

IRREDUCIBLE CHARACTERS OF SEMISIMPLE LIE GROUPS II. THE KAZHDAN-LUSZTIG CONJECTURES

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1. Introduction. Let G be a connected semisimple Lie group. In [15], a study of the irreducible characters of G was begun, using the ideas developed by Jantzen for Verma modules and extended to group representations in [13]. In particular, the explicit determination of these characters was reduced to the problem of decomposing certain representations $\{U_\alpha(X)\}$ attached to an irreducible representation X ; by this was meant the determination of the irreducible composition factors of each $U_\alpha(X)$, and their multiplicities, in terms of the Langlands classification of irreducible representations. It was conjectured ([15], Conjecture 3.15) that each $U_\alpha(X)$ is completely reducible.

Since [15] was written, Kazhdan and Lusztig have given in [9] a conjecture for the characters of irreducible quotients of Verma modules. (By [4] or [8], this is equivalent to finding the irreducible characters of a complex semisimple Lie group G). The present paper generalizes this conjecture to representations of real linear groups, and reduces its proof to the complete reducibility conjecture mentioned above; in fact, the conjectured formulas are equivalent to the complete reducibility conjecture. (The assumption of linearity is invoked only in section 7 below, for minor technical reasons; it can almost certainly be eliminated with a little cleverness.)

The idea of the argument is very simple. Let $\mathfrak{g}$ be the complexified Lie algebra of G , θ a Cartan involution, and $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ a θ -stable Levi decomposition of a θ -stable parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$. The idea is to try to compute $H^i(\mathfrak{u}, X)$ as an $\mathfrak{l}$ module whenever X is an irreducible Harish-Chandra module. This is done first when X is an irreducible representation induced from a discrete series on a cuspidal parabolic subgroup of G (Theorem 6.13). Once the answer is known for some X , one can compute $H^i(\mathfrak{u}, U_\alpha(X))$ (Theorem 7.2). If $U_\alpha(X)$ is completely reducible, this information (via [14]) allows one to determine the irreducible constituents of $U_\alpha(X)$ and their multiplicities (Proposition 5.5); and induction can proceed. The (conjectured) character formulas can be computed from the multiplicities in the various $U_\alpha(X)$, as in [15], or can be written in terms of the $H^i(\mathfrak{u}, X)$ by an Euler-Poincaré principle (Theorem 8.1—see also the remarks at the end of section 8).

Because the computations in general are quite complicated, we will first treat the Verma module case, assuming the infinitesimal character to be integral. Most of the ideas are already apparent there.

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I would like to thank D. Kazhdan and G. Lusztig for providing part of [9] in manuscript form. Conversations with D. Barbasch, D. Peterson, and G. Zuckerman were also helpful. Many results which were first proved in [13] have been referred to by their appearance in [15], since [13] was unpublished when this paper was written. Apologies are extended to B. Speh for these omissions.

2. Definition of $U_\alpha(X)$. We recall here some definitions from [15].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. (From section 4 on, $\mathfrak{g}$ will be the complexified Lie algebra of our real semisimple Lie group G .) Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{Z}(\mathfrak{g})$ its center, and

$$\xi : \mathfrak{Z}(\mathfrak{g}) \rightarrow S(\mathfrak{h})^W$$

the Harish-Chandra isomorphism; here $W = W(\mathfrak{g}/\mathfrak{h})$ is the Weyl group, and $S(\mathfrak{h})$ is the symmetric algebra of $\mathfrak{h}$. If $\lambda \in \mathfrak{h}^*$ (the dual of $\mathfrak{h}$), we define $\chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ by $\chi_\lambda(z) = \xi(z)(\lambda)$. If X is a $\mathfrak{g}$ module annihilated by an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{g})$, we say that X is $\mathfrak{Z}(\mathfrak{g})$ -finite; the category of such modules is closed under extensions and tensor products with finite dimensional modules, and contains the irreducible $\mathfrak{g}$ modules. If X is $\mathfrak{Z}(\mathfrak{g})$ -finite, and $\lambda \in \mathfrak{h}^*$, we define

$$P_\lambda(X) = \{x \in X \mid \text{for all } z \in \mathfrak{Z}(\mathfrak{g}) \text{ there is a positive integer } n \text{ such that } (z - \chi_\lambda(z))^n x = 0\}.$$

Then

$$X = \sum_{\lambda \in \mathfrak{h}^*/W} P_\lambda(X),$$

a finite direct sum. We say X has *generalized infinitesimal character* λ or χ_λ if $X = P_\lambda(X)$.

Write $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ for the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. Recall that $\mathfrak{h}^*$ is endowed with a natural bilinear form $\langle, \rangle$ coming from the Killing form on $\mathfrak{g}$. The weight $\lambda \in \mathfrak{h}^*$ and its associated infinitesimal character χ_λ are called *nonsingular* if $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$.

Fix now a nonsingular weight $\lambda \in \mathfrak{h}^*$. Define

$$R_\lambda = \left\{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \frac{2\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \right\}$$

$$R_\lambda^+ = \{ \alpha \in R_\lambda \mid \langle \alpha, \lambda \rangle > 0 \}$$

$$W_\lambda = W(R_\lambda) \subseteq W(\mathfrak{g}/\mathfrak{h})$$

$$\Pi_\lambda = \{ \text{simple roots of } R_\lambda^+ \};$$

recall the well-known fact that R_λ is a root system. For each $\alpha \in \Pi_\lambda$, we choose (by Lemma 3.1 of [15]) a positive root system Ψ_α for $\Delta(\mathfrak{g}, \mathfrak{h})$, containing R_λ^+ , so that α is simple for Ψ_α . Next we choose an integral weight μ_α^1 , dominant for Ψ_α , so large that for every $\beta \in \Psi_\alpha$, either

$$\operatorname{Re}\langle \beta, \lambda + \mu_\alpha^1 \rangle > 0, \quad \text{or}$$

$$\operatorname{Re}\langle \beta, \lambda + \mu_\alpha^1 \rangle = 0, \quad \text{and}$$

$$\operatorname{Im}\langle \beta, \lambda + \mu_\alpha^1 \rangle > 0.$$

(When these conditions hold, we say $\lambda + \mu_\alpha^1$ is *strictly positive on* β .)

$$m = \frac{2\langle \alpha, \lambda + \mu_\alpha^1 \rangle}{\langle \alpha, \alpha \rangle},$$

and define μ_α^2 to be m times the fundamental weight of Ψ_α corresponding to the simple root α . It follows that if $\delta \in R_\lambda^+$,

$$\langle \delta, \lambda + \mu_\alpha^1 - \mu_\alpha^2 \rangle \geq 0,$$

with equality if and only if $\delta = \alpha$. Let F_α^i denote the finite dimensional irreducible $\mathfrak{g}$ module of highest weight μ_α^i with respect to Ψ_α . If X is a $\mathfrak{Z}(\mathfrak{g})$ -finite $\mathfrak{g}$ module, we define (cf. [15], section 3)

$$\begin{aligned} \psi_\alpha^1(X) &= P_{\lambda + \mu_\alpha^1}(P_\lambda(X) \otimes F_\alpha^1) \\ \psi_\alpha^2(X) &= P_{\lambda + \mu_\alpha^1 - \mu_\alpha^2}(P_{\lambda + \mu_\alpha^1}(X) \otimes (F_\alpha^2)^*) \\ \varphi_\alpha^2(X) &= P_{\lambda + \mu_\alpha^1}(P_{\lambda + \mu_\alpha^1 - \mu_\alpha^2}(X) \otimes F_\alpha^2) \\ \varphi_\alpha^1(X) &= P_\lambda(P_{\lambda + \mu_\alpha^1}(X) \otimes (F_\alpha^1)^*) \\ \psi_\alpha(X) &= \psi_\alpha^2(\psi_\alpha^1(X)) \\ \varphi_\alpha(X) &= \varphi_\alpha^1(\varphi_\alpha^2(X)) \\ \tau(X) &= \{\alpha \in \Pi_\lambda \mid \psi_\alpha(X) = 0\}. \end{aligned}$$

THEOREM 2.1. *Let X be an irreducible $\mathfrak{g}$ module with infinitesimal character λ . Suppose X is either a highest weight module, or a Harish-Chandra module for some real form of $\mathfrak{g}$. Then if $\alpha \notin \tau(X)$, $\varphi_\alpha \psi_\alpha X$ has X as its unique irreducible submodule and irreducible quotient. The sequence*

$$0 \rightarrow X \xrightarrow{d_1} \varphi_\alpha \psi_\alpha X \xrightarrow{d_2} X \rightarrow 0 \quad (2.2)$$

is a chain complex, and $U_\alpha(X)$ is defined to be its cohomology, $(\ker d_2/\text{Im } d_1)$. $U_\alpha(X)$ has finite composition series, and $\alpha \in \tau(U_\alpha(X))$.

Proof. For highest weight modules this is due to Jantzen [7], although he does not state his results in this way. For Harish-Chandra modules of linear groups it was first proved in [13]. The proof given for Harish-Chandra modules in [15], Theorem 3.9, applies to all cases. Q.E.D.

Definition 2.3. In the complex (2.2), put

$$K_\alpha = K_\alpha(X) = \ker d_2, \quad Q_\alpha = Q_\alpha(X) = \frac{\varphi_\alpha \psi_\alpha(X)}{d_1 X}.$$

Thus we have short exact sequences

$$\begin{aligned} (a) \quad & 0 \rightarrow K_\alpha \rightarrow \varphi_\alpha \psi_\alpha X \rightarrow X \rightarrow 0 \\ (b) \quad & 0 \rightarrow X \rightarrow \varphi_\alpha \psi_\alpha X \rightarrow Q_\alpha \rightarrow 0 \\ (c) \quad & 0 \rightarrow X \rightarrow K_\alpha \rightarrow U_\alpha \rightarrow 0 \\ (d) \quad & 0 \rightarrow U_\alpha \rightarrow Q_\alpha \rightarrow X \rightarrow 0. \end{aligned} \tag{2.4}$$

These will play a crucial role in our calculations.

Finally, we restate Conjecture 3.15 of [15].

CONJECTURE 2.5. *In the setting of Theorem 2.1, $U_\alpha(X)$ is a completely reducible $\mathfrak{g}$ module.*

Although the preceding formalism is the appropriate one for defining the τ -invariant, and works well for linear groups, certain technical problems (cf. [15], Lemma 4.9) make it convenient to consider non-integral roots as well. If $\lambda \in \mathfrak{h}^*$ is regular, we define Δ_λ^+ to be the set of roots which are strictly positive on λ ; recall that this means

$$\Delta_\lambda^+ = \{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \text{Re} \langle \alpha, \lambda \rangle > 0, \text{ or } \text{Re} \langle \alpha, \lambda \rangle = 0 \text{ and } \text{Im} \langle \alpha, \lambda \rangle > 0 \}. \tag{2.6}$$

Let α be a simple root of Δ_λ^+ . If α is integral, we can and do assume $\mu_\alpha^1 = 0$; thus $\psi_\alpha = \psi_\alpha^2$, etc. In this situation we may write $\mu_\alpha = \mu_\alpha^2$. If α is not integral, choose a sum of roots $\mu_\alpha \in \mathfrak{h}^*$ so that $\lambda + \mu_\alpha$ is regular and dominant for $s_\alpha(\Delta_\lambda^+)$; here s_α is the simple reflection in $W(\mathfrak{g}/\mathfrak{h})$ corresponding to α . If X is a $\mathfrak{Z}(\mathfrak{g})$ -finite $\mathfrak{g}$ module, define ([13], Section 5)

$$\begin{aligned} \psi_\alpha(X) &= P_{\lambda+\mu_\alpha}(P_\lambda(X) \otimes F_\alpha) \\ \varphi_\alpha(X) &= P_\lambda(P_{\lambda+\mu_\alpha}(X) \otimes F_\alpha^*). \end{aligned}$$

Here of course F_α is the finite dimensional irreducible $\mathfrak{g}$ module of extremal weight μ_α .

THEOREM 2.7. *Let X be an irreducible $\mathfrak{g}$ module with infinitesimal character λ , and suppose $\alpha \in \Delta_\lambda^+$ is simple but not integral. If X is either a highest weight*

module or a Harish-Chandra module for some real form of $\mathfrak{g}$, then $\psi_\alpha(X)$ is irreducible, and

$$\varphi_\alpha \psi_\alpha(X) \cong X.$$

Proof. For highest weight modules this is due to Jantzen [7]. For Harish-Chandra modules it is essentially Theorem 5.20(a) of [13]. Q.E.D.

In both cases the result follows formally from the existence of some kind of character theory for the categories in question.

3. Case of Verma modules. Retaining the notation of section 2, we assume in this section that λ is regular and integral; thus $R_\lambda = \Delta(\mathfrak{g}, \mathfrak{h})$. Put $\Delta^+ = R_\lambda^+$. Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to $-\Delta^+$. For $w \in W(\mathfrak{g}, \mathfrak{h})$, define

$$M_w = U(\mathfrak{g}) \bigotimes_{\mathfrak{b}} \mathbb{C}_{w\lambda + \rho};$$

here $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let L_w be the unique irreducible quotient of M_w .

If X is any $\mathfrak{g}$ module on which $\mathfrak{h}$ acts semisimply, then the cohomology groups $H^i(\mathfrak{u}, X)$ are semisimple $\mathfrak{h}$ modules. For $\mu \in \mathfrak{h}^*$, we let $H^i(\mathfrak{u}, X)^\mu$ denote the μ weight space. An easy application of the Euler-Poincaré principle gives the following well-known result.

PROPOSITION 3.1. *The formal character $\text{ch } L_w$ as an $\mathfrak{h}$ module satisfies*

$$\text{ch } L_w = \sum_{y \in W} \left(\sum_{i=0}^{\dim \mathfrak{u}} (-1)^i \dim H^i(\mathfrak{u}, L_w)^{y\lambda + \rho} \right) \text{ch } M_y.$$

Now Kazhdan and Lusztig (cf. [9]) define a family of polynomials $\{p_{y,w} \mid y, w \in W\}$ in a variable q . If $\leq$ denotes the Bruhat ordering on W , these satisfy

- (a) $p_{y,w} = 0$ unless $y \leq w$
- (b) $p_{w,w} = 1$ (3.2)
- (c) If $y < w$, $\deg p_{y,w} \leq 1/2(l(w) - l(y) - 1)$. (Here l is the length function on W .)

Next they define a new relation $<$ on W by

$$y < w \text{ if and only if } y < w \text{ and } \deg p_{y,w} = 1/2(l(w) - l(y) - 1).$$

If $y < w$, $\mu(y, w)$ denotes the leading coefficient of $p_{y,w}$. The $p_{y,w}$ also have the following property:

Suppose $y \leq w$, s is a simple reflection in W , and $ws < w$. Then

$$(d) \quad p_{y,w} = q^{1-a} p_{ys,ws} + q^a p_{y,ws} - \sum_{\substack{z \\ y \leq z < ws \\ zs < z}} \mu(z, ws) q^{1/2(l(w) - l(z))} p_{y,z} \quad (3.2)$$

where $a = 1$ if $ys < y$ and $a = 0$ if $ys > y$. Clearly these relations allow one to compute $p_{y,w}$ by induction on $l(w)$. The Kazhdan-Lusztig conjecture reads (setting $\epsilon_w = (-1)^{l(w)}$):

CONJECTURE 3.3. ([9], Conjecture 1.5) $\text{ch } L_w = \sum_{y \leq w} \epsilon_y \epsilon_w P_{y,w}(1) \text{ch } M_y$.

Comparing this with Proposition 3.1, we see that it would be enough to prove

CONJECTURE 3.4. *The dimension of $H^i(\mathfrak{u}, L_w)^{y\lambda+\rho}$ is the coefficient of $q^{1/2(l(w)-l(y)-i)}$ in $p_{y,w}$.*

(In fact, the two conjectures can be shown to be equivalent.)

THEOREM 3.5. *Conjecture 2.5 implies Conjecture 3.4.*

Proof. We proceed by induction on $l(w)$. If $w = 1$, $L_w = M_w$ is an irreducible Verma module. If $H^i(\mathfrak{u}, L_w)^{y\lambda+\rho} \neq 0$, then clearly L_w has a weight of the form

$$\mu = y\lambda + \rho - \sum \beta_j,$$

with $\{\beta_j\}$ a set of i positive roots. Also

$$y\lambda = \lambda - \sum_{\alpha \in \Delta^+} n_\alpha \alpha$$

with $n_\alpha \geq 0$, since λ is dominant; and some n_α is positive unless $y = 1$. On the other hand, every weight of $L_w = M_1$ is of the form

$$\lambda + \rho + \sum_{\alpha \in \Delta^+} m_\alpha \alpha$$

with $m_i \geq 0$. This forces $\{\beta_j\} = \emptyset$, $n_\alpha = 0$ for all α ; so $i = 0$ and $y = 1$. Obviously

$$\dim H^0(\mathfrak{u}, M_1)^{\lambda+\rho} = 1,$$

so Conjecture 3.4 is true when $w = 1$ (as is well known).

Suppose then that Conjecture 3.4 holds for $l(w') < l(w)$, and $l(w) > 0$. Then we can find a simple root $\alpha \in \Pi$ such that the corresponding simple reflection s satisfies $l(ws) = l(w) - 1$. Put $X = L_{ws}$; then by [7] $\alpha \notin \tau(X)$. The first step is to compute $H^i(\mathfrak{u}, U_\alpha(X))$. For simplicity we write $\varphi = \varphi_\alpha$, $U = U_\alpha(X)$, $Q = Q_\alpha(X)$, etc. with definitions as in section 2. To define U , we can and do choose $\mu_\alpha^1 = 0$; write $F = F_\alpha^2$, the representation of extremal weight μ_α^2 ; we write $\mu = \mu_\alpha^2$. Fix $y \in W$, and suppose for definiteness that $ys < y$. Choose a b -stable filtration $0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = F$, with $\dim(F_i/F_{i-1}) = 1$ for $1 \leq i \leq n$. This gives rise to a b -stable filtration of $\varphi\psi X \cong P_\lambda(\psi X \otimes F)$, and hence to a spectral sequence for $H^i(\mathfrak{u}, \varphi\psi X)^{y\lambda+\rho}$. A typical E_2 term is

$$\left[H^i(\mathfrak{u}, \psi X) \otimes F_i/F_{i-1} \right]^{y\lambda+\rho}.$$

Now every weight of $H^i(\mathfrak{u}, \psi X)$ is of the form

$$y'(\lambda - \mu) + \rho = y(\lambda - \mu) + \rho + \sum_{\substack{\alpha \in \Delta \\ \langle \alpha, y\lambda \rangle < 0}} m_\alpha \alpha,$$

with $m_\alpha \geq 0$, and the weight of F_i/F_{i-1} is of the form

$$y\mu + \sum_{\substack{\beta \in \Delta \\ \langle \beta, y\lambda \rangle < 0}} m_\beta \beta$$

with $m_\beta \geq 0$. So this E_2 term is zero unless

$$\begin{aligned} y\lambda + \rho &= y(\lambda - \mu) + \rho + \sum m_\alpha \alpha + y\mu + \sum m_\beta \beta \\ &= y\lambda + \rho + \sum m_\alpha \alpha + \sum m_\beta \beta. \end{aligned}$$

Clearly this forces $y' = y$, and the weight of F_i/F_{i-1} must be $y\mu$. So the spectral sequence collapses, giving

$$H^i(\mathfrak{u}, \varphi\psi X)^{y\lambda+\rho} \cong H^i(\mathfrak{u}, \psi X)^{y(\lambda-\mu)+\rho} \otimes \mathbb{C}_{y\mu}. \quad (3.6)$$

Now we use the dual filtration of F^* to compute $H^i(\mathfrak{u}, \psi X)^{y(\lambda-\mu)+\rho}$; the spectral sequence collapses to the long exact sequence

$$\begin{aligned} \dots \rightarrow H^i(\mathfrak{u}, X)^{ys\lambda+\rho} \otimes \mathbb{C}_{-ys\mu} &\rightarrow H^i(\mathfrak{u}, \psi X)^{y(\lambda-\mu)+\rho} \rightarrow \\ H^i(\mathfrak{u}, X)^{y\lambda+\rho} \otimes \mathbb{C}_{-y\mu} &\rightarrow \dots \end{aligned} \quad (3.7)$$

On the other hand, we have a long exact sequence from 2.4(a):

$$\begin{array}{ccccccc} \dots \rightarrow H^i(\mathfrak{u}, K)^{y\lambda+\rho} & \rightarrow & H^i(\mathfrak{u}, \varphi\psi X)^{y\lambda+\rho} & \rightarrow & H^i(\mathfrak{u}, X)^{y\lambda+\rho} & \rightarrow & \dots \\ & & \downarrow \cong & & \searrow \cong & & \\ \dots \rightarrow H^i(\mathfrak{u}, X)^{ys\lambda+\rho} \otimes \mathbb{C}_{-ys\mu} \otimes \mathbb{C}_{y\mu} & \rightarrow & H^i(\mathfrak{u}, \psi X)^{y(\lambda-\mu)+\rho} \otimes \mathbb{C}_{y\mu} & \rightarrow & H^i(\mathfrak{u}, X)^{y\lambda+\rho} & & \\ & & \otimes \mathbb{C}_{-y\mu} \otimes \mathbb{C}_{y\mu} & \rightarrow & \dots & & \end{array} \quad (3.8)$$

Here the second row is just (3.7); the first isomorphism is (3.6); and the second is the identity map. As will see in section 4 (Theorem 4.2) this diagram is commutative up to a nonzero scalar. By a simple diagram chase, we deduce

$$H^i(\mathfrak{u}, K)^{y\lambda+\rho} \cong H^i(\mathfrak{u}, X)^{ys\lambda+\rho} \otimes \mathbb{C}_{y\mu - ys\mu}. \quad (3.9)$$

(The isomorphism is not canonical, but we will not need that.) Now look at the

long exact sequence from 2.4(c). Combining it with (3.9) we get

$$\dots \rightarrow H^i(\mathfrak{u}, X)^{y\lambda+\rho} \rightarrow H^i(\mathfrak{u}, X)^{ys\lambda+\rho} \otimes C_{y(\mu-s\mu)} \rightarrow H^i(\mathfrak{u}, U)^{y\lambda+\rho} \rightarrow \dots \quad (3.10)$$

By inductive hypothesis,

$$\begin{aligned} \dim H^i(\mathfrak{u}, X)^{y\lambda+\rho} &= \text{coefficient of } q^{1/2(l(ws)-l(y)-i)} \text{ in } p_{y,w} \\ &= \text{zero unless } i \text{ has the same parity as } l(y) - l(ws). \end{aligned}$$

Furthermore

$$\begin{aligned} \dim H^i(\mathfrak{u}, X)^{ys\lambda+\rho} &= \text{coefficient of } q^{1/2(l(ws)-l(ys)-i)} \text{ in } p_{ys,w} \\ &= \text{zero unless } i \text{ has the opposite parity from } l(y) - l(ws). \end{aligned}$$

This shows that the long exact sequence (3.10) collapses into a family of short exact sequences; and if we define a polynomial r in q so that the coefficient of $q^{1/2(l(w)-l(y)-i)}$ is $\dim H^i(\mathfrak{u}, U)^{y\lambda+\rho}$, then

$$r(q) = p_{ys,ws} + qp_{y,ws}. \quad (3.11)$$

Suppose now that U is completely reducible (i.e. that Conjecture 2.5 holds). We will compute the constituents of U and their multiplicities. So let $z \in W$. If L_z is to occur in U , we must have $\alpha \in \tau(L_z)$, i.e. $zs < z$. In that case the multiplicity of L_z in U is just

$$\dim H^0(\mathfrak{u}, U)^{z\lambda+\rho} = \dim H^0(\mathfrak{u}, X)^{zs\lambda+\rho} + \dim H^1(\mathfrak{u}, X)^{z\lambda+\rho}.$$

The first term is zero unless $z = w$, in which case it is 1; in that case the second term is zero (since it is a coefficient of $p_{w,ws} = 0$). If $z \neq w$, the second term is the coefficient of $q^{1/2(l(ws)-l(z)-1)}$ in $p_{z,ws}$; so we have

$$U = L_w \oplus \sum_{\substack{z \\ y \leq z < ws \\ zs < z}} \mu(z, ws) L_z \quad (3.12)$$

Combining (3.12) and (3.11), we get the statement of Conjecture 3.4 for L_w . (Here we use the identity 3.2(d).) Q.E.D.

In addition to Conjecture 2.5, there were four important parts to this argument. The first was the computation of $H^i(\mathfrak{u}, X)$ when X is a “nice irreducible.” Next, we needed the commutativity of a certain diagram. Keeping track of various parities was necessary to make long exact sequences collapse. Finally, we needed a way to determine the irreducible constituents of U from the groups $H^i(\mathfrak{u}, U)$; in this case H^0 sufficed. These steps can all be carried out,

with varying degrees of difficulty, for Harish-Chandra modules: they are given by Theorem 6.13, Theorem 4.2, Definition 7.1, and Proposition 5.5 respectively. (The only one of these requiring linearity of G is Definition 7.1.)

4. Homological algebra. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$; we assume that $\mathfrak{h} \subseteq \mathfrak{l}$. We do *not* assume any longer that λ is integral (although it is still assumed regular), and we assume no relation between Δ_λ^+ and the roots of $\mathfrak{h}$ in $\mathfrak{u}$. We fix throughout this section a simple root α of Δ_λ^+ . Notation is as in section 2, although we may again drop the subscript α ; in particular $s = s_\alpha \in W$ corresponds to α . We will be interested in the functors $\varphi^{\mathfrak{l}}$ and $\psi^{\mathfrak{l}}$ which are the analogues for $\mathfrak{l}$ of φ and ψ . More precisely, let

$$\rho_{\mathfrak{u}} = 1/2 \sum_{\beta \in \Delta(\mathfrak{u}, \mathfrak{h})} \beta;$$

let $F_\lambda^{\mathfrak{l}}$ be the finite dimensional representation of $\mathfrak{l}$ of extremal weight μ_α ; and define

$$\psi_\lambda^{\mathfrak{l}}(Y) = P_{\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}(P_{\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}(Y) \otimes F_\lambda^{\mathfrak{l}})$$

etc., whenever Y is a $\mathfrak{Z}(\mathfrak{l})$ -finite $\mathfrak{l}$ -module. The extra $\rho_{\mathfrak{u}}$ is included because of

THEOREM 4.1. ([2]) *Let X be a $\mathfrak{Z}(\mathfrak{g})$ -finite $\mathfrak{g}$ module with generalized infinitesimal character λ (i.e. $X = P_\lambda(X)$). Then $Y = H^i(\mathfrak{u}, X)$ is a $\mathfrak{Z}(\mathfrak{l})$ -finite $\mathfrak{l}$ module, and*

$$Y = \bigoplus_{w \in W^{\mathfrak{l}} \setminus W} P_{w\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}(Y);$$

here $W^{\mathfrak{l}} = W(\mathfrak{l}/\mathfrak{h})$, the Weyl group of $\mathfrak{h}$ in $\mathfrak{l}$.

Since $w_0 \rho_{\mathfrak{u}} = \rho_{\mathfrak{u}}$ for $w_0 \in W^{\mathfrak{l}}$, $P_{w\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}} = P_{w_0 w \lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}$; so $P_{w\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}$ is well defined for $w \in W^{\mathfrak{l}} \setminus W$.

THEOREM 4.2. *Let X be a $\mathfrak{Z}(\mathfrak{g})$ -finite $\mathfrak{g}$ module with generalized infinitesimal character λ . With notation as above, suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, $\mathfrak{l} \supseteq \mathfrak{h}$, and $\alpha \in \Delta_\lambda^+$ is simple and integral.*

(a) *Suppose $\alpha \in \Delta(\mathfrak{u})$. There is a natural isomorphism*

$$P_{\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}(H^i(\mathfrak{u}, \varphi\psi X)) \cong \varphi_\lambda^{\mathfrak{l}}(H^i(\mathfrak{u}, \psi X))$$

of $\mathfrak{l}$ modules, and a natural long exact sequence

$$\dots \rightarrow \psi_{s\lambda}^{\mathfrak{l}}(H^i(\mathfrak{u}, X)) \rightarrow P_{\lambda - \mu - \rho_{\mathfrak{u}}}^{\mathfrak{l}}(H^i(\mathfrak{u}, \psi X)) \rightarrow \psi_\lambda^{\mathfrak{l}}(H^i(\mathfrak{u}, X)) \rightarrow \dots$$

of $\mathfrak{l}$ modules. Suppose $\mathfrak{Z}(\mathfrak{g})$ actually acts by scalars on X . Let $\varphi\psi X \rightarrow X$ be the natural map corresponding to the identity under $\text{Hom } \mathfrak{g}(\varphi\psi X, X) \cong \text{Hom } \mathfrak{g}(\psi X, \psi X)$

(cf. [17]). If we consider the induced map on cohomology, then the following diagram commutes up to a non-zero scalar:

$$\begin{array}{ccc} P_{\lambda-\rho_u}^I(H^i(u, \varphi\psi X)) & \xrightarrow{h_1} & P_{\lambda-\rho_u}^I(H^i(u, X)) \\ f_1 \downarrow \cong & & \uparrow f_2 \\ \varphi_\lambda^I(H^i(u, \psi X)) & \xrightarrow{h_2} & \varphi_\lambda^I \psi_\lambda^I(H^i(u, X)) \end{array}$$

Here the map on the right is induced by the natural isomorphism

$$\mathrm{Hom}_I(\varphi_\lambda^I \psi_\lambda^I Y, P_{\lambda-\rho_u}^I Z) \cong \mathrm{Hom}_I(\psi_\lambda^I Y, \psi_\lambda^I Z)$$

and the identity.

(b) Suppose $\alpha \in \Delta(I)$. Then there is a natural isomorphism

$$P_{\lambda-\rho_u}^I(H^i(u, \varphi\psi X)) \cong \varphi_\lambda^I \psi_\lambda^I(H^i(u, X)).$$

Defining maps $\varphi\psi X \rightarrow X$, $\varphi_\lambda^I \psi_\lambda^I Y \rightarrow Y$ as in (a), we get a commutative diagram

$$\begin{array}{ccc} P_{\lambda-\rho_u}^I(H^i(u, \varphi\psi X)) & \longrightarrow & P_{\lambda-\rho_u}^I(H^i(u, X)) \\ \cong \downarrow & & \downarrow \cong \\ \varphi_\lambda^I \psi_\lambda^I(H^i(u, X)) & \longrightarrow & P_{\lambda-\rho_u}^I(H^i(u, X)); \end{array}$$

and similarly for the natural maps $X \rightarrow \varphi\psi X$.

(c) Suppose $-\alpha \in \Delta(u)$. Then the formulae of (a) hold, with all arrows reversed.

Proof. We consider only (a); (b) and (c) are quite similar. We first construct the long exact sequence. To do this, choose a q -stable filtration

$$0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = (F_\alpha)^*$$

such that F_i/F_{i-1} is an irreducible I module. By Theorem 4.1,

$$P_{\lambda-\mu-\rho_u}^I(H^i(u, \psi X)) \cong P_{\lambda-\mu-\rho_u}^I(H^i(u, X \otimes (F_\alpha)^*)).$$

We compute the left side using the spectral sequence of the filtration. A typical E_2 term is

$$P_{\lambda-\mu-\rho_u}^I(H^i(u, X) \otimes F_i/F_{i-1}).$$

By a theorem of Kostant ([10], Theorem 5.1), and Theorem 4.1, this is zero unless there is a weight γ of F_i/F_{i-1} , and some $w \in W$, such that for some $w_0 \in W^I$,

$$\lambda - \mu - \rho_u = w_0(w\lambda - \rho_u + \gamma).$$

Since the set of weights of F_i/F_{i-1} is W^1 stable, we may assume $w_0 = 1$. We can write

$$\gamma = w\left(-\mu + \sum_{\beta \in \Delta_\lambda^+} n_\beta \beta\right),$$

with $n_\beta \geq 0$. Thus

$$w^{-1}(\lambda - \mu) = \lambda - \mu + \sum n_\beta \beta.$$

Since $\lambda - \mu$ is also dominant (but singular) for Δ_λ^+ , this forces

$$w^{-1}(\lambda - \mu) = \lambda - \mu,$$

and thus $w^{-1} = 1$ or s . Furthermore we must have $\gamma = -\mu$ or $-s\mu$. Since these weights are extremal in F_α^* , this gives

$$F_i/F_{i-1} \cong (F_\lambda^1)^* \text{ or } (F_{s\lambda}^1)^*$$

respectively. Thus the spectral sequence does collapse to the desired long exact sequence. (Since

$$-\mu = -s\mu - m\alpha, m > 0,$$

and α is a root of $\mathfrak{h}$ in $\mathfrak{u}$, the $(F_{s\lambda}^1)^*$ term of the filtration may be chosen to precede the $(F_\lambda^1)^*$ term. This accounts for the direction of the arrows.)

We only sketch the proof that the last diagram in (a) commutes. This proceeds by induction on i . Suppose first that $i = 0$. Then

$$H^0(\mathfrak{u}, Z) \cong Z^\mathfrak{u},$$

the space of $\mathfrak{u}$ -invariant vectors in Z . Let ω be an element of

$$P_{\lambda - \rho_\mathfrak{u}}^1((\varphi\psi X)^\mathfrak{u}),$$

and put $F = F_\alpha$. We can regard $\varphi\psi X$ as contained in $(X \otimes F^*) \otimes F$, and hence write

$$\omega = \sum (x_{ij} \otimes v_i^*) \otimes v_j$$

Filter F^* as in (a) and define k so that

$$F_k/F_{k-1} \cong (F_\lambda^1)^*.$$

Let F^i be the dual filtration of F . The argument for the first part of (a) shows that the natural inclusion

$$P_{\lambda - \rho_\mathfrak{u}}^1[(F^{k-1} \otimes \psi X)^\mathfrak{u}] \rightarrow P_{\lambda - \rho_\mathfrak{u}}^1((\varphi\psi X)^\mathfrak{u})$$

is an isomorphism; so we may assume that all v_j occurring in the expression for ω lie in F^{k-1} . More precisely, we may assume that the v_j and v_i^* are dual bases of F and F^* obtained from bases of the $\mathfrak{l}$ -primary subspaces, consistent with the filtrations; in particular suppose $\{v_j\}_{j=1}^t$ is a basis for the $\mathfrak{l}$ type of extremal weight μ . Then the map

$$P_{\lambda-\rho_u}^{\mathfrak{l}}((\varphi\psi X)^u) \cong \varphi_{\lambda}^{\mathfrak{l}}((\psi X)^u)$$

takes ω to

$$\tilde{\omega} = \sum_{1 \leq j \leq t} (x_{ij} \otimes v_i^*) \otimes v_j.$$

Again the argument for the first part of (a) shows that we may assume $v_i^* \in F_k$ whenever $x_{ij} \neq 0$. Since $\{v_i^*\}_{i=1}^t$ is a basis for the $\mathfrak{l}$ type of F^* of extremal weight $-\mu$, the map

$$\varphi_{\lambda}^{\mathfrak{l}}((\psi X)^u) \xrightarrow{h_2} \varphi_{\lambda}^{\mathfrak{l}} \psi_{\lambda}^{\mathfrak{l}}(X^u)$$

under consideration takes $\tilde{\omega}$ to

$$\tilde{\tilde{\omega}} = \sum_{1 \leq i, j \leq t} (x_{ij} \otimes v_i^*) \otimes v_j;$$

The two remaining maps

$$P_{\lambda-\rho_u}^{\mathfrak{l}}((\varphi\psi X)^u) \xrightarrow{h_1} X^u$$

and

$$\varphi_{\lambda}^{\mathfrak{l}} \psi_{\lambda}^{\mathfrak{l}}(X^u) \xrightarrow{f_2} X^u$$

are just contractions: they take ω and $\tilde{\omega}$ to

$$\sum_i x_{ii}, \quad \sum_{1 \leq i \leq t} x_{ii}$$

respectively. Unfortunately it is far from obvious that these differ by a scalar. To see this, we need to construct an explicit inverse for the isomorphism f_1 from $P_{\lambda-\rho_u}^{\mathfrak{l}}(\varphi\psi X^u)$ to $\varphi_{\lambda}^{\mathfrak{l}}((\psi X)^u)$. To do this regard $\varphi_{\lambda}^{\mathfrak{l}}((\psi X)^u)$ as contained in $F \otimes \psi X$ via the natural inclusion $F_{\lambda}^{\mathfrak{l}} \subseteq F$. Let $\lambda = \lambda_0, \lambda_1, \dots, \lambda_N$ be representatives of the distinct W orbits of the form $\lambda - \mu + \gamma$, with γ a weight of F . For each i , $1 \leq i \leq N$, choose $z_i \in \mathfrak{Z}(\mathfrak{g})$ so that $\lambda_i(z_i) = 0$, but $\lambda(z_i) = 1$. Choose $n_i > 0$ so large that $(z_i)^{n_i}$ annihilates the λ_i generalized eigenspace of $\mathfrak{Z}(\mathfrak{g})$ in $F \otimes Y$ whenever Y has infinitesimal character $\lambda - \mu$, and $\mathfrak{Z}(\mathfrak{g})$ acts by scalars in Y . (This is possible by [10].) For $\tilde{\omega} \in \varphi_{\lambda}^{\mathfrak{l}}((\psi X)^u)$, define

$$g(\tilde{\omega}) = \left(\prod_{i=1}^N z_i^{n_i} \right) \cdot \tilde{\omega} \in F \otimes \psi X.$$

We will prove three things about g : first that it actually maps $\varphi_\lambda^l((\psi X)^u)$ into $P_{\lambda-\rho_u}^l(\varphi\psi X)^u$; next that $f_1 \circ g$ is a scalar (possibly zero, but depending only on g, λ , and μ); and finally that $h_1 \circ g = f_2 \circ h_2$. At that point we will only need to show that the scalar is non-zero. Suppose not; since f_1 is an isomorphism, this forces $g = 0$. If we take X for example in category $\mathcal{O}$ of Bernstein, Gelfand, and Gelfand, then the $\mathfrak{l}$ module $H^0(u, X)$ lies in category $\mathcal{O}$ for $\mathfrak{l}$. It follows easily (cf. [7]) that f_2 is necessarily an isomorphism. Since $0 = h_1 \circ g = f_2 \circ h_2$, it follows that h_2 is necessarily zero. But by taking X to be the $\mathfrak{h}$ -finite dual of an appropriate Verma module, we can easily arrange

$$\psi_{s\lambda}^l(H^0(u, X)) = 0$$

$$P_{\lambda-\mu-\rho_u}^l(H^0(u, \psi X)) \neq 0,$$

forcing $h_2 \neq 0$; details are left to the reader. So it remains only to prove the stated properties of g .

First, we want

$$g(\varphi_\lambda^l(\psi X)^u) \subseteq P_{\lambda-\rho_u}^l(\varphi\psi X)^u.$$

Evidently g commutes with the action of $\mathfrak{l}$, so it suffices to show that $g(\tilde{\omega})$ is annihilated by u . Suppose not. Write

$$\tilde{\omega} = \sum_{1 \leq i \leq t} y_i \otimes v_i, \text{ with } y_i \in (\psi X)^u.$$

If $x \in u$, then

$$\begin{aligned} x(g(\tilde{\omega})) &= x(\prod z_i^{n_i}) \cdot \tilde{\omega} \\ &= (\prod z_i^{n_i})(x\tilde{\omega}) \\ &= g(x\tilde{\omega}) \end{aligned}$$

with an obvious extension of the domain of definition of g . Now

$$\begin{aligned} x \cdot \tilde{\omega} &= \sum x \cdot y_i \otimes v_i + y_i \otimes xv_i \\ &= \sum y_i \otimes xv_i \in (\psi X)^u \otimes F^k; \end{aligned}$$

for the action of u takes F^j to F^{j+1} . Choose a maximal set $\{x_1, \dots, x_r\} \subseteq u$ so that

$$0 \neq g((x_1 \dots x_r) \cdot \tilde{\omega}).$$

Clearly such a maximal set exists, since

$$(x_1 \dots x_r) \cdot \tilde{\omega} = \sum y_i \otimes (x_1 \dots x_r)v_i \in (\psi X)^u \otimes F^{k+r-1}$$

and $F^{k+r-1} = 0$ for large r . We want to show that the set is empty, i.e. $r = 0$. Put

$\omega_1 = g((x_1 \dots x_r)\tilde{\omega})$. Now if $x_0 \in \mathfrak{u}$,

$$x_0 \cdot \omega_1 = x_0 \cdot g((x_1 \dots x_r) \cdot \tilde{\omega}) = g((x_0 \dots x_r)\tilde{\omega}) = 0$$

by the maximality of r . Also ω_1 lies in $\varphi\psi X \subseteq F \otimes \psi X$, since g is just the projection on this space. So

$$\omega_1 \in (\varphi\psi X)^{\mathfrak{u}}.$$

If we decompose ω_1 according to the action of $\mathfrak{Z}(\mathfrak{l})$, it is easy to see that the characters occurring are of the form $\lambda - \mu - \rho_{\mathfrak{u}} + \gamma$, with γ a weight of $\mathfrak{h}$ in F^{k+r-1} ; for this holds already for $(x_1 \dots x_r)\tilde{\omega}$. So we get

$$P_{\lambda - \mu - \rho_{\mathfrak{u}} + \gamma}^1(\varphi\psi X)^{\mathfrak{u}} \neq 0.$$

By Theorem 4.1, there is a $w \in W$ such that

$$w\lambda = \lambda - \mu + \gamma.$$

In particular

$$\langle \lambda - \mu + \mu, \lambda - \mu + \mu \rangle = \langle \lambda - \mu + \gamma, \lambda - \mu + \gamma \rangle.$$

Since γ is a weight of F , which has extremal weight μ , this implies that $\gamma = w_1 \mu$, and

$$\langle \lambda - \mu, \mu \rangle = \langle \lambda - \mu, w_1 \mu \rangle.$$

Since $\lambda - \mu$ and μ are dominant for Δ_{λ}^+ , and $\lambda - \mu$ is singular only with respect to the simple root α , we get

$$\gamma = \mu \quad \text{or} \quad \gamma = s_{\alpha} \mu = \mu - m\alpha.$$

Since γ is a weight of F^{k+r-1} , the second case is impossible and the first forces $r = 0$.

We turn now to the proof that $f_1 \circ g$ is a scalar. So we continue to fix $\tilde{\omega} = \sum_{1 \leq i \leq t} y_i \otimes v_i$, with $y_i \in (\psi X)^{\mathfrak{u}}$. Choose a basis $\{X_q, Y_r, L_s\}$ of $\mathfrak{g}$, with $x_q \in \mathfrak{u}$, $y_r \in \bar{\mathfrak{u}}$, $L_s \in \mathfrak{l}$; here $\bar{\mathfrak{q}} = \mathfrak{l} + \bar{\mathfrak{u}}$ is the parabolic opposite to $\mathfrak{q}$. Write

$$\prod_{i=1}^N z_i^{n_i} = \sum a_{\alpha\beta\gamma} Y_{\alpha} L_{\beta} X_{\gamma},$$

using multi-index notation. Then $g(\tilde{\omega})$ can be written as a sum of terms of the form

$$Y_{\alpha} L_{\beta} y_i \otimes Y_{\alpha'} L_{\beta'} X_{\gamma'} v_i$$

in a computable way. Keeping track of the weights of the center of $\mathfrak{l}$, we see that $f_1 \circ g(\tilde{\omega})$ depends only on the terms of the form $L_{\beta} y_i \otimes Y_{\alpha'} L_{\beta'} X_{\gamma'} v_i$. More precisely, there are elements $u_{ij} \in U(\mathfrak{l})$, $1 \leq i, j \leq t$, depending only on $\mathfrak{g}$, λ , and

μ , such that if $y \in (\psi X)^u$,

$$(f_1 \circ g)(P_{\lambda-\rho_u}^l(y \otimes v_i)) = \sum_{j=1}^t (u_{ij}y) \otimes v_j.$$

Here we must bear in mind that $P_{\lambda-\rho_u}^l$ can be written as an element of $\mathfrak{Z}(\mathfrak{l})$, just as we used g as a projection for $\mathfrak{g}$ earlier; so we can also find $u'_{ij} \in U(\mathfrak{l})$ such that

$$P_{\lambda-\rho_u}^l(y \otimes v_i) = \sum_{j=1}^t u'_{ij}y \otimes v_j$$

whenever $y \in P_{\lambda-\mu-\rho_u}^l((\psi X)^u)$. We want to show that there is a constant c such that

$$u_{ij} - cu'_{ij} \in U(\mathfrak{l}) \cdot I_{\lambda-\mu-\rho_u},$$

where $I_{\lambda-\mu-\rho_u} \subseteq \mathfrak{Z}(\mathfrak{l})$ is just the kernel of the infinitesimal character $\lambda - \mu - \rho_u$; clearly this will prove that $f_1 \circ g = c \cdot id$. To see this, we choose X so that $(\psi X)^u$ is an irreducible $\mathfrak{l}$ module whose annihilator is *precisely* $U(\mathfrak{l}) \cdot I_{\lambda-\mu-\rho_u}$. This can be achieved using an appropriate highest weight module for X , so that $(\psi X)^u$ is an irreducible Verma module for $\mathfrak{l}$ (cf. [3]). In that case $(\psi X)^u \otimes F_{\lambda}^l$ has a unique irreducible composition factor of infinitesimal character $\lambda - \rho_u$; as $f_1 \circ g$ is an intertwining operator for $\mathfrak{l}$, this implies that

$$f_1 \circ g \circ P_{\lambda-\rho_u}^l = cP_{\lambda-\rho_u}^l.$$

Hence $u_{ij} - cu'_{ij} \in \text{Ann}((\psi X)^u) = U(\mathfrak{l}) \cdot I_{\lambda-\mu-\rho_u}$, as desired.

Last, we must show that $h_1 \circ g = f_2 \circ h_2$. So suppose

$$\tilde{\omega} = \sum_{1 \leq i \leq t} (x_{ij} \otimes v_j^*) \otimes v_i \in P_{\lambda-\rho_u}^l((\psi X)^u \otimes F_{\lambda}^l).$$

On the one hand

$$\begin{aligned} (h_1 \circ g)(\tilde{\omega}) &= h_1(\prod z_i^{n_i} \tilde{\omega}) \\ &= \prod z_i^{n_i} (h_1 \tilde{\omega}) \\ &= \prod z_i^{n_i} \left(\sum_{1 \leq i \leq t} x_{ii} \right) = \sum_{1 \leq i \leq t} x_{ii} \end{aligned}$$

if we extend h_1 to all of $F \otimes F^* \otimes X$ in the obvious way; the last equality is because X has infinitesimal character λ . We have essentially already seen that

$$f_2 \circ h_2(\tilde{\omega}) = \sum_{1 \leq i \leq t} x_{ii},$$

completing the proof of commutativity for H^0 .

To extend to higher cohomology, we use a standard dimension shifting argument (cf. [2], [14]). Suppose the result known for H^{i-1} , with $i > 0$. One

knows that $U(\mathfrak{l})$ is free over the image of the Harish-Chandra homomorphism from $\mathfrak{Z}(\mathfrak{g})$ to $\mathfrak{Z}(\mathfrak{l})$; choose a free basis $\{a_i\}$, and let A be its linear span. By induction on degree, one verifies an isomorphism

$$U(\mathfrak{g}) \cong U(\mathfrak{u}) \otimes A \otimes U(\bar{\mathfrak{u}}) \otimes \mathfrak{Z}(\mathfrak{g})$$

as a left $\mathfrak{u}$ module and right $\mathfrak{Z}(\mathfrak{g})$ module. It follows that if X is a $U(\mathfrak{g})$ module with infinitesimal character λ , then

$$I = \text{Hom}_{\mathfrak{Z}(\mathfrak{g})}(U(\mathfrak{g}), X)$$

is a $U(\mathfrak{g})$ module with infinitesimal character λ , containing X , which is injective as a $U(\mathfrak{u})$ module. The short exact sequence

$$0 \rightarrow X \rightarrow I \rightarrow Q \rightarrow 0$$

gives rise to

$$0 \rightarrow \psi X \rightarrow \psi I \rightarrow \psi Q \rightarrow 0$$

$$0 \rightarrow \varphi \psi X \rightarrow \varphi \psi I \rightarrow \varphi \psi Q \rightarrow 0.$$

Since $F^* \otimes I$ is injective as a $U(\mathfrak{u})$ module for formal reasons, and ψI is a direct factor of $F^* \otimes I$, ψI is injective; and similarly $\varphi \psi I$ is. Thus in the associated long exact sequences, the maps such as

$$H^{i-1}(\mathfrak{u}, Q) \rightarrow H^i(\mathfrak{u}, X)$$

are surjective. With some straightforward verifications, the commutativity of the diagram for $H^{i-1}(\mathfrak{u}, Q)$ gives that for $H^i(\mathfrak{u}, X)$. Q.E.D.

COROLLARY 4.3. *Suppose X is an irreducible Harish-Chandra module for $\mathfrak{g}$ (Definition 5.2) with respect to a subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ and Cartan involution θ . Suppose $\mathfrak{q}$ and $\mathfrak{l}$ are θ -stable. Then $H^i(\mathfrak{u}, X)$ is a (possibly reducible) Harish-Chandra module for $\mathfrak{l}$. In the setting of Theorem 4.2 and (2.4), we have*

(a) $P_{\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}(H^i(\mathfrak{u}, K_{\alpha}(X)))$ and $\varphi_{\lambda}^{\mathfrak{l}} \psi_{s\lambda}^{\mathfrak{l}}(H^i(\mathfrak{u}, X))$ have the same composition factors and multiplicities whenever $\alpha \in \Delta(\mathfrak{u})$; and

(b) $P_{\lambda - \rho_{\mathfrak{u}}}^{\mathfrak{l}}(H^i(\mathfrak{u}, Q_{\alpha}(X)))$ and $\varphi_{\lambda}^{\mathfrak{l}} \psi_{s\lambda}^{\mathfrak{l}}(H^i(\mathfrak{u}, X))$ have the same composition factors and multiplicities whenever $-\alpha \in \Delta(\mathfrak{u})$.

Proof. The first assertion is [14], Corollary 3.10. For (a), for example, we compare the long exact sequence of 4.2(a) with the long exact sequence in cohomology coming from 2.4(a), and use the commutativity of the last diagram in 4.2(a). By [17], Theorem 1.2, the map f_2 in that diagram is an isomorphism. A simple diagram chase gives the result; and (b) is identical. Q.E.D.

The corresponding result for non-integral walls is quite trivial, but we state it for reference.

PROPOSITION 4.4. *Let X be a $\mathfrak{Z}(\mathfrak{g})$ -finite $\mathfrak{g}$ module with generalized infinitesimal character λ . With other notation as at the beginning of this section,*

suppose $\alpha \in \Delta_\lambda^+$ is simple and non-integral. Then (with the notation established for Theorem 2.7)

$$P_{\lambda - \rho_u + \mu}^1(H^i(u, \psi X)) \cong \psi_\lambda^1(H^i(u, X)).$$

Proof. This is obtained from the collapse of the spectral sequence in exactly the same way as the first assertion of 4.2(a). Details are left to the reader. Q.E.D.

5. The special cohomology class of a Harish-Chandra module. In this section we consider the problem of recognizing a Harish-Chandra module on the basis of knowledge of some of its cohomology. To indicate what we need, we first state a result for highest weight modules, which was used implicitly in section 3.

PROPOSITION 5.1. *In the setting of section 3, suppose X is a $\mathfrak{g}$ module of finite length. Then the multiplicity $m(L_w, X)$ of L_w as a composition factor of X satisfies*

$$m(L_w, X) \leq \dim H^0(u, X)^{w\lambda + \rho}.$$

If X is completely reducible, equality holds.

This is obvious. The version we need for real groups will be based on the results of [14].

Henceforth we suppose, as in the introduction, that $\mathfrak{g}$ is the complexified Lie algebra of a real semisimple Lie group G ; K is a maximal compact subgroup of G , with complexified Lie algebra $\mathfrak{k}$, and the Cartan involution is written θ .

Definition 5.2. A Harish-Chandra module for G is a $\mathfrak{g}$ module of finite length which is a direct sum of finite dimensional semisimple $\mathfrak{k}$ modules. We may sometimes implicitly assume that G has finite center, and that the $\mathfrak{k}$ modules lift to representations of the group K ; but this is not particularly important.

We define a “norm” $\| \cdot \|$ on the set $\hat{\mathfrak{k}}$ of irreducible $\mathfrak{k}$ -modules as in [14], Definition 5.1; and define the lowest $\mathfrak{k}$ -types of a Harish-Chandra module X to be the smallest $\mathfrak{k}$ -types (with respect to $\| \cdot \|$) occurring in X .

Fix an irreducible Harish-Chandra module Y , and let μ be a lowest $\mathfrak{k}$ -type of Y . Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra of $\mathfrak{g}$ associated to μ by Proposition 4.1 of [14]. By Theorem 7.16 of [14], the conjugacy class of $\mathfrak{q}$ depends only on Y . Let $\tilde{\mu}^{\mathfrak{l}}$ be the highest $\mathfrak{l} \cap \mathfrak{k}$ -type of μ with respect to $\mathfrak{u} \cap \mathfrak{k}$ (i.e. the representation on $\mathfrak{u} \cap \mathfrak{k}$ invariant vectors in μ), and put

$$\mu^{\mathfrak{l}} = \tilde{\mu}^{\mathfrak{l}} \otimes (\wedge^R(\mathfrak{u} \cap \mathfrak{p}))^*;$$

here $R = \dim(\mathfrak{u} \cap \mathfrak{p})$.

THEOREM 5.3. ([14], Theorem 10.1, [13], Theorem 4.5). *With notation as above, μ occurs exactly once in Y , and $\mu^{\mathfrak{l}}$ occurs exactly once in $H^R(\mathfrak{u}, Y)$. Let $Y^{\mathfrak{l}}$ denote the unique irreducible subquotient of $H^R(\mathfrak{u}, Y)$ containing the $\mathfrak{l} \cap \mathfrak{k}$ -type $\mu^{\mathfrak{l}}$. Then*

Y^1 is independent of μ ; and the position of Y^1 in the Langlands classification of irreducible Harish-Chandra modules can be computed from that of Y .

The specification of Y^1 in terms of Y is made explicit in Theorem 4.5 of [13]. The Langlands classification will be discussed in more detail in section 6.

Definition 5.4. With notation as above, the representation Y^1 is called the *special cohomology class* of Y .

PROPOSITION 5.5. Suppose X is a Harish-Chandra module for $\mathfrak{g}$, and Y is an irreducible Harish-Chandra module for $\mathfrak{g}$ of non-singular infinitesimal character λ . Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra associated to Y . Then the multiplicity $m(Y, X)$ of Y as a composition factor of X satisfies

$$m(Y, X) \leq m(Y^1, H^R(\mathfrak{u}, X)).$$

If X is semisimple, equality holds.

Proof. Let $\mathfrak{h}^1$ be a maximally split θ -stable Cartan subalgebra of $\mathfrak{l}$; write $\mathfrak{k}^1 = \mathfrak{t}^1 + \mathfrak{a}^1$, with $\mathfrak{t}^1 = \mathfrak{h}^1 \cap \mathfrak{k}$, $\mathfrak{a}^1 = \mathfrak{h}^1 \cap \mathfrak{p}$. (Here $\mathfrak{p}$ is the -1 eigenspace of θ in $\mathfrak{g}$.) Extend $\mathfrak{t}^1$ to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l} \cap \mathfrak{k}$ (and hence of $\mathfrak{k}$). We may write $\mu \in \mathfrak{t}^*$ for the highest weight of the $\mathfrak{k}$ -type μ . In that case the element $\tilde{\lambda} \in \mathfrak{t}^*$ associated to μ by Proposition 4.1 of [14] has the following properties: first, $\tilde{\lambda}$ actually lies in $(\mathfrak{t}^1)^*$, and (by definition) the roots of $\mathfrak{t}$ in $\mathfrak{l}$ are precisely the roots of $\mathfrak{t}$ in $\mathfrak{g}$ orthogonal to $\tilde{\lambda}$, i.e.

$$\Delta(\mathfrak{l}, \mathfrak{t}) = \{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \alpha, \tilde{\lambda} \rangle = 0 \}. \quad (5.6)$$

Next, $\tilde{\lambda}$ is close to μ , i.e. there is a constant $A_{\mathfrak{g}}$ depending only on $\mathfrak{g}$, such that

$$|\tilde{\lambda} - \mu| \leq A_{\mathfrak{g}}. \quad (5.7)$$

(This is immediate from Proposition 4.1 of [14].) Finally, in the Langlands classification, Y is associated to the Cartan subalgebra $\mathfrak{h}^1$ and a weight of the form

$$(\tilde{\lambda}, \nu) \in (\mathfrak{t}^1)^* + (\mathfrak{a}^1)^* = (\mathfrak{h}^1)^*$$

([14], Theorems 1.2 and 7.16).

We want to study $H^R(\mathfrak{u}, X)$ using the spectral sequence of Theorem 3.9 in [14]; in particular we are interested in the $\mathfrak{l} \cap \mathfrak{k}$ -type μ^1 . The E_1 terms of this spectral sequence look essentially like

$$H^p(\mathfrak{u} \cap \mathfrak{k}, X) \otimes (\wedge^q(\mathfrak{u} \cap \mathfrak{p}))^*.$$

(Actually this is a sum of several E_1 terms; but this point is unimportant.) Since we are interested in

$$H^R(\mathfrak{u}, X)^{\mu^1}$$

(i.e. the $\mu^1 \cap \mathfrak{f}$ -type), it would be convenient to know that

$$H^p(\mathfrak{u} \cap \mathfrak{f}, X) \otimes (\wedge^q(\mathfrak{u} \cap \mathfrak{p})^*)^{\mu^1} = 0 \quad (5.8)$$

whenever $p \geq 1$. Because of (5.7) above, this will be the case if $\tilde{\lambda}$ is very regular with respect to $\mathfrak{u} \cap \mathfrak{f}$, i.e. if

$$\langle \alpha, \tilde{\lambda} \rangle \text{ is large when } \alpha \in \Delta(\mathfrak{u} \cap \mathfrak{f}, \mathfrak{t}).$$

So the first step is to arrange that, using Zukerman's translation principle ([17]). So we choose an integral weight $\gamma \in (\mathfrak{h}^1)^*$ with the following properties:

- (a) $\langle \alpha, \gamma|_{\mathfrak{t}} \rangle$ is large when $\alpha \in \Delta(\mathfrak{u}, \mathfrak{t})$
 - (b) γ is dominant for $\Delta_{(\tilde{\lambda}, \nu)}^+(\mathfrak{g}, \mathfrak{h}^1)$
 - (c) $\langle \alpha, \gamma|_{\mathfrak{t}} \rangle = 0$ when $\alpha \in \Delta(\mathfrak{l}, \mathfrak{t})$
- (5.9)

We leave to the reader the easy verification that this is possible; one can take for γ a rational approximation of $\text{Re}(\tilde{\lambda}, \nu)$, multiplied by a large integer. Now apply the translation functor from infinitesimal character $(\tilde{\lambda}, \nu)$ to $(\tilde{\lambda}, \nu) + \gamma$. The spectral sequence technique used in section 4 shows that, on the level of cohomology, this amounts to applying the corresponding translation functor for $\mathfrak{l}$. Furthermore the effect of this translation on Langlands parameters is computed in [13], Corollary 5.17 (see also [15], Corollary 4.8); the conclusion is that the translation affects neither hypotheses nor conclusion in our theorem. (We needed 5.9(c) to insure that the θ -stable parabolic associated to the translate of Y is still $\mathfrak{q}$.) So we may assume (5.8). One concludes immediately that the "edge homomorphism"

$$\pi_0^\mu : (H^0(\mathfrak{u} \cap \mathfrak{f}, X) \otimes (\wedge^R(\mathfrak{u} \cap \mathfrak{p})^*)^{\mu_1}) \rightarrow H^R(\mathfrak{u}, X)^{\mu_1}$$

is surjective. By Theorem 3.5 of [14], this computes the action of $U(\mathfrak{g})^{\mathfrak{f}}$ on $H^0(\mathfrak{u} \cap \mathfrak{f}, X)^{\mu}$ modulo the kernel of π_0^μ ; and in particular, the $U(\mathfrak{g})^{\mathfrak{f}}$ module $H^0(\mathfrak{u} \cap \mathfrak{f}, Y)^{\mu}$ occurs at least $m(Y^1, H^R(\mathfrak{u}, X))$ times in the $U(\mathfrak{g})^{\mathfrak{f}}$ module $H^0(\mathfrak{u} \cap \mathfrak{f}, X)^{\mu}$. By a theorem of Harish-Chandra, the $U(\mathfrak{g})^{\mathfrak{f}}$ action on one $\mathfrak{f}$ -type of an irreducible Harish-Chandra module is irreducible and determines the whole module uniquely. Hence Y occurs at least $m(Y^1, H^R(\mathfrak{u}, X))$ times in X . The last assertion of the theorem follows trivially. Q.E.D.

6. The cohomology of irreducible induced modules. The next step in our program of generalizing the Verma module arguments is to compute $H^1(\mathfrak{u}, X)$ as an $\mathfrak{l}$ module whenever X is a "standard irreducible," i.e. an analogue of the irreducible Verma modules (cf. Definition 6.7 below). The main case to bear in mind is when X is a discrete series representation. From here on, we will be making constant use of the Langlands classification of irreducible Harish-Chandra modules, in the form described in [13] and section 4 of [15]. We recall briefly the definitions. The real Lie algebra of a Lie group is denoted by the

corresponding lower case German letter, with a subscript zero; complexification is indicated by dropping the zero as before.

Let $H^1 \subseteq G$ be a θ -stable Cartan subgroup; put $T^1 = H^1 \cap K$, $A^1 = H^1 \cap \exp(\mathfrak{p}_0)$. Then $H^1 = T^1 \cdot A^1$, a direct product. (One should remember that T^1 need not be abelian.) Let $M^1 A^1$ be the Langlands decomposition of G^{A^1} , the centralizer of A^1 in G . T^1 is a Cartan subgroup of M^1 .

Definition 6.1. An M^1 -regular pseudocharacter, or simply pseudocharacter of H^1 , is an ordered pair $\gamma = (\Gamma, \bar{\gamma})$, with Γ an irreducible representation of H^1 , and $\bar{\gamma} \in \mathfrak{h}^*$. Further, we require the following compatibility: $\bar{\gamma}|_{\mathfrak{t}^1}$ should be purely imaginary, and regular with respect to $\Delta(\mathfrak{m}^1, \mathfrak{t}^1)$. Let $\Delta^+(\mathfrak{m}^1, \mathfrak{t}^1)$ be the positive system defined by $\bar{\gamma}$, and write $\rho_{\mathfrak{m}^1}$, $\rho_{\mathfrak{m}^1 \cap \mathfrak{t}}$ for the obvious half sums of positive roots. Then we want

$$d\Gamma = \bar{\gamma} + \rho_{\mathfrak{m}^1} - 2\rho_{\mathfrak{m}^1 \cap \mathfrak{t}}.$$

We write $(\hat{H}^1)'$ for the set of pseudocharacters of H^1 . The pair $(\Gamma, \bar{\gamma})$ defines in a natural way a discrete series representation $\delta = \delta(\gamma)$ of M^1 , and a character

$$\nu = \Gamma|_{A^1} = \exp(\bar{\gamma}|_{\mathfrak{a}^1})$$

of A^1 .

Definition 6.2. Let $\gamma \in (\hat{H}^1)'$. Choose a parabolic subgroup $P^1 = M^1 A^1 N^1$ associated to H^1 in such a way that if $\nu = \bar{\gamma}|_{\mathfrak{a}^1}$, then

$$\operatorname{Re}\langle \alpha, \nu \rangle \leq 0$$

for every root α of $\mathfrak{a}^1$ in $\mathfrak{n}^1$. The Langlands principal series with parameter γ , $\pi(\gamma)$, is defined by

$$\Pi(\gamma) = \operatorname{Ind}_P^G \delta(\gamma) \otimes \nu \otimes 1.$$

Its irreducible subrepresentations $\bar{\pi}^i(\gamma)$ are called the Langlands subrepresentations of $\pi(\gamma)$.

THEOREM 6.3. ([11]; [15], Theorem 4.2; [13], Theorem 2.9). Every irreducible Harish-Chandra module X for $\mathfrak{g}$ is infinitesimally equivalent to some $\bar{\pi}^i(\gamma)$; and (H^1, γ) is unique up to conjugation. If $\pi(\gamma)$ has nonsingular infinitesimal character, then it has exactly one irreducible subrepresentation, which we call $\bar{\pi}(\gamma)$.

One should bear in mind that the infinitesimal character of $\pi(\gamma)$ corresponds to $\bar{\gamma} \in (\mathfrak{h}^1)^*$ via the Harish-Chandra homomorphism. The next definition is just (2.6); we repeat it for emphasis.

Definition 6.4. Suppose $\gamma \in (\hat{H}^1)'$ is regular (i.e. $\pi(\gamma)$ has nonsingular infinitesimal character). Set

$$\Delta_\gamma^+ = \{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^1) \mid \operatorname{Re}\langle \alpha, \gamma \rangle < 0, \text{ or } \operatorname{Re}\langle \alpha, \gamma \rangle = 0, \text{ and } \operatorname{Im}\langle \alpha, \gamma \rangle > 0 \}.$$

Recall from [13], Proposition 6.1 or [15], Proposition 4.5, the element $m_\alpha \in H^1$ associated to every real root α of $\mathfrak{h}^1$ in $\mathfrak{g}$; if $G = \mathrm{SL}(2, \mathbb{R})$, m_α is the element of order two, and m_α is defined in general by embedding a covering group of $\mathrm{SL}(2, \mathbb{R})$ in G using α .

Definition 6.5. The real root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^1)$ is said to *satisfy the parity condition* if the eigenvalues of $\gamma(m_\alpha)$ are of the form

$$\epsilon_\alpha \exp(\pm 2\pi i \langle \gamma, \alpha \rangle / \langle \alpha, \alpha \rangle);$$

here $\epsilon_\alpha = \pm 1$ is defined as in the remarks after Proposition 5.14 of [13].

PROPOSITION 6.6 (cf. [13], Proposition 6.1). Suppose $\gamma \in (\hat{H}^1)'$ is regular, and satisfies

- (a) For all $\alpha \in \Delta_\gamma^+$, either α is real, or $\theta\alpha \in \Delta_\gamma^+$
- (b) If $\alpha \in \Delta_\gamma^+$ is real, then α does not satisfy the parity condition

Then $\pi(\gamma) = \bar{\pi}(\gamma)$ is irreducible.

Definition 6.7. The representations $\pi(\gamma)$ arising in Proposition 6.6 are called *standard irreducible modules*.

To describe the cohomology of the standard irreducible modules, another definition is convenient.

Definition 6.8. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$, with $\mathfrak{l} \supseteq \mathfrak{h}^1$. Suppose $\gamma \in (\hat{H}^1)'$. Let $\gamma_{\mathfrak{q}}$ be the regular character of H^1 with respect to $\mathfrak{l}$ such that

- (a) $\gamma_{\mathfrak{q}} = (\Gamma_{\mathfrak{q}}, \bar{\gamma}_{\mathfrak{q}})$
- (b) $\bar{\gamma}_{\mathfrak{q}} = \bar{\gamma} - \rho(\mathfrak{u}) = \bar{\gamma} - 1/2 \sum_{\alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^1)} \alpha$
- (c) $\Gamma|_{T^1}$ and $\Gamma_{\mathfrak{q}}|_{T^1}$ differ by a “sum of roots,” i.e. a tensor product of several one-dimensional representations of T^1 on root subspaces of $\mathfrak{g}$.

Clearly $\gamma_{\mathfrak{q}}$ is unique if it exists, because of the compatibility required between $\Gamma_{\mathfrak{q}}$ and $\gamma_{\mathfrak{q}}$. We leave to the reader the easy verification that $\gamma_{\mathfrak{q}}$ does indeed exist.

Finally, we need a way to keep track of induction arguments, substituting for the length function on W in the Verma module case. The first part of the next definition is a slight modification (by a constant) of the one in [15], Section 5.

Definition 6.9. Suppose $\gamma \in (\hat{H}^1)'$ is regular. Set $l(\bar{\pi}(\gamma)) = l(\gamma) = 1/2 | \{ \alpha \in \Delta_\gamma^+ | \theta\alpha \notin \Delta_\gamma^+ \} | + 1/2 \dim \mathfrak{a}^1 - c_0$, with c_0 equal to half the dimension of the split part of the fundamental Cartan subalgebra of $\mathfrak{g}$. If $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra, and $\mathfrak{h}^1 \subseteq \mathfrak{l}$, define

$$\begin{aligned} l_{\mathfrak{q}}(\gamma) = & | \{ \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^1) | \theta\alpha = \alpha, \text{ and either } \alpha \text{ is compact and } \langle \alpha, \gamma \rangle < 0, \\ & \text{or } \alpha \text{ is noncompact and } \langle \alpha, \gamma \rangle > 0 \} | \\ & + 1/2 | \{ \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^1) | \theta\alpha \neq \alpha \} | \end{aligned}$$

It is fairly easy to verify that both $l(\gamma)$ and $l_{\mathfrak{q}}(\gamma)$ are integers; for $l(\gamma)$ one has

to remember that if $\mathfrak{h}^1$ is identified with the fundamental Cartan subalgebra by a Cayley transform, then θ is modified by a Weyl group element of determinant $(-1)^{\dim \mathfrak{a}^1 - 2c_0}$.

THEOREM 6.10. *Suppose $\gamma \in (\hat{H}^1)'$ is regular, and $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra, with $\mathfrak{h}^1 \subseteq \mathfrak{l}$. Then the representation $\bar{\pi}_\ell(\gamma_\mathfrak{q})$ of $\mathfrak{l}$ occurs exactly once in $H^*(\mathfrak{u}, \bar{\pi}(\gamma))$, in degree $l_\mathfrak{q}(\gamma)$.*

Proof. Since both the Langlands classification and the cohomology groups behave well with respect to Zuckerman's translation functors (cf. [15], Corollary 4.8), it suffices to prove the result with γ replaced by $\gamma + \mu$, for some weight μ of a finite dimensional representation, such that $\gamma + \mu$ is regular and

$$\Delta_{\gamma+\mu}^+ = \Delta_\gamma^+.$$

So we may clearly assume that

$$|\operatorname{Re} \langle \alpha, \gamma_\ell \rangle| \text{ is large for } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^1) \text{ non-real.}$$

Under this hypothesis, the θ -stable parabolic subalgebra $\mathfrak{q}^1 = \mathfrak{l}^1 + \mathfrak{u}^1$ of $\mathfrak{g}$ associated to $\bar{\pi}(\gamma)$ as described in section 5, is just the one defined by $\gamma|_{\mathfrak{l}^1}$; in particular $\mathfrak{l}^1$ is the centralizer of $\mathfrak{t}^1$ in $\mathfrak{g}$, and $\mathfrak{l}^1 \subseteq \mathfrak{l}$. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ contained in $\mathfrak{l}^1$, and a Borel subalgebra of $\mathfrak{k}$ contained in $\mathfrak{q} \cap \mathfrak{k}$ and $\mathfrak{q}^1 \cap \mathfrak{l} \cap \mathfrak{k}$; let $\Delta^+(\mathfrak{k}, \mathfrak{t})$ be the corresponding positive system, which we use to identify representations of $\mathfrak{k}$ with their highest weights.

We first calculate the lowest $\mathfrak{l} \cap \mathfrak{k}$ -types of $\bar{\pi}(\gamma_\mathfrak{q})$. This is done in Theorem 7.16 of [14], in terms of the lowest $(\mathfrak{l}_1 \cap \mathfrak{k})$ -types of a certain representation of $\mathfrak{l}_1$. Using Lemma 6.33 of [14], it is easy to express that representation in our present notation; and Theorem 7.16 of [14] becomes

LEMMA 6.11. *Let $\{\tilde{\mu}_i\}$ be the lowest $\mathfrak{l}_1 \cap \mathfrak{k}$ -types of $\bar{\pi}((\gamma_\mathfrak{q})_{\mathfrak{l} \cap \mathfrak{q}^1})$. Then $\{\tilde{\mu}_i + 2\rho(\mathfrak{u}^1 \cap \mathfrak{l} \cap \mathfrak{p})\}$ is the set of lowest $\mathfrak{l} \cap \mathfrak{k}$ -types of $\bar{\pi}(\gamma_\mathfrak{q})$; here $2\rho(\mathfrak{u}^1 \cap \mathfrak{l} \cap \mathfrak{p}) = \sum_{\alpha \in \Delta(\mathfrak{u}^1 \cap \mathfrak{l} \cap \mathfrak{p}, \mathfrak{t})} \alpha$.*

The same result calculates the lowest $\mathfrak{k}$ -types of $\bar{\pi}(\gamma)$, but in a positive root system respecting $\mathfrak{q}^1$. So we choose $w \in W(\mathfrak{k}, \mathfrak{t})$ which takes the positive root system defined by $\mathfrak{q}^1 \cap \mathfrak{k}$ and $\Delta^+(\mathfrak{k}, \mathfrak{t}) \cap \Delta(\mathfrak{l}^1 \cap \mathfrak{k}, \mathfrak{t})$ into $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Then we get (since $\gamma_{\mathfrak{q}^1} = (\gamma_\mathfrak{q})_{\mathfrak{l} \cap \mathfrak{q}^1} + \rho(\mathfrak{u}) - \rho(\mathfrak{u}^1) + \rho(\mathfrak{l} \cap \mathfrak{u}^1)$).

LEMMA 6.12. *In the notation of Lemma 6.11, set $a = \rho(\mathfrak{u}) - \rho(\mathfrak{u}^1) + \rho(\mathfrak{l} \cap \mathfrak{u}^1)$. Then the lowest $\mathfrak{k}$ -types of $\bar{\pi}(\gamma)$ are*

$$\{w(\tilde{\mu}_i + 2\rho(\mathfrak{u}^1 \cap \mathfrak{p}) + a)\};$$

and every $\mathfrak{k}$ -type of $\bar{\pi}(\gamma)$ is of the form

$$w(\tilde{\mu}_i + 2\rho(\mathfrak{u}^1 \cap \mathfrak{p}) + a + Q),$$

with Q a sum of roots of $\mathfrak{t}$ in $\mathfrak{q}^1$.

Proof. The first statement is just Theorem 7.16 of [14]; and the second is established in the proof of Lemma 8.8 of [14]. Q.E.D.

We now use the spectral sequence in Theorem 3.9 of [14] (cf. also section 5) to compute

$$H^*(u, \bar{\pi}(\gamma))^{\tilde{\mu}_i + 2\rho(u^1 \cap l \cap p)},$$

i.e. the indicated $l \cap \mathfrak{k}$ -primary subspace. The E_1 terms of this spectral sequence were written down after (5.7). Suppose one of them contains the $l \cap \mathfrak{k}$ -type in question. Then there are an $l \cap \mathfrak{k}$ -type δ occurring in $H^p(u \cap \mathfrak{k}, \bar{\pi}(\gamma))$, and q roots $\beta_1 \dots \beta_q$ of t in $u \cap p$, such that

$$\tilde{\mu}_i + 2\rho(u^1 \cap l \cap p) = \delta - \sum \beta_i.$$

By Kostant's version of the Bott-Borel-Weil theorem, there is a $\mathfrak{k}$ -type μ_0 of $\bar{\pi}(\gamma)$, and a Weyl group element w_0 (having a certain property with respect to $\Delta^+(l \cap \mathfrak{k})$) such that $l(w_0) = p$, and

$$\delta = w_0(\mu_0 + \rho_c) - \rho_c;$$

here of course

$$\rho_c = 1/2 \sum_{\alpha \in \Delta^+(\mathfrak{k}, t)} \alpha.$$

Since $\gamma|_{\mathfrak{k}}$ is close to $\tilde{\mu}_i$, and is assumed to be very regular, $\tilde{\mu}_i$ and δ lie in the same Weyl chamber for $\mathfrak{k}$; so $w = w_0^{-1}$. Our equation now looks like

$$w(\tilde{\mu}_i + 2\rho(u^1 \cap l \cap p) + \rho_c + \sum \beta_i) = \mu_0 + \rho_c.$$

Inserting the information from Lemma 6.12, we get

$$\tilde{\mu}_i + 2\rho(u^1 \cap l \cap p) + \rho_c + \sum \beta_i = \tilde{\mu}_i + 2\rho(u^1 \cap p) + Q + a + w^{-1}\rho_c,$$

with Q a sum of roots in $\mathfrak{q}^1$. Clearly

$$\begin{aligned} a &= \sum_{\substack{\alpha \in \Delta(u) \\ \alpha \notin \Delta(u^1)}} \alpha = - \sum_{\substack{a \in \Delta(u^1) - \Delta(u^1 \cap l) \\ a \notin \Delta(u)}} \alpha \\ w^{-1}\rho_c - \rho_c &= - \sum_{\substack{a \in \Delta(u \cap \mathfrak{k}) \\ \alpha \notin \Delta(u^1 \cap \mathfrak{k})}} \alpha. \end{aligned}$$

So we can rewrite the above equation as

$$\sum \beta_i = \sum_{\alpha \in \Delta(u \cap u^1 \cap p)} \alpha + Q.$$

Clearly this is impossible unless

$$\{\beta_i\} = \Delta(u \cap u^1 \cap \mathfrak{p}), Q = 0.$$

So the $\mathfrak{l} \cap \mathfrak{k}$ type $\tilde{\mu}_i + 2\rho(u^1 \cap \mathfrak{l} \cap \mathfrak{p})$ occurs exactly once in E^1 , and hence exactly once in $H^*(u, \bar{\pi}(\gamma))$; the degree in which it occurs is

$$|\{\alpha \in \Delta(u, t^1) | \alpha \text{ is compact, and } \langle \alpha, \gamma|_{t^1} \rangle < 0,$$

$$\text{or } \alpha \text{ is noncompact, and } \langle \alpha, \gamma|_{t^1} \rangle > 0\}.$$

It is an easy exercise to check that this is precisely $l_a(\gamma)$.

It remains only to check that the irreducible representation of $\mathfrak{l}$ $\pi_t^1(\gamma_a)$ occurring in $H^*(u, \bar{\pi}(\gamma))$ and containing the lowest $\mathfrak{l} \cap \mathfrak{k}$ -types of $\bar{\pi}_t(\gamma_a)$, actually coincides with $\bar{\pi}_t(\gamma_a)$. By the Langlands classification, we can write

$$\pi_t^1(\gamma_a) \cong \bar{\pi}_t((\gamma_a|_{T^1}, \nu(\gamma)));$$

where $\nu(\gamma) \in (\mathfrak{a}^1)^*$. (This assume the easily verified fact that the $\mathfrak{l} \cap \mathfrak{k}$ -type we have been dealing with is $\mathfrak{l} \cap u^1$ -minimal in the cohomology ([14], Definition 3.11).) By Theorem 4.1, there is an element $w \in W(\mathfrak{g}/\mathfrak{h}^1)$ such that

$$(\bar{\gamma}|_{t^1}, \nu(\gamma)) = w(\bar{\gamma}).$$

Since $H^*(u, X)$ behaves well under translation, we may assume this holds for all γ in some Weyl chamber. Hence

$$\bar{\gamma}|_{t^1} = (w\bar{\gamma})|_{t^1}$$

for all $\bar{\gamma} \in (\mathfrak{h}^1)^*$; clearly this forces $w \in W(\mathfrak{l}^1/\mathfrak{h}^1)$. By shifting γ again, we can arrange for the stabilizer of $\Gamma_a|_{T^1}$ in $W(\mathfrak{l}^1/\mathfrak{h}^1)$ to be all of $W(\mathfrak{l}^1/\mathfrak{h}^1)$ (cf. [15], proof of 6.1). For such γ , we deduce

$$\bar{\pi}_t(\gamma_a|_{T^1}, \nu(\gamma)) \cong \bar{\pi}_t(\gamma_a)$$

(since $\bar{\pi}_t(\gamma_a)$ depends only on the conjugacy class of γ_a). This equality persists to all γ by translation, as desired. Q.E.D.

Up to this point, our arguments have been fairly routine extensions of those of [14]. The next result seems to be fundamentally deeper, however. (Even if a more elementary proof exists, the ideas used below are interesting in their own right.)

THEOREM 6.13. *Suppose $\gamma \in (\hat{H}^1)'$ is regular, and $\bar{\pi}(\gamma) = \pi(\gamma)$ is a standard irreducible (Definition 6.7). Then $\pi(\gamma)$ has no cohomology other than that specified in Theorem 6.10. More precisely, let $\mathfrak{a} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{h}^2 \subset \mathfrak{l}$ a θ -stable Cartan subalgebra, and $(\gamma_a^2) \in (H^2)'_{\mathfrak{l}}$ a regular character for $\mathfrak{l}$ (so that $\gamma^2 \in (\hat{H}^2)'$ is a regular character for $\mathfrak{g}$). Then $\bar{\pi}_t(\gamma_a^2)$ occurs in $H^*(u, \bar{\pi}(\gamma))$*

if and only if (H^1, γ) is conjugate to (H^2, γ^2) ; and in that case it occurs exactly once, in degree $l_q(\gamma^2)$.

The proof requires some preparation. We will need to study extensions in the category of $(\mathfrak{g}, \mathfrak{k})$ modules (cf. [1]). This problem is related to cohomology by

THEOREM 6.14 (Zuckerman). *Suppose $\gamma \in (\hat{H}^1)'$ is regular. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the parabolic associated to γ (as in section 5). Then there is a spectral sequence $\text{Ext}_{\mathfrak{l} \cap \mathfrak{k}}^p(H^{R-q}(\mathfrak{u}, X), \pi(\gamma_{\mathfrak{q}})) \Rightarrow \text{Ext}_{\mathfrak{g}, \mathfrak{k}}^{p+q}(X, \pi(\gamma))$. Here $R = \dim(\mathfrak{u} \cap \mathfrak{p})$, and X is any $(\mathfrak{g}, \mathfrak{k})$ module; the $E_2^{p,q}$ term is zero unless p and q are both non-negative.*

The proof of this result relies on Zuckerman's "derived functor" construction of representations, which has not yet been published. So we leave the result unproved; since our goal is simply to formulate a conjecture, this is not very serious.

LEMMA 6.15. *Suppose $\mathfrak{g}$ is split, and $\mu \in \hat{\mathfrak{k}}$ is a fine $\mathfrak{k}$ -type ([14], Definition 6.11). Suppose X is a $(\mathfrak{g}, \mathfrak{k})$ module of finite length, containing the $\mathfrak{k}$ -type μ exactly once; and suppose $\mathcal{B}(\mathfrak{g})$ acts by scalars in X , according to a regular infinitesimal character. Let $\pi(\gamma)$ be the principal series representation of $\mathfrak{g}$ containing μ , such that $U(\mathfrak{g})^{\mathfrak{k}}$ acts by the same scalars in the μ $\mathfrak{k}$ -types of X and $\pi(\gamma)$. Then there is a $\mathfrak{g}$ -module map from X to $\pi(\gamma)$ which is non-zero on the $\mathfrak{k}$ -type μ .*

Proof. Recall that $\pi(\gamma)$ is defined in such a way that $\bar{\pi}(\gamma)$ is its unique irreducible subrepresentation. By the assumptions of the lemma, $\bar{\pi}(\gamma)$ must be equivalent to the unique irreducible constituent of X containing μ . By Casselman's (apparently unpublished) realizability theorem, we can find a representation ξ of G on a Hilbert space of which X is the space of K -finite vectors. Let $G = KAN$ be an Iwasawa decomposition; suppose $\lambda \in \mathfrak{a}^*$ is dominant and gives the infinitesimal character of X . Let $\bar{N}$ be the unipotent group "opposite" to N , and let ρ be half the sum of the roots of $\mathfrak{a}$ in $\mathfrak{n}$. Consider the asymptotic expansions of matrix coefficients of ξ along the positive Weyl chamber A^+ (cf. [16], Theorem 9.1.1.1). Every exponent occurring in such an expansion is of the form $w\lambda - Q - \rho$, with Q a non-negative integral combination of roots of $\mathfrak{a}$ in $\mathfrak{n}$; in particular, the weights $\lambda + \alpha - \rho$, for α a root of $\mathfrak{a}$ in $\mathfrak{n}$, do not occur.

For fixed $w \in X$, consider the map

$$v \rightarrow p_{v,w} \in S(\mathfrak{a}^*)$$

taking v to the (polynomial) coefficient of $e^{\lambda - \rho}$ in the expansion of $\langle \xi(a)v, w \rangle$. It follows easily (just as in the proof of Casselman's subrepresentation theorem) that if $Y_1 \in \bar{\mathfrak{n}}$, $Y_2 \in \mathfrak{a}$, then

$$p_{Y_1 v, w} = 0$$

$$p_{Y_2 v, w} = [(\lambda - \rho)(Y_2) + \partial(Y_2)] p_{v, w}.$$

Since $\mathfrak{Z}(\mathfrak{g})$ acts by scalars on X according to a regular infinitesimal character, a standard argument now shows that we must in fact have Y_2 acting semisimply on the $p_{v,w}$; so $\partial(Y_2)$ always acts trivially, and therefore $p_{v,w}$ is constant. Thus we have maps

$$\begin{aligned}\varphi_w : X/\bar{\pi}X &\rightarrow \mathbb{C} \\ \varphi_w(Y \cdot v) &= (\lambda - \rho)(Y)v \quad (Y \in \mathfrak{a});\end{aligned}$$

of course $\varphi_w(v)$ is just $p_{v,w}$.

By the definitions in Langlands' classification (cf. [11]), the exponent $\lambda - \rho$ occurs in the expansion of the matrix coefficients of the μ $\mathfrak{k}$ -type in $\bar{\pi}(\gamma)$, and hence also in X . So we can find a $w \in X$ so that φ_w is non-trivial on the μ $\mathfrak{k}$ -type. By Frobenius reciprocity, this gives a map (for some $\delta_1 \in \hat{M}$)

$$X \rightarrow \text{Ind}_{MAN}^G \delta_1 \otimes \lambda \otimes 1$$

which is non-trivial on the μ $\mathfrak{k}$ -type. Since λ is negative for $\bar{N}$, the left side is of the form $\pi(\gamma')$. Clearly $\bar{\pi}(\gamma')$ is equivalent to the unique irreducible constituent of X containing μ , and hence to $\bar{\pi}(\gamma)$; so $\pi(\gamma)$ is equivalent to $\pi(\gamma')$, proving the lemma. Q.E.D.

COROLLARY 6.16. *Suppose $X_0 \approx \bar{\pi}(\gamma^0)$ and $X_1 \cong \bar{\pi}(\gamma^1)$ are distinct irreducible $(\mathfrak{g}, \mathfrak{k})$ modules with regular infinitesimal character, and $\text{Ext}_{\mu, \mathfrak{k}}^1(X_0, X_1) \neq 0$. Then for some $i \in \{0, 1\}$, X_i is a subquotient of $\pi(\gamma^{1-i})$.*

Proof. By translation, we may assume that X_0 and X_1 have distinct lowest $\mathfrak{k}$ -types (argue as at the end of the proof of Theorem 6.10). Let $(\mu_i, \mathfrak{q}^i)$ be lowest $\mathfrak{k}$ -types of X_i and associated θ -stable parabolic subalgebras; by a further translation we may assume $\mathfrak{l}^i$ is split. Suppose for definiteness that $\|\mu_1\| \leq \|\mu_0\|$; since

$$\text{Ext}_{\mathfrak{g}, \mathfrak{k}}^1(X_0, X_1) \cong \text{Ext}_{\mu, \mathfrak{k}}^1(X_1, X_0)$$

([15], Lemma 3.16), this causes no loss of generality. Let E be a proper extension of X_0 and X_1 ; thus

$$0 \rightarrow X_1 \rightarrow E \rightarrow X_0 \rightarrow 0$$

is exact, and X_1 is the unique irreducible submodule of E . Obviously E has no self-intertwining operators other than scalars, so $\mathfrak{Z}(\mathfrak{g})$ acts by scalars on E . By the proof of Theorem 5.2 of [14],

$$H^R(\mathfrak{u}^1, E)$$

contains the $\mathfrak{l}^1 \cap \mathfrak{k}$ type $\mu_1^{\mathfrak{l}^1}$ exactly once; and this is accounted for by the occurrence of $\bar{\pi}_1(\gamma_a^{\mathfrak{l}^1})$ as a composition factor. Furthermore Theorem 2.6 of [2] (cf. [14], Theorem 3.3) implies that $\mathfrak{Z}(\mathfrak{l})$ acts semisimply in $H^R(\mathfrak{u}^1, E)$. Since $\mathfrak{l}^1$ is

split and μ_1^1 is fine (by [14]) we can apply Lemma 6.15. The conclusion is

$$\text{Hom}_{\mathfrak{l}, \mathfrak{l} \cap \mathfrak{t}}(H^R(u^1, E), \pi_{\mathfrak{l}}(\gamma_{q^1})) \neq 0.$$

By a trivial "corner argument," Theorem 6.14 says

$$\text{Hom}_{\mathfrak{g}, \mathfrak{t}}(E, \pi(\gamma^1)) \neq 0.$$

Since $\bar{\pi}(\gamma^1) = X_1$ is the unique irreducible subrepresentation of γ^1 , X_1 is the unique irreducible subrepresentation of E , and $X_0 \not\cong X_1$, any non-zero map is injective; so X_0 occurs in $\pi(\gamma_1)$. Q.E.D.

Proof of Theorem 6.13. We proceed by downward induction on $\dim u$; and for fixed u , by downward induction on the degree i in which $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$ occurs. Notice that if we can only show that (H^2, γ^2) is conjugate to (H, γ) , then we are done by Theorem 6.10.

Suppose first that $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$ is actually a quotient of $H^i(u, \bar{\pi}(\gamma))$. Let $q^2 = \mathfrak{l}^2 + u^2$ be the parabolic subalgebra of $\mathfrak{l}$ associated to γ_q^2 , and $q^1 = \mathfrak{l}^2 + (u^2 + u) = \mathfrak{l}^2 + u^1$ the corresponding parabolic subalgebra of $\mathfrak{g}$; put $R = \dim(u^2 \cap \mathfrak{p})$. By translation, we may as well assume that γ^2 is very regular, and that $\mathfrak{l}^2$ is split. The regularity of γ^2 , together with the spectral sequence in the proof of Proposition 5.5, shows that if Y is any Harish-Chandra module for $\mathfrak{l}$, then $\bar{\pi}_{\mathfrak{l}^2}(\gamma_{q^1}^2) = \bar{\pi}_{\mathfrak{l}^2}((\gamma_q^2)_{q^2})$ cannot occur in $H^j(u^2, Y)$ unless $j \leq R$. The long exact sequence in u^2 cohomology associated to the quotient map

$$H^i(u, \bar{\pi}(\gamma)) \rightarrow \bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$$

now shows that

$$H^R(u^2, H^i(u, \bar{\pi}(\gamma)))$$

contains $\bar{\pi}_{\mathfrak{l}^2}(\gamma_{q^1}^2)$ as a subquotient. Now the Hochschild-Serre spectral sequence

$$H^p(u^2, H^q(u, \bar{\pi}(\gamma))) \Rightarrow H^{p+q}(u^1, \bar{\pi}(\gamma))$$

respects the $\mathfrak{l}^2$ module structure. We want to use this to investigate the occurrence of $\bar{\pi}_{\mathfrak{l}^2}(\gamma_{q^1}^2)$ in $H^{R+i}(u^1, \bar{\pi}(\gamma))$. This will be non-zero unless $\bar{\pi}_{\mathfrak{l}^2}(\gamma_{q^1}^2)$ occurs in either

- (a) $H^{R+i}(u^2, H^{i-j+1}(u, \bar{\pi}(\gamma)))$, $j \geq 2$, or
- (b) $H^{R-j}(u^2, H^{i+j-1}(u, \bar{\pi}(\gamma)))$, $j \geq 2$.

The first case we already know to be impossible. In case (b), the inductive hypothesis would give (H^2, γ^2) conjugate to (H^1, γ) , and we would be done. So we may assume $H^{R+i}(u^1, \bar{\pi}(\gamma))$ contains $\bar{\pi}_{\mathfrak{l}^2}(\gamma_{q^2}^2)$. If $u^2 \neq 0$, then $\dim u^1 > \dim u$, and we are again done by induction. So $u^2 = 0$, and $\mathfrak{l} = \mathfrak{l}^2$; thus $\mathfrak{l}$ is split, and $\mathfrak{h}^2$ is a maximally split Cartan subalgebra of $\mathfrak{l}$.

Changing notation, we now let $q^1 = \mathfrak{l}^1 + u^1$ be the parabolic associated to γ ; again a translation allows us to assume that $\mathfrak{l}^1$ is split, with maximally split

Cartan subalgebra $\mathfrak{h}^1$. Write

$$\mathfrak{h}^i = \mathfrak{t}^i + \alpha^i$$

$$\gamma = (\lambda, \nu)$$

$$\gamma^2 = (\lambda^2, \nu^2)$$

with $\lambda \in (\mathfrak{t}^1)^*$, etc. The spectral sequence in the proof of Proposition 5.5 shows that the lowest $\mathfrak{l} \cap \mathfrak{k}$ -type of $\bar{\pi}_\mathfrak{l}(\gamma_\alpha^2)$ has approximately the same length as some $\mathfrak{k}$ -type of $\bar{\pi}(\gamma)$. More precisely, this implies that there is a constant C depending only on $\mathfrak{g}$, such that

$$|\lambda| \leq |\lambda^2| + C. \quad (*)$$

Let $\varphi: \mathfrak{h}^1 \rightarrow \mathfrak{h}^2$ be the $\mathfrak{g}$ -inner automorphism that takes $(\mathfrak{h}^1, \Delta_\gamma^+)$ to $(\mathfrak{h}^2, \Delta_{\lambda^2}^+)$; then applying $(*)$ to all translates of γ , we get whenever $\mu \in (\mathfrak{h}^1)^*$ is a Δ_γ^+ dominant, real combination of roots, then

$$|\mu|_{\mathfrak{t}^1} \leq |\varphi\mu|_{\mathfrak{t}^2}.$$

Apply this to μ such that $\mu|_{\alpha^1} = 0$; by the special nature of γ , there are many such μ which are dominant (but singular) for Δ_γ^+ . We get

$$|\mu| = |\mu|_{\mathfrak{t}^1} \leq |\varphi\mu|_{\mathfrak{t}^2} \leq |\varphi\mu| = |\mu|;$$

so in this case $\varphi\mu|_{\alpha^2} = 0$. It follows that $\varphi(\alpha^1) \supseteq \alpha^2$. By a standard argument, we can therefore find an element $x \in K$ such that

$$\text{Ad}(x) \cdot \alpha^1 \supseteq \alpha^2.$$

Replacing (H^1, γ) by its conjugate by x , we get $\alpha^1 \supseteq \alpha^2$, and hence $M^1 A^1 \subseteq M^2 A^2$. In particular $\mathfrak{t}_0^1 \subseteq \mathfrak{m}_0^2$; so (replacing H^1 by another conjugate),

$$\mathfrak{t}^1 \subseteq \mathfrak{t}^2,$$

and therefore

$$\mathfrak{l}^1 \supseteq \mathfrak{l}.$$

We claim that $\mathfrak{l}^1 = \mathfrak{l}$. Assuming this for a moment, let us complete the proof of the theorem. It follows easily that λ and λ^2 have the same length. We can now apply the arguments in the proof of Theorem 5.10 to compute the multiplicity of the lowest $\mathfrak{l} \cap \mathfrak{k}$ -type of $\bar{\pi}_\mathfrak{l}(\gamma_\alpha^2)$ in $H^*(\mathfrak{u}^2, \bar{\pi}(\gamma))$; and just as in the proof of that theorem, we conclude that (after a translation) this is zero unless γ^2 is conjugate to γ , as desired.

We have shown that any quotient of $H^i(\mathfrak{u}, \bar{\pi}(\gamma))$ is of the form $\bar{\pi}_\mathfrak{l}(\gamma_\alpha^2)$, with γ^2 conjugate to γ . Finally we claim that $H^i(\mathfrak{u}, \bar{\pi}(\gamma))$ is completely reducible.

Suppose not; then we can find a quotient representation $\bar{\pi}_l(\gamma_q^2)$ of $H^l(u, \bar{\pi}(\gamma))$, and another constituent $\bar{\pi}_l(\gamma_q^3)$ of $H^l(u, \bar{\pi}(\gamma))$, such that

$$\text{Ext}_{l, l \cap \mathfrak{f}}^1(\bar{\pi}_l(\gamma_q^2), \bar{\pi}_l(\gamma_q^3)) \neq 0.$$

Since $\bar{\pi}_l(\gamma_q^2)$ occurs exactly once in the cohomology, we may assume that these representations of l are distinct. By Corollary 6.16, either

(a) $\bar{\pi}_l(\gamma_q^3)$ is a subquotient of $\pi_l(\gamma_q^2)$, or

(b) $\bar{\pi}_l(\gamma_q^2)$ is a subquotient of $\pi_l(\gamma_q^3)$.

It is easy to check that $\pi_l(\gamma_q^2)$ is a standard irreducible, since $\pi(\gamma)$ is and γ is conjugate to γ^2 . So case (a) is impossible. In case (b), write

$$\gamma^3 = (\lambda^3, \nu^3)$$

according to the decomposition $\mathfrak{h}^3 = \mathfrak{t}^3 + \mathfrak{a}^3$ of the corresponding Cartan subalgebra. Then (see for example [14], Lemma 8.8) we have

$$|\lambda_q^3| < |\lambda_q^2|;$$

by a translation we can arrange to make this difference very large. But $|\lambda_q^2|$ is close to the norm of the lowest $\mathfrak{k}$ -type of $\bar{\pi}(\gamma)$; and $|\lambda_q^3|$ is close to the norm of the lowest $l \cap \mathfrak{k}$ -type of $\bar{\pi}_l(\gamma_q^3)$, which in turn is close to the norm of some $\mathfrak{k}$ -type of $\bar{\pi}(\gamma)$ by the spectral sequence after 5.7. So again we have a contradiction, proving Theorem 6.13.

It remains only to verify our assertion that $l^1 = l$. It is convenient to prove something much more general. Suppose $\gamma \in (\hat{H}^1)'$ is a pseudocharacter, and $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^1)$ is a real root satisfying the parity condition. By Proposition 4.5 of [15], there is a pseudocharacter $C_\alpha(\gamma) \in (\hat{H}^2)'$ such that γ occurs on the right and $C_\alpha(\gamma)$ on the left of a character identity of the form given by Theorem 4.4 of [15]. Here H^2 is the more compact Cartan obtained from H^1 by a Cayley transform through α . $C_\alpha(\gamma)$ is *not* unique. The real roots of $\Delta(\mathfrak{g}, \mathfrak{h}^2)$ may be identified with the real roots of $\Delta(\mathfrak{g}, \mathfrak{h}^1)$ orthogonal to α . So the following definition makes at least a preliminary sort of sense.

Definition 6.17. Suppose $\gamma \in (\hat{H}^1)'$. A sequence $(\beta_1, \beta_2, \dots, \beta_r)$ of orthogonal real roots is called *admissible* for γ if one can make choices so that $C_{\beta_r} C_{\beta_{r-1}} \dots C_{\beta_1}(\gamma)$ is defined. The *compactness* of γ , $c(\gamma)$, is the dimension of $\mathfrak{t}^1$ plus the order of a maximal admissible sequence for γ . By “maximal” we mean that $\gamma' = C_{\beta_r} \dots C_{\beta_1}(\gamma)$ is defined, but no real root for γ' and the corresponding Cartan satisfies the parity condition. So $c(\gamma)$ is not obviously well defined. Our first task is to prove this.

LEMMA 6.18. Suppose $\gamma \in (\hat{H}^1)'$, $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}^1)$ is real and satisfies the parity condition. Fix a choice $C_\alpha(\gamma) = \gamma'$ in $(\hat{H}^2)'$. If $\beta \in \Delta(\mathfrak{g}, \mathfrak{h}^1)$ is real and orthogonal to α , let $\beta' \in \Delta(\mathfrak{g}, \mathfrak{h}^2)$ be the corresponding real root. Then the $\epsilon_{\beta'}$ and $\gamma'(m_{\beta'})$

involved in the parity condition satisfy

(a) If α and β span an $A_1 \times A_1$, or a G_2 , then

$$\epsilon_{\beta'} = \epsilon_{\beta}$$

$$\gamma'(m_{\beta'}) = \gamma(m_{\beta})$$

(b) If α and β span a B_2 , and both are long,

$$\epsilon_{\beta'} = -\epsilon_{\beta}$$

$$\gamma'(m_{\beta'}) = \gamma(m_{\beta})$$

(c) If α and β span a B_2 , and both are short,

$$\epsilon_{\beta'} = \epsilon_{\beta}$$

$$\gamma'(m_{\beta'}) = \gamma(m_{\beta})\gamma(m_{\alpha+\beta}^{-2}),$$

for an appropriate choice of $m_{\alpha+\beta}$ (which is defined only up to inverse).

Proof. Recall that $-\epsilon_{\alpha}$ is defined as the parity of

$$\frac{2\langle \alpha^2, 2\rho_c \rangle}{\langle \alpha^2, \alpha^2 \rangle}$$

where $2\rho_c$ is the sum of a set of positive compact imaginary roots of $\mathfrak{h}^2$, and α^2 is the root of $\mathfrak{h}^2$ corresponding to α (cf. [13], remarks proposition 5.14). Obviously this is independent of which positive root system is chosen. A very routine argument therefore reduces us to the case when G is split, with $\mathfrak{h}^1$ a split Cartan subalgebra spanned by α and β . Thus G is $SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$, $SP(2, \mathbf{R})$, the split real form of G_2 , or a covering group of one of these. We can therefore compute explicitly; details are left to the reader. Q.E.D.

Lemma 6.18 guarantees that the notion of a maximal admissible sequence $(\beta_1, \dots, \beta_r)$ depends only on the roots, and not on choices of $C_{\beta_1} \dots C_{\beta_r}(\gamma)$. We turn now to the proof that $c(\gamma)$ is well defined. Just as in the proof of the lemma, we may assume G is split, and $\mathfrak{h}^1$ is a split Cartan subalgebra. We may also assume G is simple. It is convenient to treat several cases separately. Notice that ϵ_{α} is -1 for all α .

Case I. $\gamma(m_{\alpha}) = \pm 1$ for all α . (This always holds if G is linear.) Only integral roots can possibly satisfy the parity condition, so we let R denote the system of integral roots of $\mathfrak{h}^1$ in $\mathfrak{g}$. We write $\check{R} = \{\check{\alpha}\}$ for the coroots. By Lemma 6.15 of [14], if $\check{\alpha} + \check{\beta} = \check{\delta}$,

$$m\check{\delta} = m_{\check{\alpha}}m_{\check{\beta}}.$$

Since ϵ_α is always -1 , it follows that if we put

$$\check{R}_0 = \{ \check{\alpha} \mid \alpha \text{ does not satisfy the parity condition} \}$$

$$\check{R}_1 = \{ \check{\alpha} \mid \alpha \text{ does satisfy the parity condition} \},$$

then for $i, j \in \{0, 1\}$,

$$\check{R}_i + \check{R}_j \subseteq \check{R}_{i+j \pmod{2}}.$$

Thus we can think of R_0 and R_1 as the compact and noncompact roots of a compact Cartan subalgebra $\mathfrak{t}^0$ in an equal rank semisimple symmetric pair $(\mathfrak{g}^0, \mathfrak{k}^0)$. By Lemma 2.61 of [12], for example, and Lemma 6.18, $c(\gamma)$ is nothing but the rank of this symmetric pair, ie. the dimension of a maximal reductive abelian subalgebra orthogonal to $\mathfrak{k}^0$. In particular it is well defined.

Case II. $\gamma(m_\alpha^2) = \pm 1$ for all α , but $\gamma(m_\alpha)$ is not always ± 1 . By Lemma 6.15 of [14], $\gamma(m_\alpha) = \pm 1$ if α is short in type B_n, C_n , or F_4 , and $\gamma(m_\alpha) = \pm i$ otherwise. It will therefore be convenient for us to refer to all root lengths in A_n, D_n, G_2 , and E_n as long; we apologize for G_2 . A long root α satisfies the parity condition if and only if

$$\frac{2\langle \alpha, \gamma \rangle}{\langle \alpha, \alpha \rangle} \in 1/2 \mathbb{Z} - \mathbb{Z}.$$

If α is long, $\gamma(m_\alpha^2) = -1$. So in the setting of 6.18(c), the sequence (α, β) is admissible if and only if α and β are integral, and (β, α) is not admissible. Let R be the system of roots which are either short and integral or long and half integral (i.e. integral for 2γ). It is an easy exercise to verify that this is a root system; we need only show that if α is long and half integral, and β is short and integral, then $s_\beta(\alpha)$ is half integral (which is obvious), and $s_\alpha(\beta)$ is integral (which is easy to check, since $2\langle \alpha, \gamma \rangle / \langle \beta, \beta \rangle = 2$. $2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle$ is integral). Put

$$R_0 = \{ \alpha \in R \mid \alpha \text{ does not satisfy the parity condition} \}$$

$$= \{ \alpha \in R \mid \alpha \text{ is long and integral or short and} \\ \text{does not satisfy the parity condition} \}.$$

$$R_1 = \{ \alpha \in R \mid \alpha \text{ does satisfy the parity condition} \}$$

$$= \{ \alpha \in R \mid \alpha \text{ is long and not integral or short and} \\ \text{does satisfy the parity condition} \}.$$

It is routine to check that

$$R_i + R_j \subseteq R_{i+j \pmod{2}}.$$

So again we can think of R_0 and R_1 as the compact and noncompact roots of an equal rank symmetric pair $(\mathfrak{g}^0, \mathfrak{k}^0)$; and the remarks above show that $c(\gamma)$ is the rank of $\mathfrak{g}^0/\mathfrak{k}^0$.

Case III. Otherwise. If $\gamma(m_\alpha^2) \neq \pm 1$, then Lemma 6.15 of [14] shows that α is strongly orthogonal to all other roots of the same length. This can happen only in type C_n . We can identify $\mathfrak{h}^1$ with $\mathbb{R}^n$, in which case the roots are $\pm e_i \pm e_j$. Put $\alpha_i = 2e_i$. We choose all the m_{α_i} consistently, so that $m_{\alpha_i}^2 = m_{\alpha_j}^2$ for all i and j . (Recall that m_α is defined only up to inverse.) Choose a square root $\exp(\pi it)$ of $\gamma(m_\alpha^2)$, with $-\frac{1}{2} < t < \frac{1}{2}$, $t \neq 0$. Then define $\delta_i = 0$ or 1 by

$$\gamma(m_{\alpha_i}) = (-1)^{\delta_i} \exp(\pi it);$$

then α_i satisfies the parity condition if and only if

$$\frac{2\langle \gamma, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_i \pm t \pmod{2}.$$

Furthermore

$$m_{\pm e_i \pm e_j} = m_{\alpha_i} m_{\alpha_j}^{-1},$$

so $\beta = \pm e_i \pm e_j$ satisfies the parity condition if and only if $2\langle \beta, \gamma \rangle / \langle \beta, \beta \rangle$ is an integer of the same parity as $\delta_i + \delta_j$. Finally, if α and β are orthogonal but not strongly orthogonal, and α satisfies the parity condition, then (α, β) is an admissible sequence if and only if

$$\frac{2\langle \gamma, \beta \rangle}{\langle \beta, \beta \rangle} = \pm 2t + \delta_i + \delta_j \pmod{2},$$

by 6.18(c). In particular one can check easily that in this case, one of (α, β) and (β, α) is admissible if and only if one of $(\alpha \pm \beta, \alpha \mp \beta)$ is admissible. Verification that $c(\gamma)$ is well defined is now a straightforward problem, which we leave to the reader. (It can be formulated in terms of $\mathfrak{g}^0, \mathfrak{k}^0$ as in the other cases, but this is not particularly necessary.)

So $c(\gamma)$ is well defined. The reason for introducing it is

PROPOSITION 6.19. *Suppose $\gamma^i \in (\hat{H}^i)'$, $i = 1, 2$ are regular pseudocharacters with regular infinitesimal character, $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic subalgebra containing H^2 , and $\bar{\pi}_1(\gamma_q^2)$ occurs in $H^{i'}(\mathfrak{u}, \bar{\pi}(\gamma^1))$. Then $c(\gamma^1) = c(\gamma^2)$.*

In the setting of Theorem 6.13, clearly $c(\gamma) = \dim \mathfrak{t}^1$, and $c(\gamma^2) \geq \dim \mathfrak{t}^2$. We had $\mathfrak{t}^2 \supseteq \mathfrak{t}^1$, so this result will give $\mathfrak{t}^2 = \mathfrak{t}^1$, and $\mathfrak{l} = \mathfrak{l}^1$ as desired. This will therefore complete the proof of Theorem 6.13.

The idea of the proof of Proposition 6.19 is to relate cohomology to Ext using Theorem 6.14. So we begin with some formal results, which are observations of G. Zuckerman.

Definition 6.20. *Block equivalence* of irreducible Harish-Chandra modules for $\mathfrak{g}$ is the equivalence relation generated by $X \sim Y$ if

$$\mathrm{Ext}_{\mathfrak{g}, \mathfrak{r}}^1(X, Y) \neq 0.$$

Equivalence classes are called *blocks*. If B is a block and X is a Harish-Chandra module, we say that X is of *type* B if every irreducible composition factor of X is in B .

LEMMA 6.21. *Suppose X is a Harish-Chandra module for $\mathfrak{g}$. Then*

$$X = \sum_{\text{blocks } B} X_B,$$

a finite direct sum; here X_B is of type B .

Proof. This is obvious.

COROLLARY 6.22 (Zuckerman). *Suppose B and B' are distinct blocks, X is of type B , and X' is of type B' . Then*

$$\mathrm{Ext}_{\mathfrak{g}, \mathfrak{r}}^i(X, X') = 0.$$

Proof. This follows from the interpretation of Ext in terms of long exact sequences of Harish-Chandra modules (cf. [1]), and Lemma 6.21. Q.E.D.

LEMMA 6.23. *Suppose $\gamma^i \in (\hat{H}^i)'$ is regular, ($i = 1, 2$), and $\bar{\pi}(\gamma^2)$ occurs in $\pi(\gamma^1)$. Then $c(\gamma^1) = c(\gamma^2)$.*

Proof. We proceed by induction on $l(\gamma^1)$. The argument is very similar to parts of section 5 of [15], to which we will often refer; so many details are left to the reader. Suppose first that there is some root $\alpha \in \Delta_{\gamma^1}^+$ such that α is complex, and $\theta\alpha \notin \Delta_{\gamma^1}^+$. By [15], we may choose α to be simple. Choose a weight μ of a finite dimensional representation so that $\gamma^1 + \mu$ is regular and dominant for $s_{\alpha}(\Delta_{\gamma^1}^+)$. Then $\pi(\gamma^1)$ is obtained from $\pi(\gamma^1 + \mu)$ by coherent continuation by μ , $l(\gamma^1 + \mu) = l(\gamma^1) - 1$, and $c(\gamma^1 + \mu) = c(\gamma^1)$. So $\bar{\pi}(\gamma^2)$ must arise from some constituent $\bar{\pi}(\gamma^3)$ of $\pi(\gamma^1 + \mu)$ by coherent continuation by μ . Such continuation was analyzed in [15]. Using the inductive hypothesis, the fact that $l(\gamma^2) < l(\gamma^1)$ ([15], Lemma 5.5), and Corollary 4.8, Lemma 4.9, and Theorem 4.12 of [15], one deduces immediately that $c(\gamma^2) = c(\gamma^3)$. (The only non-trivial fact is that if γ and γ' occur in the same character identity as in [15], Theorem 4.4, then $c(\gamma) = c(\gamma')$; but this is almost the definition of $c(\gamma)$.) By induction, $c(\gamma^3) = c(\gamma^1 + \mu)$, proving the lemma in this case.

So we may assume that no such roots α exist. This implies that the system of real roots is spanned by simple roots. Suppose there is a real root α satisfying the parity condition. After a series of continuations across real walls not satisfying the parity condition (which are handled just as the complex roots before) we

may assume α is simple. We can now set up a character identity ([15], Theorem 4.4) which transfers the problem to a more compact Cartan, and smaller $l(\gamma^1)$.

If no such real root α exists, then $\pi(\gamma^1)$ is a standard irreducible, and γ^2 must be conjugate to γ^1 ; so $c(\gamma^1) = c(\gamma^2)$. Q.E.D.

COROLLARY 6.24. *Suppose $\gamma^i \in (\hat{H}^i)'$ is regular ($i = 1, 2$), and*

$$\text{Ext}_{\mathfrak{u}, \mathfrak{f}}^i(\bar{\pi}(\gamma^1), \pi(\gamma^2)) \neq 0.$$

Then $c(\gamma^1) = c(\gamma^2)$.

Proof. Since $\pi(\gamma^2)$ has $\bar{\pi}(\gamma^2)$ as its unique irreducible subrepresentation, Lemma 6.21 implies that it must be of type B for some block B . By Corollary 6.22, $\bar{\pi}(\gamma^1)$ also belongs to B . But then Corollary 6.16 and Definition 6.20 imply that $c(\gamma^1) = c(\gamma^2)$. Q.E.D.

Proof of Proposition 6.19. We proceed by downward induction on the dimension of $\mathfrak{u}$; then for fixed $\mathfrak{u}$, by upward induction on the number N of roots of $\mathfrak{h}^2$ in $\mathfrak{u}$ not belonging to $\Delta_{\gamma^2}^+$; then for fixed $\mathfrak{u}$ and N , by downward induction on i . By Lemma 6.20, $\bar{\pi}_l(\gamma_q^2)$ belongs to the same l -block as some quotient $\bar{\pi}_l(\gamma_q^3)$ of $H^i(\mathfrak{u}, \bar{\pi}(\gamma^1))$. By Corollary 6.24, $c(\gamma^2) = c(\gamma^3)$. Since $\bar{\gamma}^2$ and $\bar{\gamma}^3$ define the same infinitesimal character of l , they are conjugate under the adjoint group L_G . Since this preserves $\mathfrak{u}$, $N(\gamma^2) = N(\gamma^3)$. So we may as well replace γ^2 by γ^3 . Arguing exactly as at the beginning of the proof of Theorem 6.13, we are reduced to the case when l is split, and $\mathfrak{h}^2 \subseteq l$ is a maximally split Cartan subalgebra.

Define a positive root system Δ_l^+ for $\mathfrak{h}^2$ in $\mathfrak{g}$ to consist of the roots of $\mathfrak{h}^2$ in $\mathfrak{u}$ together with the real roots in $\Delta_{\gamma^2}^+$. Then

$$N = |\Delta_l^+ \cap (-\Delta_{\gamma^2}^+)|.$$

Suppose first that $N = 0$, i.e. $\Delta_l^+ = \Delta_{\gamma^2}^+$. Then (at least after a slight translation to “generic” γ^2) $\mathfrak{q}$ is the θ -stable parabolic subalgebra associated to γ^2 ; and Theorem 6.14 gives a spectral sequence

$$\text{Ext}_{l, l \cap \mathfrak{f}}^p(H^{R-q}(\mathfrak{u}, \bar{\pi}(\gamma^1)), \pi_l(\gamma_q^2)) \Rightarrow \text{Ext}_{\mathfrak{g}, \mathfrak{f}}^{p+1}(\bar{\pi}(\gamma^1), \pi(\gamma^2)).$$

Consider the $E_2^{0, R-i}$ term; this is

$$\text{Hom}_l(H^i(\mathfrak{u}, \bar{\pi}(\gamma^1)), \pi_l(\gamma_q^2)),$$

which is non-zero since $\bar{\pi}_l(\gamma_q^2)$ is assumed to be a quotient of the cohomology group. This will “persist to E_∞ ” unless either

- (a) $E_2^{-p, R-i+p-1} \neq 0$, some $p \geq 2$, or
- (b) $0 \neq E_2^{p, R-i-p+1} = \text{Ext}_{l, l \cap \mathfrak{f}}^p(H^{i+p-1}(\mathfrak{u}, \bar{\pi}(\gamma^1)), \bar{\pi}_l(\gamma_q^2))$, some $p \geq 2$.

Case (a) is obviously impossible. In case (b), $i + p - 1 > i$. By Corollary 6.24, $H^{i+p-1}(\mathfrak{u}, \bar{\pi}(\gamma^1))$ must have a constituent $\bar{\pi}_l(\gamma_q^3)$ such that $c(\gamma^3) = c(\gamma^2)$. Since γ^3

and γ^2 have the same infinitesimal character for $\mathfrak{l}$, it follows as before that $N(\gamma^3) = N(\gamma^2)$. By induction $c(\gamma^3) = c(\gamma^1)$; so in this case $c(\gamma^1) = c(\gamma^2)$. So we may as well assume that the $E_2^{0,R-i}$ term persists to E_∞ , i.e. that

$$\text{Ext}_{\mathfrak{g}, \mathfrak{t}}^{R-i}(\bar{\pi}(\gamma^1), \bar{\pi}(\gamma^2)) \neq 0.$$

By Corollary 6.24, $c(\gamma^1) = c(\gamma^2)$.

Suppose now that $N > 0$, and that the result is known for smaller N . Choose a simple root $\alpha \in \Delta_{\gamma^2}^+$ such that $\alpha \notin \Delta_1^+$. Necessarily α is not real. Suppose first that α is not integral. Choose a weight $\mu^2 \in \hat{H}^2$ of a finite dimensional representation of G , such that $\gamma^2 + \mu^2$ is regular and dominant for $s_\alpha(\Delta_{\gamma^2}^+)$. Let $\mu^1 \in \hat{H}^1$ be the weight of a finite dimensional representation of G corresponding to μ^2 under a G_C -inner automorphism taking $(H^2, \Delta_{\gamma^2}^+)$ to $(H^1, \Delta_{\gamma^1}^+)$, and let $\alpha^1 \in \Delta_{\gamma^1}^+$ be the simple root corresponding to α under the same automorphism. Define φ, ψ using μ^2 as in section 2. We write (loosely) $\bar{\pi}(\gamma^1 + \mu^1)$ for the representation $\psi(\bar{\pi}(\gamma^1))$; by Corollary 4.8 and Lemma 4.9 of [15], this notation is precisely correct unless α^1 is noncompact or real satisfying the parity condition. By Proposition 4.4, $\bar{\pi}_{\mathfrak{t}}(\gamma_q^2 + \mu^2)$ occurs as a quotient of $H^i(\mathfrak{u}, \bar{\pi}(\gamma^1 + \mu^1))$. Furthermore

$$\Delta_{\gamma^2 + \mu^2}^+ = s_\alpha(\Delta_{\gamma^2}^+)$$

has only $N - 1$ roots not in Δ_1^+ ; so by induction

$$c(\gamma^2 + \mu^2) = c(\gamma^1 + \mu^1)$$

(By $\gamma^1 + \mu^1$, we actually mean the pseudocharacter corresponding to the representation we are calling $\bar{\pi}(\gamma^1 + \mu^1)$.) Since $c(\gamma^i) = c(\gamma^i + \mu^i)$, we are done by induction.

Suppose next that α is integral. Set

$$n = \frac{2\langle \alpha, \gamma^2 \rangle}{\langle \alpha, \alpha \rangle},$$

and define φ, ψ , etc. corresponding to the simple root α as in sections 2 and 4. We want to use the long exact sequences of Theorem 4.2; we use the notation there freely. Notice that we are in the case $-\alpha \in \Delta(\mathfrak{u})$, with $X = \bar{\pi}(\gamma^1)$.

Suppose first that $\psi X = 0$. By the first long exact sequence of 4.2(c), we have

$$\psi_\lambda^1(H^i(\mathfrak{u}, X)) \cong \psi_{s\lambda}^1(H^{i-1}(\mathfrak{u}, X)).$$

In combination with Corollary 4.8 of [15], this implies that $\bar{\pi}_{\mathfrak{t}}(\gamma_q^2 - n\alpha)$ is a quotient of $H^{i-1}(\mathfrak{u}, X)$. Now $N(\gamma^2 - n\alpha) = N(\gamma^2) - 1$; so by induction $c(\gamma^2 - n\alpha) = c(\gamma^1)$. Since $c(\gamma^2 - n\alpha) = c(\gamma^2)$, we are done in this case.

The remaining possibility is $\psi X \neq 0$. We have a short exact sequence 2.4(b)

$$0 \rightarrow X \rightarrow \varphi\psi X \rightarrow Q \rightarrow 0.$$

Applying Corollary 4.3(b), we find that $H^i(u, Q)$ contains $\bar{\pi}_1(\gamma_q^2 - n\alpha)$ as a subquotient; so some irreducible constituent $Y = \bar{\pi}(\gamma^4)$ of Q is such that $\bar{\pi}_1(\gamma_q^2 - n\alpha)$ occurs in $H^i(u, Y)$. By [15], Theorem 3.9, X and Y belong to the same block B ; so by Corollary 6.24, $c(\gamma^1) = c(\gamma^4)$. Since $c(\gamma^2) = c(\gamma^2 - n\alpha)$, it is enough to prove the theorem with $(\gamma^4, \gamma^2 - n\alpha)$ replacing (γ^1, γ^2) . But since $N(\gamma^2 - n\alpha) = N(\gamma^2) - 1$, we are done by induction.

With the proof of Proposition 6.19, that of Theorem 6.13 is complete as well.

7. The cohomology of irreducibles. In this section we will make explicit the implications of Conjecture 2.5 for cohomology. As remarked in the introduction, a technical problem has made it necessary to assume henceforth that G is linear. There is no problem in extending the proofs to the nonlinear case, but I have been unable to formulate the parity condition in the result properly. For linear groups, this is based on the following variant of Definition 6.9.

Definition 7.1. Suppose G is linear, and $\gamma \in (\hat{H}^1)'$ is regular. Put

$$l^I(\bar{\pi}(\lambda)) = l^I(\lambda) = \frac{1}{2} |\{ \alpha \in \Delta_\gamma^+ \mid \alpha \text{ is integral and } \theta\alpha \notin \Delta_\gamma^+ \}| + \frac{1}{2} \dim \mathfrak{a}^1 - c_0,$$

where c_0 is as in Definition 6.9.

THEOREM 7.2. Suppose G is linear and Conjecture 2.5 holds for G and the various Levi factors of θ -stable parabolic subalgebras of $\mathfrak{g}$. Suppose $\gamma^i \in (\hat{H}^i)'$, $i = 1, 2$, are regular pseudocharacters of θ -stable Cartan subgroups H^i , with regular infinitesimal characters. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}^2$. Then

(a) $H^i(u, \bar{\pi}(\gamma^1))$ contains $\bar{\pi}_1(\gamma_q^2)$ as a composition factor only if $(l^I(\gamma^1) - l^I(\gamma^2)) - (l_q(\gamma^2) - i)$ is even.

(b) If $\bar{\pi}(\gamma^1)$ and $\bar{\pi}(\gamma^2)$ are distinct and $l(\gamma^1) \geq l(\gamma^2)$, then $\text{Ext}_{\mathfrak{g}, \mathfrak{r}}^1(\bar{\pi}(\gamma^1), \bar{\pi}(\gamma^2)) \neq 0$ only if $l^I(\gamma^1) - l^I(\gamma^2)$ is odd.

(c) $H^i(u, \bar{\pi}(\gamma^1))$ is completely reducible as an $\mathfrak{l}$ module.

(d) In the setting of Corollary 4.3, suppose $\alpha \in \Delta_{\gamma^2}^+$ is a simple integral root, with $n = 2\langle \alpha, \gamma^2 \rangle / \langle \alpha, \alpha \rangle$.

(d1) If $\alpha \in \Delta(u)$, then the multiplicity of $\bar{\pi}_1(\gamma_q^2)$ in $H^i(u, U_\alpha(X))$ is its multiplicity in $H^{i+1}(u, X)$ plus the multiplicity of $\bar{\pi}_1(\gamma_q^2 - n\alpha)$ in $H^i(u, X)$.

(d2) If $\alpha \in \Delta(\mathfrak{l})$, then the multiplicity of $\bar{\pi}_1(\gamma_q^2)$ in $H^i(u, U_\alpha(X))$ is zero unless $\psi_\lambda^1(\bar{\pi}_1(\gamma_q^2)) = 0$ (i.e., α lies in the τ invariant with respect to $\mathfrak{l}$), and in that case it is the multiplicity of $\bar{\pi}_1(\gamma_q^2)$ in

$$\phi_\lambda^1 \psi_\lambda^1(H^i(u, X)) \oplus H^{i-1}(u, X) \oplus H^{i+1}(u, X).$$

(d3) If $-\alpha \in \Delta(u)$, then the multiplicity of $\bar{\pi}_1(\gamma_q^2)$ in $H^i(u, U_\alpha(X))$ is its multiplicity in $H^{i-1}(u, X)$ plus the multiplicity of $\bar{\pi}_1(\gamma_q^2 - n\alpha)$ in $H^i(u, X)$.

(e) In the setting of (d), $\bar{\pi}(\gamma^3)$ occurs in $U_\alpha(X)$ only if $l^I(\gamma^3) - l^I(\gamma^1)$ is odd.

Proof. By now this is quite routine. We proceed by induction on $\dim \mathfrak{g}$, then for fixed $\mathfrak{g}$ on $l(\gamma^1)$. Consider first (a). If $\bar{\pi}(\gamma^1)$ is a standard irreducible, this is a

rather weak version of Theorem 6.13. So suppose $\bar{\pi}(\gamma^1)$ is not a standard irreducible. Then we can find a simple root $\alpha \in \Delta_{\gamma^1}^+$ such that $\theta\alpha \notin \Delta_{\gamma^1}^+$; and if α is real, then α satisfies the parity condition. If α is not integral, Proposition 4.4 computes the desired cohomology in terms of that of some $\bar{\pi}(\gamma^3)$, with $l(\gamma^3) = l(\gamma^1) - 1$, and one easily checks that this computation preserves the parity condition we want. So assume α is integral. By [15], Theorem 4.12, we can find γ^3 with $l(\gamma^3) = l(\gamma^1) - 1$, such that $\bar{\pi}(\gamma^1)$ occurs in $U_{\alpha}(\bar{\pi}(\gamma^3))$, and also $l'(\gamma^3) = l'(\gamma^1) - 1$ since α is integral. Since we are assuming U_{α} is completely reducible, we may as well replace $\bar{\pi}(\gamma^1)$ by $U_{\alpha}(\bar{\pi}(\gamma^3))$. Since $l(\gamma^3) < l(\gamma^1)$, we can apply (d) to study $H^1(u, U_{\alpha}(\bar{\pi}(\gamma^3)))$; we deduce at once (from applying (a) to γ^3 and (e) to the algebra $\mathfrak{l}$) that the desired parity condition holds.

Consider next (b). By Lemma 5.5 of [15], $\bar{\pi}(\gamma^1)$ is not a constituent of $\pi(\gamma^2)$. There is a short exact sequence.

$$0 \rightarrow \bar{\pi}(\gamma^2) \rightarrow \pi(\gamma^2) \rightarrow Q \rightarrow 0.$$

The long exact sequence in Ext, together with the fact that

$$\text{Hom}_{\mathfrak{g}, \mathfrak{r}}(\bar{\pi}(\gamma^1), Q) = 0,$$

implies that

$$\text{Ext}_{\mathfrak{g}, \mathfrak{r}}^1(\bar{\pi}(\gamma^1), \pi(\gamma^2)) \neq 0.$$

Let (for a moment) $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ -stable parabolic subalgebra associated to γ^2 ; after a shift we may assume $\mathfrak{l}$ is split. By Theorem 6.14, either

- (1) $\text{Ext}_{\mathfrak{l}, \mathfrak{l} \cap \mathfrak{r}}^1(H^R(u^2, \bar{\pi}(\gamma^1)), \pi_{\mathfrak{l}}(\gamma_q^2)) \neq 0$, or
- (2) $\text{Hom}_{\mathfrak{l}, \mathfrak{l} \cap \mathfrak{r}}(H^{R-1}(u^2, \bar{\pi}(\gamma^1)), \pi_{\mathfrak{l}}(\gamma_q^2)) \neq 0$.

In case (2), $H^{R-1}(u^2, \bar{\pi}(\gamma^1))$ must contain $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$. Since $R = l_q(\gamma^2)$, (a) implies that $l'(\gamma^1) - l'(\gamma^2)$ is odd. We claim that case (1) cannot arise. For let Y be the direct summand of $H^R(u^2, \bar{\pi}(\gamma^1))$ with the same infinitesimal character as $\pi_{\mathfrak{l}}(\gamma_q^2)$, and let E be the extension of $\pi_{\mathfrak{l}}(\gamma_q^2)$ and Y corresponding to the element of Ext. By Proposition 5.5, Y cannot contain $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$. We want to apply Lemma 6.15 to E . By a shift, we can arrange as usual for Y not to contain the lowest $\mathfrak{l} \cap \mathfrak{k}$ -types of $\pi_{\mathfrak{l}}(\gamma_q^2)$. $\mathfrak{Z}(\mathfrak{l})$ acts semisimply on cohomology and hence by scalars on Y . To apply the lemma, we need it to act by scalars on E . So suppose $z \in \mathfrak{Z}(\mathfrak{l})$ and z annihilates $\pi_{\mathfrak{l}}(\gamma_q^2)$. Then the action of z in E defines a map

$$Y \cong E / \pi_{\mathfrak{l}}(\gamma_q^2) \rightarrow E.$$

The composition of this with the quotient map $E \rightarrow Y$ is just the action of z on Y , and hence zero, so we get a map

$$Y \rightarrow \pi_{\mathfrak{l}}(\gamma_q^2).$$

Since Y does not contain $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$, such a map is zero, so z acts by zero. Lemma

6.15 now provides a map

$$E \xrightarrow{\varphi} \pi_{\mathfrak{l}}(\gamma_q^2)$$

which is nonzero on $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$. The restriction to $\pi_{\mathfrak{l}}(\gamma_q^2)$ gives an intertwining map

$$\pi_{\mathfrak{l}}(\gamma_q^2) \xrightarrow{\psi} \pi_{\mathfrak{l}}(\gamma_q^2)$$

which is nonzero on $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$. Since this is the unique irreducible subrepresentation of $\pi_{\mathfrak{l}}(\gamma_q^2)$, it is clear that ψ must be equal to the scalar to which it restricts on $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$. (This uses the fact that $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$ occurs exactly once in $\pi_{\mathfrak{l}}(\gamma_q^2)$.) Thus some multiple of φ is a splitting of the exact sequence

$$0 \rightarrow \pi_{\mathfrak{l}}(\gamma_q^2) \rightarrow E \rightarrow Y \rightarrow 0,$$

contradicting the assumption that E was not split. This proves (b).

For (c), suppose $\bar{\pi}_{\mathfrak{l}}(\gamma_q^2)$ and $\bar{\pi}_{\mathfrak{l}}(\gamma_q^3)$ both occur in $H^i(\mathfrak{u}, \bar{\pi}(\gamma^1))$. We want to show that

$$\text{Ext}_{\mathfrak{u}, \mathfrak{l} \cap \mathfrak{f}}^1(\bar{\pi}_{\mathfrak{l}}(\gamma_q^2), \bar{\pi}_{\mathfrak{l}}(\gamma_q^3)) = 0;$$

this will suffice. Suppose not. Then γ^2 and γ^3 differ by an $L_{\mathbb{C}}$ -inner automorphism. By (a),

$$(l^I(\gamma^2) - l_{\mathfrak{a}}(\gamma^2)) - (l^I(\gamma^3) - l_{\mathfrak{a}}(\gamma^3)) \quad (*)$$

is even. By (b), it suffices to show that

$$l_{\mathfrak{l}}^I(\gamma^2) - l_{\mathfrak{l}}^I(\gamma^3)$$

is even, with obvious notation. We prove this whenever γ^2 and γ^3 differ by an $L_{\mathbb{C}}$ -inner automorphism, and $(*)$ holds. We proceed by induction on the number N of roots in $\Delta_{\gamma^2}^+$ in $-\Delta(\mathfrak{u}, \mathfrak{h}^2)$. Suppose first that $N = 0$, i.e., that $\Delta(\mathfrak{u}, \mathfrak{h}^2) \subseteq \Delta_{\gamma^2}^+$. Since γ^2 and γ^3 differ by an $L_{\mathbb{C}}$ -inner automorphism, this implies that $\Delta_{\gamma^3}^+ \supseteq \Delta(\mathfrak{u}, \mathfrak{h}^3)$. In this case both $l_{\mathfrak{a}}(\gamma^2)$ and $l_{\mathfrak{a}}(\gamma^3)$ are easily seen to be $\dim(\mathfrak{u} \cap \mathfrak{f})$. Furthermore,

$$l^I(\gamma^2) - l_{\mathfrak{l}}^I(\gamma^2) = |\{\alpha \in \Delta(\mathfrak{u}, \mathfrak{h}^2) \mid \alpha \text{ is integral, and exactly one of } \alpha \text{ and } \theta\alpha$$

$$\text{belongs to } \Delta_{\gamma^2}^+\} |$$

$$= 0,$$

and similarly for γ^3 . The result follows in this case.

Suppose next that $N > 0$. We can find a simple root $\alpha^2 \in \Delta_{\gamma^2}^+$ such that $\alpha^2 \in -\Delta(\mathfrak{u}, \mathfrak{h}^2)$. If α^3 is the corresponding root for γ^3 , then $\alpha^3 \in -\Delta(\mathfrak{u}, \mathfrak{h}^3)$. Suppose we shift γ^2 and γ^3 across the α^2 and α^3 walls using corresponding

weights of finite-dimensional representations. If α^2 is non-integral, nothing changes. If it is integral, then we change $l^i(\gamma_i)$ by 0 if $\theta\alpha^i = \alpha^i$ and by ± 1 otherwise, and $l_q(\gamma^i)$ by ± 1 if $\theta\alpha^i = \alpha^i$ and by 0 otherwise. So in any case $l^i(\gamma^i) - l_q(\gamma^i)$ changes by ± 1 , and $*$ remains valid. Furthermore, N decreases by 1, and $l^i_1(\gamma^i)$ is unchanged, so by induction $l^i_1(\gamma^2) = l^i_1(\gamma^3)$. This proves (c).

Parts (1) and (3) of (d) are immediate from Corollary 4.3; we use the long exact sequences in cohomology attached by 2.4 (c) and (d) just as in the proof of Theorem 3.5. Consider then part 2 of (d). Recall the exact sequences 2.4 (a), (c):

$$0 \rightarrow K \rightarrow \varphi\psi X \rightarrow X \rightarrow 0$$

$$0 \rightarrow X \rightarrow K \rightarrow U \rightarrow 0.$$

From 4.2(b), we have the isomorphism

$$P_{\lambda-\rho_u}^1(H^i(u, \varphi\psi X)) \cong \varphi_\lambda^1 \psi_\lambda^1(H^i(u, X)).$$

Applying this with X replaced by U_α , we get at once the first assertion of (d2). Now write

$$P_{\lambda-\rho_u}^1(H^i(u, X)) = Y_n^i \oplus Y_s^i = Y^i.$$

Here Y_n^i is the direct sum of all the irreducible constituents not annihilated by ψ_λ^1 , and Y_s^i is the sum of those which are; this is possible by (c). The image of the natural map

$$\varphi_\lambda^1 \psi_\lambda^1(Y^i) \rightarrow Y^i \quad (*)$$

is clearly Y_n^i . Considering the long exact sequence in cohomology for 2.4(a) and the commutative diagram in 4.2(b), we deduce at once that there is a short exact sequence

$$0 \rightarrow Y_s^{i-1} \rightarrow P_{\gamma-\rho_u}^1(H^i(u, K)) \rightarrow K(Y_n^i) \rightarrow 0;$$

here of course $K(Y_n^i)$ denotes the kernel of the natural map $(*)$ above. The long exact sequence in cohomology for 2.4(c) gives

$$\begin{aligned} \cdots &\xrightarrow{\rho_1^{i-1}} P_{\lambda-\rho_u}^1(H^{i-1}(u, K)) \xrightarrow{\rho_2^{i-1}} P_{\lambda-\rho_u}^1(H^{i-1}(u, U)) \xrightarrow{\rho_3^{i-1}} Y_n^i \oplus Y_s^i \\ &\rightarrow P_{\lambda-\rho_u}^1(H^i(u, K)) \rightarrow \cdots \end{aligned}$$

By the first assertion of (d2), the image of ρ_3^{i-1} is contained in Y_s^i . By (a), Y_s^i and Y_n^{i-1} have no common constituents. If Y_s^i and $K(Y_n^i)$ have a common constituent, it must be in $U(Y_n^i)$ (the cokernel of the natural inclusion $Y_n^i \hookrightarrow K(Y_n^i)$). By (e) applied to $\mathbb{I}$, this would mean that some pair of constituents $\bar{\pi}_1(\delta_q^j)$, $j = 1, 2$, of Y^i satisfy

$$l^i_1(\delta_q^1) - l^i_1(\delta_q^2) \text{ is odd.}$$

This is ruled out by the proof of (c). So Y_s^i and $P_{\lambda - \rho_u}^i(H^i(u, K))$ have no common constituents. So the image of ρ_3^{i-1} is precisely Y_s^i . This implies that the cokernel Z of ρ_1^{i-1} appears in a short exact sequence

$$0 \rightarrow Y_s^{i-2} \rightarrow Z \rightarrow U(Y_n^{i-1}) \rightarrow 0.$$

The last assertion of (d2) is immediate.

For (e), suppose $\bar{\pi}(\gamma^3)$ occurs in $U_\alpha(X)$. Since $U_\alpha(X)$ is assumed to be completely reducible, Theorem 3.9 of [15] implies that $\text{Ext}^1(\bar{\pi}(\gamma^3), X) \neq 0$. So this is a special case of (b). Q.E.D.

THEOREM 7.3. *The formulas of 7.2(d) provide an inductive method for computing $H^*(u, \bar{\pi}(\gamma^1))$, assuming Conjecture 2.5; i.e., the multiplicity of each $\bar{\pi}(\gamma_\alpha^2)$ can be computed.*

Proof. We must simply exhibit the required algorithm, which of course proceeds by induction on $l(\gamma^1)$. If $\bar{\pi}(\gamma^1)$ is a standard irreducible module, the cohomology is computed by Theorem 6.13. Suppose then that $\bar{\pi}(\gamma^1)$ is not a standard irreducible. We now argue as in Section 5 of [15]. Suppose first that there is a simple complex root $\alpha \in \Delta_\gamma^+$ with $\theta\alpha \notin \Delta_\gamma^+$. Choose a weight μ of a finite-dimensional representation of G such that $\gamma^1 + \mu$ is dominant and regular for $s_\alpha(\Delta_\gamma^+)$; if

$$\frac{2\langle \alpha, \gamma^1 \rangle}{\langle \alpha, \alpha \rangle} = n$$

is an integer, we take $\mu = -n\alpha$. Then $l(\gamma^1 + \mu) = l(\gamma^1) - 1$, so $H^*(u, \bar{\pi}(\gamma^1 + \mu))$ is known by induction. If α is not integral, then $\bar{\pi}(\gamma^1) = \varphi_\alpha(\bar{\pi}(\gamma^1 + \mu))$ and $H^*(u, \bar{\pi}(\gamma^1))$ is computed from $H^*(u, \bar{\pi}(\gamma^1 + \mu))$ by Proposition 4.4. So suppose α is integral; put $X = \bar{\pi}(\gamma^1 - n\alpha)$. By [15], Theorem 4.12, $U_\alpha(X)$ contains $\bar{\pi}(\gamma^1)$ exactly once, and every other constituent $\bar{\pi}(\gamma^3)$ of $U_\alpha(X)$ satisfies $l(\gamma^3) < l(\gamma^1)$. By 7.2(d), we know $H^*(u, U_\alpha(X))$. By induction, we know $H^*(u, \bar{\pi}(\gamma^3))$ for every γ^3 with $l(\gamma^3) < l(\gamma^1)$. Since $U_\alpha(X)$ is assumed to be completely reducible, we will be done if we can compute the multiplicity of $\bar{\pi}(\gamma^3)$ in $U_\alpha(X)$. But Proposition 5.5 tells how to do that in terms of the (known) cohomology of $U_\alpha(X)$ (with respect to some other parabolic q').

If no such root α exists, then the set of real roots of Δ_γ^+ is spanned by simple roots, and some real root must satisfy the parity condition; for otherwise $\bar{\pi}(\gamma^1)$ is a standard irreducible. Let L^1 be the connected subgroup of G containing H^1 , corresponding to the real roots of H^1 in G . The set of non-real roots in Δ_γ^+ is θ -stable and hence defines a θ -stable parabolic subalgebra $\mathfrak{q}^1 = \mathfrak{l}^1 + \mathfrak{u}^1$ of $\mathfrak{g}$. The algorithm to be described depends on the existence of roots with certain properties, so we first need to get some *a priori* restrictions on γ^2 . Define

$$\mathcal{G}(\gamma^1) = \{ \gamma \in (\hat{H}^1)' \mid \gamma \text{ is regular, } \gamma - \gamma^1 \text{ is a sum of roots, and}$$

$$\Delta_\gamma^+ \text{ differs from } \Delta_\gamma^+ \text{ only in real roots} \}.$$

($\mathfrak{g}$ stands for γ , or possibly “good”.) Next define

$$\mathfrak{F} = \{ \delta \in (\hat{H}^3)' \mid H^3 \subseteq L^1, \text{ and } \Delta_\delta^+ \text{ contains } \Delta(u, \mathfrak{h}^3) \}.$$

(Of course, $\mathfrak{F}$ stands for “fair”.)

LEMMA 7.4. *Suppose $\mathfrak{h}$ and $\mathfrak{h}^3$ are θ -stable Cartan subalgebras of $\mathfrak{l}^1$, $\delta \in \hat{H}'$ and $\delta^3 \in (\hat{H}^3)'$ belong to $\mathfrak{F}$, and $\bar{\pi}(\delta^3)$ occurs in $\pi(\delta)$.*

(a) *If $\bar{\pi}(\delta)$ occurs in $\pi(\gamma^1)$, then there is a $\gamma \in \mathfrak{g}(\gamma^1)$ such that $\bar{\pi}(\delta^3)$ occurs in $\pi(\gamma)$.*

(b) *If $\bar{\pi}(\delta^3)$ occurs in $\pi(\gamma^1)$, then there is a $\gamma \in \mathfrak{g}(\gamma^1)$ such that $\bar{\pi}(\delta)$ occurs in $\pi(\gamma)$.*

Proof. By Corollary 4.18 and Proposition 4.19 of [13], the composition series of $\pi(\delta)$, for any $\delta \in \mathfrak{F}$, can be computed in terms of $\mathfrak{l}^1$; more precisely, in terms of the composition series of $\pi_{\mathfrak{l}^1}(\delta_{\mathfrak{q},1})$. So we can immediately reduce to the case $\mathfrak{l}^1 = \mathfrak{g}$.

Consider first (a). We proceed by *downward* induction on $l(\delta)$. Suppose first that there is a complex simple root $\alpha \in \Delta_\delta^+$ with $\theta\alpha \in \Delta_\delta^+$. Choose a sum of roots μ so that $\delta + \mu$ is regular and dominant for $s_\alpha(\Delta_\delta^+)$; if $n = 2\langle \alpha, \delta \rangle / \langle \alpha, \alpha \rangle$ is an integer, we take $\mu = -n\alpha$. Then $l(\delta + \mu) = l(\delta) + 1$, so the result is known for $\delta + \mu$. We introduce two “translation functors” F^+ and F^- ; if α is integral, these are both $\varphi_\alpha \psi_\alpha$. If not, F^+ is defined to be the sum of the identity and ψ_α (cf. Section 2); similarly F^- is the identity plus φ_α . These are exact functors, and on the level of characters satisfy

$$F^+(\Theta(\delta)) = \Theta(\delta) + \Theta(\delta + \mu) = F^-(\Theta(\delta + \mu))$$

$$F^+(\Theta(\gamma^1)) = \Theta(\gamma^1) + \Theta(\gamma^1 + \tilde{\mu}^1) = F^-(\Theta(\gamma^1 + \tilde{\mu}^1)). \quad (*)$$

(Of course, $\Theta(\delta)$ is the character of $\pi(\delta)$.) Here $\tilde{\mu}^1 \in \hat{H}^1$ is the sum of roots corresponding to μ . These formulas are essentially obvious; proofs may be found in [13], Section 5. Finally, by Corollary 4.8 and Theorem 4.12 of [15], $\bar{\pi}(\delta + \mu)$ occurs in $F^+(\bar{\pi}(\delta))$.

By assumption $\bar{\pi}(\delta)$ occurs in $\pi(\gamma^1)$, so $\bar{\pi}(\delta + \mu)$ must occur in either $\pi(\gamma^1)$ or $\pi(\gamma^1 + \tilde{\mu}^1)$ because of (*). Now (*) also implies that we can find a constituent $\bar{\pi}(\delta^4)$ of $\pi(\delta + \mu)$ such that $\bar{\pi}(\delta^3)$ occurs in $F^-(\bar{\pi}(\delta^4))$. By induction, we can choose $\gamma^3 \in \mathfrak{g}(\gamma^1 + \tilde{\mu}^1) = \mathfrak{g}(\gamma^1)$ so that $\bar{\pi}(\delta^4)$ occurs in $\pi(\gamma^3)$. By (*) with γ^3 replacing γ^1 , $\bar{\pi}(\delta^3)$ will then occur in $\pi(\gamma)$, with $\gamma = \gamma^3$ or $\gamma^3 + \tilde{\mu}^3$; in either case $\gamma \in \mathfrak{g}(\gamma^1)$.

So we may suppose there are no such roots α . Thus the imaginary roots of $\mathfrak{h}$ are spanned by simple roots for Δ_δ^+ . If there are any such imaginary roots, there is a noncompact simple one α (since $\mathfrak{g}$ is split). Let $\delta^5 \in (\hat{H}^5)'$ be obtained from δ and H by Cayley transform through α ; this can be arranged so that $\Theta(\delta^5) \supseteq \Theta(\delta)$ (i.e., the difference is the character of a representation). In this case $l(\delta^5) = l(\delta) + 1$, and $\bar{\pi}(\delta^5)$ occurs in $\varphi_\alpha \psi_\alpha(\bar{\pi}(\delta))$. By (*), $\bar{\pi}(\delta^5)$ occurs in

$\pi(\gamma^1)$ or $\pi(\gamma^1 - n\alpha^1)$. Also $\bar{\pi}(\delta^3)$ occurs in $\pi(\delta)$ and hence in $\pi(\delta^5)$; so by induction, it occurs in some $\pi(\gamma)$, with $\gamma \in \mathcal{G}(\gamma^1 - n\alpha^1) = \mathcal{G}(\gamma^1)$.

So we may suppose there are no imaginary roots of $\mathfrak{h}$, i.e., that $\mathfrak{h}$ is split. Then the Langlands classification forces (H, δ) to be conjugate to (H^1, γ^1) , and the result is trivial.

The argument for (b) is similar; it also proceeds by *downward* induction on $l(\delta^3)$. Suppose first that there is a simple complex root $\alpha^3 \in \Delta_{\delta^3}^+$ with $\theta\alpha^3 \in \Delta_{\delta^3}^+$. If α^3 is not integral, a straightforward argument applies; we leave the details to the reader. So suppose $n = 2\langle\alpha^3, \delta^3\rangle/\langle\alpha^3, \alpha^3\rangle$ is an integer. If $\bar{\pi}(\delta^3 - n\alpha^3)$ also occurs in $\pi(\delta)$, we are done by induction; so suppose this is *not* the case. Let $\alpha \in \Delta_{\delta}^+$ be the root corresponding to α^3 . By say Lemma 3.11 and Theorem 4.12 of [15], $\bar{\pi}(\delta^3 - n\alpha^3)$ will occur with positive multiplicity in $\varphi_{\alpha}\psi_{\alpha}(\pi(\delta))$. (This uses the assumption that it does not occur in $\pi(\delta)$.) If α is compact imaginary, or real and does not satisfy the parity condition, then $\varphi_{\alpha}\psi_{\alpha}(\pi(\delta))$ is a multiple of $\pi(\delta)$ and hence could not contain $\bar{\pi}(\delta^3 - n\alpha^3)$, so these possibilities do not arise. Otherwise there is a δ^5 so that

- (1) $\varphi_{\alpha}\psi_{\alpha}(\Theta(\delta)) \subseteq \Theta(\delta) + \Theta(\delta^5)$
- (2) $\bar{\pi}(\delta)$ occurs in $\pi(\delta^5)$ or $\varphi_{\alpha}\psi_{\alpha}(\bar{\pi}(\delta^5))$.

(This is proved in [15], Section 4.) By (1), $\bar{\pi}(\delta^3 - n\alpha^3)$ occurs in $\pi(\delta^5)$, so by induction $\bar{\pi}(\delta^5)$ occurs in $\pi(\gamma^3)$ for some $\gamma^3 \in \mathcal{G}(\gamma^1)$. If $\bar{\pi}(\delta)$ occurs in $\pi(\delta^5)$, we are done by part (a) of the lemma. Otherwise (2) implies that $\bar{\pi}(\delta)$ occurs in $\varphi_{\alpha}\psi_{\alpha}(\bar{\pi}(\delta^5))$, so (*) implies that $\bar{\pi}(\delta)$ occurs in $\pi(\gamma^3)$ or $\pi(\gamma^3 - n\alpha)$.

So we may suppose that no such root exists. If $\mathfrak{h}^3$ is not split, then as before there is a noncompact simple imaginary root α^3 , and in that case Theorem 4.12 again gives a δ^4 with $l(\delta^4) = l(\delta^3) + 1$ and $\bar{\pi}(\delta^4)$ occurring in $\varphi_{\alpha}\psi_{\alpha}(\bar{\pi}(\delta^3))$. So we can repeat the argument of the last case.

So finally we may suppose $\mathfrak{h}^3$ is split. In this case necessarily $\bar{\pi}(\delta^3) = \bar{\pi}(\delta)$, and the result is obvious. Q.E.D.

Next we need a more complicated version of the same result.

LEMMA 7.5. *Suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a θ -stable parabolic, $\mathfrak{h} \subseteq \mathfrak{l} \cap \mathfrak{l}^1$ is a θ -stable Cartan subalgebra, $\delta \in \hat{H}'$ belongs to $\mathfrak{F}$, and $\bar{\pi}(\delta)$ occurs in $\pi(\gamma^1)$. Suppose $\delta^3 \in (\hat{H}^3)'$, $\mathfrak{h}^3 \subseteq \mathfrak{l}$, and $\bar{\pi}_1(\delta_q^3)$ is a constituent of $\pi_1(\delta_q)$. Then (possibly after conjugating (H^3, δ^3) by L) $\mathfrak{h}^3 \subseteq \mathfrak{l}^1$, $\delta^3 \in \mathfrak{F}$, and there is a $\gamma \in \mathcal{G}(\gamma^1)$ so that $\bar{\pi}(\delta^3)$ occurs in $\pi(\gamma)$.*

Proof. We proceed by induction on $l(\delta)$. Suppose first that there is a simple complex root $\alpha \in \Delta_{\delta}^+$ with $\theta\alpha \notin \Delta_{\delta}^+$. Since $\delta \in \mathfrak{F}$, α is a root of $\mathfrak{h}$ in $\mathfrak{l}^1$. If α is non-integral, we leave the details to the reader; so suppose $n = 2\langle\alpha, \delta\rangle/\langle\alpha, \alpha\rangle$ is an integer. Then $\delta - n\alpha \in \hat{H}'$ belongs to $\mathfrak{F}$, and $l(\delta - n\alpha) = l(\delta) - 1$. Since $\bar{\pi}(\delta - n\alpha)$ is a constituent of $\pi(\delta)$, Lemma 7.5 implies that $\bar{\pi}(\delta - n\alpha)$ occurs in $\pi(\gamma^3)$, for some $\gamma^3 \in \mathcal{G}(\gamma^1)$. Let α^3 be the simple root of $\Delta_{\delta^3}^+$ corresponding to α . If α is not a root of $\mathfrak{h}$ in $\mathfrak{l}$, then by translation $\bar{\pi}_1(\delta_q^3 - n\alpha^3)$ occurs in $\pi_1(\delta_q - n\alpha)$. By induction, we can conjugate (H^3, δ^3) by L so that $\mathfrak{h}^3 \subseteq \mathfrak{l}^1$, $\delta^3 - n\alpha^3 \in \mathfrak{F}$, and there is a $\gamma^4 \in \mathcal{G}(\gamma^3) = \mathcal{G}(\gamma^1)$ so that $\bar{\pi}(\delta^3 - n\alpha^3)$

occurs in $\pi(\gamma^4)$. If α^3 is compact imaginary, or real and does not satisfy the parity condition, then $\delta^3 - n\alpha^3$ is conjugate to δ^3 , and we are done. Otherwise (by Theorem 4.12 of [15]) we can find a $\delta^4 \in \mathcal{F}$ such that either $\bar{\pi}(\delta^3)$ and $\bar{\pi}(\delta^3 - n\alpha^3)$ both occur in $\pi(\delta^4)$ or $\bar{\pi}(\delta^4)$ occurs in both $\pi(\delta^3)$ and $\pi(\delta^3 - n\alpha^3)$. (If α^3 is complex, δ^4 is either δ^3 or $\delta^3 - n\alpha^3$; otherwise it is obtained by a Cayley transform.) By Lemma 7.4, it follows that there is a $\gamma \in \mathcal{G}(\gamma^1)$ such that $\bar{\pi}(\delta^3)$ occurs in $\pi(\gamma)$, as desired.

So we may assume α is a root of $\mathfrak{h}$ in $\mathfrak{l}$. In that case, the formula $(*)$ in the proof of Lemma 7.4 implies that there is a constituent $\bar{\pi}_l(\delta_q^4)$ of $\pi_l(\delta_q - n\alpha)$ such that $\bar{\pi}_l(\delta_q^3)$ occurs in

$$\varphi_\alpha^l \psi_\alpha^l(\bar{\pi}_l(\delta_q^4)).$$

By induction, we may assume that $\mathfrak{h}^4 \subseteq \mathfrak{l}^1$, $\delta^4 \in \mathcal{F}$, and $\bar{\pi}(\delta^4)$ occurs in $\pi(\gamma^3)$ for some $\gamma^3 \in \mathcal{G}(\gamma^1)$. By Theorem 4.12 of [15], either $\bar{\pi}_l(\delta_q^3)$ occurs in $\pi_l(\delta_q^4)$ (and we are done by induction) or (after conjugating (H^3, δ^3) by L) $\delta^3 \in \mathcal{F}$, and $\bar{\pi}(\delta^4)$ occurs in $\pi(\delta^3)$. In that case we are done by Lemma 7.4.

So we may assume that no such simple root α exists. In that setting, Corollary 4.18 and Proposition 4.19 of [13] imply that the composition series of $\pi_l(\delta_q)$ and $\pi(\delta)$ depend only on the real roots of $\mathfrak{h}$ in $\mathfrak{l} \cap \mathfrak{l}^1$, and in particular (up to conjugacy), $\mathfrak{h}^3 \subseteq \mathfrak{l} \cap \mathfrak{l}^1$, $\delta^3 \in \mathcal{F}$, and $\bar{\pi}(\delta^3)$ occurs in $\pi(\delta)$. So we are done by Lemma 7.4. Q.E.D.

LEMMA 7.6. *Suppose $\delta \in \mathcal{F}$, $\bar{\pi}(\delta)$ occurs in $\pi(\gamma^1)$, and $\bar{\pi}_l(\gamma_q^2)$ occurs in $H^i(\mathfrak{u}, \bar{\pi}(\delta))$. Then (after replacing (H^2, γ^2) by a conjugate under G) $\gamma^2 \in \mathcal{F}$, and there is a $\gamma \in \mathcal{G}(\gamma^1)$ such that $\bar{\pi}(\gamma^2)$ occurs in $\pi(\gamma)$.*

Proof. We proceed by induction on $l(\delta)$. Suppose first that there is a simple complex root $\alpha \in \Delta_\delta^+$ with $\theta\alpha \notin \Delta_\delta^+$; since $\delta \in \mathcal{F}$, α is a root of $\mathfrak{l}^1$. If α is not integral, an easy argument applies. So suppose $2\langle\alpha, \delta\rangle/\langle\alpha, \alpha\rangle = n$ is an integer. Clearly $\delta - n\alpha \in \mathcal{F}$, and $l(\delta - n\alpha) = l(\delta) - 1$. Since $\bar{\pi}(\delta - n\alpha)$ occurs in $\pi(\delta)$, Lemma 7.4 provides a $\gamma^3 \in \mathcal{G}(\gamma^1)$ such that $\bar{\pi}(\delta - n\alpha)$ occurs in $\pi(\gamma^3)$. According to Conjecture 2.5, $\bar{\pi}(\delta)$ is a direct summand of $U_\alpha(\bar{\pi}(\delta - n\alpha))$, so $\bar{\pi}_l(\gamma_q^2)$ occurs in $H^i(\mathfrak{u}, U_\alpha(\bar{\pi}(\delta - n\alpha)))$. (This invocation of the conjecture could be eliminated with a little more work, which is why it was not mentioned in the hypotheses of the lemma.) Now apply Theorem 7.2(d); either $\bar{\pi}_l(\gamma_q^2)$ occurs in $H^*(\mathfrak{u}, \bar{\pi}(\delta - n\alpha))$ (in which case we are done by induction), or there is a constituent $\bar{\pi}_l(\gamma_q^4)$ which “gives rise” to $\bar{\pi}_l(\gamma_q^2)$ in an obvious sense. By induction, we can conjugate (H^4, γ^4) by G so that $\gamma^4 \in \mathcal{F}$, and $\bar{\pi}(\gamma^4)$ occurs in $\pi(\gamma^5)$ for some $\gamma^5 \in \mathcal{G}(\gamma^1)$.

Let $\alpha^2 \in \Delta_{\gamma^2}^+$ correspond to α , and suppose first that $\alpha^2 \notin \Delta(\mathfrak{l}, \mathfrak{h}^2)$. Then

$$\gamma^4 = \gamma^2 - n\alpha^2.$$

Arguing as in the first part of the proof of Lemma 7.5, we can easily complete the proof in this case. So suppose $\alpha^2 \in \Delta(\mathfrak{l}, \mathfrak{h}^2)$. Then $\bar{\pi}_l(\gamma_q^2)$ occurs in $U_{\alpha^2}(\bar{\pi}_l(\gamma_q^4))$.

So either $\bar{\pi}_l(\gamma^2)$ occurs in $\pi_l(\gamma^4)$, or (after conjugating (H^2, γ^2) by L) $\gamma^2 \in \mathfrak{F}$, and $\bar{\pi}(\gamma^4)$ occurs in $\pi(\gamma^2)$ (by Theorem 4.12 of [15]). In the first case we are done by Lemma 7.5, and in the second by Lemma 7.4.

So we may suppose there are no such roots α ; thus the real roots of $\mathfrak{h}$ are spanned by the simple roots. If $\bar{\pi}(\delta)$ is a standard irreducible, we are done by Theorem 6.13. Otherwise, after crossing some non-integral real walls, we can find a simple real root α satisfying the parity condition. Passing to a more compact Cartan H^3 by a Cayley transform through α , we get $\delta^3 \in (\hat{H}^3)'$, $l(\delta^3) = l(\delta) - 1$, and $\delta^3 \in \mathfrak{F}$, such that $\bar{\pi}(\delta)$ occurs in $U_\alpha(\bar{\pi}(\delta^3))$. Now we argue exactly as in the previous case. Q.E.D.

Recall that we are trying to compute the multiplicity of $\bar{\pi}_l(\gamma^2)$ in $H^i(u, \bar{\pi}(\gamma^1))$. By Lemma 7.6, we can and do assume that $H^2 \subseteq L^1$, and $\gamma^2 \in \mathfrak{F}$.

LEMMA 7.7. *Suppose the multiplicity of $\bar{\pi}_l(\gamma^3)$ in $H^i(u, \bar{\pi}(\gamma^1))$ is known whenever $\gamma^3 \in (\hat{H}^2)'$, $\gamma^3 - \gamma^2$ is a sum of roots, and Δ_γ^+ differs from Δ_γ^+ only in non-integral roots of $\mathfrak{h}^2$ in $\mathfrak{l}^1$. Then the multiplicity of all such $\bar{\pi}_l(\gamma^3)$ in $H^i(u, \bar{\pi}(\gamma))$ can be computed for all $\gamma \in \mathfrak{G}(\gamma^1)$, assuming Conjecture 2.5.*

Proof. Clearly we can reduce to the case when γ is dominant for $s_\alpha(\Delta_\gamma^+)$, with $\alpha \in \Delta_\gamma^+$ real and simple. A simple translation argument then reduces us to the case when $n = 2\langle \alpha, \gamma^1 \rangle / \langle \alpha, \alpha \rangle$ is an integer, and $\gamma = \gamma^1 - n\alpha$.

If α does not satisfy the parity condition, then γ^1 is conjugate to γ and there is nothing to prove. So suppose α satisfies the parity condition. Let H^3 be the Cartan obtained from H^1 by Cayley transform through α , and $\gamma^3 \in (\hat{H}^3)'$ such that

$$l(\gamma^3) = l(\gamma^1) - 1$$

$$U_\alpha(\bar{\pi}(\gamma^3)) \text{ contains } \bar{\pi}(\gamma^1) \text{ and } \bar{\pi}(\gamma).$$

By Theorem 7.2, we know $H^*(u, U_\alpha(\bar{\pi}(\gamma^3)))$, and the components of $U_\alpha(\bar{\pi}(\gamma^3))$ are computed by Proposition 5.5. Except for $\bar{\pi}(\gamma^1)$ and $\bar{\pi}(\gamma)$, their cohomology is known by inductive hypothesis (as they occur in $\pi(\gamma^3)$), so we can compute

$$H^*(u, \bar{\pi}(\gamma^1) \oplus \bar{\pi}(\gamma)).$$

This proves the lemma. Q.E.D. (We do not rule out the rather unlikely event that γ is conjugate to γ^1 , by the way.)

At this point we should list some information which is available. The main point is that we need to know the composition series of $\pi(\gamma^1)$, and therefore the character of $\bar{\pi}(\gamma^1)$, in order to complete the computation of $H^*(u, \bar{\pi}(\gamma^1))$. Unfortunately, the algorithm given in Section 5 of [15] for computing this composition series relies on a knowledge of certain $U_\alpha(\bar{\pi}(\gamma^1))$, and the inductive hypotheses here do not provide us with that information. So we will refine the argument of [15] slightly.

Let R be the set of integral real roots of $\mathfrak{h}^1$, R^+ (or $R^+(\gamma^1)$) the positive

system defined by γ^1 , and Π (or $\Pi(\gamma^1)$) the set of simple roots of R^+ . Write $R = R_1 \times \cdots \times R_r$, a product of simple root systems; we use similar notation for R^+ and Π . As is pointed out before Definition 5.1 of [15], Π_i contains a root satisfying the parity condition (for γ^1) if and only if R_i does. (This is also obvious from the proof given in Section 6 that $c(\gamma)$ is well defined.) Let R^0 be the direct factor of R consisting of those R_i containing a root satisfying the parity condition; similarly, define $(R^0)^+, \Pi^0$. In [15], a certain subset D_{γ^1} of Π^0 is defined. It is clear that $R^0(\gamma) = R^0(\gamma^1)$ for all $\gamma \in \mathfrak{g}(\gamma^1)$, so $\Pi^0(\gamma^1)$ can be naturally identified with $\Pi^0(\gamma)$; this identification is accomplished by an element of $W(R)$, the Weyl group of R . In our present notation,

$$\begin{aligned} D_{\gamma^1} &= \bigcup_{\gamma \in \mathfrak{g}(\gamma^1)} \{ \alpha \in \Pi^0(\gamma) \mid \alpha \text{ satisfies the parity condition} \} \\ &= \{ \alpha \in \Pi^0 \mid \text{for some } w \in W(R^0), w\alpha \text{ satisfies the parity condition} \}. \end{aligned}$$

By Lemma 5.3 of [15], $\Pi^0 - D_{\gamma^1}$ consists of the short simple roots in certain $R_i \subseteq R^0$ of type B_n, C_n , or F_4 .

LEMMA 7.8. *Suppose $H^2 \subseteq L^1$, $\gamma^2 \in (\hat{H}^2)'$ is in $\mathfrak{F}$, and either $\bar{\pi}_1(\gamma_q^2)$ occurs in $H^1(u, \bar{\pi}(\gamma^1))$, or $\bar{\pi}_{l^1}(\gamma_q^2)$ occurs in $\pi_{l^1}(\gamma_q^1)$. Let $\alpha \in \Delta_{\gamma^2}^+$ be an integral root of $\mathfrak{h}^2$ in l^1 . Then either α is real and does not satisfy the parity condition, or the root $\tilde{\alpha}$ in R corresponding to α lies in R^0 .*

Proof. By Lemma 7.6, we may as well assume $\bar{\pi}_{l^1}(\gamma_q^2)$ occurs in $\pi_{l^1}(\gamma_q^1)$. If α is complex, we may as well replace it by an adjacent imaginary root (which exists since l^1 is split). So α is either real or imaginary. The result now follows from Lemma 5.4 of [15]. Q.E.D.

We turn now to the computation of the composition series of the various $\pi(\gamma)$, $\gamma \in \mathfrak{g}(\gamma^1)$. For this purpose, as was pointed out in Section 5 of [15], we may assume that $\mathfrak{g} = l^1$, and of course that $\mathfrak{g}$ is simple. There is an obvious analogue of Lemma 7.7 for this problem—at each stage of the argument, in trying to compute a certain multiplicity in $\pi(\gamma^1)$, we are free to replace γ^1 by any $\gamma \in \mathfrak{g}(\gamma^1)$. The algorithm in [15] proceeds nicely (since we know all $U_\alpha(\bar{\pi}(\gamma^2))$ whenever H^2 is not split by induction) until the following situation arises: $\alpha \in \Delta_{\gamma^2}^+$ is complex, integral, and simple, $\theta\alpha \in \Delta_{\gamma^2}^+$, and the root $\tilde{\alpha} \in \Pi^0(\gamma^1)$ corresponding to α does not satisfy the parity condition. In that case the algorithm given requires knowing the multiplicity of $\bar{\pi}(\gamma^2 - n\alpha)$ in $U_\alpha(\bar{\pi}(\gamma^1))$. Now if $\tilde{\alpha} \in D_{\gamma^1}$, we can simply replace γ^1 by some $\gamma \in \mathfrak{g}(\gamma^1)$ with the corresponding simple root $\tilde{\alpha} \in \Pi^0(\gamma)$ satisfying the parity condition. So we may suppose $\tilde{\alpha} \notin D_{\gamma^1}$. This means that the corresponding simple factor R_i is of type B, C , or F_4 , and thus that $\mathfrak{g}$ is of type B, C , or F_4 . We treat these cases separately.

Case I. Type C_n . We know that D_{γ^1} contains a long simple root; say $\beta \in \Pi^0(\gamma)$ satisfies the parity condition, with $\gamma \in \mathfrak{g}(\gamma^1)$, β long, and

$2\langle\beta, \gamma\rangle/\langle\beta, \beta\rangle = m$. Since all the long roots are orthogonal to each other, one can check that $\gamma - m\beta$ is conjugate to γ , and therefore (by the proof of Lemma 7.7) the composition series of $\pi(\gamma)$ is computable.

Case II. Type B_n . Since $\tilde{\alpha} \notin D_{\gamma^1}$, α must be short. Since H^2 is not split, we can find a root β , simple among the integral roots in $\Delta_{\gamma^2 - n\alpha}^+$, such that $\theta\beta \in \Delta_{\gamma^2 - n\alpha}^+$; and if β is imaginary, we may assume it is noncompact. Obviously $\beta \neq -\alpha$. Let α^1, β^1 be the roots in $\Pi^0(\gamma^1)$ corresponding to $-\alpha$ and β in $\Delta_{\gamma^2 - n\alpha}^+$. We claim that $\beta^1 \in D_{\gamma^1}$. Suppose not; then β^1 is short. Hence it cannot lie in the simple factor R_i containing α^1 (since this is of type B), so the long root $\alpha^1 + \beta^1$ is not integral. This means that $2\langle\alpha^1, \gamma^1\rangle/\langle\alpha^1, \alpha^1\rangle$ and $2\langle\beta^1, \gamma^1\rangle/\langle\beta^1, \beta^1\rangle$ have opposite parity. By Lemma 6.15 of [14], $m_{\alpha^1} = m_{\beta^1}$, so since α^1 does not satisfy the parity condition, β^1 does. So $\beta^1 \in D_{\gamma^1}$. Replace γ^1 by some γ with the corresponding root $\beta^2 \in \Pi^0(\gamma)$ satisfying the parity condition. Then $\beta \notin \tau(\pi(\gamma^2 - n\alpha))$, but $\beta \in \tau(\pi(\gamma))$; so by Theorem 4.14 of [15], the multiplicity of $\pi(\gamma^2 - n\alpha)$ in $U_\alpha(\pi(\gamma))$ is known. This allows the algorithm to proceed.

Case III. Type F_4 . We begin as in Case II by choosing β and defining α^1 and β^1 as in that case. If $\beta^1 \in D_{\gamma^1}$, we can repeat the argument given there, so suppose we cannot arrange that. Then β^1 is short; both α^1 and β^1 are short roots in simple factors of R of type B, C , or F_4 . The existence of two such roots and the fact that R has rank at most four means that R must be $C_3, B_2 + B_2, C_3 + A_1, C_4$, or F_4 . Realize F_4 in $\mathbb{R}^4$ as the elements $\pm e_i \pm e_j, \pm e_1 \pm e_2 \pm e_3 \pm e_4$ in the standard basis. Two orthogonal long roots in F_4 are clearly conjugate under W to $\{2e_1, 2e_2\}$. If $\gamma \in (\mathbb{R}^4)^* \cong \mathbb{R}^4$ is integral for both of them, then γ_1 and γ_2 (the first two coordinates of γ) are integers, so γ is integral for $e_1 + e_2$. So two long roots in R must belong to the same simple factor. So R cannot be of type $B_2 + B_2$, and if it is of type $C_3 + A_1$, then the A_1 is short. This last possibility would give four orthogonal roots, exactly one of which is short, and this is impossible in F_4 . So R is C_3, C_4 , or F_4 .

Suppose first R is of type C_4 ; after conjugating by W , we may assume R consists of the roots $\pm e_i \pm e_j$. Thus the coordinates of γ^1 are integers. Since $e_1 + e_2 + e_3 + e_4 \notin R$, their sum is odd. Shift γ^1 by a weight of a finite-dimensional representation so that all m_α act trivially (as we may, since G is linear). Choose i and j so that $\gamma_i^1 + \gamma_j^1$ is odd; this is possible since the sum of the γ_i^1 is odd. Then $e_i + e_j$ satisfies the parity condition, so $D_{\gamma^2} = \Pi$, and we are done.

Suppose next that R is of type C_3 or F_4 . Let $P^2 = M^2 A^2 N^2$ be a cuspidal parabolic subgroup corresponding to H^2 . Let $\delta^2 \in \hat{M}^2, \nu^2 \in \hat{A}^2$ be the discrete series and character associated to γ^2 ; thus

$$\pi(\gamma^2) = \text{Ind}_{P^2}^{G^2} \delta^2 \otimes \nu^2 \otimes 1,$$

at least after changing N^2 . Choose $\gamma^3 \in (\hat{H}^1)^\vee$ so that $\delta^2 \otimes \nu^2$ occurs in

$$\text{Ind}_{P \cap M^2 A^2}^{M^2 A^2} \gamma^3 \otimes 1;$$

here $P = H^1N$ is a Borel subgroup. (This is possible by the subquotient theorem for M^2 .) Then $\bar{\pi}(\gamma^2)$ occurs in $\pi(\gamma^3)$, so by the “converse” part of Lemma 7.4, if $\bar{\pi}(\gamma^2)$ occurs in $\pi(\gamma^1)$, then $\bar{\pi}(\gamma^3)$ occurs in $\pi(\gamma)$ for some $\gamma \in \mathcal{G}(\gamma^1)$. By the Langlands classification theorem, γ is conjugate to γ^3 . We claim that the simple roots of $\mathfrak{h}^1$ in $\mathfrak{m}^2$ all belong to D_{γ^3} . Since γ^3 is essentially defined in terms of M^2 , this is really a question only about M^2 . So it is enough to show that if G is an arbitrary split group and $\pi(\gamma)$ contains a discrete series representation, then D_γ consists of all the simple roots. By the argument just given, it is enough to prove this for one particular γ containing a discrete series. Such γ are known explicitly by Schmid’s character identities, so the result is easily checked. (It is also contained in Lemma 7.9.) So D_{γ^3} contains all the simple roots of M^2 . But γ^3 is conjugate to γ , and $D_{\gamma^1} = D_\gamma$ contains only long roots by hypothesis. So the simple roots of M^2 are a subset of the long simple roots of F_4 , so M_2 is of type A_1 or A_2 . Since $SL(3, \mathbb{R})$ is not equal rank, in fact M^2 must correspond to a single long root. The Cartan involution θ on $\mathfrak{h}^2$ is given by

$$\theta x = -s_\delta(x),$$

where δ is the imaginary root. We may assume δ is positive for $\Delta_{\gamma^2 - n\alpha}^+$.

We now think of our previous realization of F_4 as the root system of $\mathfrak{h}^2$. We may take

$$\Delta_{\gamma^2 - n\alpha}^+ = \{e_i \pm e_j \mid i \leq j\} \cup \{e_1 \pm e_2 \pm e_3 \pm e_4\},$$

which has simple roots

$$r_1 = e_1 - e_2 - e_3 - e_4$$

$$r_2 = 2e_4$$

$$r_3 = e_3 - e_4$$

$$r_4 = e_2 - e_3.$$

The integral roots may be taken to be spanned by r_2, r_3 , and r_4 in case R is of type C_3 . Our old simple roots $-\alpha$ and β are r_3 and r_4 (not necessarily in that order). We want to examine all the possibilities for the imaginary root δ . It is long and has nonzero inner product with r_3 and r_4 . The possibilities are

$$2e_3$$

$$e_1 - e_2 + e_3 - e_4$$

$$e_1 + e_2 - e_3 + e_4.$$

In the first case, $s_{r_4}(\delta) = 2e_4$ is simple. It follows that a principal series representation can be gotten from $\pi(\gamma^2)$ by crossing two integral walls, so $l'(\gamma^2 - n\alpha) - l'(\gamma^1)$ is even. By Theorem 7.2(e), $\bar{\pi}(\gamma^2 - n\alpha)$ occurs in $U_\alpha(\bar{\pi}(\gamma^1))$ with multiplicity zero, and the algorithm for computing the multiplicity of $\bar{\pi}(\gamma^2)$ in $\pi(\gamma^1)$ can proceed.

The remaining cases cannot arise if R is of type C_3 , since δ must be integral; so we may suppose R is of type F_4 , i.e., that γ^1 is integral. In case $\delta =$

$e_1 - e_2 + e_3 - e_4$, we have

$$\begin{aligned}\theta r_1 &= -s_\delta(r_1) \\ &= -(r_1 - \delta) \\ &= 2e_3,\end{aligned}$$

which is positive. So we could have chosen $\beta = r_1$, which is long, so then $\beta^1 \in D_{\gamma^1}$. Similarly, if $\delta = e_1 + e_2 - e_3 + e_4$, then

$$\theta r_2 = e_1 + e_2 - e_3 - e_4,$$

so in this case we could have chosen $\beta = r_2$. This completes Case III.

So we may assume that the composition series of $\pi(\gamma^1)$ is known. By inverting the argument which has just been giving us so much trouble, this immediately allows us to compute $U_\alpha(\pi(\gamma^1))$ whenever α is a simple real root of $\mathfrak{h}^1$ not satisfying the parity condition.

We turn now to the algorithm for computing the multiplicity of $\pi_i(\gamma_a^2)$ in $H^i(u, \pi(\gamma^1))$; recall that we are assuming $\mathfrak{h}^2 \subseteq \mathfrak{l}^1$, and $\gamma^2 \in \mathcal{F}$ by Lemma 7.6. We proceed by *downward* induction on $l(\gamma^2)$, then by downward induction on $\dim u$, and finally by downward induction on i . By the argument used at the beginning of the proof of Theorem 6.13, we are reduced to the case when $\mathfrak{l}$ is split, with maximally split Cartan subalgebra $\mathfrak{h}^2$. (The spectral sequence appearing there collapses by the parity result in Theorem 7.2(a).)

Suppose first that there is a complex root α of $\mathfrak{h}^2$ in $\mathfrak{l}^1$, simple with respect to $\Delta_{\gamma^2}^+$, such that $\theta\alpha \in \Delta_{\gamma^2}^+$. If α is non-integral, a routine argument applies. So suppose $2\langle\alpha, \gamma^2\rangle/\langle\alpha, \alpha\rangle = n$ is an integer. Since the roots of $\mathfrak{h}^2$ in $\mathfrak{l}$ are real, $\pm\alpha$ is in $\Delta(u, \mathfrak{h}^2)$. Suppose for definiteness that $\alpha \in \Delta(u)$; the other case is similar. Let α^1 be the root of $\mathfrak{h}^1$ in $\mathfrak{l}^1$ corresponding to α . Suppose first that α^1 satisfies the parity condition, so that $\psi_\alpha(\pi(\gamma^1)) = 0$. By the first exact sequence of 4.2(a), the multiplicity of $\pi_i(\gamma_a^2)$ in $H^i(u, \pi(\gamma^1))$ is equal to that of $\pi_i(\gamma_a^2 - n\alpha)$ in $H^{i+1}(u, \pi(\gamma^1))$. Since $l(\gamma^2 - n\alpha) = l(\gamma^2) + 1$, this is known by induction.

Next, suppose α^1 does not satisfy the parity condition, so that $U_\alpha(\pi(\gamma^1))$ is defined. As remarked earlier, we know the constituents of $U_\alpha(\pi(\gamma^1))$; all occur in $\pi(\gamma^1)$ but are not equal to $\pi(\gamma^1)$, so their cohomology is known by inductive hypothesis. On the other hand, the multiplicity of $\pi_i(\gamma_a^2 - n\alpha)$ in $H^i(u, U_\alpha(\pi(\gamma^1)))$ must also be equal to its multiplicity in $H^{i-1}(u, \pi(\gamma^1))$ (which we know by induction), plus the multiplicity of $\pi_i(\gamma_a^2)$ in $H^i(u, \pi(\gamma^1))$. So we can solve for the multiplicity we want.

So we may assume that no such root α exists. This means that the positive imaginary roots of Δ_I^+ of $\mathfrak{h}^2$ in $\mathfrak{l}^1$ are spanned by simple imaginary roots of $\Delta_{\gamma^2}^+$. Define

$$N = |\{\alpha \in \Delta(u, \mathfrak{h}^2) \mid \alpha \text{ is not a root of } \mathfrak{h}^2 \text{ in } \mathfrak{l}^1, \theta\alpha = \alpha, \text{ and either}$$

$$\alpha \text{ is compact and } \alpha \notin \Delta_{\gamma^2}^+, \text{ or } \alpha \text{ is noncompact and } \alpha \in \Delta_{\gamma^2}^+\} |$$

$$+ \tfrac{1}{2} |\{\alpha \in \Delta(u, \mathfrak{h}^2) \mid \theta\alpha \neq \alpha\} | M = |\Delta_I^+|.$$

By Lemma 7.5, we can find some γ in $\mathfrak{g}(\gamma^1)$ so that $\bar{\pi}(\gamma^2)$ occurs in $\pi(\gamma)$. Let $R(\gamma, \gamma^2)$ be the roots of $\mathfrak{h}^1$ in $\mathfrak{l}^1$ corresponding to Δ_I^+ under the inner automorphism taking $\Delta_{\gamma^2}^+$ to Δ_{γ}^+ , and let

$$W(\gamma, \gamma^2) = W(R(\gamma, \gamma^2))$$

$$\mathfrak{g}(\gamma, \gamma^2) = \{\gamma^3 \in (\hat{H}^1)' \mid \gamma - \gamma^3 \text{ is a sum of roots, and}$$

$$\bar{\gamma} \text{ is conjugate to } \bar{\gamma}^3 \text{ under } W(\gamma, \gamma^2)\}.$$

LEMMA 7.9. *With assumptions as above, suppose $\gamma^4 \in (\hat{H}^2)'$, $\gamma^4 - \gamma^2$ is a sum of roots, and $\Delta_{\gamma^4}^+$ differs from $\Delta_{\gamma^2}^+$ only in non-integral and imaginary roots of $\mathfrak{h}^2$ in $\mathfrak{l}^1$. If $\bar{\pi}_i(\gamma_a^4)$ occurs in $H^i(u, \bar{\pi}(\gamma^3))$ for some $\gamma^3 \in \mathfrak{g}(\gamma, \gamma^2)$, then*

- (a) $\gamma^4 - \gamma^2$ is a sum of imaginary roots, and $\bar{\gamma}^4$ is conjugate to $\bar{\gamma}^2$ under $W(\Delta_I)$.
- (b) $N \leq i \leq N + M$, at least if Conjecture 2.5 holds. Finally, we may choose $\gamma^3 \in \mathfrak{g}(\gamma, \gamma^2)$ so that every simple root of $R^+(\gamma^3, \gamma^2)$ satisfies the parity condition.

This will be proved in a moment. Assuming it, we complete the algorithm for computing the multiplicity of $\bar{\pi}_i(\gamma_a^2)$ in $H^i(u, \bar{\pi}(\gamma^1))$. By Lemma 7.7, it suffices to compute the multiplicity of $\bar{\pi}_i(\gamma_a^4)$ in $H^i(u, \bar{\pi}(\gamma^3))$ when γ^3 is chosen as in the last assertion of Lemma 7.9, $\gamma^4 - \gamma^2$ is a sum of imaginary roots, and $\bar{\gamma}^4$ is conjugate to $\bar{\gamma}^2$ under $W(\Delta_I)$. Suppose $\alpha \in \Delta_{\gamma^4}^+$ is simple and imaginary; let α^1 be the corresponding root of $\Delta_{\gamma^2}^+$. Since α^1 satisfies the parity condition, $\psi_\alpha(\bar{\pi}(\gamma^3)) = 0$. Put $n = 2\langle \alpha, \gamma^4 \rangle / \langle \alpha, \alpha \rangle$; then by the argument given a moment ago, the multiplicity of $\bar{\pi}_i(\gamma_a^4)$ in $H^i(u, \bar{\pi}(\gamma^3))$ is that of $\bar{\pi}_i(\gamma_a^4 - n\alpha)$ in $H^{i \pm 1}(u, \bar{\pi}(\gamma^1))$; the $+1$ occurs if $\alpha \notin \Delta(u)$, and the -1 if $\alpha \in \Delta(u)$. Now apply 7.9(b); by a simple argument, $\bar{\pi}_i(\gamma_a^4)$ can occur in $H^{N+j}(u, \bar{\pi}(\gamma^3))$ only if

$$j = |-\Delta_I^+(\gamma^4) \cap \Delta(u)|.$$

So it is enough to know the multiplicity of $\bar{\pi}_i(\gamma_a^4)$ in the Euler characteristic of $H^*(u, \bar{\pi}(\gamma^3))$. But this is computable from the results of Section 8, since we know the character of $\bar{\pi}(\gamma^3)$. This completes the algorithm, proving Theorem 7.3. Q.E.D.

Proof of Lemma 7.9. We will be a little sketchy from time to time, since the proof given here seems to be unnecessarily complicated. For (a), we use the fact that γ^4 and γ^2 must define the same infinitesimal character; so $\bar{\gamma}^4$ and $\bar{\gamma}^2$ are conjugate under $W(\mathfrak{g}, \mathfrak{h}^2)$. Since they differ by a sum of roots, they must therefore be conjugate under the Weyl group of the integral roots. Since the integral positive roots they define differ only in imaginary roots, and these are spanned by simple roots, (a) follows.

For the last two statements, we try to reduce to the case when H^2 is compact. First, modify γ by a translation so that $\langle \alpha, \gamma \rangle$ is fairly small when $\alpha \in R(\gamma, \gamma^2)$, and $\langle \beta, \gamma \rangle$ has very large real part whenever $\beta \in \Delta_{\gamma^2}^+$ does not belong to $R(\gamma, \gamma^2)$. For notational convenience, we will assume $G = L^1$; a formally

identical argument applies to the general case. Let $P^1 = H^1 N^1$ be the Borel subgroup defined by Δ_γ^+ , and

$$P^2 = M^2 A^2 N^2$$

the cuspidal parabolic corresponding to $R(\gamma, \gamma^2)$; thus if $H^2 = T^2 A^2$, T^2 is a compact Cartan subgroup of M^2 .

LEMMA 7.10. *With notation and hypotheses as above, suppose $\delta \in (\hat{H}^3)'$, and $\bar{\pi}(\delta)$ occurs in $\pi(\gamma^3)$, some $\gamma^3 \in \mathfrak{g}(\gamma, \gamma^2)$. Then either*

(a) *the lowest K -type of $\bar{\pi}(\delta)$ is large, so that $H^*(u, \bar{\pi}(\delta))$ cannot contain $\bar{\pi}_i(\gamma_a^4)$ when γ^4 is as in 7.9(a); or*

(b) *after conjugation, $H^3 \subseteq M^2 A^2$, and $\bar{\pi}_{M^2 A^2}(\delta)$ occurs in $\rho = \text{Ind}_{P^1 \cap M^2 A^2}^{M^2 A^2}(\gamma^3 \otimes 1)$ exactly as often as $\bar{\pi}(\delta)$ occurs in $\pi(\gamma^3)$.*

We only sketch the proof of this fact. Suppose $\bar{\pi}_{M^2 A^2}(\delta)$ is a constituent of ρ . Then there is an intertwining operator

$$I : \text{Ind}_{P^2}^{G^2} \bar{\pi}_{M^2 A^2}(\delta) \otimes 1 \rightarrow \text{Ind}_{P^2}^{G^2} \bar{\pi}_{M^2 A^2}(\delta) \otimes 1;$$

here $\overline{P^2}$ is the parabolic opposite to P^2 , and I is just an integral operator. Its image is irreducible and equal to $\bar{\pi}(\delta)$ by the Langlands theory. Its kernel can be estimated by replacing $\bar{\pi}_{M^2 A^2}(\delta)$ by the full induced representation ρ . In that case I factors into a product involving the various roots of H^1 not in $R(\gamma^1, \gamma^2)$. The kernel of each factor is induced from a discrete series representation of an $SL(2, \mathbb{R})$ Levi factor, and this will have a large lowest K -type by our hypotheses on γ^3 . (For details see [13], Section 3; such arguments were first used by B. Speth in her thesis.) In particular, the kernel of I consists of representations with large lowest K -type. So induction from P^2 to G in our setting takes irreducibles to irreducibles plus terms we don't care about, and the lemma follows. Q.E.D.

Define

$$\mathfrak{S}(\gamma, \gamma^2) = \{ \delta \in (\hat{H}^3)' \mid H^3 \subseteq M^2 A^2, \text{ and } \bar{\pi}_{M^2 A^2}(\delta) \text{ occurs in}$$

$$\text{Ind}_{P^1 \cap M^2 A^2}^{M^2 A^2} \gamma^3 \otimes 1 \text{ for some } \gamma^3 \in \mathfrak{g}(\gamma, \gamma^2) \}.$$

LEMMA 7.11. *Suppose Conjecture 2.5 holds. Let $\mathfrak{q} = \mathfrak{h}^2 + \tilde{\mathfrak{u}}$ be the intersection of $\mathfrak{q}$ with $\mathfrak{m}^2 + \alpha^2$. Suppose $\delta \in (\hat{H}^3)'$ belongs to $\mathfrak{S}(\gamma, \gamma^2)$. Let $\gamma^4 \in (\hat{H}^2)'$ be as in Lemma 7.9(a). If H^3 is not split, then the multiplicity of $\bar{\pi}_i(\gamma_a^4)$ in $H^{N+j}(u, \bar{\pi}(\delta))$ is equal to the multiplicity of γ_a^4 (as a weight of H^2) in $H^j(\tilde{\mathfrak{u}}, \bar{\pi}_{M^2 A^2}(\delta))$.*

Proof. We assume that the reader can extend the theory of this paper to the disconnected group $M^2 A^2$. (Helpful hints can be found in [13]; Conjecture 2.5 can be reduced to the connected case.) We proceed by induction on $l(\delta)$. If $H^3 = H^2$, the result can be read off from our computation of special cohomology groups in section 6. Suppose α is a simple root of $\mathfrak{h}^3$ in $\mathfrak{m}^2 + \alpha^2$. By

Theorem 4.12 of [15],

$$\psi_\alpha(\bar{\pi}(\delta)) \neq 0 \Leftrightarrow \psi_\alpha^{M^2 A^2}(\bar{\pi}_{M^2 A^2}(\delta)) \neq 0.$$

In this case, the argument given for Lemma 7.10 and the relation of U_α to coherent continuation show that if $\delta' \in \mathfrak{S}(\gamma, \gamma^2)$, then $\bar{\pi}(\delta')$ occurs in $U_\alpha(\bar{\pi}(\delta))$ exactly as often as $\bar{\pi}_{M^2 A^2}(\delta')$ occurs in $U_\alpha^{M^2 A^2}(\bar{\pi}_{M^2 A^2}(\delta))$. An inspection of the arguments of this section now shows that the two multiplicities in question can be computed by precisely the same algorithm; so they coincide. Q.E.D.

The only reason for the restriction that H^3 should not be split in the preceding lemma is that we have not yet completed the algorithm for computing $H^*(u, \bar{\pi}(\gamma^3))$ when $\gamma^3 \in \mathfrak{g}(\gamma, \gamma^2)$. We have seen how to compute things of the form $H^*(u, \bar{\pi}(\gamma^3) \oplus \bar{\pi}(\gamma^5))$ when $\gamma^5 = \gamma^3 - n\alpha$ ($\alpha \in R(\gamma^3, \gamma^2)$ simple, $2\langle \alpha, \gamma^3 \rangle / \langle \alpha, \alpha \rangle = n$). (Such a root α always exists, by the argument before Definition 5.1 in [15].) So we have

$$m(\bar{\pi}_1(\gamma_q^4), H^{N+j}(u, \bar{\pi}(\gamma^3) \oplus \bar{\pi}(\gamma^5))) = m(\gamma_q^4, H^j(\tilde{u}, \bar{\pi}_{M^2 A^2}(\gamma^3) \oplus \bar{\pi}_{M^2 A^2}(\gamma^5))).$$

Now $\dim \tilde{u} = M$, so the right side is zero unless $0 \leq j \leq M$. This proves 7.9(b).

For the last assertion, Lemma 7.10 again reduces us to the case $G = M^2$. Let $\gamma^3 \in (\hat{H}^1)'$ correspond to the finite-dimensional representation of extremal weight γ^2 . Trivially every simple root satisfies the parity condition for γ^3 . On the other hand, γ_q^2 occurs in $H^*(u, \bar{\pi}(\gamma^3))$, so by Lemma 7.5, $\gamma^3 \in \mathfrak{g}(\gamma) = \mathfrak{g}(\gamma, \gamma^2)$. Q.E.D.

8. Euler-Poincaré character formulas. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic subalgebra of $\mathfrak{g}$, and X a $(\mathfrak{g}, \mathfrak{k})$ module of finite length. Let $\Theta = \Theta(X)$ be the global character of X , which we can regard as a locally L^1 function on G . For $x \in L$, define

$$\delta_u(x) = \sum_i (-1)^i \text{tr}(\text{ad } x|_{\wedge^i \mathfrak{u}}).$$

THEOREM 8.1. *With hypotheses as above,*

$$\Theta(X)|_L = \left[\sum_i (-1)^i \Theta(H^i(\mathfrak{u}, X)) \right] / \delta_u.$$

Here $\Theta(H^i(\mathfrak{u}, X))$ is the global character of the $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$ module $H^i(\mathfrak{u}, X)$ (which exists by Corollary 4.3).

The proof will be substantially copied from the proof of the Osborne conjecture (an analogous formula where the parabolic is real) given by Hecht and Schmid. We will assume some familiarity with the kind of arguments in [5] or [6]. The main technical lemma peculiar to this case is

LEMMA 8.2. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, and $\Lambda \subseteq \mathfrak{h}$ the lattice of weights of finite-dimensional representations of $\mathfrak{g}$. Let X_1 and X_2 be two $(\mathfrak{g}, \mathfrak{k})$ modules of*

finite length, of regular infinitesimal character $\lambda_0 \in \mathfrak{h}^*$. Let $(\mathfrak{h}^*)^+$ be the positive Weyl chamber defined by $\Delta_{\lambda_0}^+$, and

$$\{X_i(\lambda) \mid \lambda \in (\Lambda + \lambda_0) \cap (\mathfrak{h}^*)^+\}$$

the coherent families of $(\mathfrak{g}, \mathfrak{k})$ modules attached to X_1 and X_2 (cf. [13], Section 5). Suppose that for every λ , $X_1(\lambda)$ and $X_2(\lambda)$ have the same $\mathfrak{k}$ multiplicities. Then X_1 and X_2 have the same composition factors and multiplicities.

Proof. We proceed by induction on the length of X_1 and argue as in the last part of the proof of Theorem 6.10. Clearly, it is enough to show that X_1 and X_2 have a common composition factor. Let $\{\bar{\pi}(\gamma^i) \mid 1 \leq i \leq r, \gamma^i \in (\hat{H}^i)'\}$ be the composition factors of X_1 with multiplicities. Let $\mathfrak{l}^i$ be the centralizer of $\mathfrak{t}^i$ in $\mathfrak{g}$. For $w \in W(\mathfrak{g}/\mathfrak{h}^i)$, but w not in $W(\mathfrak{l}^i/\mathfrak{h}^i)$, the set of solutions of

$$w\gamma - \gamma \in \alpha^i$$

in $(\mathfrak{h}^i)^*$ is a proper subspace. So by a coherent translation, we may assume that in this setting,

$$w\gamma^i - \gamma^i|_{\mathfrak{l}^i}$$

is very large. This hypothesis is stable under small coherent translations. Similarly, we may assume that the only roots annihilated by $\gamma^i|_{\mathfrak{l}^i}$ are real, so the parabolic associated to γ^i is of the form $\mathfrak{q}^i = \mathfrak{l}^i + \mathfrak{u}^i$, with $\mathfrak{l}^i$ as above. We may also assume (by coherent translation) that

$$|\gamma^i|_{\mathfrak{l}^i} - |\gamma^j|_{\mathfrak{l}^j}$$

is large unless $(\bar{\gamma}^i, \mathfrak{h}^i)$ and $(\bar{\gamma}^j, \mathfrak{h}^j)$ are conjugate by an automorphism from $G_{\mathbb{C}}$ preserving θ .

Order the γ^i so that $|\gamma^1|_{\mathfrak{l}^1}$ is minimal; this will be preserved by small translations by our hypotheses. Make such a translation so that the stabilizer of $\Gamma^1|_{T^1}$ in $W(\mathfrak{l}^1/\mathfrak{h}^1)$ is all of $W(\mathfrak{l}^1/\mathfrak{h}^1)$ (cf. [15], proof of 6.1). Let μ be a lowest $\mathfrak{k}$ -type of X_1 . By our hypotheses and the results of [14] (for example, Lemma 8.8), it is easy to check that μ must be $\mathfrak{u}^1$ -minimal in X^1 ([14], Definition 3.11). So it is $\mathfrak{u}^1$ -minimal in X^2 as well. By [14], Theorem 5.2, X^2 has a constituent $\bar{\pi}(\delta^1)$, $\delta^1 \in (\hat{H}^1)'$, such that δ^1 differs from γ^1 only on A^1 . Since X^1 and X^2 have the same infinitesimal character, $\bar{\delta}^1$ and γ^1 are conjugate by some $w \in W(\mathfrak{g}/\mathfrak{h}^1)$. Thus

$$w\gamma^1 - \gamma^1 \in \mathfrak{a}^1.$$

By hypothesis this forces $w \in W(\mathfrak{l}^1/\mathfrak{h}^1)$, and since $W(\mathfrak{l}^1/\mathfrak{h}^1)$ stabilizes $\gamma^1|_{T^1}$, it follows that $\bar{\pi}(\delta^1) = \bar{\pi}(\gamma^1)$. As remarked at the beginning of the proof, we are done by induction. Q.E.D.

COROLLARY 8.3. *Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ -stable parabolic, and $\gamma \in \hat{H}'$ a regular pseudocharacter with regular infinitesimal character. Then*

$$\sum_i (-1)^i \Theta(H^i(\mathfrak{u}, \pi(\gamma))) = \sum_{\substack{\gamma^2 \in \hat{H}' \\ \gamma^2 \text{ conjugate to } \gamma}} (-1)^{l_a(\gamma^2)} \Theta(\gamma_q^2)$$

if $H \subseteq L$; and the left side is zero if H is not conjugate under G to a Cartan in L .

Proof. Assume first that $H \subseteq L$. Since both sides behave well under coherent continuation across walls (which is obvious from the definition of such continuation and Theorem 4.1), we may assume that the set of nonreal roots in Δ_γ^+ is invariant under θ . Let $\mathfrak{t}^0$ be the compact part of the center of $\mathfrak{l}$. After a slight shift of γ , we may assume that if w_1 and w_2 are in $W(\mathfrak{g}/\mathfrak{h})$ and

$$(w_1 \gamma)|_{\mathfrak{t}^0} = (w_2 \gamma)|_{\mathfrak{t}^0}, \quad (*)$$

then $w_2^{-1}w_1$ belongs to $W(\mathfrak{l}/\mathfrak{h})$. It follows now from Theorem 4.1 that two representations of $\mathfrak{l}$ occurring on the left in the expression in the corollary either have the same infinitesimal character, or have different restrictions to $\mathfrak{t}^0$. Suppose then that we can prove the desired equality after restriction to $\mathfrak{l} \cap \mathfrak{k}$. By the preceding remark, it would persist after projection on a fixed infinitesimal character on both sides. Since both sides behave well under coherent continuation, we would have (for each infinitesimal character of $\mathfrak{l}$) two coherent families of virtual $(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})$ modules with the same $\mathfrak{l} \cap \mathfrak{k}$ -multiplicities at “most” points. By the proof of Lemma 8.2, they would have to coincide.

So we are reduced to proving that the $\mathfrak{l} \cap \mathfrak{k}$ multiplicities on the two sides are equal. They depend only on $\gamma|_{\mathcal{T}}$; this is clear on the right, and on the left it follows from the spectral sequence appearing in the proof of Proposition 5.5. So we may assume that no real roots satisfy the parity condition. In light of our previous assumptions on γ , this makes $\pi(\gamma)$ a standard irreducible. The result is then immediate from Theorem 6.13.

So finally suppose that H is not conjugate under G to a Cartan in L . Let $\mathfrak{h}^1 \subseteq \mathfrak{l}$ be some Cartan, and $\gamma^1 \in (\mathfrak{h}^1)^*$ a $G_{\mathbb{C}}$ conjugate of γ . We shift γ so that γ^1 satisfies the condition (*) imposed on γ in the previous case and argue in precisely the same way. Q.E.D.

Proof of Theorem 8.1. It is enough to verify the formula on each Cartan subgroup H of L . Both sides are exact functors of X , so it is enough to check the result on generators of the Grothendieck group. Both sides behave well under coherent continuation, so (using the results of [13], Section 5) we may assume X has regular infinitesimal character. By downward induction on $\dim \mathfrak{u}$ and induction on $\dim \mathfrak{g}$, the argument from the beginning of the proof of Theorem 6.13 reduces us to the case when L is split and H is a maximally split Cartan subalgebra in L . We take for generators of the Grothendieck group the representations $\pi(\gamma)$. So assume $X = \pi(\gamma^1)$ with $\gamma^1 \in (\hat{H}^1)'$. By Corollary 8.3,

both sides of the formula are zero unless H^1 is conjugate to a Cartan in L ; so we assume $H^1 \subseteq L$. Fix x_0 in the regular set in H ; then both sides of the formula in question are of the form

$$f(x_0 \exp(Y)) = \left[\sum_{w \in W(\mathfrak{g}/\mathfrak{h})} a_w \exp\langle w\lambda, Y \rangle \right] / d$$

for Y small; here d is a denominator, and $\lambda \in \mathfrak{h}^*$ is some $G_{\mathbb{C}}$ conjugate of γ . By a coherent continuation across walls argument, it is enough to verify the equality of the coefficients a_w , when $w\lambda$ lies in a set of Weyl chambers depending on x_0 in a natural way. We may as well assume that $|\alpha(x_0)| \neq 1$ when α is not imaginary; let

$$S^+ = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid |\alpha(x_0)| > 1\}.$$

We confine our attention to those $w\lambda$ such that for all $\alpha \in S^+$

$$\operatorname{Re}\langle \alpha, w\lambda \rangle > 0, \quad \text{or}$$

$$\operatorname{Re}\langle \alpha, w\lambda \rangle = 0 \quad \text{and} \quad \operatorname{Im}\langle \alpha, w\lambda \rangle > 0.$$

These are the terms which grow most rapidly at ∞ in the component of the regular set containing x_0 . It follows from the temperedness of discrete series characters and Corollary 8.3 that there are no such terms on either side in 8.1 unless H^1 is conjugate to H . In that case the formula is easily verified by computing both sides (using Corollary 8.3). Q.E.D.

Theorem 8.1 tells us nothing about characters on split Cartan subgroups, even when all $H^1(\mathfrak{u}, X)$ are known. These can be computed as follows: The terms with largest growth appear only if X has constituents of the form $\bar{\pi}(\gamma)$, for $\gamma \in (\hat{H})'$ and H split. Then they are the same as if X were the sum of the corresponding principal series representations. The terms of lower growth can be computed using Harish-Chandra's matching conditions and the formulae on more compact Cartan subgroups.

REFERENCES

1. A. BOREL AND N. R. WALLACH, *Seminar on the cohomology of discrete subgroups of semisimple groups*, to appear.
2. W. CASSELMAN AND M. S. OSBORNE, *The $\mathfrak{n}$ -cohomology of representations with an infinitesimal character*, *Comp. Math.* **31** (1975), 219–227.
3. J. DIXMIER, *Algèbres Enveloppantes*, Cahiers Scientifiques XXXVII, Gauthier-Villars, Paris, 1974.
4. T. ENRIGHT, *The representations of complex semisimple Lie groups*, preprint, 1978.
5. H. HECHT, *On characters and asymptotics of representations of a real reductive Lie group*, preprint, 1977.
6. H. HECHT AND W. SCHMID, *A proof of Osborne's conjecture*, to appear.
7. J. C. JANTZEN, *Moduln mit einem höchsten Gewicht*, Habilitationsschrift, Universität Bonn, 1977.
8. A. JOSEPH, *Dixmier's problem for Verma and principal series submodules*, preprint, 1978.
9. D. KAZHDAN AND G. LUSZTIG, *Representations of Coxeter groups and Hecke algebras*, preprint, 1979.

10. B. KOSTANT, *On the tensor product of a finite and an infinite dimensional representation*, J. Func. Anal. **20** (1975), 257–285.
11. R. P. LANGLANDS, *On the classification of irreducible representations of real algebraic groups*, mimeographed notes, Institute for Advanced Study, 1973.
12. W. SCHMID, *On the characters of the discrete series (the Hermitian symmetric case)*, Inv. Math. **30** (1975), 47–144.
13. B. SPEH AND D. VOGAN, *Reducibility of generalized principal series representations*, preprint, 1978.
14. D. VOGAN, *The algebraic structure of the representations of semisimple Lie groups I*, Ann. of Math. **109** (1979), 1–60.
15. ———, *Irreducible characters of semisimple Lie groups I*, Duke Math. J. **46** (1979), 61–108.
16. G. WARNER, *Harmonic Analysis on Semisimple Lie Groups*, Springer-Verlag, New York, 1972.
17. G. ZUCKERMAN, *Tensor products of finite and infinite dimensional representations of semisimple Lie groups*, Ann. of Math. **106** (1977), 295–308.

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