\check{\mathfrak{h}}^a$.

(c) $[\bar{g}^a(cl(\phi))]^{\vee} = \bar{g}^a(d(cl(\phi)))$

(d) *If $\phi \in \mathfrak{D}(c_{\mathfrak{s}})$, then*

$$d\big(c_{\mathfrak{s}}(\,cl(\phi))\big) = c^{\mathfrak{s}}\big(d(\,cl(\phi))\big),$$

and similarly for $c^{\mathfrak{s}}$.

(e) $d(cl(w \times \phi)) = w \times d(cl(\phi))$.

Theorem 1.15 will be deduced fairly easily from the formal properties (c)–(e) of the bijection, using [24]. In (b), we could just as well have taken γ^1 to be almost minimal. Probably the result is true without hypothesis (b), but proving this would require a little more work.

Proof. By (a), $\check{\gamma}^1$ is almost minimal. Choose a maximal $\check{\delta}(\gamma^1)$-admissible subspace $\mathfrak{s}^0$ of $\mathfrak{h}^a$; then the corresponding subspace $\mathfrak{s}^0$ of $\check{\mathfrak{h}}^a$ is maximal $\epsilon(\check{\gamma}^1)$-admissible by (a). Fix $\gamma^2 \in c_{\mathfrak{s}^0}(\gamma^1)$. By Proposition 9.2, there is a $\check{\gamma}^2 \in c^{\mathfrak{s}^0}(\check{\gamma}^1)$, such that

$$\big[\,\bar{g}^a(\gamma^2)\big]^{\vee} = \bar{g}^a(\check{\gamma}^2).$$

By (10.1)(a) and Proposition 4.14, we get

$$\begin{aligned}
&\text{(a)} && \left[\bar{g}^a(\gamma^i)\right]^{\vee} = \bar{g}^a(\check{\gamma}^i) \\
&\text{(b)} && W_1(\gamma^i)^a = W_1(\check{\gamma}^i)^a \\
&\text{(c)} && \gamma^1 \text{ and } \check{\gamma}^2 \text{ are almost maximal} \\
&\text{(d)} && \check{\gamma}^1 \text{ and } \gamma^2 \text{ are almost minimal.}
\end{aligned} \tag{10.2}$$

We define the duality map d by

$$d\left[cp_{\mathfrak{s}^1}(w^1 \times \gamma^1) \cap rp^{\mathfrak{s}^2}(w^2 \times \gamma^2)\right] = rp^{\mathfrak{s}^1}(w^1 \times \check{\gamma}^1) \cap cp_{\mathfrak{s}^2}(w^2 \times \check{\gamma}^2)$$

whenever both sides are defined and non-empty. Using (10.2), Theorem 8.5, and Propositions 8.6 and 8.7, we see that this is a well defined bijection. By Proposition 9.2, it satisfies (10.1)(c). By Lemmas 7.11 and 7.22, it satisfies (10.1)(e). Part (d) is a little tricky, and requires two lemmas.

LEMMA 10.4. *Suppose $\mathfrak{s} = \mathfrak{u} \oplus \mathfrak{v} \subseteq \mathfrak{h}^a$ (orthogonal direct sum). Then the domain of the composition $c_{\mathfrak{v}} \cdot cp_{\mathfrak{u}}$ is*

$$\mathfrak{D}(c_{\mathfrak{v}} \cdot cp_{\mathfrak{u}}) = \mathfrak{D}(c_{\mathfrak{s}}) \cap \mathfrak{D}(c_{\mathfrak{u}}) \subseteq \mathfrak{D}(c_{\mathfrak{s}})$$

Suppose that γ is almost maximal, and lies in this domain. Then

$$c_{\mathfrak{v}}(cp_{\mathfrak{u}}(\delta)) = \{\phi \in cp_{\mathfrak{s}}(\gamma) \cap \mathfrak{D}(c^{\mathfrak{v}}) \mid c^{\mathfrak{v}}(g^a(\phi)) \cap cp_{\mathfrak{u}}(\mathfrak{g}^a(\gamma)) \neq \varnothing\}.$$

Analogous results hold for $c^{\mathfrak{v}} \circ rp^{\mathfrak{u}}$.

Proof. By Lemma 7.13, $c_{\mathfrak{v}}(cp_{\mathfrak{u}}(\gamma)) \subseteq cp_{\mathfrak{s}}(\gamma)$. So by Proposition 7.23,

$$c_{\mathfrak{v}}(cp_{\mathfrak{u}}(\gamma)) \subseteq \{\phi \in cp_{\mathfrak{s}}(\gamma) \cap \mathfrak{D}(c^{\mathfrak{v}}) \mid c^{\mathfrak{v}}(g^a(\phi)) \cap cp_{\mathfrak{u}}(g^a(\gamma)) \neq \varnothing\}.$$

So suppose ϕ belongs to the right hand side. The proof that ϕ belongs to the left side is a straightforward imitation of the proof of Proposition 8.7, except at one point. In that proposition, we knew that γ^2 (the analogue of ϕ) was almost minimal; but ϕ need not be. This was used in one place. We constructed an element ψ, and wanted to know that

$$cl(w \times \psi) = cl(\gamma^2)$$

for some $w \in W^{i\mathbb{R}}(\gamma^2)^a$. This was done in two steps (see (8.13c) and (8.14)), both of which used the almost maximality of ψ. In the present case, the assumption

$\phi \in cp_{\mathfrak{s}}(\gamma)$ (which is a single orbit of $W^{i\mathbb{R}}(\phi)^a$) will give

$$cl(w \times \psi) = cl(\phi), \qquad \text{some} \quad w \in W^{i\mathbb{R}}(\psi)^a$$

for the analogously constructed ψ. Details are left to the reader. Q.E.D.

LEMMA 10.5. *Suppose ϕ and ψ are regular characters, and $\phi \in \mathfrak{D}(c_{\mathfrak{s}})$. Then the following conditions are equivalent.*

(a) $\psi \in c_{\mathfrak{s}}(cl(\phi))$

(b) $\psi \in \mathfrak{D}(c^{\mathfrak{s}})$, $\psi \in c_{\mathfrak{s}}(cp(\phi))$, *and* $\phi \in c^{\mathfrak{s}}(rp(\psi))$.

Proof. By Proposition 7.23, (a)$\Rightarrow$(b). (Notice that $c_{\mathfrak{s}}(cp(\phi))$ is defined, by Lemma 8.2). So assume (b). Then we can write

$$\mathfrak{h}^a = \mathfrak{h}^+ \oplus \mathfrak{s} \oplus \mathfrak{h}^-,$$

an orthogonal direct sum, with

$$\begin{aligned} \mathfrak{h}^+ &= +1 \text{ eigenspace of } \theta^a(\phi) \\ \mathfrak{h}^+ \oplus \mathfrak{s} &= +1 \text{ eigenspace of } \theta^a(\psi) \\ \mathfrak{s} \oplus \mathfrak{h}^- &= -1 \text{ eigenspace of } \theta^a(\phi) \\ \mathfrak{h}^- &= -1 \text{ eigenspace of } \theta^a(\psi). \end{aligned} \tag{10.6}$$

Since $\psi \in c_{\mathfrak{s}}(cp(\phi))$, there is a w^+ such that

$$w^+ \in W(\mathfrak{h}^+), \qquad w^+ \times \psi \in c_{\mathfrak{s}}(cl(\phi)). \tag{10.7a}$$

Since $\phi \in c^{\mathfrak{s}}(rp(\psi))$, Proposition 7.23 provides an element

$$\gamma \in c_{\mathfrak{s}}(\phi) \cap rp(\psi);$$

so there is a w^- with

$$w^- \in W(\mathfrak{h}^-), \qquad (w^-)^{-1} \times \psi \in c_{\mathfrak{s}}(cl(\phi)). \tag{10.7b}$$

By Proposition 7.8, (10.7a) – (10.7b) imply that there is a w_0 with

$$w_0 \in W(\mathfrak{s}), \qquad (w^- w_0 w^+) \times cl\psi = cl\psi. \tag{10.7c}$$

By (10.6) and (10.7a)–(10.7c),

$$w^- \in W^{\mathbb{R}}(\psi)^a, \qquad w_0 w^+ \in W^{i\mathbb{R}}(\psi)^a. \tag{10.7d}$$

By Proposition 4.14 and (10.7c)–(10.7d), we deduce that

$$(w^-)^{-1} \times cl\psi = cl\psi.$$

By (10.7b), therefore,

$$\psi \in c_{\mathcal{S}}(cl(\varphi)).\quad \text{Q.E.D.}$$

We return now to the proof of (10.1d). The idea of the proof is this. Duality d is defined to respect Cayley transforms based on the almost maximal or almost minimal elements; so we must express $c_{\mathcal{S}}$ in terms of these. Lemmas 10.4 and 10.5 accomplish this. So suppose $\phi \in \mathfrak{D}(c_{\mathcal{S}})$, and ψ is another regular character. We must show that

$$\psi \in c_{\mathcal{S}}(cl(\psi)) \Leftrightarrow d(\psi) \in c^{\mathcal{S}}(d(cl(\phi))). \tag{$*$}$$

By (10.1c), we may as well assume that $\psi \in \mathfrak{D}(c^{\mathcal{S}})$, and $d(cl(\psi)) \in \mathfrak{D}(c_{\mathcal{S}})$. Write

$$cl(\phi) = cp_{\mathcal{S}^1}(w^1 \times \gamma^1) \cap rp^{s^2}(w^2 \times \gamma^2).$$

Thus

$$cp(\phi) = cp_{\mathcal{S}^1}(w^1 \times \gamma^1).$$

By Lemma 10.4, the definition of d, and (10.1c), we conclude that

$$d[c_{\mathcal{S}}\, cp(\phi)] = c^{\mathcal{S}}\, rp(d(cl(\phi))).$$

Using also the analogous formula for $c_{\mathcal{S}}$, we see that ($*$) follows from Lemma 10.5. Q.E.D.

Now we can direct all our efforts to proving the existence of a $\check{\gamma}^1$ satisfying (10.1a). To do this, we need to describe abstractly which bigradings can arise from regular characters.

PROPOSITION 10.8. *Suppose $H = TA$ is a θ-stable Cartan subgroup of G, and $\phi \in \hat{H}'_{\chi}$. Write*

$$g = g(\phi) = (\theta, \epsilon(\phi), \check{\delta}(\psi)) = (\theta, \epsilon, \check{\delta}) \in \mathcal{G}(R(\phi))$$

for the bigrading of Proposition 4.11. *Write $\check{\rho}^{i\mathbb{R}}, \rho^{\mathbb{R}}$ for the half sums of positive integral imaginary coroots (respectively, real roots); and identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using $\langle\ ,\ \rangle$.*

(a) *There is an element*

$$x \in \check{P}_1(R(\phi)) \cap (i\mathfrak{t}_0)$$

(*Definition* 3.3) *such that*

$$\langle \alpha, x + \check{\rho}^{i\mathbb{R}} \rangle \equiv \epsilon(\alpha) \qquad (\text{mod } 2), \quad \alpha \in R^{i\mathbb{R}}(\phi).$$

(b) *There is an element*

$$y \in P_1(R(\phi)) \cap \mathfrak{a}_0^*$$

such that

$$\langle \check{\alpha}, y + \rho^{\mathsf{R}} \rangle \equiv \check{\delta}(\alpha) \qquad (\mathrm{mod}\ 2), \quad \alpha \in R^{\mathsf{R}}(\phi).$$

The proof involves the following fairly well-known fact.

LEMMA 10.9. *Suppose* $\mathfrak{g}$ *is a complex reductive Lie algebra,* $\mathfrak{h} \subseteq \mathfrak{g}$ *is a Cartan subalgebra,* θ *is an involution of* $\Delta(\mathfrak{g}, \mathfrak{h})$*, and* $(\Delta^{i\mathsf{R}})^+$ *is a set of positive imaginary roots (Definition* 3.10*). Then there is an involution* σ *of* $\mathfrak{g}$ *such that*

(a) $\mathfrak{h}$ *is* σ*-stable, and* $\sigma|_{\mathfrak{h}} = \theta$

(b) *If* α *is a* $(\Delta^{i\mathsf{R}})^+$*-simple root, then* $\sigma X_\alpha = -X_\alpha$.

Proof. We proceed by induction on $|\Delta^{i\mathsf{R}}|$. Suppose first that this set is empty. Then we can find a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ stable under $-\theta$. By the functoriality of the correspondence between Dynkin diagrams and semisimple Lie algebras, we get an involution σ_0 of $\mathfrak{g}$ such that $\sigma_0\mathfrak{h} = \mathfrak{h}$, and $\sigma_0|_{\mathfrak{h}} = -\theta$. We may arrange for σ_0 to permute the simple root vectors X_α for a Chevalley basis; then σ_0 will commute with the principal automorphism σ_1 of $\mathfrak{g}$ (see [10], Proposition 14.3) which is -1 on $\mathfrak{h}$. Then $\sigma = \sigma_0\sigma_1$ has the desired properties.

Next, suppose $\Delta^{i\mathsf{R}}$ is non-empty. Choose a simple imaginary root β, and set $\tilde{\theta} = s_\beta\theta$. Then the $\tilde{\theta}$-imaginary roots are the θ-imaginary roots orthogonal to β. By induction, we can find an involution $\tilde{\sigma}$ of $\mathfrak{g}$ satisfying the conditions of the lemma for $\tilde{\theta}$ and $\tilde{\sigma}$. Write

$$\Phi_\beta : \mathrm{SL}(2, \mathrm{C}) \to \mathrm{Ad}(\mathfrak{g})$$

for the root SL(2) (compare (2.3)–(2.4)) with

$$\Phi_\beta\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in \mathrm{Ad}(\mathfrak{h})$$

Since $\tilde{\theta}\beta = -\beta$, $\tilde{\sigma}$ preserves the image of Φ_β, and pulls back to an involution of $\mathrm{SL}(2, \mathrm{C})$ which is $h \to h^{-1}$ on diagonal matrices. Therefore there is a non-zero $a \in \mathrm{C}$ such that

$$\tilde{\sigma} \cdot [\Phi_\beta(g)] = \Phi_\beta\left[\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} g \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}\right]. \tag{10.10}$$

(Here we write $\cdot$ for the action of $\tilde{\sigma}$.)

Let x be a square root of $-a^2$, and define

$$\sigma_\beta = \Phi_\beta\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix},$$

a representative of s_β in $W(\mathfrak{g}, \mathfrak{h})$. We will choose σ to be one of the two automorphisms

$$\sigma_1 = \sigma_\beta \tilde{\sigma}, \qquad \sigma_2 = \tilde{\sigma}\sigma_\beta .$$

Obviously σ_i preserves $\mathfrak{h}$, and $\sigma_i|_{\mathfrak{h}} = s_\beta \tilde{\theta} = \theta$. To see that σ_i is an involutive automorphism, we compute (say for σ_1)

$$\begin{aligned} \sigma_1^2 &= \sigma_\beta \tilde{\sigma} \sigma_\beta \tilde{\sigma} \\ &= \sigma_\beta [\tilde{\sigma} \cdot \sigma_\beta] \tilde{\sigma}^2 \\ &= \sigma_\beta [\tilde{\sigma} \cdot \sigma_\beta]. \end{aligned}$$

By (10.10),

$$\begin{aligned} \tilde{\sigma} \cdot \sigma_\beta &= \Phi_\beta\left[\begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & x \\ -x^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}\right] \\ &= \Phi_\beta \begin{pmatrix} 0 & a^2 x^{-1} \\ -a^{-2}x & 0 \end{pmatrix}. \end{aligned}$$

Since $x^2 = -a^2$, $x = -a^2 x^{-1}$; so

$$\tilde{\sigma} \cdot \sigma_\beta = \Phi_\beta \begin{pmatrix} 0 & -x^{-1} \\ x & 0 \end{pmatrix} = \sigma_\beta^{-1}.$$

Therefore,

$$\sigma_1^2 = \sigma_\beta [\sigma \cdot \sigma_\beta] = \sigma_\beta \sigma_\beta^{-1} = 1,$$

as required. Define gradings

$$\epsilon_i \in E(\Delta^{i\mathbb{R}}(\theta)), \qquad \tilde{\epsilon} \in E(\Delta^{i\mathbb{R}}(\tilde{\theta})).$$

(Definition 3.13) by

$$(-1)^{\epsilon_i(\alpha)} = \text{eigenvalue of } \sigma_i \text{ on } X_\alpha ,$$

and similarly, for $\tilde{\epsilon}$. Since

$$\sigma_1 = \sigma_\beta \sigma_\alpha \sigma_\beta^{-1},$$

it is clear that

$$\epsilon_1 = s_\beta \times \epsilon_2 .$$

We claim that also

$$\tilde{\epsilon} = c^\beta(\epsilon_i) \tag{10.11a}$$

(Definition 5.2). By Lemma 5.12 and the choice of $\tilde{\sigma}$, it will follow that one of the ϵ_i satisfies

$$\epsilon_i(\alpha) = 1, \qquad \text{all} \quad \alpha \in (\Delta^{i\mathrm{R}})^+ \text{ simple.}$$

Then $\sigma = \sigma_i$ satisfies the conclusions of the lemma. Now by Definition 5.2 and the definition of σ_i, (10.11a) is equivalent to

$$\mathrm{Ad}(\sigma_\beta)X_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \text{ and } \beta \text{ are strongly orthogonal} \\ -X_\alpha & \text{if } \alpha \text{ and } \beta \text{ are orthogonal but not} \\ & \text{strongly orthogonal.} \end{cases} \tag{10.11b}$$

Now α and β are strongly orthogonal exactly when $\Phi_\beta(SL(2, \mathbf{C}))$ fixes X_α, so the first case is obvious. In the second case, X_α is the zero weight vector in a three dimensional representation of $\Phi_\beta(SL(2, \mathbf{C}))$. Since σ_β acts by -1 on such a weight vector (as is obvious in the adjoint representation, for example), the second case follows. Q.E.D.

The dual of Lemma 10.9 is

LEMMA 10.12. *Suppose $H = TA$ is a θ-stable Cartan subgroup of G, and $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is a fixed positive system. Suppose $\lambda_0 \in \mathfrak{h}^*$ is a $\Delta^+(\mathfrak{g}, \mathfrak{h})$-dominant weight of a finite dimensional representation of G. Then there is a $\gamma_0 = (\Gamma_0, \bar{\gamma}_0) \in \hat{H}'$ such that*

(a) $\gamma_0 = \lambda_0 + \rho(\Delta^+(\mathfrak{g}, \mathfrak{h}))$

(b) *If α is a $(\Delta^{\mathrm{R}})^+$-simple root, then $\check{\delta}(\gamma_0)(\alpha) = 1$ (Definition 4.5).*

Proof. Suppose first that there are no real roots. Let $\Lambda_0 \in \hat{H}$ be a weight of a finite dimensional representation of G, with $d\Lambda_0 = \lambda_0$. Define

$$\Gamma_0|_T = \Lambda_0|_T + 2\rho(\Delta^+(\mathfrak{p}, \mathfrak{t}))$$

$$\Gamma_0|_A = \Lambda_0|_A$$

$$\bar{\gamma}_0 = \lambda_0 + \rho(\Delta^+(\mathfrak{g}, \mathfrak{h})).$$

Obviously $\gamma_0 = (\Gamma_0, \bar{\gamma}_0) \in \hat{H}'$, and satisfies the lemma. We now proceed by induction on $|\Delta^{\mathrm{R}}|$, using the Cayley transform c^β of Definition 7.16 in place of the construction used in the previous lemma. Details (which are easier in this case) are left to the reader. Q.E.D.

Lemma 10.12 is easier than Lemma 10.9 because it is obvious that every real reductive group has a fundamental series of representations. Lemma 10.9 would have been just as easy if we had invoked the dual fact that every real reductive group has a quasisplit inner form. We essentially proved that fact at the beginning of the proof of Lemma 10.9.

A more direct proof can be given for Lemma 10.12. It is shown in [24] that the regular characters satisfying (10.12a) are parametrized naturally by $(H/H_0)^\wedge$. In this parametrization, one can take γ_0 corresponding to the trivial character of H/H_0.

Proof of Proposition 10.8. For (a), use Lemma 10.9 to find an involution σ of $\mathfrak{g}$ such that $\sigma|_{\mathfrak{h}} = \theta|_{\mathfrak{h}}$, and

$$\epsilon(\sigma)(\alpha) = 1, \qquad \alpha \in R^{i\mathbb{R}}(\phi)^+ \text{simple.}$$

(Here we define $\epsilon(\sigma)$ like $\epsilon(\phi)$, using σ in place of θ. Thus if β is any imaginary root, $\sigma X_\beta = (-1)^{\epsilon(\sigma)} X_\beta$.) Then

$$\epsilon(\sigma)(\alpha) \equiv \langle \alpha, \check{\rho}^{i\mathbb{R}} \rangle \qquad (\mathrm{mod}\ 2). \tag{10.13}$$

The automorphism $\sigma\theta$ of $\mathfrak{g}$ is trivial on $\mathfrak{h}$, so it is of the form $\mathrm{Ad}(\exp(x_0))$ for some x_0 belonging to the span of the roots in $\mathfrak{h}$; that is

$$\theta = \sigma\,\mathrm{Ad}(\exp x_0) \tag{10.14a}$$

Now

$$\begin{aligned} 1 = \theta^2 &= \left[\sigma\,\mathrm{Ad}(\exp x_0)\right]^2 \\ &= \sigma^2\left[\sigma \cdot \mathrm{Ad}(\exp x_0)\right]\mathrm{Ad}(\exp x_0) \\ &= \mathrm{Ad}(\exp(\theta x_0 + x_0)). \end{aligned}$$

Thus

$$\left\langle \alpha, \frac{\theta x_0 + x_0}{2\pi i} \right\rangle \in \mathbf{Z}, \qquad \text{all} \quad \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}). \tag{10.14b}$$

Put

$$x = \frac{\theta x_0 + x_0}{2\pi i} \in \check{P}_1(\Delta(\mathfrak{g}, \mathfrak{h})); \tag{10.14c}$$

the last inclusion is (10.14b). Obviously x satisfies the first condition of (10.8)(a). For the second, suppose α is an imaginary root, and X_α a root vector. Then $\theta\alpha = \alpha$, so

$$\begin{aligned} \langle \alpha, x_0 \rangle &= \tfrac{1}{2}\langle \alpha, \theta x_0 + x_0 \rangle \\ &= i\pi\langle \alpha, x \rangle. \end{aligned} \tag{10.14d}$$

Therefore

$$\begin{aligned}(-1)^{\epsilon(\alpha)}X_\alpha &= \theta X_\alpha \\ &= \sigma\big[\mathrm{Ad}(\exp(x_0))X_\alpha\big] \\ &= \sigma\big[\exp\langle\alpha, x_0\rangle X_\alpha\big] \\ &= \sigma\big[(-1)^{\langle\alpha,x\rangle}X_\alpha\big] \qquad \text{by (10.14d)} \\ &= (-1)^{\langle\alpha,x\rangle}(-1)^{\epsilon(\sigma)(\alpha)}X_\alpha \\ (-1)^{\epsilon(\alpha)}X_\alpha &= (-1)^{\langle\alpha,x+\check{\rho}^{i\mathsf{R}}\rangle}X_\alpha \qquad \text{by (10.13).}\end{aligned}$$

This proves the second condition of (10.8)(a).

The proof of (b) is analogous, but requires a technical observation (formula (10.16) below) to get started. Choose an orthogonal set $\{\beta_1, \ldots, \beta_m\} \subseteq R^{\mathsf{R}}(\phi)$ such that

$$m \text{ is maximal for orthogonal sets of roots, and} \tag{10.15a}$$

$$\{\check{\beta}_i\} \text{ is strongly orthogonal in } \check{R}^{\mathsf{R}}(\phi). \tag{10.15b}$$

If $\check{\beta}_i \pm \check{\beta}_j \in \check{\Delta}(\mathfrak{g}, \mathfrak{h})$, then

$$\langle\check{\beta}_i \pm \check{\beta}_j, \bar{\phi}\rangle = \langle\check{\beta}_i, \bar{\phi}\rangle \pm \langle\check{\beta}_j, \bar{\phi}\rangle \in \mathbf{Z},$$

so $\check{\beta}_i \pm \check{\beta}_j \in R^{\mathsf{R}}(\phi)$. This is not the case, by (10.15b); so $\{\check{\beta}_i\}$ is strongly orthogonal in $\Delta(\mathfrak{g}, \mathfrak{h})$.

By Proposition 6.4(e), there is a principal cograding $\check{\delta}_1$ of $\Delta^{\mathsf{R}}(\mathfrak{g}, \mathfrak{h})$ so that

$$\check{\delta}_1(\beta_i) = 1, \qquad \text{all } i. \tag{10.15c}$$

By the definition of principal cograding, there is a system $(\Delta^+)^{\mathsf{R}}(\mathfrak{g}, \mathfrak{h})$ of positive real roots, such that if we write ρ_1^{R} for half its sum, then

$$\check{\delta}_1(\alpha) \equiv \langle\check{\alpha}, \rho_1^{\mathsf{R}}\rangle \qquad (\text{mod } 2). \tag{10.15d}$$

By (10.15a)–(10.15c), there is a maximal orthogonal set in $R^{\mathsf{R}}(\phi)$ which is $\check{\delta}_1|_{R^{\mathsf{R}}(\phi)}$-admissible. By the proof of Proposition 6.4(e), it follows that $\check{\delta}_1|_{R^{\mathsf{R}}(\phi)}$ is principal. The cograding $\check{\delta}_0 \in \check{E}(R^{\mathsf{R}}(\phi))$ defined by

$$\check{\delta}_0(\alpha) \equiv \langle\check{\alpha}, \rho^{\mathsf{R}}\rangle \qquad (\text{mod } 2) \tag{10.15e}$$

is obviously principal; so it differs from $\check{\delta}_1|_{R^{\mathsf{R}}(\phi)}$ by an element w of $W(R^{\mathsf{R}}(\phi))$. After acting on $(\Delta^+)^{\mathsf{R}}(\mathfrak{g}, \mathfrak{h})$ by w, we conclude that

$$\langle\rho_1^{\mathsf{R}}, \check{\alpha}\rangle \equiv \langle\rho^{\mathsf{R}}, \check{\alpha}\rangle \qquad (\text{mod } 2), \quad \text{all} \quad \alpha \in R^{\mathsf{R}}(\phi). \tag{10.16}$$

Now extend $(\Delta^+)^{\mathsf{R}}(\mathfrak{g},\mathfrak{h})$ to any positive system $\Delta^+(\mathfrak{g},\mathfrak{h})$, and fix any γ_0 as in Lemma 10.12. By (10.12b)

$$\check{\delta}(\gamma_0)(\alpha) \equiv \langle \check{\alpha}, \rho_1^{\mathsf{R}} \rangle \qquad \left(\alpha \in \Delta^{\mathsf{R}}(\mathfrak{g},\mathfrak{h})\right).$$

By (10.16),

$$\check{\delta}(\gamma_0)(\alpha) \equiv \langle \check{\alpha}, \rho^{\mathsf{R}} \rangle \qquad \left(\alpha \in R^{\mathsf{R}}(\phi)\right). \tag{10.17}$$

By Definition 4.5,

$$\begin{aligned}(-1)^{\check{\delta}(\phi)(\alpha)} &= (-1)^{\check{\delta}(\gamma_0)(\alpha)}\Gamma_0(m_\alpha)\Phi(m_\alpha)^{-1}(-1)^{\langle \check{\alpha},\, \bar{\gamma}_0 - \bar{\phi}\rangle} \\ (-1)^{\check{\delta}(\phi)(\alpha)} &= \Gamma_0(m_\alpha)\Phi(m_\alpha)^{-1}(-1)^{\langle \check{\alpha}, \gamma_0 - \phi + \rho^{\mathsf{R}}\rangle}\end{aligned} \tag{10.18a}$$

for $\alpha \in R^{\mathsf{R}}(\phi)$; the second equality uses (10.17). Since G is linear and T is compact, there is a weight $\Lambda \in \check{H}$ of a finite dimensional representation of G, such that

$$\Lambda|_T = [\Gamma_0 - \Phi]|_T. \tag{10.18b}$$

Write $\lambda = d\Lambda$. By (4.4c),

$$\Lambda(m_\alpha) = (-1)^{\langle \check{\alpha}, \lambda \rangle}. \tag{10.18c}$$

Now (10.18) gives

$$(-1)^{\check{\delta}(\phi)(\alpha)} = (-1)^{\langle \check{\alpha}, \lambda - (\gamma_0 - \phi) + \rho^{\mathsf{R}} \rangle} \tag{10.19}$$

Let y be the projection of $\lambda - (\bar{\gamma}_0 - \bar{\phi})$ on the span of the roots. By (10.18b), $y \in \mathfrak{a}_0^*$. As the differential of a weight of a finite dimensional representation, λ is integral; and $\bar{\gamma}_0$ is as well by Lemma 10.12(a). So the integral roots for y are exactly $R(\phi)$, proving the first condition of Proposition 10.8(b). The second condition is (10.19). Q.E.D.

Finally, we need to describe the $\mathfrak{R}$ groups which can be attached to regular characters.

PROPOSITION 10.20. *Suppose $H = TA$ is a θ-stable Cartan subgroup of G, and $\gamma \in \hat{H}'_\chi$. Use the notation of Proposition* 4.11. *Choose a set $\{\alpha_1, \ldots, \alpha_r\}$ of superorthogonal imaginary roots for $\epsilon(\gamma)$ as in Proposition* 3.14, *and $\{\beta_1, \ldots, \beta_s\}$ superorthogonal real roots for $\check{\delta}(\gamma)$ similarly. Define*

$$\begin{aligned}\Phi &= \{\beta \in \Delta(\mathfrak{g},\mathfrak{h}) \mid \beta|_{\mathfrak{t}} \in \operatorname{span}\{\alpha_i\}\} \\ &= \{\beta \mid s_{\alpha_1} \cdots s_{\alpha_r}\theta\beta = -\beta\} \\ \Psi &= \{\beta \in \Delta(\mathfrak{g},\mathfrak{h}) \mid \beta|_{\mathfrak{a}} \in \operatorname{span}\{\beta_j\}\}\end{aligned}$$

For $\beta \in \Phi$, put

$$\sigma_\beta = \prod (s_{\alpha_i})^{\langle \alpha_i, \check{\beta} \rangle} \in W^{i\mathbf{R}}(\gamma)$$

and for $\beta \in \Psi$, put

$$\tau_\beta = \prod_j (s_{\beta_j})^{\langle \check{\beta}_j, \beta \rangle} \in W^{\mathbf{R}}(\gamma)$$

Put

$$Q_+ = \text{group generated by } \{\sigma_\beta \mid \beta \in \Phi\}$$

$$\check{Q}_+ = \text{group generated by } \{\tau_\beta \mid \beta \in \Psi\}$$

$$W_+^{i\mathbf{R}}(\gamma) = Q_+ W_0^{i\mathbf{R}}(\gamma)$$

$$W_+^{R}(\gamma) = \check{Q}_+ W_0^{\mathbf{R}}(\gamma).$$

(a) *For all $\beta \in \Phi$, $\sigma_\beta \in W_1^{i\mathbf{R}}(\gamma)$; that is*

$$W_0^{i\mathbf{R}}(\gamma) \subseteq W_+^{i\mathbf{R}}(\gamma) \subseteq W_1^{iR}(\gamma) \subseteq W_2^{iR}(\gamma)$$

(b) $$W_0^{\mathbf{R}}(\gamma) \subseteq W_+^{\mathbf{R}}(\gamma) \subseteq W_1^{\mathbf{R}}(\gamma) \subseteq W_2^{\mathbf{R}}(\gamma)$$

If G is small (Definition 9.4), then

(c) $$W_1^{\mathbf{R}}(\gamma) = W_2^{\mathbf{R}}(\gamma)$$

(d) $$W_1^{i\mathbf{R}}(\gamma) = W_+^{i\mathbf{R}}(\gamma).$$

If G is large (Definition 9.4), then

(e) $$W_1^{i\mathbf{R}}(\gamma) = W_2^{i\mathbf{R}}(\gamma)$$

(f) $$W_1^{\mathbf{R}}(\gamma) = W_+^{\mathbf{R}}(\gamma),$$

In order to reduce this result to subgroups of G, we need an elementary and well-known structural fact.

LEMMA 10.21. *Suppose $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$, $\mathfrak{a}_0 \subseteq \mathfrak{p}_0$ are abelian subalgebras of $\mathfrak{g}_0$. Write $G^{\mathfrak{t}_0}$, $G^{\mathfrak{a}_0}$ for their respective centralizers.*
(a) *If G is small, so is $G^{\mathfrak{t}_0}$.*
(b) *If G is large, so is $G^{\mathfrak{a}_0}$.*

Proof. For (a), recall that G is small exactly when $\mathrm{Ad}(G)$ is connected. Now $\mathrm{Ad}(G^{\mathfrak{t}_0})$ is obviously a homomorphic image of $\mathrm{Ad}(G)^{\mathfrak{t}_0}$; so it is enough to prove that $G^{\mathfrak{t}_0}$ is connected whenever G is. But this is immediate from the Cartan decomposition $G = K \cdot \exp(\mathfrak{p}_0)$ and the corresponding fact for compact groups.

For (b), we may as well assume $G = \mathrm{Ad}(G) \subseteq \mathrm{Ad}(\mathfrak{g}) = G^{\mathbf{C}}$. Extend $\mathfrak{a}_0$ to a maximal abelian subalgebra $\mathfrak{a}_0^m$ of $\mathfrak{p}_0$. Define

$$F = \left\{ x \in \exp(i\mathfrak{a}_0^m) \,|\, x^2 = 1 \right\} \subseteq G^{\mathbf{C}}.$$

Suppose $x \in F$. Since $x \in \exp(i\mathfrak{a}_0^m)$, $\bar{x} = x^{-1}$; here bar is the complex structure. Since $x^2 = 1$, it follows that $x = \bar{x}$. So x normalizes $\mathfrak{g}_0$, and (since G is large), $F \subseteq G$; and therefore also $F \subseteq G^{\mathfrak{a}_0}$.

Now suppose $g \in G^{\mathbf{C}}$ normalizes $G^{\mathfrak{a}_0}$; we will show that

$$\mathrm{Ad}(g)|_{\mathfrak{g}_0^{\mathfrak{a}_0}} \in \mathrm{Ad}(G_0^{\mathfrak{a}_0}F) \subseteq \mathrm{Ad}(G^{\mathfrak{a}_0}), \tag{$*$}$$

proving that $G^{\mathfrak{a}_0}$ is large. Extend $\mathfrak{a}_0^m$ to a Cartan subalgebra $\mathfrak{h}_0^m = \mathfrak{t}_0^m + \mathfrak{a}_0^m \subseteq \mathfrak{g}_0^{\mathfrak{a}_0}$. Obviously it is maximally split, and therefore unique up to conjugation in $G_0^{\mathfrak{a}_0}$; so we may assume (after changing g inside $gG_0^{\mathfrak{a}_0}$) that g normalizes $\mathfrak{h}_0^m$. By Proposition 4.16

$$W(G, H^m) = W(G_0, H^m \cap G_0);$$

so we may even assume that g centralizes $\mathfrak{h}_0^m$. Therefore g belongs to $(H^m)^{\mathbf{C}}$. Write $g = \exp(X)$, $X \in \mathfrak{h}^m$.

Since $\mathrm{Ad}(g)$ normalizes $\mathfrak{m}_0 = \mathfrak{g}_0^{\mathfrak{a}^m}$, the (necessarily imaginary) roots of $\mathfrak{h}^m$ in $\mathfrak{m}$ must take imaginary values on X. So we may modify X by an element of $\mathfrak{t}_0^m$ (which doesn't change gG_0^0) and assume that X centralizes $\mathfrak{m}$. Choose a system of positive restricted roots (roots of $\mathfrak{a}^m$ in $\mathfrak{g}$) so that $\Delta(\mathfrak{g}^{\mathfrak{a}}, \mathfrak{a}^m)$ is spanned by simple roots $\bar{\alpha}_1, \ldots, \bar{\alpha}_r$. List the remaining simple restricted roots as $\bar{\alpha}_{r+1}, \ldots, \bar{\alpha}_{r+s}$. Write

$$\mathfrak{g}_0 = \mathfrak{m}_0 \oplus \sum_{\bar{\alpha} \in \Delta(\mathfrak{g}, \mathfrak{a}^m)} \mathfrak{g}_0^{\bar{\alpha}}$$

$$\mathfrak{g}_0^{\mathfrak{a}_0} = \mathfrak{m}_0 \oplus \sum_{\bar{\alpha} \in \Delta(\mathfrak{g}^{\mathfrak{a}}, \mathfrak{a}^m)} \mathfrak{g}_0^{\bar{\alpha}}.$$

Now the group $M_0 \subseteq G_0^{\mathfrak{a}_0}$ has an open orbit on each $\mathfrak{g}_0^{\bar{\alpha}}$; so since g centralizes $\mathfrak{m}$, $\mathrm{Ad}(g)$ is a non-zero real number $\bar{\alpha}(g)$ on each $\mathfrak{g}_0^{\bar{\alpha}}$, for $\bar{\alpha} \in \Delta(\mathfrak{g}^{\mathfrak{a}}, \mathfrak{a}^m)$. (For the roots not in $\mathfrak{g}^{\mathfrak{a}}$, g may not preserve the *real* root space $\mathfrak{g}_0^{\bar{\alpha}}$, so we cannot define $\bar{\alpha}(g)$ in general.) After modifying g by A^m, we may assume that all the $\bar{\alpha}(g)$ are ± 1. Now choose $Y \in i\mathfrak{a}_0^m$ so that

$$\langle \bar{\alpha}_i, Y \rangle = \sqrt{-1}\,\pi, \qquad i = 1, \ldots, r \quad \text{and} \quad \bar{\alpha}_i(g) = -1$$

$$\langle \bar{\alpha}_i, Y \rangle = 0 \qquad \text{otherwise}$$

Then $g' = \exp(Y)$ has the following properties. First, $\mathrm{Ad}(g')$ agrees with $\mathrm{Ad}(g)$

on all the $\mathfrak{g}_0^{\alpha_i}$ ($i = 1, \cdots, r$) and so on all of $\mathfrak{g}_0^{\mathfrak{a}_0}$. Second, g' obviously belongs to F. This proves (*). Q.E.D.

The following proof assumes some familiarity with the structure of $\mathfrak{R}$ groups in products of $SL(2, \mathbf{R})$'s—see [25], Section 1.5.

Proof of Proposition 10.20. Clearly it suffices to prove (c)–(f). For (c), let L be the centralizer of T_0 in G. By Lemma 10.21(a), L is also small. We can replace γ by a regular character $\gamma_{\mathfrak{q}}$ for L, defined in Definition 7.3. (Of course, we have to choose a $\mathfrak{q}$.) This does not change $W_2^{\mathbf{R}}$ (Lemma 7.4(a)) and it is easy to check that it does not change $W_1^{\mathbf{R}}$ either. So we may as well assume that H is maximally split on G. Since $\mathfrak{R}(G) = \{1\}$, (c) follows from Proposition 9.6(g). For (e), we consider the centralizer of A_0 in G, and use Proposition 9.5 instead.

Now consider (d). Let $\mathfrak{t}^1$ be the orthogonal complement of the span of the α_i, and let G^1 be the centralizer of $\mathfrak{t}^1$ in G. By Proposition 3.14,

$$W_1^{i\mathbf{R}}(\gamma) = W_0^{i\mathbf{R}}(\gamma)\big(W_1^{i\mathbf{R}}(\gamma) \cap W(\mathfrak{g}^1, \mathfrak{h})\big).$$

In this way, and again using Definition 7.3, we can reduce to the case $G = G^1$; for this does not change the root system Φ. Since α_i are superorthogonal, we now have

$$R^{i\mathbf{R}}(\gamma) = \{\pm\alpha_i\}$$

and all of these are noncompact. Write $\mathfrak{u}$ for the span of the α_i. Define a new Cartan subgroup

$$H^1 = c^{\mathfrak{u}}(H) = T^1 A^1.$$

(Definition 7.15); this $\mathfrak{t}^1$ is really the same as the one defined above. Since T_0^1 is central in G, H^1 is split; that is, all roots are real. Write $\{\tilde{\alpha}_i\}$ for the real roots corresponding to the α_i. Define maps

$$\xi : T^1 \to \{\pm 1\}^r, \qquad \xi(t) = (\tilde{\alpha}_i(t))$$

$$\zeta : W_1^{i\mathbf{R}}(\gamma) \to \{\pm 1\}^r, \qquad \zeta(w) = \big((-1)^{m_i}\big) \qquad \text{if} \quad w = \prod s_{\alpha_i}^{m_i}.$$

Both these maps are related only to the group G^A, which is locally a product of r copies of $SL(2, \mathbf{R})$. By [25], (1.5.21), ξ and ζ have the same image. Since G is small, T^1 is generated by T_0^1, $Z(G)$, and the various $m_{\tilde{\beta}}$, for $\tilde{\beta} \in \Delta(\mathfrak{g}, \mathfrak{h}^1)$. Clearly ξ is trivial on T_0^1 and $Z(G)$; and

$$\begin{aligned}\xi(m_{\tilde{\beta}}) &= (\tilde{\alpha}_i(m_{\tilde{\beta}}))\\ &= \big((-1)^{\langle \tilde{\alpha}_i, \check{\tilde{\beta}}\rangle}\big)\\ &= \zeta(\sigma_\beta).\end{aligned}$$

with $\beta \in \Delta(\mathfrak{g}, \mathfrak{h})$ the root corresponding to $\tilde{\beta}$. This proves (d).

For (f), set $\mathfrak{a}^1$ = orthogonal complement of the β_j. Just as in (d), we may pass to the centralizer of $\mathfrak{a}^1$. As this centralizer has A^1 as a direct factor, we may even assume $\mathfrak{a}^1 = \{0\}$. Thus

$$R^{\mathsf{R}}(\gamma) = \Delta^{\mathsf{R}}(\mathfrak{g}, \mathfrak{h}) = \{\pm\beta_j\},$$

and

$$\check{\delta}(\gamma)(\beta_j) = 1, \qquad \text{all } j.$$

Again it is helpful to write $\mathfrak{s}$ for the span of the β_j, and to introduce

$$c_{\mathfrak{s}}(H) = T^1$$

as in Definition 7.1. Write $\tilde{\beta}_j$ for the (noncompact) roots of $\mathfrak{t}^1$ in $\mathfrak{g}$ corresponding to the β_j. Define

$$\xi : T \to \{\pm 1\}^r, \qquad \xi(t) = (\beta_i(t))$$

using the β_i as before. The study of $W_1^{\mathsf{R}}(\gamma)$ can be reduced to the centralizer of T_0 in G, which is a product of copies of SL(2, R). If we identify W^{R} with the dual group of $\{\pm 1\}^r$ in the obvious way, then

$$W_1^{\mathsf{R}}(\gamma) = \operatorname{Ann}\xi(T)$$

([25], Lemma 1.5.22). We can also define

$$\zeta : \left(W(G, T^1) \cap W(\{\pm\tilde{\beta}_i\})\right) \to \{\pm 1\}^r$$

as above; and ζ and ξ have the same image. So we have to show that the annihilator of the image of ζ is generated exactly by the various τ_β. So suppose $w \in W(\{\pm\tilde{\beta}_i\})$. Since G is large, $w \in W(G, T^1)$ if and only if w preserves the notion of compact. Write

$$w = \prod_{j \in A} s_{\tilde{\beta}_j}.$$

Then

$$w\tilde{\beta} = \tilde{\beta} - \sum_{j \in A} \langle \check{\tilde{\beta}}_j, \tilde{\beta} \rangle \tilde{\beta}_j.$$

Since all the $\tilde{\beta}_j$ are noncompact, we see that

$$w \in W(G, T^1) \Leftrightarrow \sum_{j \in A} \langle \check{\beta}_j, \tilde{\beta} \rangle \in 2\mathsf{Z}, \qquad \text{all } \tilde{\beta}.$$

On the other hand, in the pairing of W^{R} with $\{\pm 1\}^r$ mentioned earlier,

$$\langle \zeta(w), \tau_\beta \rangle = \prod_{j \in A} (-1)^{\langle \check{\beta}_j, \beta \rangle};$$

so

$$w \in W(G, T^1) \Leftrightarrow \langle \zeta(w), \tau_\beta \rangle = 1, \qquad \text{all } \beta,$$

as we wished to show. Q.E.D.

11. Existence of $\check{G}$.

THEOREM 11.1. *Suppose $\gamma \in \hat{H}'_\chi$. Let $\check{G}^{\mathbf{C}}$ denote a complex simply connected semisimple Lie group with root system $\check{R}^a$ (see (2.6)). More precisely, we fix in the Lie algebra $\check{\mathfrak{g}}$ of $\check{G}^{\mathbf{C}}$ a Cartan subalgebra $\check{\mathfrak{h}}^a$; and we fix once and for all an identification of $\check{\mathfrak{h}}^a$ with the span of R^a in $\mathfrak{h}^a$, taking $\Delta(\check{\mathfrak{g}}^{\mathbf{C}}, \check{\mathfrak{h}}^a)$ to $\check{R}^a$. In particular, we identify W^a with $W(\check{\mathfrak{g}}^{\mathbf{C}}, \check{\mathfrak{h}}^a)$.*

(a) *There is a real form $\check{\mathfrak{g}}_0$ of $\check{\mathfrak{g}}$, with Cartan involution $\check{\theta}$ and a $\check{\theta}$-stable Cartan subalgebra $\check{\mathfrak{h}}_0$, such that we have an isomorphism*

$$\check{\mathfrak{h}} \xrightarrow{\ i\ } \textit{span of } R(\gamma) \textit{ in } \mathfrak{h}$$

satisfying

(1) $i(\Delta(\check{\mathfrak{g}}, \check{\mathfrak{h}})) = \check{R}(\gamma)$

(2) $i(\check{\theta}|_{\check{\mathfrak{h}}}) = -\theta|_{\textit{span of } R(\gamma)}$

If we write ϵ for the grading of the $\Delta(\check{\mathfrak{g}}, \check{\mathfrak{h}})$-imaginary roots given by $\check{\theta}$, then ϵ induces a cograding $i(\epsilon)$ on $R^{\mathbf{R}}(\gamma)$ by (1). *We can arrange*

(3) $i(\epsilon) = \check{\delta}(\gamma)$

(b) *Define*

$$\check{G}^2 = \left\{ x \in \check{G}^{\mathbf{C}} \mid \mathrm{Ad}(x) \textit{ normalizes } \check{\mathfrak{g}}_0 \right\}$$

$$\check{H}^2 = \textit{centralizer of } \check{\mathfrak{h}} \textit{ in } \check{G}^2$$

Then we can find a regular character $\check{\gamma}^2$ of $\check{H}^2$, with regular integral infinitesimal character satisfying

$$[g^a(\gamma)]^{\vee} = g^a(\check{\gamma}^2)$$

(*Definition* 3.23).

(c) *There is a group $\check{G} \subseteq \check{G}^2$ of finite index having the following property. Write $\check{H} = \check{H}^2 \cap \check{G}$, $\check{\gamma} = \check{\gamma}^2|_{\check{H}}$. Then*

$$W_1^{\mathbf{R}}(\check{\gamma})^a = W_1^{i\mathbf{R}}(\gamma)^a.$$

In particular, if γ is almost minimal, we get

$$[\bar{g}^a(\gamma)]^{\vee} = \bar{g}^a(\check{\gamma}).$$

Proof. Let $\check{\mathfrak{h}}$ be any Cartan subalgebra of $\check{\mathfrak{g}}$. Using appropriate inner automorphisms to identify $\check{\mathfrak{h}}$ with $\check{\mathfrak{h}}^a$ and $\mathfrak{h}$ with $\mathfrak{h}^a$, we get

$$i : \check{\mathfrak{h}} \to \text{span of } R(\gamma) \text{ in } \mathfrak{h},$$

$$i : \Delta(\check{\mathfrak{g}}, \check{\mathfrak{h}}) \to R(\gamma)$$

Since $R(\gamma)$ is θ-stable, we can define $\check{\theta}$ on $\check{\mathfrak{h}}$ to satisfy (a2) of the theorem. Now use Lemma 10.9 to construct an involution σ of $\mathfrak{g}$, satisfying

$$\sigma|_{\check{\mathfrak{h}}} = \theta$$

$$\sigma X_{\check{\alpha}} = -X_{\check{\alpha}}, \qquad \check{\alpha} \in \left[\Delta^{i\mathsf{R}}(\check{\mathfrak{g}}, \check{\mathfrak{h}})\right]^+ \text{ simple.}$$

Here we use the identification

$$i : \Delta^{i\mathsf{R}}(\check{\mathfrak{g}}, \check{\mathfrak{h}}) \cong \check{R}^{\mathsf{R}}(\gamma)$$

to define $(\Delta^{i\mathsf{R}})^+$. If $\epsilon(\sigma)$ is the grading of $\Delta^{i\mathsf{R}}$ defined by σ, then the corresponding cograding $\check{\delta}_0$ of $R^{\mathsf{R}}(\gamma)$ obviously satisfies

$$\check{\delta}_0(\alpha) \equiv \langle \check{\alpha}, \rho^{\mathsf{R}} \rangle \qquad (\text{mod } 2).$$

By Proposition 10.8(b), there is an element $y \in \mathfrak{a}_0^*$, satisfying

$$\begin{aligned} &\langle \check{\alpha}, y \rangle \in \mathbf{Z}, \qquad \text{all} \quad \alpha \in R(\gamma) \\ &\check{\delta}(\alpha) \equiv \check{\delta}_0(\alpha) + \langle \alpha, \gamma \rangle \qquad (\text{mod } 2). \end{aligned} \tag{11.2}$$

Let $\check{y}$ denote the element of $\check{\mathfrak{h}}$ corresponding to y under the map i. Then

$$\begin{aligned} &\sigma\check{y} = \check{\theta}\check{y} = \check{y} \\ &\langle \beta, \check{y} \rangle \in \mathbf{Z}, \qquad \text{all} \quad \beta \in \Delta(\check{\mathfrak{g}}, \check{\mathfrak{h}}) \end{aligned} \tag{11.3}$$

Define

$$\theta = \sigma \circ \operatorname{Ad}(\exp i\pi\check{y}).$$

By the first part of (11.3), σ and $\operatorname{Ad}(\exp i\pi y)$ commute; so by the second, θ is an involution. Because of the second formula in (11.2), it is easy to check (a3) of the theorem. Any involution of a complex semisimple Lie algebra is the complexified Cartan involution for a unique real form, so the rest of (a) follows. The proof of (b) is entirely similar, using Lemma 10.12 and Proposition 10.8(a). At some point one has to lift an integral weight of $\check{\mathfrak{h}}$ to a weight of a finite dimensional representation of $\check{G}^{\mathbf{C}}$; there one uses the assumption that $G^{\mathbf{C}}$ is simply connected. Details may be left to the reader.

For (c), a generalization of Proposition 9.6(e) is needed.

LEMMA 11.4. *Suppose $H^2 = TA^2$ is a θ-stable Cartan subgroup of a real reductive group G^2, and $\gamma^2 \in (H^2)'_{\chi}$. Then we can find a subgroup $G^1 \subseteq G^2$, of finite index, with the following properties. Write*

$$H^1 = H^2 \cap G^1, \qquad \gamma^1 = \gamma^2|_{H^1}.$$

(a) $W_1^{\mathsf{R}}(\gamma^1) = W_1^{\mathsf{R}}(\gamma^2)$

(b) *There is a natural surjective homomorphism*

$$s : G^1 \to \left[W_2^{\mathsf{R}}(\gamma^1)/W_1^{\mathsf{R}}(\gamma^1)\right]^{\wedge}$$

(c) *Suppose $G \subseteq G^1$ is a subgroup of finite index, and $H = H^1 \cap G$, $\gamma = \gamma^1|_H$. Then*

$$W_1^{\mathsf{R}}(\gamma) = \left\{ w \in W_2^{\mathsf{R}}(\gamma^1) \,|\, [s(g)](w) = 1, \textit{all } g \in G \right\}$$

(d) *Suppose X is any subgroup of $W_2^{\mathsf{R}}(\gamma^1)$ containing $W_1^{\mathsf{R}}(\gamma^1)$. Then there is a subgroup G as in* (c), *such that $W_1^{\mathsf{R}}(\gamma) = X$.*

Proof. There are natural maps $\pi : G^2 \to G^2/G_0$, $i : H^2/(H^2 \cap G_0) \to G^2/G_0$ which are surjective and injective respectively. Put

$$G^1 = \pi^{-1}\big(i\big(H^2/H^2 \cap G_0\big)\big).$$

Then $H^1 = H^2$, and for every $G \subseteq G^1$ as in (c), there is a natural surjection

$$G \to H/H \cap G_0. \tag{11.5}$$

Put L = centralizer of T_0 in G^1. Choose a θ-stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subseteq \mathfrak{g}$, and use it to construct a regular character $\gamma_{\mathfrak{q}}^1$ for L (Definition 7.3). It is clear from the definition that if $w \in W^{\mathsf{R}}(\gamma^1) \subseteq W(L, H)$, then

$$(w \times \gamma^1)_{\mathfrak{q}} = w \times (\gamma_{\mathfrak{q}}^1), \qquad (w \cdot \gamma_{\mathfrak{q}}^1) = w \cdot (\gamma_{\mathfrak{q}}^1).$$

By Lemma 7.4,

$$W_i^{\mathsf{R}}(\gamma_{\mathfrak{q}}^1) = W_i^{\mathsf{R}}(\gamma^1) \qquad (i = 1, 2).$$

Fix $h \in H^1$. Then for $w \in W^{\mathsf{R}}(\gamma^1)$,

$$\left[(w \times \Gamma_{\mathfrak{q}}^1)(h)\right]\left[(w \cdot \Gamma_{\mathfrak{q}}^1)(h)\right]^{-1} = \left[(w \times \Gamma^1)(h)\right]\left[(w \cdot \Gamma^1)(h)\right]^{-1} \tag{11.6}$$

by Definition 7.3. Denote their common value by $[\tilde{s}(h)](w)$. Then $\tilde{s}$ defines a group homomorphism

$$\tilde{s} : H^1 \to \left[W_2^{\mathsf{R}}(\gamma)\right]^{\wedge}$$

(It is not obvious that $\tilde{s}(h)$ is a character of $W_2^{\mathbb{R}}(\gamma)$; but by (11.6), it is enough to check this for L. There it follows from the definition of the homomorphism of Proposition 9.6(e).) If G is as in (c), then clearly

$$\begin{aligned} W_1^{\mathbb{R}}(\gamma) &= \operatorname{Ann}(\tilde{s}(H)) \\ &= \{ w \in W_2^{\mathbb{R}}(\gamma) \mid [\tilde{s}(h)](w) = 1, \text{ all } h \in H^0 \}. \end{aligned} \tag{11.7}$$

Taking $G = G_0$ (the identity component of G^1) and applying Proposition 10.20(c), we conclude in particular that $\tilde{s}$ is trivial on $H^1 \cap G_0$; so it is defined on $H^1/H^1 \cap G_0$. Taking $G = G^1$, we see that $\operatorname{Ann}(\tilde{s}(H)) = W_1^{\mathbb{R}}(\gamma)$; so

$$\tilde{s} : H/H \cap G_0 \to [W_2^{\mathbb{R}}(\gamma)/W_1^{\mathbb{R}}(\gamma)]^{\wedge},$$

a surjection. Composing $\tilde{s}$ with the surjection of (11.5), we get the map s of the Proposition. Parts (a)–(c) are clear from (11.7), and (d) is a consequence of (c): take

$$G = s^{-1}(\operatorname{Ann} X). \quad \text{Q.E.D.}$$

To prove Theorem 11.1(c), recall from Proposition 10.20(a) that

$$W_+^{i\mathbb{R}}(\gamma) \subseteq W_1^{i\mathbb{R}}(\gamma) \subseteq W_2^{i\mathbb{R}}(\gamma).$$

The group $W_2^{i\mathbb{R}}(\gamma)$ depends only on the grading $\epsilon(\gamma)$ (Definition 3.13); so condition (b) of Theorem 11.1 says that

$$W_2^{i\mathbb{R}}(\gamma) = W_2^{\mathbb{R}}(\check{\gamma}^2).$$

The group $W_+^{i\mathbb{R}}(\gamma)$ is defined using all of $\Delta(\mathfrak{g}, \mathfrak{h})$ (and not just the integral roots). If we had used only $R(\gamma)$, we would obviously have gotten a smaller group

$$X \subseteq W_+^{i\mathbb{R}}(\gamma).$$

By the symmetry of the definitions in Proposition 10.20, and Theorem 11.1(b), we see that (if we transfer everything to $\mathfrak{h}^a$)

$$X^a = W_+^{\mathbb{R}}(\check{\gamma})^a.$$

This shows that

$$W_+^{\mathbb{R}}(\check{\gamma}^2)^a \subseteq W_1^{i\mathbb{R}}(\gamma)^a \subseteq W_2^{\mathbb{R}}(\check{\gamma}^2)^a. \tag{11.8}$$

By its definition, $\check{G}^2$ is large (Definition 9.4). By Proposition 10.20(f),

$$W_1^{\mathbb{R}}(\check{\gamma}^2) = W_+^{\mathbb{R}}(\check{\gamma}^2).$$

By Lemma 11.4(d) and (11.8), we can find $G \subseteq G^2$ of finite index, so that

$$W_1^{\mathbb{R}}(\check{\gamma})^a = W_1^{i\mathbb{R}}(\gamma)^a. \quad \text{Q.E.D.}$$

THEOREM 11.9. *Suppose G is a real reductive group (cf. (2.1)) and $B = \{\bar{\pi}(\gamma_1), \ldots, \bar{\pi}(\gamma_r)\}$ is a block of irreducible admissible representations of G having the fixed nonsingular infinitesimal character χ of (2.6b). Write $\check{\mathfrak{h}}^a$ for the span of the abstract integral roots R^a. Fix a complex simply connected semisimple Lie group $\check{G}^{\mathbf{C}}$, with Lie algebra $\check{\mathfrak{g}}$, having abstract Cartan subalgebra $\check{\mathfrak{h}}^a \subseteq \check{\mathfrak{g}}$ and root system*

$$\Delta(\check{\mathfrak{g}}, \check{\mathfrak{h}}^a) = \check{R}^a.$$

(We use this identification constantly below.) Then there is a real form $\check{\mathfrak{g}}_0$ of $\check{\mathfrak{g}}$, and a subgroup $\check{G}$ of $\check{G}^{\mathbf{C}}$, with Cartan involution $\check{\theta}$; and a block $\check{B} = \{\bar{\pi}(\check{\gamma}_1), \ldots, \bar{\pi}(\check{\gamma}_r)\}$ of irreducible admissible representations of $\check{G}$, having the following properties. Write $d\colon B \to \check{B}$ for the bijection taking γ_i to $\check{\gamma}_i$.

(a) *$\bar{\pi}(\check{\gamma}_i)$ has nonsingular integral infinitesimal character—that is, the same infinitesimal character as some finite dimensional representation of $\check{G}$.*

(b) *The strong bigradings of R^a and $\check{R}^a$ defined by γ_i and $\check{\gamma}_i$ (Definition 4.12) are dual in the sense of Definition 3.23:*

$$\left[\bar{g}^a(\gamma_i)\right]^{\vee} = \bar{g}^a(\check{\gamma}_i), \qquad \text{all } i.$$

(c) *If $\mathfrak{s} \subseteq \check{\mathfrak{h}}^a \cong \mathfrak{h}^a$, then $\gamma_i \in \mathfrak{D}(c_{\mathfrak{s}})$ if and only if $\check{\gamma}_i \in \mathfrak{D}(c^{\mathfrak{s}})$ (Definitions 7.10 and 7.16); and in that case*

$$d\big(c_{\mathfrak{s}}(\,cl(\gamma_i))\big) = c^{\mathfrak{s}}(\,cl(\check{\gamma}_i))$$

That is,

$$c_{\mathfrak{s}}(\,cl(\gamma_i)) = \left\{\, cl\gamma_j \mid cl\check{\gamma}_j \in c^{\mathfrak{s}}(\,cl(\check{\gamma}_i))\right\}.$$

Analogous results hold for $c^{\mathfrak{s}}$.

(d) *If $w \in W^a$, then*

$$d\big(\,cl(w \times \gamma_i)\big) = cl(w \times \check{\gamma}_i).$$

This is immediate from Theorems 10.1 and 11.1: given B, we fix an almost minimal $\gamma_{i_0} \in B$, and construct $\check{G}$, $\check{\gamma}_{i_0}$ using Theorem 11.1, to satisfy (a) and (b) for $i = i_0$. Then the existence of the bijection d with the stated properties is Theorem 10.1.

12. The Kazhdan–Lusztig conjecture

Definition 12.1. Fix a block B of regular characters for G with infinitesimal character χ, and a constant $c_0 \in (1/2)\mathbf{Z}$. If $H = TA$ is a θ-stable Cartan subgroup of G, and $\gamma \in \hat{H}'_\chi$, define the *integral length* of γ by

$$l^I(\gamma) = \tfrac{1}{2}|\{\alpha \in R^+(\bar{\gamma}) \mid \theta\alpha \notin R^+(\bar{\gamma})\}| + \tfrac{1}{2}\dim A - c_0.$$

We assume c_0 is such that $l^I(\gamma)$ is always an integer. In [25], Definition 8.1.4, $l^I(\gamma)$ was defined with c_0 equal to half the dimension of the split part of a fundamental Cartan in G. It was asserted incorrectly there that this makes $l^I(\gamma)$ an integer; that is the case if γ is integral, but not in general. That there is such a choice of c_0 follows from Lemma 8.6.13 and Theorem 9.2.11 of [25]; these show that for any c_0,

$$l^I(\gamma) \equiv l^I(\gamma') \quad (\text{mod } \mathbf{Z}), \quad \text{all} \quad \gamma, \gamma' \in B.$$

It is inconvenient to normalize l^I in any uniform way: to make some formulas nicer, we want to use different kinds of normalizations for G and $\check{G}$.

Definition 12.2. Suppose $\gamma \in \hat{H}'_\chi$. Write $\Pi(\gamma)$ for the simple roots of $R^+(\gamma)$. The *τ-invariant of* γ is the subset $\tau(\gamma) \subseteq \Pi(\gamma)$ defined by

$$\begin{aligned}\tau(\gamma) = \{\alpha \in \Pi(\gamma) \mid & \ \alpha \text{ is imaginary and } \epsilon(\gamma)(\alpha) = 0; \text{ or} \\ & \ \alpha \text{ is complex and } -\theta\alpha \in R^+(\gamma); \text{ or } \alpha \text{ is real} \\ & \text{ and } \check{\delta}(\gamma)(\alpha) = 1\}.\end{aligned}$$

The *abstract τ-invariant* is the corresponding subset $\tau^a(\gamma) \subseteq \Pi^a$ (cf. (2.6b)). We may identify either of these sets with the corresponding simple reflections in $S(\gamma) \subseteq W(\gamma)$ or $S^a \subseteq W^a$, respectively—cf. (2.5b), (2.6b).

Definition 12.3 ([24], Definition 6.4). In the setting of Definition 12.1, consider the ring $\mathbf{Z}[u, u^{-1}]$ of finite Laurent series. The *Hecke module of B*, $\mathfrak{M}(B)$, (or simply $\mathfrak{M}$) is the free $\mathbf{Z}[u, u^{-1}]$ module with basis $\{cl(\gamma) \mid \gamma \in B\}$. An element of $\mathfrak{M}(B)$ is thus a formal expression

$$\sum a_\gamma \gamma, a_\gamma \in \mathbf{Z}[u, u^{-1}], \qquad \gamma \in B,$$

with conjugate γ's identified. For each $s \in S^a$, we define a $\mathbf{Z}[u, u^{-1}]$-linear map

$$T_s : \mathfrak{M}(B) \to \mathfrak{M}(B),$$

as follows. It is enough to define each $T_s\gamma$ ($\gamma \in B$). Write $\alpha \in \Pi(\gamma)$ for the simple root corresponding to γ. There are several cases.

(a) α imaginary, $\epsilon(\alpha) = 0$. $T_s\gamma = u\gamma$.
(a*) α real, $\check{\delta}(\alpha) = 0$. $T_s\gamma = -\gamma$.
(b) α complex, $\theta\alpha \in R^+(\gamma)$. $T_s\gamma = s \times \gamma$
(b*) α complex, $\theta\alpha \notin R^+(\gamma)$. $T_s\gamma = u(s \times \gamma) + (u-1)\gamma$
(c) α imaginary, $\epsilon(\alpha) = 1$, $s \notin W_1^{i\mathbf{R}}(\gamma)$. (In the terminology of [25], α is type I).

Then $c^\alpha(\gamma)$ consists of a single element (cf. [25], Definition 8.3.6), and we set

$$T_s\gamma = s \times \gamma + c^\alpha(\gamma)$$

(c*) α real, $\check{\delta}(\alpha) = 1$, $s \in W_1^{i\mathbf{R}}(\gamma)$. (Then α is Type II in the sense of [25]). In this case $c_\alpha(\gamma)$ has just one element ([25], Definition 8.3.16), and we set

$$T_s\gamma = (u-1)\gamma - s \times \gamma + (u-1)c_\alpha(\gamma).$$

(d) α imaginary, $\epsilon(\alpha) = 1$, $s \in W_1^{i\mathbf{R}}(\gamma)$. (In this case α is type II). Then $c^\alpha(\gamma)$ consists of two elements $\gamma_\pm^\alpha$, with $\gamma_\pm^\alpha = s \times \gamma_\mp^\alpha$. We define

$$T_s\gamma = \gamma + \gamma_+^\alpha + \gamma_-^\alpha$$

(d*) α real, $\check{\delta}(\alpha) = 1$, $s \in W_1^{\mathbf{R}}(\gamma)$. (Now α is type I.) Then $c_\alpha(\gamma)$ consists of two elements $\gamma_\alpha^\pm$, with $\gamma_\alpha^\pm = s \times \gamma_\alpha^\mp$. Put

$$T_s\gamma = (u-2)\gamma + (u-1)(\gamma_\alpha^+ + \gamma_\alpha^-).$$

We will explain the pairing of cases ((a) with (a*), etc.) in due course.

Although we have no need of it, the Hecke algebra is floating around in the background somewhere; so we should recall its definition. For more about it in this context see [15].

Definition 12.4. *The Hecke algebra of* W^a, $\mathcal{H}^a$, is the $\mathbf{Z}[u,u^{-1}]$ algebra with generators $\{T_w \mid w \in W^a\}$, subject to the relations

$$T_{w_1w_2} = T_{w_1}T_{w_2} \qquad (w_i \in W^a,\ l(w_1w_2) = l(w_1) + l(w_2))$$

$$(T_s + 1)(T_s - u) = 0 \qquad (s \in S^a).$$

Here l denotes the length function on W.

It turns out that $\mathcal{H}^a$ is a free $\mathbf{Z}[u,u^{-1}]$ module with basis $\{T_w \mid w \in W\}$. The first relation in Definition 12.4 shows that the various T_s ($s \in S^a$) generate $\mathcal{H}^a$.

PROPOSITION 12.5. *The operators T_s of Definition* 12.3 *come from an action of the Hecke algebra $\mathcal{H}^a$ on $\mathfrak{M}(B)$.*

For λ^a integral, this is proved in [18]. The general case is a consequence of the following result.

LEMMA 12.6. *In the setting of Definition* 12.3, *we can find a second group G^2 and a block B^2 for G^2, such that*

(a) *The representations in B^2 have integral infinitesimal character.*

(b) *The abstract ordered root systems*

$$\left(R^a, (R^a)^+\right) \quad \text{and} \quad \left(\Delta\left(\mathfrak{g}^2, (\mathfrak{h}^2)^a\right), \Delta^+\left(\mathfrak{g}^2, (\mathfrak{h}^2)^a\right)\right)$$

are isomorphic. Henceforth we identify them, and so also the abstract Weyl groups.

(c) *There is a bijection* $i: B \to B^2$ *which commutes with Cayley transforms and the cross action of* W^a, *and satisfies*

$$\bar{g}^a(\gamma) = \bar{g}^a(i(\gamma)) \qquad (\gamma \in B)$$

(*Definition* 4.12).

(d) *The induced isomorphism* $i: \mathfrak{M}(B) \to \mathfrak{M}(B^2)$ *respects the operators* T_s.

Proof. Apply Theorem 11.9 twice; this gives G^2 satisfying (a)–(c). Now (d) follows by inspection of the definition of T_s. Q.E.D.

We will use this lemma from time to time to extend purely combinatorial results proved for the integral case in [18] or [24] (sometimes using algebraic geometry in a way which does *not* obviously extend) to the general case; the proof of Proposition 12.5 is typical. In the future we will simply refer to [18] or [24], even though the statements there cover only the integral case. We turn now to the definition of the Bruhat order.

Definition 12.7. In the setting of Definition 12.1, suppose $\phi, \phi' \in B$, and $s \in B^a$. We say that $\phi \xrightarrow{s} \phi'$ if and only if

(a) $l^I(\phi') = l^I(\phi) - 1$, and

(b) ϕ' appears in $T_s\phi$ (Definition 12.3).

Suppose $\phi \xrightarrow{s} \phi'$. It is easy to check that if $\alpha \in R^+(\phi)$ is the corresponding simple root, then either

$$\begin{aligned} &\text{(a)} \qquad \alpha \text{ is complex, } \theta\alpha \notin R^1(\phi), \text{ and } \phi' \in cl(s \times \phi); \text{ or} \\ &\text{(b)} \qquad \alpha \text{ is real, } \check{\delta}(\phi)(\alpha) = 1, \text{ and } \phi' \in c_\alpha(cl(\phi)) \end{aligned} \tag{12.8}$$

(Definition 7.5). In this case we also have the relations

$$s \times \phi \to s \times \phi', \qquad s \times \phi \to \phi', \qquad \phi \to s \times \phi'.$$

(The existence of case (b) is of course the source of all our woes.) These conditions can also be formulated in terms of the $R^+(\phi')$-simple root α' corresponding to s: we get

$$\begin{aligned} &\text{(a)} \qquad \alpha' \text{ is complex, } \theta\alpha' \in R^+(\phi'), \text{ and } \phi \in cl(s \times \phi'); \text{ or} \\ &\text{(b)} \qquad \alpha' \text{ is imaginary, } \epsilon(\phi')(\alpha) = 1, \text{ and } \phi \in c^\alpha(cl(\phi')). \end{aligned} \tag{12.8$'$}$$

Definition 12.9. In the setting of Definition 12.1, the *Bruhat order* on B is the smallest ordering such that

$$m(\bar{\pi}(\phi), \pi(\gamma)) \neq 0 \to \phi \leqslant \gamma$$

(notation 1.2). More explicitly, $\phi \leqslant \gamma$ if and only if there is a sequence $\phi = \phi_0, \phi_1, \ldots, \phi_r = \gamma$, such that $m(\bar{\pi}(\phi_{i-1}), \pi(\phi_i)) \neq 0$ for $i = 1, \ldots, r$.

LEMMA 12.10. *The Bruhat order is an order relation*: *that is*,
(a) $\phi \leqslant \gamma \leqslant \phi \Rightarrow \phi = \gamma$. *We also have*
(b) $\phi \leqslant \gamma \Rightarrow l^I(\phi) \leqslant l^I(\gamma)$
(c) $\phi \leqslant \gamma$ *and* $l^I(\phi) = l^I(\gamma) \Rightarrow \phi = \gamma$.

Proof. Obviously it suffices to prove (b) and (c); and for these, we may assume $m(\bar{\pi}(\phi), \pi(\gamma)) \neq 0$. Then they are precisely Proposition 8.6.19 of [25].

LEMMA 12.11. *The Bruhat order may be defined using the formal character multiplicities* $M(\pi(\phi), \bar{\pi}(\gamma))$ *in place of* $m(\bar{\pi}(\phi), \pi(\gamma))$.

Proof. The definition of $<$ may be reformulated as follows: $\phi \leqslant \gamma$ if and only if we *cannot* index B as

$$B = \{\xi_1, \xi_2, \ldots, \xi_r\},$$

with γ preceding ϕ and still have the matrix

$$m = (m_{ij}) = \big(m(\bar{\pi}(\xi_i), \pi(\xi_j))\big)$$

be upper triangular. (The hard part is only if; so suppose it is false that $\phi \leqslant \gamma$. To construct the required indexing of B, list first all the elements less than γ (compatibly with $<$), then γ, then all the elements less than ϕ but not less than γ, then ϕ, then everything else. This has γ preceding ϕ, but makes m upper triangular.) Set

$$M = \big(M(\pi(\xi_i), \bar{\pi}(\xi_j))\big).$$

By (1.3), $m = M^{-1}$; so m is upper triangular if and only if M is. Q.E.D.

In the case of Verma modules, the Bruhat order is usually defined directly on the Weyl group, and the formulation of Definition 12.9 is a theorem (see [8]). Essentially because Cayley transforms of regular characters are multi-valued, no easy generalization of this approach is available to us. Even *a posteriori*, I know of no simple combinatorial description of the Bruhat order on a block. To make computations, a larger relation was introduced in [24]; we use a slight variant of it here.

Definition 12.12. Suppose $\gamma \in B$. Define $\mathcal{G}^r(\gamma)$ to be the smallest subset of B containing γ, and closed under the cross action of simple real reflections $s \in S(\gamma)$ (cf. (2.5), (2.6)). (This is smaller than the r-packet $rp(\gamma)$ of Definition 8.1, which was closed under the cross action of $W^{\mathbb{R}}(\gamma)$. $W^{\mathbb{R}}(\gamma)$ is not generated by simple real reflections in general, because the simple roots of $(R^{\mathbb{R}})^+(\gamma)$ need not be simple for all of $R^+(\gamma)$.) The *Bruhat r-order* is the smallest relation on B containing the Bruhat order, and constant on the various $\mathcal{G}^r(\gamma)$. We write it as $\overset{r}{<}$. For duality, we will also have to consider the sets $\mathcal{G}^c(\gamma)$ (closed under the cross action of simple imaginary roots) and the *c-order* $\overset{c}{<}$.

These two orders do admit straightforward descriptions which we will record here.

LEMMA 12.13. *The Bruhat r-order is the smallest order relation on B which is constant on the* $\mathcal{G}^r(\gamma)$ *(Definition 12.12), and has the following property. Suppose* $\phi, \gamma \in B$ *are related as follows: there is an* $s \in S^a$, $\gamma' \in B$, *with* $\gamma \stackrel{s}{\to} \gamma'$; *and at least one of the following conditions is satisfied.*

(a) $\phi \stackrel{r}{\to} \gamma'$, *or*

(b) *there is a* $\phi' \in B$, *with* $\phi \stackrel{s}{\to} \phi'$, *and* $\phi' \stackrel{r}{<} \gamma'$. *Then* $\phi \stackrel{r}{<} \gamma$.

The Bruhat c-order has an analogous description with $(\stackrel{c}{>}, \stackrel{s}{\leftarrow})$ *replacing* $(\stackrel{r}{<}, \stackrel{s}{\to})$.

In order not to interrupt the development, we postpone the proof to the end of the section.

LEMMA 12.14 ([24], Lemma 6.8, and [18]). *In the setting of Definition 12.3, there is a unique* **Z***-linear map* $D : \mathfrak{M}(B) \to \mathfrak{M}(B)$ *with the following properties. Define* $R_{\phi\gamma} \in \mathbf{Z}[u, u^{-1}]$ *by*

$$D\gamma = u^{-l^I(\gamma)} \sum_{\phi \in B} (-1)^{l^I(\phi) - l^I(\gamma)} R_{\phi\gamma} \phi.$$

Then we require

(a) $D(um) = u^{-1} D(m)$, *all* $m \in \mathfrak{M}$

(b) $D((T_s + 1)m) = u^{-1}(T_s + 1) D(m)$, *all* $m \in \mathfrak{M}$

(c) $R_{\gamma\gamma} = 1$

(d) $R_{\phi\gamma} \neq 0$ *only if* $\phi \stackrel{r}{\leqslant} \gamma$ *(Definition 12.12).*

This map has also the following properties.

(e) $R_{\phi\gamma}$ *is a polynomial in u, of degree at most* $l^I(\gamma) - l^I(\phi)$.

(f) D^2 *is the identity.*

(g) *The specialization of D to* $u = 1$ *is the identity.*

LEMMA 12.15 ([24], Corollary 6.12 and Theorem 7.1). *In the setting of Lemma 12.14, fix* $\gamma \in B$. *Then there is a unique element*

$$C_\gamma = \sum_{\phi \in B} P_{\phi\gamma} \phi \qquad \left(P_{\phi\gamma} \in \mathbf{Z}[u, u^{-1}]\right)$$

of $\mathfrak{M}(B)$, *with the following properties*

(a) $DC_\gamma = u^{-l^I(\gamma)} C_\gamma$

(b) $P_{\gamma\gamma} = 1$

(c) $P_{\phi\gamma} \neq 0$ *only if* $\phi \stackrel{r}{\leqslant} \gamma$

(d) *If* $\phi \neq \gamma$, *then* $P_{\phi\gamma}$ *is a polynomial in u, of degree at most* $1/2(l^I(\gamma) - l^I(\phi) - 1)$.

In [24], two methods for computing the polynomials $P_{\phi\gamma}$ are discussed. Both are quite cumbersome in practice.

CONJECTURE 12.16 (The Kazhdan–Lusztig conjecture for real reductive groups). Fix a block B of regular characters for G, with nonsingular infinitesimal character χ ((2.10)–(2.12)); and define polynomials $P_{\phi\gamma}(u)$ by Lemma 12.15 above. Then

$$M(\phi,\gamma) = (-1)^{l'(\phi)-l'(\gamma)}P_{\phi\gamma}(1)$$

(notation (1.1), (12.1)); that is,

$$\bar{\pi}(\gamma) = \sum_{\phi\in B}(-1)^{l'(\phi)-l'(\gamma)}P_{\phi\gamma}(1)\pi(\phi)$$

in the Grothendieck group of admissible finite length representations of G.

THEOREM 12.17 (Lusztig, Beilinson–Bernstein; see [24], Theorems 1.6 and 1.12). *Conjecture* 12.16 *is true whenever the infinitesimal character of χ is integral*; *or, more generally, whenever there are no non-integral roots in the rational span of the integral roots. The polynomials $P_{\phi\gamma}$ have non-negative coefficients.*

To say that χ is integral means $R^a = \Delta(\mathfrak{g},\mathfrak{h}^a)$ (notation 2.6)). The hypothesis for the "more generally" statement may be formulated as

$$\mathbf{Q}R^a \cap \Delta(\mathfrak{g},\mathfrak{h}^a) = R^a.$$

This case may be reduced to the first by the methods of [21]. The same arguments reduce Conjecture 12.16 to the case

$$\mathbf{Q}R^a = \Delta(\mathfrak{g},\mathfrak{h}^a).$$

This is the case which Bernstein informs me he can handle; so when the dust settles, Conjecture 12.16 will presumably be proved. In any case, Theorem 12.17 already covers all cases when $G^{\mathbf{C}}$ is of type A_n, for example.

We will now give the proof of Lemma 12.13. The non-formal part is contained in the next result.

LEMMA 12.18. *Suppose ϕ and γ are distinct elements of B, and $m(\bar{\pi}(\phi),\pi(\gamma))$ $\neq 0$. Then we can find a $\gamma' \in B$ and $s \in S^a$ with $\gamma \xrightarrow{s} \gamma'$ (Definition* 12.7). *For any such s, at least one of the following conditions is satisfied.*

(a) $m(\bar{\pi}(\phi),\pi(\gamma')) \neq 0$

(b) *There is a $\phi' \in B$, $\phi \xrightarrow{s} \phi'$ and $m(\bar{\pi}(\phi'),\pi(\gamma')) \neq 0$.*

(c) *The relation $\gamma \xrightarrow{s} \gamma'$ is of the form* (12.8b); *and either* (a) *or* (b) *holds with $s \times \gamma'$ replacing γ'.*

Conversely, suppose ϕ and γ are distinct, $\gamma \xrightarrow{s} \gamma'$, and at least one of (a)–(c) *holds. Then either $\phi < \gamma$, or s is a real reflection for γ, and $\phi < s \times \gamma$.*

This simply summarizes formally the algorithm for controlling the composition series of $\pi(\gamma)$ given in [25], proof of Proposition 8.6.19.

Proof of Lemma 12.13. Write $\overset{l}{<}$ for the relation described by the lemma; here l stands for lemma. We will prove that

$$\phi \overset{r}{<} \gamma \Leftrightarrow \phi \overset{l}{<} \gamma \tag{12.19}$$

by induction on $l^I(\gamma)$. So suppose $\phi \overset{r}{<} \gamma$; we want to show that $\phi \overset{l}{<} \gamma$. By definition of $\overset{r}{<}$, we can find $\psi \in B$, $\tilde{\gamma} \in \mathcal{G}^r(\gamma)$, such that

$$\begin{aligned} &\text{(a)} \qquad m(\overline{\pi}(\psi), \pi(\tilde{\gamma})) \neq 0, \text{ and} \\ &\text{(b)} \qquad \text{either } \phi \overset{r}{<} \psi, \text{ or } \mathcal{G}^r(\phi) = \mathcal{G}^r(\psi). \end{aligned} \tag{12.20}$$

We claim that $\psi \overset{l}{<} \gamma$. Since $\overset{l}{<}$ is defined to be constant on $\mathcal{G}^r(\gamma)$, we may as well assume $\gamma = \tilde{\gamma}$. By Lemma 12.18, ψ and γ are in the relationship required by Lemma 12.13; so $\psi \overset{l}{<} \gamma$. Now $l^I(\psi) < l^I(\gamma)$; so (12.20)(b) and the inductive hypothesis guarantee that either $\phi \overset{l}{<} \psi$, or $\mathcal{G}^r(\phi) = \mathcal{G}^r(\psi)$. Now the desired relation $\phi \overset{l}{<} \gamma$ follows from the transitivity of $\overset{l}{<}$.

Conversely, suppose $\phi \overset{l}{<} \gamma$. By the definition in Lemma 12.13, we can find a $\psi \in B$, $\tilde{\gamma} \in \mathcal{G}^r(\gamma)$, such that

$$\begin{aligned} &\text{(a)} \quad \psi \text{ and } \tilde{\gamma} \text{ are in the relationship described by Lemma 12.13, and} \\ &\text{(b)} \quad \text{either } \phi \overset{l}{<} \psi, \text{ or } \mathcal{G}^r(\phi) = \mathcal{G}^r(\psi). \end{aligned} \tag{12.21}$$

By the converse direction of Lemma 12.18, $\psi \overset{r}{<} \gamma$; and the argument is completed as above. The statements about $\overset{c}{<}$ are proved in the same way. Q.E.D.

13. Duality for Hecke modules and proof of the main theorem.

LEMMA 13.1. *In the setting of Lemmas* 12.14 *and* 12.15, *we have the following additional information*

(a) *If* $P_{\phi\gamma} \neq 0$, *then* $\phi \overset{c}{\leqslant} \gamma$

(b) *If* $R_{\phi\gamma} \neq 0$, *then* $\phi \overset{c}{\leqslant} \gamma$.

Proof. Since all of the objects in question depend only on the combinatorial structure of the block (see Lemma 12.13), Lemma 12.6 allows us to assume that $R^a = \Delta(\mathfrak{g}, \mathfrak{h}^a)$, and therefore (by Theorem 12.17) that $P_{\phi\gamma}$ has non-negative coefficients, and

$$P_{\phi\gamma}(1) = \pm M(\phi, \gamma).$$

Now (a) is a consequence of Lemma 12.11. In fact we get the stronger statement that $\overset{c}{\leqslant}$ may be defined by $P_{\phi\gamma}$ in place of $M(\phi, \gamma)$. Write $Q_{\phi\gamma}$ for the inverse matrix to $P_{\phi\gamma}$. By the proof of Lemma 12.11, $Q_{\phi\gamma}$ can be used to define $\overset{c}{\leqslant}$; so

$$\gamma = \sum_{\phi \overset{c}{\leqslant} \gamma} Q_{\phi\gamma}(u) C_\phi$$

By Lemma 12.15,

$$\begin{aligned} D\gamma &= \sum_{\phi \overset{c}{\leqslant} \gamma} u^{-l^I(\phi)} Q_{\phi\gamma}(u^{-1}) C_\phi \\ &= u^{-l^I(\gamma)} \sum_{\phi \overset{c}{\leqslant} \gamma} u^{l^I(\gamma)-l^I(\phi)} Q_{\phi\gamma}(u^{-1}) C_\phi . \end{aligned}$$

Comparing this with Lemma 12.14 gives

$$R_{\phi\gamma} = (-1)^{l^I(\phi)-l^I(\gamma)} \sum_{\phi \overset{c}{\leqslant} \psi \overset{c}{\leqslant} \gamma} u^{l^I(\gamma)-l^I(\psi)} Q_{\psi\gamma}(u^{-1}) P_{\phi\psi}(u)$$

Now assertion (b) of the lemma is immediate. Q.E.D.

Definition 13.3. In the setting of Definition 12.3, the *dual Hecke module* $\check{\mathfrak{M}}(B)$ (or simply $\check{\mathfrak{M}}$) is the free $\mathbf{Z}[u,u^{-1}]$ module having the same basis as $\mathfrak{M}(B)$ (that is, $\{cl(\gamma) \mid \gamma \in B\}$), endowed with the following structures. When we want to regard γ as a basis element of $\check{\mathfrak{M}}$, we write it as $\check{\gamma}$. Put

(a) $l^I(\check{\gamma}) = -l^I(\gamma)$

Define a $\mathbf{Z}[u,u^{-1}]$-linear pairing from $\mathfrak{M}(B) \oplus \check{\mathfrak{M}}(B)$ into $\mathbf{Z}[u,u^{-1}]$ by

(b) $$\langle \gamma, \check{\phi} \rangle = \begin{cases} 0, & \phi \neq \gamma \\ (-1)^{l^I(\gamma)}, & \phi = \gamma \end{cases}$$

Use a superscript t to the left to denote the transpose with respect to this pairing: if A is a $\mathbf{Z}$-linear operator on $\mathfrak{M}(B)$, then ${}^tA$ is the unique $\mathbf{Z}$-linear operator on $\check{\mathfrak{M}}(B)$ satisfying

(c) $\langle Ax, y \rangle = \langle x, {}^tAy \rangle$ $(x \in \mathfrak{M}, y \in \check{\mathfrak{M}})$

For $s \in S^a$, define $\check{T}_s : \check{\mathfrak{M}} \to \check{\mathfrak{M}}$ by

(d) $\check{T}_s = -{}^tT_s + (u-1)$.

Recall the map D of Lemma 12.14. Write bar for the unique automorphism of $\mathbf{Z}[u,u^{-1}]$ sending u to u^{-1}. Define $\check{D} : \check{\mathfrak{M}} \to \check{\mathfrak{M}}$ by the requirement

(e) $\overline{\langle Dx, y \rangle} = \langle x, \check{D}y \rangle$ $(x \in U, y \in \check{\mathfrak{M}})$

To see that (e) makes sense, fix $y \in \check{\mathfrak{M}}$, and consider the map

$$f_y : \mathfrak{M} \to \mathbf{Z}[u,u^{-1}], \qquad f_y(x) = \overline{\langle Dx, y \rangle} .$$

Clearly f_y is $\mathbf{Z}[u,u^{-1}]$-linear; so it is given by

$$f_y(x) = \langle x, z \rangle$$

for a unique $z \in \check{\mathfrak{M}}$.

LEMMA 13.4. *The map* $\check{D}: \check{\mathfrak{M}} \to \check{\mathfrak{M}}$ *of Definition* 13.3(e) *has the following properties. Define elements* $R_{\check{\phi}\check{\gamma}} \in \mathbf{Z}[u, u^{-1}]$ *by*

$$\check{D}\check{\gamma} = u^{-l'(\check{\gamma})} \sum_{\phi \in B} (-1)^{l'(\check{\phi}) - l'(\check{\gamma})} R_{\check{\phi}\check{\gamma}} \check{\phi}$$

Then

(a) $\check{D}(um) = u^{-1}\check{D}(m)$, *all* $m \in \check{\mathfrak{M}}$.

(b) $\check{D}(\check{T}_s + 1) = u^{-1}(\check{T}_s + 1)$

(c) $\check{R}_{\check{\gamma}\check{\gamma}} = 1$

(d) $\check{R}_{\check{\phi}\check{\gamma}} \neq 0$ *only if* $\phi \overset{c}{\geqslant} \gamma$ (*Definition* 12.12).

Proof. Part (a) is equivalent to

$$\langle x, D(um) \rangle = \langle x, u^{-1}D(m) \rangle, \qquad \text{all} \quad x \in \mathfrak{M}, \quad m \in \check{\mathfrak{M}}.$$

By Definition 12.10(e), this equation can be rewritten as

$$\overline{\langle uD(x), m \rangle} = \overline{\langle D(u^{-1}, x), m \rangle}, \qquad \text{all} \quad x \in \mathfrak{M}, \quad m \in \check{\mathfrak{M}}.$$

In this form it is a consequence of Lemma 12.14(a). By a similar calculation, (b) amounts to

$$(-T_s + u)D(x) = D\big[(-T_s + u)(u^{-1}x)\big], \qquad \text{all} \quad x \in \mathfrak{M}; \tag{13.5}$$

for ${}^t(\check{T}_s) = -T_s + (u - 1)$. To prove this, write the right side as

$$\begin{aligned} &-D\big[(T_s + 1)(u^{-1}x)\big] + D\big[(u + 1)(u^{-1}x)\big] \\ &= -u^{-1}(T_s + 1)D(u^{-1}x) + (1 + u)\,Dx && \text{(by (12.14)(b) and (a))} \\ &= -(T_s + 1)\,Dx + (1 + u)\,Dx && \text{(by (12.14)(a))} \\ &= (-T_s + u)\,Dx. \end{aligned}$$

This proves (13.5), and so (b). For (c) and (d), we may as well compute the matrix of $\check{D}$. We have

$$\begin{aligned} \check{R}_{\check{\phi}\check{\gamma}} &= u^{l'(\check{\gamma})}(-1)^{l'(\check{\phi}) - l'(\check{\gamma})} && \big[\text{coefficient of } \check{\phi} \text{ in } \check{D}\check{\gamma}\big] \\ &= u^{-l'(\gamma)}\langle \phi, \check{D}\check{\gamma} \rangle \\ &= u^{-l'(\gamma)}\overline{\langle D\phi, \check{\gamma} \rangle} \\ &= u^{-l'(\gamma)}(-1)^{l'(\phi) - l'(\gamma)} && \overline{\big[\text{coefficient of } \gamma \text{ in } D\phi\big]} \\ &= u^{-l'(\gamma)}\overline{\big[u^{-l'(\phi)}R_{\gamma\phi}(u)\big]} \\ \check{R}_{\check{\phi}\check{\gamma}} &= u^{l'(\phi) - l'(\gamma)}R_{\gamma\phi}(u^{-1}). \end{aligned} \tag{13.6}$$

Part (c) follows from (13.6) and Lemma 12.14(c), and part (d) from(13.6) and Lemma 13.1(b). Q.E.D.

Although we will not use the fact, Lemma 12.14(e) and (13.6) shows that $\check{R}_{\check{\phi}\check{\gamma}}$ is a polynomial, of degree at most $l^I(\check{\gamma}) - l^I(\check{\phi})$: here we are also using Definition 13.3(a).

LEMMA 13.7. *In the setting of Definition* 13.3, *there are unique elements* $\{\check{C}_\gamma \mid \gamma \in B\}$ *of* $\check{\mathfrak{M}}$ *such that* (*if we define* C_ϕ *as in Lemma* 12.15)

$$\langle C_\phi, \check{C}_\gamma \rangle = \begin{cases} (-1)^{l^I(\gamma)}, & \phi = \gamma \\ 0 & \phi \neq \gamma \end{cases} \tag{1}$$

More explicitly, define $P_{\phi\gamma}$ *as in Lemma* 12.15, *and write* $(Q_{\phi\gamma})$ *for the matrix inverse of* $(P_{\phi\gamma})$. *Then*

$$\check{C}_\gamma = \sum_{\phi \overset{c}{\geqslant} \gamma} (-1)^{l^I(\phi) - l^I(\gamma)} Q_{\gamma\phi} \check{\phi}. \tag{2}$$

Write

$$\check{P}_{\check{\phi}\check{\gamma}} = (-1)^{l^I(\phi) - l^I(\gamma)} Q_{\gamma\phi}, \tag{3}$$

so that

$$\check{C}_\gamma = \sum_{\phi \in B} \check{P}_{\check{\phi}\check{\gamma}} \check{\phi}. \tag{4}$$

Then

(a) $\check{D}\check{C}_\gamma = u^{-l^I(\check{\gamma})} \check{C}_\gamma$

(b) $\check{P}_{\check{\gamma}\check{\gamma}} = 1$

(c) $\check{P}_{\check{\phi}\check{\gamma}} \neq 0$ *only if* $\phi \overset{c}{\geqslant} \gamma$

(d) *If* $\phi \neq \gamma$, *then* $\check{P}_{\check{\phi}\check{\gamma}}$ *is a polynomial in* u, *of degree at most* $1/2(l^I(\check{\gamma}) - l^I(\check{\phi}) - 1)$.

Proof. The statements through (4) are obvious, since P is upper triangular for $\overset{c}{\geqslant}$ (Lemma 13.1(a)). For (a), we compute

$$\begin{aligned} \langle C_\phi, \check{D}\check{C}_\gamma \rangle &= \overline{\langle DC_\phi, \check{C}_\gamma \rangle} \\ &= \overline{u^{-l^I(\phi)} \langle C_\phi, \check{C}_\gamma \rangle} \\ &= u^{l^I(\phi)} \overline{\langle C_\phi, \check{C}_\gamma \rangle} \\ &= u^{-l^I(\check{\phi})} \overline{\langle C_\phi, \check{C}_\gamma \rangle} \end{aligned}$$

by Lemma 12.15(a), and Definition 13.3(a), (e). Now (a) follows from (1) of the lemma. Since P is upper triangular for $\overset{c}{\geqslant}$ with 1's on the diagonal, $Q = P^{-1}$ is as well; so (b) and (c) follow. Part (d) amounts to

$$\deg Q_{\phi\gamma} \leqslant \tfrac{1}{2}\big(l^I(\gamma) - l^I(\phi) - 1\big), \qquad \phi \neq \gamma; \tag{13.8}$$

we will prove this by induction on $l^I(\gamma)$. Since $P_{\gamma\gamma} = Q_{\phi\phi} = 1$, the $\phi\gamma$ entry of the equation $PQ = 1$ is

$$\begin{aligned} 0 &= \sum_{\phi \overset{c}{\leqslant} \psi \overset{c}{\leqslant} \gamma} Q_{\phi\psi} P_{\psi\gamma} \\ 0 &= Q_{\phi\gamma} + P_{\phi\gamma} + \sum_{\phi \overset{c}{\leqslant} \psi \overset{c}{\leqslant} \gamma} Q_{\phi\psi} P_{\psi\gamma} \end{aligned} \tag{13.9}$$

By inductive hypothesis and Lemma 12.15(d),

$$\begin{aligned} \deg(Q_{\phi\psi} P_{\psi\gamma}) &\leqslant \tfrac{1}{2}\big(l^I(\gamma) - l^I(\psi) + l^I(\psi) - l^I(\phi) - 2\big) \\ &= \tfrac{1}{2}\big(l^I(\gamma) - l^I(\phi) - 1\big) - \tfrac{1}{2} \end{aligned}$$

whenever $\phi \overset{c}{<} \psi \overset{c}{<} \gamma$. Now Lemma 12.15(d) shows that every term of (13.9) except perhaps $Q_{\phi\gamma}$ satisfies the degree estimate in question; so (13.8) follows. Q.E.D.

PROPOSITION 13.10. *Fix a block $B = \{cl(\gamma_1), \ldots, cl(\gamma_r)\}$ of regular characters for the real reductive group G, having nonsingular infinitesimal character. Fix a second group $\check{G}$ and a block $\check{B} = \{cl(\check{\gamma}), \ldots, cl(\check{\gamma})\}$ satisfying the conclusions* (a)–(d) *of Theorem* 11.9.

(a) *The integral length function for $\check{B}$ (Definition* 12.1) *may be normalized so that $l^I(\gamma) = -l^I(\check{\gamma})$ for all $\gamma \in B$.*

(b) *If we identify $\mathfrak{M}(B)$ with $\mathfrak{M}(\check{B})$ (Definitions* 12.3 *and* 13.3) *by identifying their bases as indicated by the notation, then the operators T_s for $\mathfrak{M}(\check{B})$ are identified with $\check{T}_s$.*

(c) *If $\phi, \gamma \in B$, then $\phi \overset{r}{>} \gamma$ if and only if $\check{\gamma} \overset{c}{>} \check{\phi}$.*

Proof. Part (a) is an easy consequence of Theorem 11.9(b) and Definition 12.1; in fact one only needs to know that $\theta^a(\gamma) = -\theta^\alpha(\check{\gamma})$. Part (c) follows from Lemma 12.13; for (11.9)(c), (d), show that

$$\gamma \xrightarrow{\ s\ } \gamma' \Leftrightarrow \check{\gamma}' \xrightarrow{\ s\ } \check{\gamma} \tag{13.11}$$

(Definition 12.17). To prove (b), notice that (coefficient of $\check{\phi}$ in ${}^tT_s\check{\gamma}) = (-1)^{l^I(\phi) - l^I(\gamma)}$ (coefficient of γ in $T_s\phi$), by Definition 13.3. This allows us to compute $\check{T}_s$ case by case, using Definition 12.3. By Theorem 11.9(b), γ is case (x) of that definition if and only if $\check{\gamma}$ is in case (x*). As an example, suppose γ is in case (d); write $c^\alpha(\gamma) = \gamma_\pm^a$. Then γ appears with coefficient 1 in $T_s\gamma$, and with coefficient $u-1$ in $T_s\gamma_\pm^\alpha$ (case (c*)). Since also $l^I(\gamma_\pm^\alpha) = l^I(\gamma)+1$, we compute

$$'T_s\check{\gamma} = \check{\gamma} + \sum_{\phi\in c^\alpha(\gamma)} (1-u)\check{\phi}.$$

By Definition 13.3(c), we get

$$\check{T}_s\check{\gamma} = (u-2)\check{\gamma} + \sum_{\phi\in c^\alpha(\gamma)} (u-1)\check{\phi}.$$

By Theorem 11.9(c), this can be written

$$\check{T}_s\check{\gamma} = \sum_{\check{\phi}\in c_{\check{\alpha}}(\check{\gamma})} (u-1)\check{\phi} + (u-2)\check{\gamma}.$$

This is the formula for $T_s\check{\gamma}$ in Definition 12.3(d*). The other cases are similar. Q.E.D.

PROPOSITION 13.12. *In the setting of Proposition* 13.10, *suppose the integral length function on $\check{B}$ is normalized in accordance with Proposition* 13.10(a), *and $\mathfrak{M}(\check{B})$ is identified with $\check{\mathfrak{M}}(B)$ as in* (13.10)(b).

(a) *The operator $\check{D}$ of Definition* 13.3(e) *coincides with the operator D for $\check{B}$ defined by Lemma* 12.14.

(b) *The elements $\check{C}_\gamma$ of Lemma* 13.7 *coincide with the elements $C_{\check{\gamma}}$ for B of Lemma* 12.15.

Proof. In light of Proposition 13.10, Lemmas 13.4 and 13.7 show that $\check{D}$ and $\check{C}_\gamma$ satisfy the defining properties of D and $C_{\check{\gamma}}$. Q.E.D.

THEOREM 13.13. *Let G be a real reductive group, and $B = \{cl(\gamma_1), \ldots, cl(\gamma_r)\}$ a block of regular characters of G having nonsingular infinitesimal character. Let $\check{G}$ be a second reductive group, and $\check{B} = \{cl(\check{\gamma}_1), \ldots, cl(\check{\gamma}_r)\}$ a block for $\check{G}$; and assume that the bijection $\gamma_i \to \check{\gamma}_i$ satisfies conditions* (a)–(d) *of Theorem* 11.9. *Then the inverse of the Kazhdan–Lusztig matrix $(P_{\phi\gamma})$ (Lemma* 12.15) *is the matrix*

(a) $(P_{\phi\gamma})^{-1} = ((-1)^{l^I(\phi)-l^I(\gamma)}P_{\check{\gamma}\check{\phi}}$.

In particular, suppose that the Kazhdan–Lusztig conjecture (12.16) *holds for (G,B) and $(\check{G},\check{B})$, so that*

(b) $$M(\pi(\phi),\bar{\pi}(\gamma)) = (-1)^{l^I(\phi)-l^I(\gamma)}P_{\phi\gamma}(1)$$

$$M(\pi(\check{\phi}),\bar{\pi}(\check{\gamma})) = (-1)^{l^I(\phi)-l^I(\gamma)}P_{\check{\phi}\check{\gamma}}(1).$$

Then

(c) $$m(\bar{\pi}(\phi),\pi(\gamma)) = P_{\check{\gamma}\check{\phi}}(1)$$

$$= (-1)^{l^I(\phi)-l^I(\gamma)}M(\pi(\check{\gamma}),\bar{\pi}(\check{\phi}))$$

This is immediate from Proposition 13.12, Lemma 13.7(3) and (1.3). With Theorem 11.9 (and Theorem 12.17), it gives Theorem 1.15.

14. Duality and primitive ideals. In this section, we will lay the foundation for a study of certain Weyl group representations attached to blocks. This is connected with the problems studied in [1] and [2] for complex groups—Fourier inversion of unipotent orbital integrals and related matters. They will be pursued in a future paper with Barbasch.

Definition 14.1. In the setting of Lemma 12.15, suppose $\phi, \gamma \in B$ and $\phi < \gamma$. Define

$$\mu(\phi, \gamma) = \text{coefficient of } u^{1/2(l'(\gamma) - l'(\phi) - 1)} \text{ in } P_{\phi\gamma}.$$

LEMMA 14.2 ([18], Section 5). *In the setting of Lemma* 12.15, *suppose* $\gamma \in B$, $s \in S^a$.

(a) *If* $s \notin \tau^a(\gamma)$, *then* (*Definitions* 12.2, 12.7)

$$(T_s + 1)C_\gamma = \sum_{\gamma' \xrightarrow{s} \gamma} C_{\gamma'} + \sum_{\substack{\phi < \gamma \\ s \in \tau(\phi)}} \mu(\phi, \gamma) u^{1/2(l'(\gamma) - l'(\phi) - 1)} C_\phi$$

(b) *If* $s \in \tau^a(\gamma)$, *then*

$$(T_s - u)C_\gamma = 0$$

LEMMA 14.3. *In the setting of Theorem* 13.3, *suppose* $\phi, \gamma \in B$, $\phi < \gamma$, $s \in S^a$, $s \in \tau^a(\phi)$, *and* $s \notin \tau^a(\gamma)$. *Then*

$$\mu(\phi, \gamma) = \mu(\check{\gamma}, \check{\phi}).$$

Proof. By Proposition 13.12, Lemmas 14.2 and 13.7,

$$\begin{aligned}
\mu(\phi, \gamma) &= u^{-1/2(l'(\gamma) - l'(\phi) - 1)} \quad \left[\text{coefficient of } C_\phi \text{ in } (T_s + 1)C_\gamma\right] \\
&= u^{-1/2(l'(\check{\phi}) - l'(\check{\gamma}) - 1)}\left[\langle (T_s + 1)C_\gamma, \check{C}_\phi \rangle\right](-1)^{l'(\check{\phi})} \\
&= u^{-1/2(l'(\check{\phi}) - l'(\check{\gamma}) - 1)}\left[\langle C_\gamma, ({}^tT_s + 1)\check{C}_\phi \rangle\right](-1)^{l'(\check{\phi})} \\
&= u^{-1/2(l'(\check{\phi}) - l'(\check{\gamma}) - 1)}\left[\langle C_\gamma, (u - \check{T}_s)\check{C}_\phi \rangle\right](-1)^{l'(\check{\phi})} \\
&= -(-1)^{l'(\phi)} u^{-1/2(l'(\check{\phi}) - l'(\check{\gamma}) - 1)} \\
&\qquad \left[\text{coefficient of } C_{\check{\gamma}} \text{ in } (T_s + 1)C_{\check{\phi}}\right](-1)^{l'(\check{\gamma})} \\
&= \mu(\check{\gamma}, \check{\phi}).
\end{aligned}$$

Here we are using several obvious facts. First, since $\phi \neq \gamma$, we may replace $T_s - u$ by $T_s + 1$ in the next to last line. The result is trivial ($\mu(\phi, \gamma) = \mu(\check{\gamma}, \check{\phi}) = 0$) unless $l^I(\phi) - l^I(\gamma)$ is odd; so $-(-1)^{l^I(\check{\phi})}$ is equal to $(-1)^{l^I(\check{\gamma})}$; this is used at the end. Q.E.D.

Definition 14.4. Fix a block B of representations with infinitesimal character χ. Write $\mathbf{Z}(B)$ for the free $\mathbf{Z}$-module with basis B. We identify $\mathbf{Z}(B)$ with the Grothendieck group of the category of finite length admissible representations generated by the $\pi(\gamma)$, by sending γ to the class of $\pi(\gamma)$. Make W^a act on $\mathbf{Z}(B)$ by the coherent continuation representation ([25], Definition 7.2.28); write $t(w)$ for the operator corresponding to $w \in W^a$. Explicitly ([25], Chaper 8), fix $\gamma \in B$, $s \in S^a$, and label cases as in Definition 12.3. Then

(a) $$t(s)\gamma = -\gamma$$

(a*) $$t(s)\gamma = \gamma$$

(b)(b*) $$t(s)\gamma = s \times \gamma$$

(c) $$t(s)\gamma = c^\alpha(\gamma) - s \times \gamma$$

(c*) $$t(s)\gamma = s \times \gamma$$

(d) $$t(s)\gamma = \gamma_+^\alpha + \gamma_-^\alpha - \gamma$$

(d*) $$t(s)\gamma = \gamma$$

LEMMA 14.5. *In the setting of Definitions* 12.3 *and* 14.4, *let* $\mathfrak{M}(B) \xrightarrow{\epsilon} \mathbf{Z}(B)$ *be the* $\mathbf{Z}$*-linear map defined by*

$$\epsilon(u^m\gamma) = (-1)^{l^I(\gamma)}\gamma.$$

Then

(a) *For* $m \in \mathfrak{M}(B)$, $s \in S^a$

$$\epsilon(-T_s m) = t(s)\epsilon(m).$$

(b) *If the Kazhdan–Lusztig conjecture holds for* B, *then*

$$\epsilon(C_\gamma) = (-1)^{l^I(\gamma)}\overline{\pi}(\gamma)$$

(*cf.* (12.16)).

Proof. Part (a) is easily checked by comparing Definitions 12.3 and 14.4. Part (b) is a reformulation of Conjecture 12.16. Q.E.D.

Definition 14.6. In the setting of Definition 14.4, the *LR preorder* $\overset{LR}{<}$ on B is the smallest preorder relation with the following property. Fix $w \in W$, $\gamma \in B$, and write

$$t(w)\bar{\pi}(\gamma) = \sum_{\phi \in B} a_\phi \bar{\pi}(\phi).$$

Then we require that

$$a_\phi \neq 0 \Rightarrow \gamma \overset{LR}{<} \phi.$$

The cone over γ is

$$\bar{\mathcal{C}}^{LR}(\gamma) = \left\{\phi \in B \mid \gamma \overset{LR}{<} \phi\right\}.$$

Define

$$\bar{\mathcal{V}}^{LR}(\gamma) = \text{span of } \left\{\bar{\pi}(\phi) \mid \gamma \overset{LR}{<} \phi\right\} \subseteq \mathbf{Z}(B).$$

Thus $\bar{\mathcal{V}}^{LR}(\gamma)$ carries a representation of W^a, by restricting the coherent continuation representation on B. We define *LR equivalence* $\overset{LR}{\approx}$ by $\phi \overset{LR}{\approx} \gamma$ if and only if

$$\gamma \overset{LR}{<} \phi \overset{LR}{<} \gamma.$$

The *double cell of* γ is

$$\mathcal{C}^{LR}(\gamma) = \left\{\phi \in B \mid \phi \overset{LR}{\approx} \gamma\right\}.$$

We must also define

$$\mathcal{C}^{LR}_+(\gamma) = \bar{\mathcal{C}}^{LR}(\gamma) - \mathcal{C}^{LR}(\gamma),$$

and $\mathcal{V}^{LR}(\gamma)$, $\mathcal{V}^{LR}_+(\gamma)$ in analogy with $\bar{\mathcal{V}}^{LR}(\gamma)$. Clearly $\mathcal{V}^{LR}_+(\gamma)$ is $t(W^a)$-invariant, and we define a representation of W^a of $\mathcal{V}^{LR}(\gamma)$ by the natural isomorphism

$$\mathcal{V}^{LR}(\gamma) \cong \bar{\mathcal{V}}^{LR}(\gamma)/\mathcal{V}^{LR}_+(\gamma);$$

We call this a *double cell representation*. When G is complex, all of these notations can be refined into separate "left" and "right" pieces—see [2], for example. The notation here is consistent with [2], although $\overset{L}{<}$ and so on are not defined in general.

LEMMA 14.7. *Suppose the Kazhdan–Lusztig conjecture holds for B, $\gamma \in B$, and $s \in S^a$. Then, in the notation of Definition* 14.1,

$$t(s)\bar{\pi}(\gamma) = \bar{\pi}(\gamma) + \sum_{\gamma' \overset{s}{\to} \gamma} \bar{\pi}(\gamma') + \sum_{\substack{\phi < \gamma \\ s \in \tau^a(\phi)}} \mu(\phi, \gamma)\bar{\pi}(\phi)$$

if $s \notin \tau^a(\gamma)$; *and*

$$t(s)\bar{\pi}(\gamma) = -\bar{\pi}(\gamma)$$

if $s \in \tau^a(\gamma)$.

This is clear from Lemma 14.2 and 14.5.

COROLLARY 14.8. *Suppose the Kazhdan–Lusztig conjecture holds for* B. *Then* $\overset{LR}{<}$ is the smallest preorder satisfying

(a) *if* $\gamma' \xrightarrow{s} \gamma$, *then* $\gamma \overset{LR}{<} \gamma'$

(b) *if* $\gamma, \phi \in B$, $s \notin \tau^a(\gamma)$, $s \in \tau^a(\phi)$, *and* $\mu(\phi, \gamma) \neq 0$, *then* $\gamma \overset{LR}{<} \phi$.

COROLLARY 14.9. *In the setting of Theorem* 13.13 *and Definition* 14.6, *suppose the Kazhdan–Lusztig conjectures hold for* B *and* $\check{B}$. *If* $\gamma, \phi \in B$, *then*

(a) $\gamma \overset{LR}{<} \phi$ *if and only if* $\check{\phi} \overset{LR}{<} \check{\gamma}$. *In particular*

$$\left[\mathcal{C}^{LR}(\gamma)\right]^{\vee} = \mathcal{C}^{LR}(\check{\gamma}).$$

Define a pairing between $\mathbf{Z}(B)$ *and* $\mathbf{Z}(\check{B})$ *by*

$$\langle \bar{\pi}(\gamma), \bar{\pi}(\check{\phi}) \rangle = \delta_{\phi\gamma}(-1)^{l'(\gamma)}$$

Then if $m \in \mathbf{Z}(B)$, $n \in \mathbf{Z}(\check{B})$ *and* $w \in W^a$ *we have*

(b) $\langle t(w)m, n \rangle = (-1)^{l(w)} \langle m, t(w)n \rangle$.

In particular, there is an isomorphism

(c) $\mathcal{V}^{LR}(\gamma) \cong \mathcal{V}^{LR}(\check{\gamma})^* \otimes (\mathrm{sgn})$, *with* sgn *the sign representation of* W^a.

Proof. Part (a) follows from Corollary 14.8 and Lemma 14.2. It is enough to prove (b) when $w = s \in S^a$; and then it follows from Lemmas 14.7 and 14.2. Part (c) is immediate from (b). Q.E.D.

THEOREM 14.10 (King [26]). *In the setting of Definition* 14.6, *there is a natural map* (*the character polynomial*)

$$\mathrm{ch} : \mathcal{V}^{LR}(\gamma) \to S(\mathfrak{h}^a);$$

here the right side is the symmetric algebra of $\mathfrak{h}^a$. *We have*

(a) $\mathrm{ch}(\bar{\pi}(\gamma))$ *is a homogeneous*, W^a-*harmonic polynomial*.

(b) *Up to a non-zero constant*, $\mathrm{ch}(\bar{\pi}(\gamma))$ *is Joseph's Goldie rank polynomial* ([12]) *for the primitive ideal* $\mathrm{Ann}(\bar{\pi}(\gamma)) \subset U(\mathfrak{g})$.

(c) *If* $w \in W^a$,

$$\mathrm{ch}\big(t(w)\bar{\pi}(\gamma)\big) = w \cdot \mathrm{ch}\big(\bar{\pi}(\gamma)\big).$$

COROLLARY 14.11. *The double cell representation of* W^a *on* $\mathcal{V}^{LR}(\gamma)$ *contains Joseph's Goldie rank representation associated to* $\mathrm{Ann}(\bar{\pi}(\gamma))$.

One would like to show that every constituent of $\mathcal{V}^{LR}(\gamma)$ lies in the same double cell in $\hat{W}_a$ ([2], Definition 2.12). There are certain minor technical obstacles to proving this, but probably they can be overcome.

15. An L-group formulation. In this section, we will sketch without proofs another approach to the definition of $\check{G}$ in Theorem 1.15. This method is adapted only to algebraic groups (and not to the minor modifications of them usually allowed, such as the class of groups defined in section 2). Notation will not be particularly consistent with the rest of the paper, nor even with our main references [5] and [22].

Let G be a reductive algebraic group over $\mathbf{C}$. Fix once and for all Borel and Cartan subgroups

$$\begin{aligned} &B^a \supseteq H^a \text{ (the \textit{abstract Borel subgroup}, etc.)} \\ &\Delta^a = \Delta(\mathfrak{g}, \mathfrak{h}^a) \supseteq (\Delta^a) + = \Delta(\mathfrak{g}, \mathfrak{b}^a). \end{aligned} \tag{15.1a}$$

These specify a root datum ([22])

$$\Psi = \left(X^*(H^a), \Delta^a, X_*(H^a), (\Delta^a)^\vee\right). \tag{15.1b}$$

Here $X^*(H^a)$ is the lattice of rational characters of H^a, and $X_*(H^a)$ is the lattice of one parameter subgroups of H^a. We may regard $X^*(H^a)$ as contained in $(\mathfrak{h}^a)^*$, and $X_*(H_a)$ as contained in $\mathfrak{h}^a$. Fix a dual group ([5])

$${}^LG^0 \supseteq {}^L(B^a)^0 \supseteq {}^L(H^a)^0 \tag{15.2a}$$

with root datum

$$\Psi^\vee = \left(X_*(H^a), (\Delta^a)^\vee, X^*(H^a), \Delta^a\right). \tag{15.2b}$$

This means that we are fixing isomorphisms

$$X^*(H^a) \cong X_*\left({}^L(H^a)^0\right), \tag{15.2c}$$

and so on.

Roughly speaking, our objects of study will be pairs $(G^1(\mathbf{R}), \pi^1)$, with $G^1 \subseteq G$ a reductive algebraic group of the same rank as G, $G^1(\mathbf{R})$ a real form of G^1, and π^1 an irreducible admissible representation of $G^1(\mathbf{R})$. (Actually we will consider a slightly smaller class of pairs; G^1 will be among other things the identity component of the centralizer of a semisimple element of G.) To each pair we will associate a dual pair of the same sort in ${}^LG^0$. However, technicalities abound. First, we will actually study only conjugacy classes of pairs. Second, the representations will be determined only up to "translation" in the sense of Jantzen and Zuckerman (see [25], Chapter 7). A more serious problem is evident in this example: take $G = G^1 = SL(2, \mathbf{C})$, $G^1(\mathbf{R}) = SL(2, \mathbf{R})$. Let π^1 be a holomorphic discrete series representation of $G^1(\mathsf{R})$, and π^2 the corresponding antiholomorphic representation. Then the pairs $(G^1(\mathsf{R}),\pi^1)$ and $(G^1(\mathsf{R}),\pi^2)$ are conjugate under G (by the matrix $\left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$); so we cannot hope to formulate a result like Theorem 1.15 in this setting. The cure proposed here is to consider along with $G^1(\mathsf{R})$ and π^1 certain additional "rigidifying" data.

Definition 15.3. A *real datum* for G is five-tuple $g = (B, H, \sigma, t, \chi)$, such that

(a) $H \subseteq B$ are Cartan and Borel subgroups of G

(b) σ is an involution of H preserving $\Delta(\mathfrak{g},\mathfrak{h})$. Write $\mathfrak{t} = \mathfrak{h}^{\sigma}$, $\mathfrak{a} = \mathfrak{h}^{(-\sigma)}$ for the $+1$ and -1 eigenspaces of σ.

(c) Let $[X^*(H)]^{\sigma}$ denote the invariants of σ in $X^*(H)$. Put

$$\begin{aligned} X_{\sigma}(H) &= \left\{x \in \mathfrak{t} \mid \langle x, \left[X^*(H)\right]^{\sigma}\rangle \in \mathsf{Z}\right\} \\ &\supseteq \left\{x \in \mathfrak{t} \mid \langle x, X^*(H)\rangle \in \mathsf{Z}\right\} \\ &= \left[X_*(H)\right]^{\sigma} \end{aligned}$$

Then $t \in \mathfrak{t}/X_{\sigma}(H)$.

(d) With notation analogous to (*c*), $\chi \in \mathfrak{a}^*/X^{-\sigma}(H)$; here $X^{-\sigma}(H) \subseteq X^*(H)$. A *dual real datum for* ${}^LG^0$ is a five-tuple $\check{g} = ({}^LB^0, {}^LH^0, \check{\sigma}, \check{t}, \check{\chi})$, such that (in the natural isomorphism $\mathfrak{h} \cong ({}^L\mathfrak{h}^0)^*$ determined by B, ${}^LB^0$, and (15.2c)) $\check{\sigma}$ is the negative transpose of σ, $\check{t} = \chi$, and $\check{\chi} = t$.

Definition 15.4. Suppose $g = (B, H, \sigma, t, \chi)$ is a real datum for G. A *realization of* g is a four-tuple $(G^1(\mathsf{R}), B, t, \gamma)$, with the following properties. Choose representatives $t^0 \in \mathfrak{t}$, $\chi^0 \in \mathfrak{a}^*$ for t and χ.

(a) G^1 is the identity component of the centralizer of $\exp(4\pi i t^0)$

(b) G^1 is endowed with a real structure so that H is defined over R; and there is a Cartan involution θ of G^1 (for $G^1(\mathsf{R})$) which fixes H, such that $\theta|_H = \sigma$.

(c) γ is a regular character of $H(\mathsf{R})$ for $G^1(\mathsf{R})$.

(d) Write $(\check{\rho}^1)^{i\mathsf{R}}$ for half the sum of the positive imaginary (that is, σ-fixed) coroots of $\mathfrak{h}$ in $\mathfrak{g}^1$, and ϵ for the grading of the imaginary roots defined by the notion of compact root for $G^1(\mathsf{R})$. Then we require

$$\epsilon(\alpha) = \langle \alpha, (\check{\rho}^1)^{i\mathsf{R}} + 2t^0\rangle \pmod{2\mathsf{Z}}.$$

(e) Write $(\rho^1)^{\mathsf{R}}$ for half the sum of the $2\chi^0$-integral positive real roots of $\mathfrak{h}$ in $\mathfrak{g}^1$. Choose a positive system $(\tilde{\Delta}^{\mathsf{R}})^+$ for all the positive real roots, such that if $\tilde{\rho}^{\mathsf{R}}$ is half its sum, then $\langle \alpha, \rho^{\mathsf{R}} - \tilde{\rho}^{\mathsf{R}}\rangle \in 2\mathsf{Z}$, all α real and χ^0-integral; this is possible by (10.16). Now choose a θ-stable parabolic $Q = LU$ of G^1, with $L \supseteq H$ the centralizer of t. Write $\rho(\mathfrak{u})$ for half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{u}$, and $\tilde{\rho}$ for $\rho(\mathfrak{u}) + \tilde{\rho}^{\mathsf{R}}$. Write $H(\mathsf{R}) = TA$ as in (2.9), and write $2\rho(\mathfrak{u} \cap \mathfrak{p})$ for the determinant of the action of T on the -1 eigenspace of θ in $\mathfrak{u}$. Then we require that there should be a $\mu \in X^*(H)$ such that if $\gamma = (\Gamma, \bar{\gamma})$ (cf. (2.10)),

$$\bar{\gamma} = 2\chi^0 + \mu + \tilde{\rho}$$

$$\Gamma|_T = \mu|_T + 2\rho(\mathfrak{u} \cap \mathfrak{p})$$

$$\langle \bar{\gamma}, \alpha \rangle > 0 \qquad \text{for} \quad \alpha \in \Delta^+(\mathfrak{g}^1, \mathfrak{h}), 2\chi^0 - \text{integral}.$$

PROPOSITION 15.5. *Every real datum for G has a realization, which is unique up to conjugation in G and Jantzen–Zuckerman translation on the regular character involved; and the realization determines the real datum uniquely.*

So far this is not particularly difficult. The next step is to define the cross action (of the $2\chi^0$-integral Weyl group of H in G^1), and the Cayley transforms, for real data and their realizations. (The operations on realizations do not change $G^1(\mathbf{R})$.) This is in the spirit of section 7, and we leave it to the reader.

Definition 15.6. *Blocks* of real data are the smallest sets closed under conjugation in G, Cayley transforms, and the cross action.

This is a stronger requirement than the obvious one of asking only that the data have realizations with the same $G^1(\mathbf{R})$, and regular characters in the same block.

PROPOSITION 15.7. *Suppose g^1 and g^2 are real data in the same block for G, with realizations on the same $G^1(\mathbf{R})$; and suppose these realizations differ by Cayley transforms and the cross action inside $G^1(\mathbf{R})$. Then g^1 and g^2 are conjugate under G if and only if the regular characters γ^1 and γ^2 for $G^1(\mathbf{R})$ are conjugate under $G^1(\mathbf{R})$.*

This is hard; it is proved in the same way as Theorem 10.1.

Definition 15.8. Suppose g^1 and g^2 are real data in the same block for G. Choose realizations on a common $G^1(\mathbf{R})$, differing by Cayley transforms and the cross action inside $G^1(\mathbf{R})$. Write γ^1, γ^2 for the corresponding regular characters; and set

$$m(g^1, g^2) = m(\bar{\pi}(\gamma^1), \pi(\gamma^2))$$

$$M(g^1, g^2) = M(\bar{\pi}(\gamma^1), \pi(\gamma^2))$$

(notation (1.1), (1.2)). If g^1 and g^2 are in different blocks, we put $m(g^1, g^2) = M(g^1, g^2) = 0$.

This definition makes sense, even on conjugacy classes of real data, by Proposition 15.7.

PROPOSITION 15.9. *Suppose g^1 and g^2 are real data for G, and $\check{g}^1$, $\check{g}^2$ are dual real data for ${}^LG^0$ (Definition 15.3). Assume that the Kazhdan–Lusztig conjecture*

12.16 *holds for all real groups which appear. Then*

$$M(g^1, g^2) = \epsilon(g^1, g^2) m(\check{g}^2, \check{g}^1),$$

with $\epsilon(g^1, g^2) = \pm 1$; *and we can determine the sign.*

This proposition is just a formal combinatorial result about manipulating real data, and is proved by the methods of sections 12 and 13. It is interesting only because of Proposition 15.7, which relates it to multiplicities and character formulas in real reductive (algebraic) groups.

Example 15.10. Suppose $G = \mathrm{SL}(2)$. Put $B =$ upper triangular matrices, $H =$ diagonal matrices, $\sigma = 1$ on H, $\chi = 0$. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ with $\mathbf{C}$, in such a way that $\check{\alpha}$ corresponds to 1 in $\mathfrak{h}$, and α to 2 in $\mathfrak{h}^*$. Then

$$X_*(H) = \mathbf{Z}, \qquad X^*(H) = \mathbf{Z}$$

We consider real data

$$g_t = (B, H, \sigma, t, \chi),$$

with B, H, σ, and χ as above; thus

$$t \in \mathbf{C}/\mathbf{Z}.$$

If $2t \notin 1/2\mathbf{Z}$, then $G^1 = H$, and things are dull. There are four other cases: $t = 0, 1/4, 1/2, 3/4$. If $t = 1/4$ or $3/4$, then we can find a realization

$$(\mathrm{SU}(2), B, T, \gamma)$$

of g_t; here $\pi(\gamma)$ is the trivial representation of SU(2). These two labellings of the trivial representation of $SU(2)$ belong to different blocks. The dual real data have realizations

$$\left(\mathrm{PGL}(2, \mathbf{R}), {}^L B^0, 1, \check{\gamma}_{\pm}\right),$$

here $\pi(\check{\gamma} \pm)$ are irreducible principal series representations which agree on the identity component of $\mathrm{PGL}(2, \mathbf{R})$. (The need to distinguish these two representations gives rise to the need to label the trivial representation of SU(2) by $1/4$ or $3/4$.)

If $t = 0$ or $1/2$, then g_t has a realization $(\mathrm{SU}(1,1), B, t, \gamma +)$, with $\pi(\gamma +)$ a holomorphic discrete series representation for SU(1, 1). Now g_0 and $g_{1/2}$ are in the same block for G, but these two realizations do not differ by Cayley transforms and the cross action inside SU(1, 1). However, we can choose a realization $(\mathrm{SU}(1,1), B^{op}, 0, \gamma^-)$ of a conjugate of g_0; here $\pi(\gamma^-)$ is an antiholomorphic discrete series. This realization differs from the previous one of $g_{1/2}$ by the cross action inside SU(1, 1).

16. Examples.

Example 16.1. *Weyl groups of* $\mathrm{SL}(n,\mathbb{R})$. We can choose representatives for the Cartan subgroups of $\mathrm{SL}(n,\mathbb{R})$ as follows. Choose $r \leqslant [n/2]$. Then there is a unique conjugacy class of Cartan subgroups with r-dimensional compact part. As a representative, we choose $H^r = T^r A^r$; here

$$
\begin{aligned}
A^r &= \Big\{ d(e^{x_1}, e^{x_1}, e^{x_2}, e^{x_2}, \dots, e^{x_r}, e^{x_r}, e^{y_1}, \dots, e^{y_{n-2r}}) \mid \textstyle\sum 2x_i + \sum y_j \\
&\qquad = 0,\ x_i, y_j \in \mathbb{R} \Big\} \\
T^r &= \Big\{ d(t(\phi_1), \dots, t(\phi_r), \epsilon_1, \dots, \epsilon_{n-2r}) \mid \phi_i \in \mathbb{R},\ t(\phi) \\
&\qquad = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \epsilon_i = \pm 1, \prod_{i=1}^{n-2r} \epsilon_i = 1 \Big\}
\end{aligned}
\tag{16.1a}
$$

(Here and throughout, d applied to a set of numbers (or square matrices) will be the matrix with these numbers on the diagonal.) The standard identification of the complexified Cartan subalgebra $(\mathfrak{h}^r)^*$ with

$$\Big\{ v = \textstyle\sum v_i e_i \in \mathbb{C}^n \mid \sum v_i = 0 \Big\}$$

gives

$$
\begin{aligned}
(\mathfrak{a}^r)^* &= \Big\{ \sum_{i=1}^{r} x_i (e_{2i-1} + e_{2i}) + \sum_{j=1}^{n-2r} y_j e_j + 2r \Big\} \\
(\mathfrak{t}^r)^* &= \Big\{ \textstyle\sum \sqrt{-1}\, \phi_i (e_{2i-1} - e_{2i}) \Big\}.
\end{aligned}
\tag{16.1b}
$$

The Cartan involution θ^r acts by sending v to $-v$, then interchanging the first r pairs of coordinates:

$$\theta^r\Big(\textstyle\sum v_i e_i\Big) = -\left[\sum_{i=1}^{r} (v_{2i} e_{2i-1} + v_{2i-1} e_{2i}) + \sum_{j=n-2r+1}^{n} v_j e_j \right]. \tag{16.1c}$$

In the notation of Definition 3.10, we have

$$
\begin{aligned}
R &= \Delta(\mathfrak{g}, \mathfrak{h}^r) = \{ e_i - e_j \mid i \neq j \} \\
R^{\mathbb{R}} &= \{ \alpha \in R \mid \theta\alpha = -\alpha \} \\
&= \{ e_i - e_j \mid i, j \geqslant n - 2r + 1 \} \\
R^{i\mathbb{R}} &= \{ \pm(e_{2i-1} - e_{2i}) \mid 1 \leqslant i \leqslant r \}.
\end{aligned}
\tag{16.1d}
$$

If the root system is ordered lexicographically, then in the notation of Proposition 3.12,

$$
\begin{aligned}
\check{\rho}^{\mathsf{R}} &= (0, \ldots, 0, (n-2r-1)/2, (n-2r-3)/2, \ldots, -(n-2r-1)/2) \\
&= \sum_{j=1}^{n-2r} \left[(n-2r-2j+1)/2\right] e_{2r+j} \\
\rho^{i\mathsf{R}} &= (1/2, -1/2, 1/2, -1/2, \ldots, 1/2, -1/2, 0 \ldots 0) \\
&= \sum_{i=1}^{r} 1/2(e_{2i-1} - e_{2i}) \\
R^{\mathsf{C}} &= \{(e_{2k} - e_{2l}), (e_{2k-1} - e_{2l-1}) \mid \leqslant k, l \leqslant r\} \\
R^{q} &= R^{\mathsf{C}} \cup R^{\mathsf{R}} \\
R^{f} &= \{e_i - e_j \mid i, j \leqslant 2r\}
\end{aligned} \tag{16.1e}
$$

The subgroups of $SL(n,\mathsf{R})$ corresponding to R^{C}, R^q, and R^f are (up to center) $SL(r,\mathsf{C})$, $SL(r,\mathsf{C}) \times SL(n-2r,\mathsf{R})$, and $SL(2r,\mathsf{R})$. Hence

$$(W^{\mathsf{C}})^{\theta} \cong S_r, \qquad \text{the permutation group on } r \text{ letters;} \tag{16.1f}$$

it acts by permuting $\{e_1, e_3, \ldots, e_{2r-1}\}$ and $\{e_2, \ldots, e_{2r}\}$ simultaneously. Also,

$$W^{\mathsf{R}} \cong S_{n-2r}, \tag{16.1g}$$

acting by permuting the last $n-2r$ coordinates, and

$$W^{i\mathsf{R}} \cong (\mathsf{Z}/2\mathsf{Z})^r, \tag{16.1h}$$

acting by changing signs on the ϕ_i coordinates of (16.1a). To compute $W_1^{i\mathsf{R}}$—the elements of $W^{i\mathsf{R}}$ having representatives in G—we use Proposition 10.20(a), (d). The roots $\{\alpha_1, \ldots, \alpha_r\}$ of that Proposition are just

$$\alpha_i = e_{2i-1} - e_{2i} \qquad (i = 1, \ldots, r).$$

These span $\mathfrak{t}^r$, so, in the notation of Proposition 10.20, $\Phi = R$. We must compute the elements $\sigma_\beta \in W^{\mathsf{R}}(\gamma)$. Suppose first that $n > 2r$. Then if $\beta = e_{2j} - e_n$, we have

$$\langle \alpha_1, \beta \rangle = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$$

$$\sigma_\beta = s_{\alpha_i}$$

So in this case $Q_+ = W^{i\mathbb{R}}$, and (by Proposition 4.16)

$$\left.\begin{aligned} W_1^{i\mathbb{R}} &= W^{i\mathbb{R}} \\ W(G,H^r) &\cong S_r \ltimes \left(S_{n-2r} \times (\mathbb{Z}/2\mathbb{Z})^r\right) \\ &\cong W(BC_r) \times S_{n-2r}. \end{aligned}\right\} \quad n > 2r \tag{16.1i}$$

Here $W(BC_r)$ is the Weyl group of B_r or C_r; this last isomorphism is by the standard embedding of that Weyl group in S_{2r}. Next, suppose $n = 2r$. If ϵ and δ are 0 or 1, $j \neq k$, and $\beta = e_{2j-\epsilon} - e_{2k-\delta}$, then

$$\langle \alpha_i, \beta \rangle = \begin{cases} \pm 1, & i = j \text{ or } k \\ 0 & i \neq j, k \end{cases}$$

$$\sigma_\beta = s_{\alpha_j} s_{\alpha_k}.$$

Therefore

$$\begin{aligned} Q_+ &= \left\{ s_{\gamma_1} \cdots s_{\gamma_{2l}} \mid \gamma_j \in R^{i\mathbb{R}} \right\} \subseteq W^{i\mathbb{R}} \\ W_1^{i\mathbb{R}} &= Q_+ \cong (\mathbb{Z}/2\mathbb{Z})^{r-1} \\ W^{i\mathbb{R}}/W_1^{i\mathbb{R}} &\cong \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

So

$$\left.\begin{aligned} W(G,H^r) &\cong S_r \ltimes (\mathbb{Z}/2\mathbb{Z})^{r-1} \\ &\cong W(D_r) \end{aligned}\right\} \quad n = 2r \tag{16.1j}$$

(This is also clear from the fact that H^r is fundamental, so that $W(G,H^r) \cong W(K,T^r)$.) Of course replacing $\mathrm{SL}(n,\mathbb{R})$ by $\mathrm{GL}((n,\mathbb{R})$ makes $W_1^{i\mathbb{R}}$ equal to $W^{i\mathbb{R}}$ in this case by Proposition 10.20(e).

Example 16.2. *The block of the trivial representation of* $\mathrm{SL}(3,\mathbb{R})$. The Cartan subgroups H^0 (split) and H^1 (fundamental) are as in Example 16.1. Any regular character corresponding to the same infinitesimal character as the trivial representation must have differential half the sum of a set of positive roots. On H^0, $W(G,H) = W(\mathfrak{g},\mathfrak{h})$, so (up to conjugacy) we can consider only the regular characters γ^0 with differential

$$\bar{\gamma}^0 = (1, 0, -1) = \rho; \tag{16.2a}$$

we are introducing coordinates as in (16.1b). Since H^0 has four connected components, there are four such regular characters. We write them as

$$\gamma_\epsilon^0 = \gamma^0_{(\epsilon_1,\epsilon_2,\epsilon_3)} \in (\hat{H}^0)' \qquad (\epsilon_i = \pm 1).$$

with the convention that

$$\gamma_\epsilon^0(d(\delta_1, \delta_2, \delta_3)) = \Pi\epsilon_i\delta_i \qquad (\delta_i = \pm 1).$$

Since $d(\delta_i)$ is in $\mathrm{SL}(3, \mathbf{R})$ only if $\Pi\delta_i = 1$, we have

$$\gamma^0_{(\epsilon_1, \epsilon_2, \epsilon_3)} = \gamma^0_{(-\epsilon_1, -\epsilon_2, -\epsilon_3)}.$$

We have

$$\bar{\pi}\left(\gamma^0_{(1,1,1)}\right) = \text{trivial representation}; \tag{16.2b}$$

for since H^0 is split, $\pi(\gamma_\epsilon^0)$ is just the principal series representation induced from the character γ_ϵ^0 of H^0. By Theorem 8.5, the other regular characters of H^0 in the same block as γ_1^0 are obtained by the cross action of the Weyl group; so we will now compute this. Suppose $\alpha = e_1 - e_2$. Then

$$s_\alpha\rho = \rho - \alpha;$$

so by Definition 4.1

$$s_\alpha \times \gamma_\epsilon^0 = \gamma_\epsilon^0 - \alpha$$

as a character of H^0. Now

$$\alpha(d(\delta_1, \delta_2, \delta_3)) = \delta_1\delta_2^{-1};$$

so $s_\alpha \times \gamma_\epsilon^0$ looks like $\gamma^0_{(-\epsilon_1, -\epsilon_2, \epsilon_3)}$ on diagonal matrices with entries ± 1. However, we have also changed ρ to $s_\alpha\rho$. This we can reverse by conjugation by s_α; and we compute

$$s_\alpha \times \gamma_\epsilon^0 \text{ conjugate to } \gamma^0_{(-\epsilon_2, -\epsilon_1, \epsilon_3)} \qquad (\alpha = e_1 - e_2). \tag{16.2c}$$

Similarly,

$$s_\beta \times \gamma_\epsilon^0 \text{ conjugate to } \gamma^0_{\epsilon_1, -\epsilon_3, -\epsilon_2} \qquad (\beta = e_2 - e_3). \tag{16.2d}$$

Using these, we compute

$$\text{cross orbit of } \gamma_1^0 = \left\{\gamma_1^0, \gamma^0_{-1,-1,1}, \gamma^0_{1,-1,-1}\right\}. \tag{16.2e}$$

On H^1, $W(G, H) = W^{i\mathbf{R}}$ has order 2 ((16.1i)); in the coordinates (16.1b), the non-trivial element of $W(G, H)$ interchanges the first two coordinates. Since H^1 is connected, there are just three regular characters (up to conjugacy) with differential conjugate to ρ. We choose

$$\begin{aligned} \bar{\gamma}^1 &= (0, -1, 1) \\ \bar{\gamma}^2 &= (1, 0, -1) \\ \bar{\gamma}^3 &= (1, -1, 0) \end{aligned} \tag{16.2f}$$

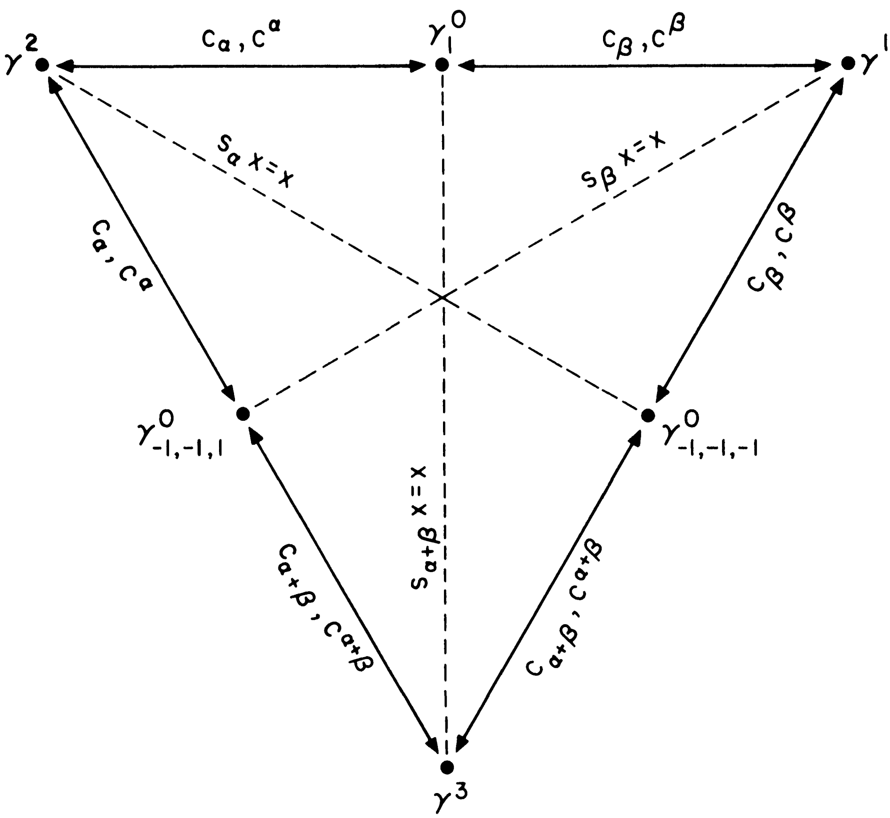

FIGURE 16.2

as representatives. Since H^1 is connected, the cross action of $W(\mathfrak{g}, \mathfrak{h})$ is just the usual action on $\bar{\gamma}$, so we will not say more about it. We would, however, like to specify Cayley transforms explicitly. So let α be the abstract simple root corresponding to $e_1 - e_2$ on $\mathfrak{h}^0$. If $c_\alpha(\gamma_\epsilon^0)$ is defined, and γ belongs to it, then the root α^1 on $\mathfrak{h}^1$ corresponding to α must be imaginary. By (16.1d), this forces $\gamma = \gamma^2$; that is,

$$c_\alpha(\gamma_\epsilon^0) = \gamma^2 \qquad \text{all} \quad \gamma_\epsilon^0 \in D(c_\alpha) \tag{16.2g}$$

The domain of c_α is easily computed to be $\{\gamma_1^0, \gamma_{-1,-1,1}^0\}$. We arrive finally at Fig. 16.2 for the block of the trivial representation in $SL(3, \mathbf{R})$. In it, the abstract Weyl group S_3 is acting on $\mathbf{R}^2$ as usual: $\alpha = e_1 - e_2$, and the reflection s_α fixes the line through $\exp(5\pi i/6)$; $\beta = e_2 - e_3$, and s_β fixes $\exp(i\pi/6)$. The cross action of the Weyl group is then the geometric one on the points representing regular characters; and Cayley transforms are shown by arrows.

We can say a little more about the nature of the representations $\bar{\pi}(\gamma)$. As was mentioned already, $\bar{\pi}(\gamma^0_{1,1,1})$ is the trivial representation. The only other unitary representation among the six is the irreducible standard representation

$$\bar{\pi}(\gamma^3) = \pi(\gamma^3), \tag{16.2h}$$

which is a tempered fundamental series representation. Both $\bar{\pi}(\gamma^1)$ and $\bar{\pi}(\gamma^2)$ are Langlands quotients of reducible (nonunitary) fundamental series representations; we have

$$\begin{aligned} \bar{\pi}(\gamma^1) &= \pi(\gamma^1) - \pi(\gamma^3) \\ \bar{\pi}(\gamma^2) &= \pi(\gamma^2) - \pi(\gamma^3) \end{aligned} \tag{16.2h}$$

in the Grothendieck group. Both $\bar{\pi}(\gamma^0_{-1,-1,1})$ and $\pi(\gamma^0_{1,-1,-1})$ are Langlands quotients of ordinary non-spherical principal series representations. Their formal characters are

$$\begin{aligned} \bar{\pi}(\gamma^0_{-1,-1,1}) &= \pi(\gamma^0_{-1,-1,1}) - \pi(\gamma^1) \\ \bar{\pi}(\gamma^0_{1,-1,-1}) &= \pi(\gamma^0_{1,-1,-1}) - \pi(\gamma^2). \end{aligned} \tag{16.2h}$$

Finally, the formal character of the trivial representation is

$$\bar{\pi}(\gamma^0_1) = \pi(\gamma^0_1) - \pi(\gamma^1) - \pi(\gamma^2) + \pi(\gamma^3). \tag{16.2h}$$

All of the formulas (16.2h) are well known, but we may as well outline their derivation from the Kazhdan–Lusztig conjecture. We first tabulate the objects of Definitions 12.2 and 12.3, normalizing l^I to be zero on γ^3.

TABLE 16.2(i)

γ	$l^I(\gamma)$	$\tau(\gamma)$	$T_{s_\alpha}(\gamma)$	$T_{s_\beta}(\gamma)$
γ^3	0	$\varnothing$	γ^1	γ^2
γ^1	1	α	$(u-1)\gamma^1 + u\gamma^3$	$\gamma^1 + \gamma^0_{1,-1,-1} + \gamma^0_1$
γ^2	1	β	$\gamma^2 + \gamma^0_{-1,-1,1} + \gamma^0_1$	$(u-1)\gamma^2 + u\gamma^3$
$\gamma^0_{-1,-1,1}$	2	α	$(u-1)\gamma^0_{-1,-1,1} - \gamma^0_1 + (u-1)\gamma^2$	$-\gamma^0_{-1,-1,1}$
$\gamma^0_{1,-1,-1}$	2	β	$-\gamma^0_{1,-1,-1}$	$(u-1)\gamma^0_{1,-1,-1} - \gamma^0_1 + (u-1)\gamma^1$
γ^0_1	2	α, β	$(u-1)\gamma^0_1 - \gamma^0_{-1,-1,1}$	$(u-1)\gamma^0_1 - \gamma^0_{1,-1,-1} + (u-1)\gamma^1$

The Bruhat r-order of Definition 2.12 turns out simply to be the partial order defined by $l^I(\gamma)$. The map D may be computed by the algorithm of [24]; one gets

TABLE 16.2(j)

$$\begin{aligned}
D\gamma^3 &= \gamma^3\\
D\gamma^1 &= u^{-1}[\gamma^1 - (u-1)\gamma^3]\\
D\gamma^2 &= u^{-1}[\gamma^2 - (u-1)\gamma^3]\\
D\gamma^0_{-1,-1,1} &= u^{-2}[\gamma^0_{-1,-1,1} - (u-1)\gamma^2 + (u^2-u)\gamma^3]\\
D\gamma^0_{1,-1,-1} &= u^{-2}[\gamma^0_{1,-1,-1} - (u-1)\gamma^1 + (u^2-u)\gamma^3]\\
D\gamma^0_1 &= u^{-2}[\gamma^0_1 - (u-1)\gamma^2 - (u-1)\gamma^1 + (u^2-2u+1)\gamma^3]
\end{aligned}$$

Of course it is elementary (but tedious) to verify that the map D defined by Table 16.2(j) satisfies the defining conditions for D in Lemma 12.14. The formulas are organized so that the polynomials $R_{\phi\gamma}$ (Lemma 12.14) may be read off easily. It should be observed that if $\phi = \gamma^2$, $\gamma = \gamma^0_{-1,-1,1}$, then $R_{\phi\gamma} = u^2 - u$. For complex groups of rank two (that is, the case originally studied by Kazhdan and Lusztig) one always had $R_{\phi\gamma} = (u-1)^k$, with $k = l^I(\gamma) - l^I(\phi)$. More generally, the constant term of $R_{\phi\gamma}$ is always ± 1 in the case of complex groups (if $R_{\phi\gamma} \neq 0$); and even that fails here.

From Table 16.2(j), it is fairly easy to compute the elements C_γ of Lemma 12.15. They are:

TABLE 16.2(k)

γ	C_γ
γ^3	γ^3
γ^1	$\gamma^1 + \gamma^3$
γ^2	$\gamma^2 + \gamma^3$
$\gamma^0_{-1,-1,1}$	$\gamma^0_{-1,-1,1} + \gamma^2$
$\gamma^0_{1,-1,-1}$	$\gamma^0_{1,-1,-1} + \gamma^1$
γ^0_1	$\gamma^0_1 + \gamma^2 + \gamma^1 + \gamma^3$

In particular, the non-zero $P_{\phi\gamma}$ are all 1 or 0, just as in the rank two complex case. The important distinction from the complex case is that $\phi < \gamma$ does not imply $P_{\phi\gamma} \neq 1$; take for example $\phi = \gamma^3$, $\gamma = \gamma^0_{-1,-1,1}$. This says that the relation "occurs in the formal character of" is not transitive; γ^3 occurs in the formal character of γ^2, and γ^2 in the formal character of $\gamma^0_{-1,-1,1}$, but γ^3 does not occur in the formal character of $\gamma^0_{-1,-1,1}$. By applying Theorem 1.15, we deduce that the relation "is a composition factor of" is not transitive in general. This phenomenon is well known, as it occurs in all of the rank one groups but $SL(2, \mathbf{R})$ and $SL(2, \mathbf{C})$.

The formulas (16.2h) follow from Table 16.2(k) by Theorem 12.17.

Example 16.3. *The block of the trivial representation of* SU(2, 1). This block is in duality with the one of Example 16.2 in the sense of Theorem 1.15. To prove that, one computes the strong bigradings attached to a non-holomorphic discrete series, and to the trivial representation (Definition 4.12), and verifies that they are dual to those attached to γ_1^0 and γ^2 for $SL(3,\mathbf{R})$ (Example 16.2). Then the duality in question follows from Theorem 10.1. Write $\check{\gamma}$ for the SU(2, 1) regular character dual to γ for $SL(3,\mathbf{R})$. Then

$$\begin{aligned}
\bar{\pi}(\check{\gamma}^2) &= \text{trivial representation}\\
\bar{\pi}(\check{\gamma}^1) &= \text{Langlands quotient of a principal series}\\
\bar{\pi}(\check{\gamma}^2) &= \text{Langlands quotient of another principal series}\\
\bar{\pi}(\check{\gamma}^0_{-1,-1,1}) &= \text{holomorphic discrete series}\\
\bar{\pi}(\check{\gamma}_{1,-1,-1}) &= \text{anti-holomorphic discrete series}\\
\pi(\check{\gamma}_1^0) &= \text{non-holomorphic discrete series.}
\end{aligned} \tag{16.3a}$$

By Theorem 10.1, Figure 16.2 applies to these representations as well. We leave to the reader the task of computing D and the various $C_{\check{\gamma}}$; by the results of section 13, one only has to perform some simple manipulations with Tables 16.2(j)-(k). An interesting feature of this example is that all of the $\pi(\check{\gamma})$ are unitary representations of SU(2, 1); and we saw that only two of the $\bar{\pi}(\gamma)$ for $SL(3,\mathbf{R})$ were unitary. For this reason and others, it seems very unlikely that the duality relationship has anything to do with unitarity.

Example 16.4. *The gradings of* D_4. We realize $R = D_4$ in $\mathbf{R}^4$ (with basis $\{e_i\}$) as usual, by

$$R = \{e_i \pm e_j \mid i \neq j\}. \tag{16.4a}$$

Then

$$W(R) = W = \begin{array}{l}\text{group of permutations and even numbers of}\\ \text{sign changes on the basis } \{e_i\};\end{array} \tag{16.4b}$$

this group has order 192. Put

$$\begin{aligned}
R^+ &= \{e_i \pm e_j \mid i < j\}\\
\Pi &= \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4\} = \{\alpha_1, \alpha_0, \alpha_2, \alpha_3\}.
\end{aligned} \tag{16.4c}$$

The numbering of the simple roots in Π is explained by the Dynkin diagram, which is

$$\begin{array}{lcl} & & \alpha_3 \\ & & / \\ \alpha_1 - \alpha_0 & & \\ & & \backslash \\ & & \alpha_2 \end{array} \tag{16.4d}$$

(This is *not* the usual numbering of the simple roots, but it is a natural one.)

By Lemma 3.14, a grading $\epsilon \in E(R)$ is determined by its restriction to $\{\alpha_i\}$, which may be arbitrary. If $\phi = (\phi_0, \phi_1, \phi_2, \phi_3) \in (\mathbf{Z}/2\mathbf{Z})^4$, then we define $\epsilon_\phi \in E(R)$ by

$$\epsilon_\phi(\alpha_i) = \phi_i\,. \tag{16.4e}$$

We may represent a grading ϵ by putting $\epsilon(\alpha_i)$ at the α_i vertex of the Dynkin diagram; thus

$$\epsilon_{(0,1,0,1)} \leftrightarrow 1\ 0\ \substack{1\\0}$$

To compute the orbit of the cross action on $E(R)$ (Definition 3.24), suppose α and β are simple roots, and $\epsilon(\alpha) = 1$. Then

$$(s_\alpha \times \epsilon)(\beta) = \begin{cases} \epsilon(\beta) & \text{if } \langle \check{\alpha}, \beta \rangle \text{ even} \\ 1 - \epsilon(\beta) & \text{if } \langle \check{\alpha}, \beta \rangle \text{ odd.} \end{cases} \tag{16.4f}$$

Thus $s_\alpha \times \epsilon$ agrees with ϵ except on the vertices adjacent to α. Using this, we compute easily that the orbits of the cross action on $E(r)$ are

$$\begin{aligned}
\mathcal{O}_p &= \left\{ 0\,1\,\substack{0\\0},\ \ 1\,1\,\substack{1\\1},\ \ 1\,0\,\substack{1\\1} \right\} \\
\mathcal{O}_{\mathrm{I}} &= \left\{ 1\,0\,\substack{0\\0},\ \ 1\,1\,\substack{0\\0},\ \ 0\,1\,\substack{1\\1},\ \ 0\,0\,\substack{1\\1} \right\} \\
\mathcal{O}_{\mathrm{II}} &= \left\{ 0\,0\,\substack{0\\1},\ \ 0\,1\,\substack{0\\1},\ \ 1\,1\,\substack{1\\0},\ \ 1\,0\,\substack{1\\0} \right\} \qquad (16.4\mathrm{g}) \\
\mathcal{O}_{\mathrm{III}} &= \left\{ 0\,0\,\substack{1\\0},\ \ 0\,1\,\substack{1\\0},\ \ 1\,1\,\substack{0\\1},\ \ 1\,0\,\substack{0\\1} \right\} \\
\mathcal{O}_t &= \left\{ 0\,0\,\substack{0\\0} \right\}
\end{aligned}$$

(Here p stands for principal, and t for trivial.) All of these arise as the grading by compact and noncompact roots in some real form G of $SO(8, \mathbf{C})$. The relevant real forms are tabulated below. In each case the set R_0 of compact roots is specified.

TABLE 16.4(h)

Orbit in $E(R)$	G	R_0 = compact roots
$\mathcal{O}_p$	$SO(4,4)$	$\{\pm e_1 \pm e_2, \pm e_3 \pm e_4\} \cong D_2 \times D_2 \cong (A_1)^4$
$\mathcal{O}_{\mathrm{I}}$	$SO(2,6)$	$\{\pm e_2 \pm e_3, \pm e_2 \pm e_4, \pm e_3 \pm e_4\} \cong D_3 \cong A_3$
$\mathcal{O}_{\mathrm{II}}$	$SO^*(8)$	$\{e_1 - e_j \mid i, j \leqslant 3\} \cup \{e_i + e_4 \mid i \leqslant 3\} \cong A_3$
$\mathcal{O}_{\mathrm{III}}$	$SO^*(8)$	$\{e_i - e_j \mid i \neq j\} \cong A_3$
$\mathcal{O}_t$	$SO(8)$	$R_0 = R = D_4$

We stated at (3.26) that the cross orbits on $E(R)$ were in one-to-one correspondence with equal rank real forms of $G^{\mathbb{C}}$. This appears to contradict Table 16.4(h), since $SO^*(8)$ is attached to both $\mathcal{O}_{\mathrm{II}}$ and $\mathcal{O}_{\mathrm{III}}$; and in any case $SO^*(8) \cong SO(2,6)$, which is attached to $\mathcal{O}_{\mathrm{I}}$. The point is that the isomorphism $SO(2,6) \cong SO^*(8)$ cannot be realized by conjugation in $SO(8,\mathbb{C})$; and neither can the isomorphism between the two versions of $SO^*(8)$ used for $\mathcal{O}_{\mathrm{II}}$ and $\mathcal{O}_{\mathrm{III}}$. By "real forms" at (3.26), we meant "real forms up to $G^{\mathbb{C}}$-conjugacy." To replace this with "real forms up to isomorphism," we must expand W to include automorphisms of the Dynkin diagram. Here $\mathcal{O}_{\mathrm{I}}$, $\mathcal{O}_{\mathrm{II}}$, and $\mathcal{O}_{\mathrm{III}}$ differ by the automorphisms which rotate the Dynkin diagram.

Set

$$\epsilon_p = 0\,1\,{}^{0}_{0}, \quad \epsilon_{\mathrm{I}} = 1\,0\,{}^{0}_{1}, \quad \epsilon_{\mathrm{II}} = 0\,0\,{}^{0}_{1}, \quad \epsilon_{\mathrm{III}} = 0\,0\,{}^{1}_{0}, \quad \epsilon_t = 0\,0\,{}^{0}_{0}$$

$$\mathcal{O}_\epsilon = W \times \epsilon \subseteq E(R) \qquad (\epsilon \in E(R)). \tag{16.4i}$$

Since the various $R_0(\epsilon)$ are tabulated in Table 16.4(h), we can immediately write down $W_0(\epsilon) = W(R_0)$. The cardinality of $W_2(\epsilon)$ may be computed by the formula

$$|\mathcal{O}_\epsilon| = |W/W_2(\epsilon)|$$

for the order of a finite homogeneous space. We would like also to specify the inclusion

$$W_2(\epsilon)/W_0(\epsilon) = \mathfrak{A}(\epsilon) \to Z(R)$$

of Proposition 3.16. First we must describe $Z(R)$. In the notation of Definition

3.3, we compute

$$\begin{aligned}
\check{L} &= \text{lattice of coroots} \\
&= \left\{(x_1, x_2, x_3, x_4) \in \mathbf{Z}^4 \mid \textstyle\sum x_i \equiv 0 \ (\mathrm{mod}\ 2\mathbf{Z})\right\} \\
\check{P} &= \left\{\lambda \in \mathbf{R}^4 \mid \lambda_i - \lambda_j \in \mathbf{Z}, \text{ all } i, j\right\} \\
&= \{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \mid \text{all } \lambda_i \in \mathbf{Z}, \text{ or all } \lambda_i \in 1/2\mathbf{Z}\} \\
Z(R) = \check{P}/\check{L} &= \left\{\bar{0}, \overline{(1,0,0,0)}, \overline{(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})}, \overline{(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})}\right\} \\
&= \{0, z_1, z_2, z_3\};
\end{aligned} \tag{16.4j}$$

here we write $\bar{x}$ for $x + \check{L}$. In the following table, the second column gives an element of $\check{P}(R)$ defining ϵ (cf. Lemma 3.14 and Propositon 3.16); $\mathfrak{A}(\epsilon)$ is exhibited as a subgroup of $Z(R)$ (Proposition 3.16); and we give representatives of $W_2(\epsilon)/W_0(\epsilon)$. For that purpose, we write $s_i = s_{\alpha_i}$.

TABLE 16.4(k) R groups for D_4

Grading	λ	$W_0(\epsilon)$	$\mathfrak{A}(\epsilon)$	Representatives of generators of $\mathfrak{A}(\epsilon)$ in $W_2(\epsilon)$; $(x_1, x_2, x_3, x_4) \to$—
ϵ_p	$(1,1,0,0)$	$(\mathbf{Z}/2\mathbf{Z})^4$	$Z(R) = \{0, z_1, z_2, z_3\}$	$(x_1, -x_2, x_3, -x_4)$, (x_3, x_4, x_1, x_2)
ϵ_{I}	$(1,0,0,0)$	$S_4 = W(A_3)$	$\{\bar{0}, z_1\}$	$x \to -x$
ϵ_{II}	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$S_4 = W(A_3)$	$\{\bar{0}, z_2\}$	$x \to -x$
ϵ_{III}	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$S_4 = W(A_3)$	$\{\bar{0}, z_3\}$	$x \to -x$
ϵ_t	$(0,0,0,0)$	$W = W(D_4)$	$\{\bar{0}\}$	—

The easiest way to calculate $\mathfrak{A}(\epsilon)$ is by first calculating $|\mathfrak{A}(\epsilon)|$ as described after 16.4(i). The elements exhibited in the last column obviously belong to $\mathfrak{A}(\epsilon)$, and so generate it by counting. Now the formulas for $\mathfrak{A}(\epsilon)$ are immediate from the definition of the map in Proposition 3.16.

Example 16.5. *Cayley transforms of the principal grading of* C_n. Realize $R = C_n$ in $\mathbf{R}^n$ as usual, with

$$\begin{aligned}
R &= \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_i\} \\
W(R) &= W = \text{permutations and sign changes of } \{e_i\} \\
&= S_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n \text{ (the hyperoctahedral group)} \\
R^+ &= \{e_i \pm e_j \mid i < j\} \cup \{2e_i\} \\
\Pi &= \{e_i - e_{i+1} \mid 1 \leqslant i \leqslant n-1\} \cup \{2e_n\}.
\end{aligned} \tag{16.5a}$$

We consider the grading $\epsilon \in E(R)$ defined in any of the following equivalent ways. Put $\lambda = (1, 1, \ldots, 1)$. Then

$$\begin{aligned} \epsilon(\alpha) &\equiv \langle \alpha, \lambda \rangle \qquad (\text{mod } 2) \\ \epsilon(\pm(e_i + e_j)) &= 1, \qquad \epsilon(e_i - e_j) = 0 \\ R_0(\epsilon) &= \{e_i - e_j\} \cong A_{n-1} \\ R_1(\epsilon) &= \{e_i + e_j\}. \end{aligned} \tag{16.5b}$$

Then

$$\begin{aligned} W_0(\epsilon) &= W(A_{n-1}) = S_n \subseteq W \\ W_2(\epsilon) &= \text{group generated by } W_0(\epsilon) \text{ and } -\mathrm{Id}. \end{aligned} \tag{16.5c}$$

There are two kinds of one dimensional Cayley transforms. The simpler is (Definition 5.2, Lemma 5.1)

$$\begin{aligned} &\alpha = 2e_m, \qquad R^\alpha = \{\pm \epsilon_i \pm e_j \mid i, j \neq m\}, \qquad \tilde{\epsilon} = \text{trivial} \\ &c^\alpha(\epsilon) = \epsilon|_R \alpha. \end{aligned} \tag{16.5d}$$

More interesting is the second kind:

$$\begin{aligned} \alpha &= e_r + e_s \\ R^\alpha &= \{\pm e_i \pm e_j \mid i, j \neq r, s\} \cup \{\pm(e_r - e_s)\} \\ &= R_s^\alpha \cup R_w^\alpha \\ \tilde{\epsilon} &= \text{trivial on } R_s^\alpha \\ \tilde{\epsilon}(e_r - e_s) &= 1 \\ c^\alpha(\epsilon)|_{R_s^\alpha} &= \epsilon|_{R_s^\alpha} \\ c^\alpha(\epsilon)(e_r - e_s) &= 1 \\ &= 1 - \epsilon(e_r - e_s). \end{aligned} \tag{16.5e}$$

It is not hard to convince oneself that every admissible sequence is (up to permutation and sign changes) conjugate under $W_0(\epsilon)$ to one of the following form:

$$\begin{aligned} S = S_{p,q,r} = \{2e_i, \ldots, 2e_p, e_{p+1} + e_{p+2}, e_{p+1} - e_{p+2}, \ldots \\ e_{p+2q-1} + e_{p+2q}, e_{p+2q-1} - e_{p+2q}, e_{p+2q+1} + e_{p+2q+2}, \\ e_{p+2q+3} + e_{p+2q+4}, \ldots, e_{p+2q+2r-1} + e_{p+1q+2r}\}. \end{aligned} \tag{16.5f}$$

For this sequence S, we have

$$\begin{aligned} R^S &= \{\pm e_i \pm e_j \mid i, j > p + 2q + 2r\} \cup \{e_{p+2q+2j-1} - e_{p+2q+2j} \mid 1 \leqslant j \leqslant r\} \\ &= R_s^S \cup R_w^S \end{aligned}$$

$$\begin{aligned} c^S(\epsilon)|_{R_s^S} &= \epsilon|_{R_s^S} \\ c^S(\epsilon)|_{R_w^S} &= (1-\epsilon)|_{R_s^S}. \end{aligned} \tag{16.5g}$$

The algorithm of Lemma 5.6 replaces $S_{p,q,r}$ by $S_{p+2,q-1,r}$; so if we put

$$S' = S_{p+2q,0,r} \tag{16.5h}$$

then S' is strongly orthogonal, and

$$c^S(\epsilon) = c^{S'}(\epsilon). \tag{16.5h}$$

Since the roots $\beta_j = e_{p+2q+2j-1} - e_{p+2q+2j}$ lie in $R_0(\epsilon)$, the corresponding reflections s_{β_j} belong to $W_0(\epsilon)$. In $c^S(\epsilon)$, however, β_j is an odd root; so $s_{\beta_j} \notin W_0(c^S(\epsilon))$. This observation leads, in the context of Lemma 6.1, to

$$\begin{aligned} W_0(c^{\mathfrak{s}}(\epsilon)) &\cong W(A_{n-(p+2q+2r)-1}) \\ c^{\mathfrak{s}}(W_0(\epsilon)) &= W(A_{n-(p+2q+2r)-1}) \times (\mathbf{Z}/2\mathbf{Z})^r \\ (\mathfrak{s} &= \text{span of } S_{p,q,r} \text{ (cf. (16.5f))}). \end{aligned} \tag{16.5i}$$

Using Lemma 9.1 (case of $c^{\mathfrak{s}}$) and Proposition 4.16, one can continue from here to determine the Weyl groups of all the Cartan subgroups of $\mathrm{Sp}(n, \mathbf{R})$. We leave this to the reader.

Example 16.6. *Connection with the "very strange formulas"*. Let G be the group $\mathrm{SL}(2, \mathbf{R})$, and consider the regular character γ^0 for the split Cartan subgroup, such that $\pi(\gamma^0)$ is irreducible and has the same infinitesimal character as the trivial representation of G. One checks easily that the cograding $\check{\delta}(\gamma^0)$ is trivial; that is, the roots do not satisfy the parity condition. Thus $\{\gamma^0\}$ is a block by itself, and $\pi(\gamma^0)$ is an irreducible (non-spherical) principal series. The dual group of $\check{G}$ of Theorem 1.15 is SU(2) and $\pi(\check{\gamma}^0)$ is the trivial representation. As SU(2) is connected, we have

$$\check{G} = \check{G}^0 \qquad \text{(Definition 9.4.)} \tag{16.6a}$$

Now consider $G^{\mathbf{C}} = \mathrm{SL}(2, \mathbf{C})$, and define

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbf{R} \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbf{C} \right\} \tag{16.6b}$$

as usual. The Iwasawa decomposition of $G^{\mathbb{C}}$ is now

$$G^{\mathbb{C}} = \check{G}^0 AN. \tag{16.6c}$$

This verifies a conjecture of P. Sally that $\check{G}^0 AN$ may be regarded as a complex simple Lie group; I am grateful to him for supplying this example.

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