Invent. math. 83, 449–505 (1986)

*Inventiones mathematicae*


# The unitary dual of GL($n$) over an archimedean field

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## 1. Introduction

Let $\mathbb{F}$ denote one of the three fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Write

$$G = \mathrm{GL}(n, \mathbb{F}) \tag{1.1}$$

for the group of invertible $n$ by $n$ matrices with entries in $\mathbb{F}$. We determine explicitly the set $\hat{G}_u$ of equivalence classes of irreducible unitary representations of $G$. For $\mathbb{F} = \mathbb{C}$, the answer has been expected for thirty years: each element of $\hat{G}_u$ is unitarily induced from (one dimensional) unitary characters and certain very simple complementary series. (A precise statement is in Theorem 6.18. Gelfand and Naimark in [14] overlooked some of the complementary series. This was pointed out by Stein in [31], and that paper may have created the impression that Gelfand and Naimark missed a great many representations. However, our result shows that Stein actually found everything that they missed.) For $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, it is still true that the "building blocks" for constructing all unitary representations are one dimensional unitary characters, and the analogues of Stein's complementary series. However, one must use not only ordinary induction, but also Zuckerman's derived functors [11, 36]. This does not seem to bode well for a generalization to non-archimedean $\mathbb{F}$, where no analogue of Zuckerman's functors is known. However, one can view them as implementing certain very special cases of Langlands functoriality; from this point of view, the results make some conjectural sense for any division algebra $\mathbb{F}$ over a local field. (They may however be too naive, at least in residual characteristic less than or equal to $n$. Since this work was completed, *M.* Tadic in [32] has announced a classification of the unitary representations of $\mathrm{GL}(n, \mathbb{F})$ for commutative $p$-adic $\mathbb{F}$. His results describe everything else in terms of the discrete series, and so avoid such pitfalls.)

If one wishes to use only unitary induction, the set of building blocks must be enlarged when $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{H}$. Suppose first that $\mathbb{F} = \mathbb{R}$. Then whenever $n$ is even, Speh [29] has described a family of unitary representations of $\mathrm{GL}(n, \mathbb{R})$

* Sloan Fellow. Supported in part by NSF grant MCS-8202127

parametrized by

$$(\mathbb{N} - (0)) \times \mathbb{R}.$$

(The last factor is just the central character.) For $n=2$, they are the discrete series; for $n>2$, they are not tempered. When $n$ is a multiple of 4, there are complementary series attached to the series induced from a product of two copies of one of Speh's representations on $\mathrm{GL}(n/2, \mathbb{R}) \times \mathrm{GL}(n/2, \mathbb{R})$.

Next, suppose $\mathbb{F} = \mathbb{H}$. In this case there is a family of unitary representations parametrized by $(\mathbb{N} - (0)) \times \mathbb{R}$ for every $n$. When $n$ is 1, it is the discrete series of $\mathrm{GL}(1, \mathbb{H})$. (Recall that this group is isomorphic to $SU(2) \times \mathbb{R}$.) For even $n$, there are complementary series attached to these representations on $\mathrm{GL}(n/2, \mathbb{H}) \times \mathrm{GL}(n/2, \mathbb{H})$. Using these extra building blocks, one can obtain all unitary representations by unitary induction.

For other real reductive groups, one expects the unipotent representations to complicate the picture substantially [1, 2]. A representation of $\mathrm{GL}(n, \mathbb{F})$ is special unipotent if and only if it is of the form

$$\mathrm{Ind}(P \uparrow G)(\chi).$$

Here $\chi$ is a character of $P$ which is trivial on the identity component of $P$. (Non-special unipotent representations have yet to be defined.) This paper may therefore (with a little twisted logic) be regarded as evidence that most of the difficulties in treating general reductive groups involve unipotent representations. At any rate, it is intended as such evidence.

Here is an incomplete outline of some previous results about this problem. When $n=2$, $\mathrm{GL}(n, \mathbb{F})$ is (up to center) locally isomorphic to $\mathrm{SO}(d+1,1)$, with $d$ equal to the dimension of $\mathbb{F}$ over $\mathbb{R}$. The unitary duals were determined by Bargmann [3] for $\mathbb{R}$, Gelfand-Naimark [13] for $\mathbb{C}$, and Hirai [18] for $\mathbb{H}$. For $n=3$, the case of $\mathbb{C}$ was treated by Tsuchikawa [33], and $\mathbb{R}$ by Vakhutinski [34]. The case of $\mathbb{H}$ is a little easier than $\mathbb{R}$, but I know of no published treatment of it for $n \geqq 3$. For $n=4$, $\mathbb{R}$ was treated by Speh [30]. For $n=4$ and 5, $\mathbb{C}$ was treated by Duflo [9]. Partial results are too numerous to discuss completely, but those of Enright [10] are among the most powerful. In unpublished joint work, Enright and Parthasarathy determined completely the spherical unitary representations of $\mathrm{GL}(n, \mathbb{C})$ with regular infinitesimal character. The Yale dissertation of S. Sahi determines the spherical unitary representations which are induced from a character (which, by the results of this paper, is all of them). The work of Guillemonat [15] is of a similar nature.

To understand the organization of the paper, keep in mind that we have four tasks: to produce a list of representations (construction); to prove that they are all unitary (unitarity); to prove that they are all irreducible (irreducibility); and to prove that any representation not on the list is not unitary (exhaustion). Crudely put, the main idea is to reduce matters to the case of spherical representations (those which have a vector fixed by a maximal compact subgroup). Each tak therefore has a "spherical" part, and a "reduction" part.

Even to state the result requires carrying out the construction step. The "spherical construction" (that is, construction of all spherical unitary representations) contains no surprises. We recall the least familiar aspect of it (Stein's complementary series) in Sect. 2, and complete it in Sect. 3. The reduction part of the construction (in which general unitary representations are constructed from spherical ones) is parametrized by $\hat{K}$, via the theory of lowest $K$-types. $\hat{K}$ is discussed in general terms in Sect. 4, and computed very explicitly for each of the fields in question in Sect. 5. Section 6 completes the reduction construction, and states the main theorem (Theorem 6.18).

The exhaustion argument depends on some complicated but formal properties of $\hat{K}$, which are discussed in Sect. 7 through 9. With these in hand, we can actually carry out the reduction part of exhaustion; this is done in Sect. 10.

The spherical case (exhaustion, irreducibility, and unitarity) is treated next, with generalities in Sect. 11, and a detailed discussion of each field in Sect. 12 through 14. This is certainly the heart of the paper; the reader wishing to find more than bells and whistles must look for it here. Sections 15 and 16 are devoted to the reduction arguments for unitarity and irreducibility, respectively. Section 17 discusses alternative constructions of the representations.

It is a pleasure to thank a few of the mathematicians who have helped in this work. Birgit Speh taught me the foundations of the subject as she was helping to build them. Over $\mathbb{C}$, the critical Proposition 12.2 was suggested by Thomas Enright; I have benefitted from many conversations with him. The fundamental idea of controlling the signature of a Hermitian form on very special $K$-types I learned from Enright (although it could be attributed to many people). I have come to understand it better through its appearance in the thesis of Susana Salamanca, and in conversations with her. Most importantly, the entire treatment of the complex case here is an extension of unpublished joint work with Dan Barbasch (including the proof of Proposition 12.2); this made everything else possible.

## 2. Stein's complementary series

We continue to write $\mathbb{F}$ for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. (For most of the next two sections, it would suffice to require $\mathbb{F}$ to be a finite dimensional division algebra over a local field.) Assume $n=2m$, with $m$ a positive integer. Write

$$P=LN=\left\{\begin{pmatrix} A & B \\ 0 & D\end{pmatrix}\middle| A,\, D\in \mathrm{GL}(m,\mathbb{F}),\, B\in M(m,\mathbb{F})\right\} \tag{2.1)(a}$$

for the indicated maximal parabolic subgroup of $G=\mathrm{GL}(2m,\mathbb{F})$. (Here $M(m,\mathbb{F})$ denotes the algebra of all $m$ by $m$ matrices over $\mathbb{F}$.) The indicated (Levi) decomposition of $P$ is as a semidirect product, with $N$ normal. More precisely,

$$L\cong \mathrm{GL}(m,\mathbb{F})\times \mathrm{GL}(m,\mathbb{F}), \tag{2.1)(b}$$

realized as block diagonal matrices, and

$$N\cong M(m,\mathbb{F}) \tag{2.1)(c}$$

is the unipotent radical of $P$.

The representations to be constructed are induced from certain characters of $P$. To describe these characters, first write

$$\delta_m\colon \mathrm{GL}(m,\mathbb{F})\to(\mathbb{R}^\times)_+ \tag{2.2}$$

for the modular function: $\delta_m(g)$ is the Jacobian of the change of variable $v \to g.v$ on $\mathbb{F}^m$, with respect to Lebesgue measure on $\mathbb{F}^m$. (If $\mathbb{F}$ has dimension $d$ over $\mathbb{R}$, then $\mathbb{F}^m$ may be identified with $\mathbb{R}^{dm}$. In that case, $\delta_m(g)$ is the determinant of the $dm$ by $dm$ real matrix representing $g$. If $\mathbb{F}=\mathbb{C}$, then

$$\delta_m(g)=|\det_{\mathbb{C}}(g)|^2.)$$

*Definition 2.3.* Fix a one dimensional unitary character $j$ of $\mathrm{GL}(m,\mathbb{F})$, and a complex number $t$. Let $P=LN$ be the group defined in (2.1). Consider the (possibly non-unitary) character

$$\phi_{2m}(j,t): P \to \mathbb{C}^x,$$

defined by

$$[\phi_{2m}(j,t)]((g,h)n)=[j(gh)][\delta_m(gh^{-1})]^t$$

(for $g$ and $h$ in $\mathrm{GL}(m,\mathbb{F})$, and $n$ in $N$). Put

$$\sigma_{2m}(j,t)=\mathrm{Ind}(P\uparrow G)(\phi_{2m}(j,t)).$$

The characters of $\mathrm{GL}(m,\mathbb{F})$ may be identified in a natural way with the characters of $\mathrm{GL}(1,\mathbb{F})$; so $j$ extends to a character (still denoted $j$) of $\mathrm{GL}(2m,\mathbb{F})$. Clearly

$$\sigma_{2m}(j,t)\cong\sigma_{2m}(1,\mathrm{t})\otimes j,$$

so we could consider only the case $j=1$ for the study of these representations.

**Lemma 2.4.** (cf. Stein [31]). *In the setting of Definition 2.3,*

a) $\sigma_{2m}(j,t)$ *is unitary and irreducible for $t$ in $i\mathbb{R}$.*

b) *There is a nondegenerate Hermitian pairing between $\sigma_{2m}(j,t)$ and $\sigma_{2m}(j,-\bar{t})$.*

c) $\sigma_{2m}(j,t)$ and $\sigma_{2m}(j,-t)$ *have the same distribution character.*

d) *When $t$ is $\frac{1}{2}$, $\sigma_{2m}(j,t)$ is reducible. It contains as a subquotient a representation $\sigma'$ induced from a one dimensional unitary character of the parabolic subgroup with Levi factor* $\mathrm{GL}(m+1,\mathbb{F})\times\mathrm{GL}(m-1,\mathbb{F})$.

e) $\sigma_{2m}(j,t)$ *is irreducible for $|t|<\frac{1}{2}$.*

*Proof.* By the remark before the lemma, we may assume that $j$ is trivial. Irreducibility follows by investigating the restriction of $\sigma_{2m}(1,t)$ to $P$ (see [31]). Another proof is in Sect. 10 through 14 (particularly Propositions 12.2, 13.4, and 14.2). For (b), let $K$ be a maximal compact subgroup of $G$. Then we can realize the non-unitarily induced representations as functions in $L^2(K)$, satisfying certain transformation properties under $K\cap P$. The same calculation proving that $\sigma_{2m}(1,t)$ is unitary for $t$ in $i\mathbb{R}$, shows that the inner product on $L^2(K)$ gives the pairing needed in (b). For (c), one computes the character as usual.

For (d), embed $\sigma=\sigma(1,\frac{1}{2})$ in a principal series representation $I$ (as can easily be done explicitly – cf. (11.12)). Then by inspection, $\sigma'$ occurs in a principal series representation $I'$ with the same character. Since $I$, $I'$, $\sigma$, and $\sigma'$ all have unique $K$-fixed vectors, and $\sigma'$ is irreducible (Sect. 11–14 again), it follows that $\sigma'$ is a subquotient of $\sigma$. That $\sigma'$ is not equal to $\sigma$ follows by inspection of characters.

The validity of (e) with some positive constant $\varepsilon$ in place of $\frac{1}{2}$ follows from the irreducibility of $\sigma_{2m}(1,0)$ (cf. (a)). That $\varepsilon$ is at most $\frac{1}{2}$ follows from (d). For $\mathbb{R}$ and $\mathbb{C}$, we will prove the irreducibility of $\sigma_{2m}(1,t)$ for $|t|<\frac{1}{2}$ in Sect. 11, by general nonsense about intertwining operators and the Langlands classification; this can be carried over to any commutative local field $\mathbb{F}$. For $\mathbb{H}$, the argument is more subtle (Sect. 14), and *I* do not know how to do it for general division algebras. (For $\mathbb{H}$, the general nonsense argument gives $\varepsilon \geqq \frac{1}{4}$.) Q.E.D.

**Proposition 2.5** (cf. Stein [31]). *For all real $t$ such that $|t| < \frac{1}{2}$, and all unitary characters $j$ of* GL$(m, \mathbb{F})$, *the representation $\sigma_{2m}(j, t)$ is infinitesimally equivalent to a unitary representation.*

*Proof.* We apply Lemma 2.4 and a standard deformation argument. Lemma 2.4(b) and (c) provide a Hermitian form on each of the representations in question. The form is non-degenerate for $|t| < \frac{1}{2}$ by Lemma 2.4(e), and positive definite for $t = 0$ by Lemma 2.4(a). It is known (and not hard to prove) that the forms may be chosen to depend continuously on $t$. They are therefore positive definite in the whole interval. Q.E.D.

*Definition 2.6.* The *Stein complementary series* of GL$(2m, \mathbb{F})$ consists of the representations $\sigma_{2m}(j, t)$, for $0 < t < \frac{1}{2}$.

Notice that these representations are induced (but not unitarily) from one-dimensional representations.

## 3. The almost spherical principal series

Fix a unitary character

(3.1)(a) $$j_1 : \mathbb{F}^\times \to \mathbb{C}^\times;$$

as in the remark after Definition 2.3, this corresponds naturally to a family of characters

(3.1)(b) $$j_m : \mathrm{GL}(m, \mathbb{F}) \to \mathbb{C}^\times,$$

characterized by the property that for $m \leqq m'$,

(3.1)(c) $$j_{m'} \,|\, _{\mathrm{GL}(m, \mathbb{F})} = j_m .$$

We refer to the collection $(j_m)$ loosely as $j$. Define

(3.2) $K(m, \mathbb{F})$ = standard maximal compact subgroup of GL$(m, \mathbb{F})$

We will discuss these groups in some detail in Sect. 5, but for now their precise nature is not important. Recall that a representation of GL$(m, \mathbb{F})$ is called *spherical* if it contains the trivial representation of $K(m, \mathbb{F})$.

*Definition 3.3.* Let $j$ be a family of characters as in (3.1). Define one dimensional representations $\mu_m$ of $K(m, \mathbb{F})$ by

$$\mu_m = j_m \,|\, _{K(m, \mathbb{F})} .$$

Write $\mu$ for the collection $\{\mu_m\}$. We call $\mu_m$ a *special one dimensional representation* of $K(m, \mathbb{F})$. A representation $\sigma$ of GL$(m, \mathbb{F})$ is called *almost spherical of type* $\mu$ if $\mu_m$ occurs in the restriction of $\sigma$ to $K(m, \mathbb{F})$; or, equivalently, if $j_m^{-1} \otimes \sigma$ is spherical.

We will often write

$$G = \mathrm{GL}(n, \mathbb{F}); \quad K = K(n, \mathbb{F}). \tag{3.4}$$

We will also need to consider the Borel subgroup

$$B = B(n, \mathbb{F}) \tag{3.5}$$

consisting of upper triangular matrices.

*Definition 3.6.* An (*ordered*) *partition* of $n$ is a sequence

$$\pi = (p_1, \ldots, p_r)$$

of positive integers, such that $\Sigma p_i = n$. (We may occasionally write this condition as $|\pi| = n$.) Define

$$\mathrm{GL}(\pi, \mathbb{F}) = \mathrm{GL}(p_1, \mathbb{F}) \times \ldots \times \mathrm{GL}(p_r, \mathbb{F}) \subset G,$$
$$K(\pi, \mathbb{F}) = K(p_1, \mathbb{F}) \times \ldots \times K(p_r, \mathbb{F}) = K \cap G(\pi)$$

to be the obvious groups of block-diagonal matrices. Next, put

$$P(\pi, \mathbb{F}) = \text{parabolic subgroup generated by } \mathrm{GL}(\pi, \mathbb{F}) \text{ and } B$$

(cf. (3.5)), and

$$N(\pi) = \text{unipotent radical of } P(\pi).$$

*Definition 3.7.* Fix $\mu = \{\mu_m\}$ as in Definition 3.3. We wish to define a special class of almost spherical representations of type $\mu$. The data are a partition $\pi = (p_i)$ of $n$, and a collection

$$\tau = (\tau_i), \quad \tau_i \in \mathrm{GL}(p_i, \mathbb{F})^\wedge,$$

such that

a) $\tau_i$ is almost spherical of type $\mu_i$, and

b) $\tau_i$ is either a unitary character or a Stein complementary series (Definition 2.6).

In terms of the family of unitary characters $(j_m)$ of (3.1), this may be made more explicit as follows. For each $i$, either

b1) there is a $v_i \in i\mathbb{R}$ such that

$$\tau_i = j_{p_i} \otimes (\delta_{p_i})^{v_i}$$

(cf. (2.2)); or

b2) $p_i = 2m$, and there are a $v_i \in i\mathbb{R}$ and $t_i \in (0, \frac{1}{2})$, such that

$$\tau_i = \sigma_{2m}(j_m \otimes (\delta_m)^{v_i}, t_i)$$

(Definition 2.3).

Finally, define

$$\sigma_\pi(\tau) = \mathrm{Ind}(P(\pi) \uparrow G)(\otimes \tau_i),$$

a *basic almost spherical representation of type* $\mu$.

**Theorem 3.8.** *Use the notation of Definitions 3.3 and 3.7.*

a) *The basic almost spherical representations* $\sigma_\pi(\tau)$ *and* $\sigma_{\pi'}(\tau')$ *are equivalent if and only if* $(\pi', \tau')$ *is a permutation of* $(\pi, \tau)$.

b) *The basic almost spherical representations are unitary.*
c) *The basic almost spherical representations are irreducible.*
d) *Any unitary almost spherical representation of* GL$(n, \mathbb{F})$ *is basic.*

*Outline of proof.* Part (a) is an easy calculation of distribution characters. Part (b) follows from Proposition 2.5. Parts (c) and (d) are of course the main points, and will occupy much of the rest of this paper (particularly Sect. 11–14). An interesting aspect of the argument is that the proof of (d) is *not* independent of the general theorem on the unitary dual: the analysis of the almost spherical representations uses a small but important part of the analysis of the general case (see the proof of Lemma 12.12).

Of course it seems natural to conjecture that Theorem 3.8 is true for all local division algebras $\mathbb{F}$. This has been proved for commutative $p$-adic $\mathbb{F}$ by S. Sahi and M. Tadic independently.

## 4. Maximal compact subgroups

Up to this point, we have described those unitary representations of $G$ containing a vector transforming in a special way under a maximal compact subgroup $K$. The general classification will be reduced to that case. To describe this a little more precisely, let us recall the Langlands classification of (not necessarily unitary) representations of a general reductive Lie group, as formulated in [35] and [36]. To each representation $\pi$ of $G$ is assigned a small finite set of "lowest $K$-types." These are representations of $K$ occuring in $\pi$. In case $G$ is GL$(n, \mathbb{F})$, the lowest $K$-type is unique. One therefore obtains an approximate partition of the representations of $G$, into parts parametrized by representations of $K$; this is precise for GL$(n, \mathbb{F})$.

The classification treats each piece of this partition separately. To each representation $\mu$ of $K$, it associates a subgroup $L$ of $G$, and a representation $\mu_L$ of $L \cap K$. The main reduction step in the classification theorem (Theorem 6.5 below) gives a bijection between representations of $G$ with lowest $K$-type $\mu$, and representations of $L$ containing the $L \cap K$-type $\mu_L$. To obtain an explicit classification, one must then treat the cases when $L = G$; but these are relatively easy.

Our analysis of the unitary dual will proceed along similar lines (cf. [23]). As noted above, a representation of GL$(n, \mathbb{F})$ has a unique lowest $K$-type. To each representation $\mu$ of $K$, we will associate a subgroup $L$ of $G$, and a representation $\mu_L$ of $L \cap K$. $L$ will be a product of various GL$(m_i, \mathbb{F}_i)$, and $\mu_L$ will be almost spherical for $L$. (Except when $\mathbb{F}$ is $\mathbb{C}$, this $L$ is different from the one in the Langlands classification; it is larger in the case of $\mathbb{H}$, and smaller in the case of $\mathbb{R}$.) The main reduction step provides a constructive bijection between unitary representations of $G$ with lowest $K$-type $\mu$, and unitary representations of $L$ containing the $L \cap K$-type $\mu_L$. Because the latter are determined by Theorem 3.8, this determines $\hat{G}_u$.

To state this reduction step more precisely, we need to describe $K$ and its representations. In this section, we will give some general results; in the next, we will consider $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{H}$ in detail.

The space $\mathbb{F}^n$ will be regarded as a right vector space of column vectors over $\mathbb{F}$; then $G$ acts (linearly) by matrix multiplication on the left. The standard anti-automorphism of $\mathbb{F}$ will be denoted by a bar. It is trivial for $\mathbb{R}$; complex conjugation for $\mathbb{C}$; and

$$(xi+yj+zk+w)^- = -xi-yj-zk+w.$$

In all cases, we have

$$(ab)^- = (b^-)(a^-).$$

Write (,) for the sesquilinear form

$$(v,w) = \Sigma\,(v_i)^-\,(w_i)$$

on $\mathbb{F}^n$. It satisfies

$$\begin{aligned}(va,w) &= a^-\,(v,w)\\(v,wb) &= (v,w)\,b\\(v,w) &= (w,v)^-\end{aligned}$$

for $v$ and $w$ in $\mathbb{F}^n$ and $a$ and $b$ in $\mathbb{F}$. The real part of (,) gives a real Euclidean structure to $\mathbb{F}^n$, corresponding to its standard identification with $\mathbb{R}^{dn}$ (mentioned after (2.2)). In particular, we have a norm

$$|v| = (v,v)^{1/2}.$$

If $A$ is any $m$ by $n$ matrix over $\mathbb{F}$, we define $A^*$ to be the $n$ by $m$ conjugate transpose matrix; it is obtained by applying bar to each matrix entry, then transposing. If $A$ is $n$ by $n$, we have

$$(Av,w) = (v,A^*w).$$

Finally, for $g$ in $G$, we define

$$\theta g = (g^*)^{-1}. \tag{4.1}$$

This is an involutive (that is, of order two) automorphism of $G$. It is also a Cartan involution, meaning that its fixed point set is a maximal compact subgroup of $G$.

*Definition 4.2.* The *standard maximal compact subgroup* $K$ of $\mathrm{GL}(n,\mathbb{F})$ consists of all elements satisfying any of the following equivalent conditions.

a) $\theta k = k$.
b) $(kv,kw) = (v,w)$, for all $v$ and $w$ in $\mathbb{F}^n$.
c) $|kv| = |v|$, for all $v$ in $\mathbb{F}^n$.

If $\mathbb{F}$ is $\mathbb{R}$, $K$ is denoted $\mathrm{O}(n)$, and called the *orthogonal group*. If $\mathbb{F}$ is $\mathbb{C}$, $K$ is denoted $\mathrm{U}(n)$, and called the *unitary group*. If $\mathbb{F}$ is $\mathbb{H}$, $K$ is denoted $\mathrm{Sp}(n)$, and called the *compact symplectic group*.

The identification of $\mathbb{F}^n$ with $\mathbb{R}^{dn}$ allows us to regard $\mathrm{GL}(n,\mathbb{F})$ as a subgroup of $\mathrm{GL}(dn,\mathbb{R})$. In this identification, we have

$$K = \mathrm{O}(dn) \cap \mathrm{GL}(n,\mathbb{F}).$$

Recall that a $(\mathfrak{g},K)$-module is a complex vector space endowed with representations of $\mathfrak{g}$ and of $K$, satisfying some compatibility and finiteness conditions (cf. [36]). The *Harish-Chandra module* of a representation of $G$ is the

$(\mathfrak{g}, K)$-module of its smooth $K$-finite vectors. From Harish-Chandra's collection of results relating group representations and Harish-Chandra modules, we want mainly the following one.

**Theorem 4.3** (Harish-Chandra [16]). *Suppose $G$ is a reductive Lie group, and $K$ a maximal compact subgroup. Passage to $K$-finite vectors defines a bijection from the set of equivalence classes of irreducible unitary representations of $G$, onto the set of equivalence classes of irreducible $(\mathfrak{g}, K)$-modules admitting a positive definite invariant Hermitian form.*

We turn now to the parametrization of representations of compact Lie groups.

*Definition 4.4.* Suppose $H$ is a compact Lie group (possibly disconnected). Let $T_0$ be a maximal torus in the identity component $H_0$. The complexified Lie algebra then has a root space decomposition

$$\mathfrak{h} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta(\mathfrak{h}, \mathfrak{t})} \mathfrak{h}_\alpha .$$

Fix a set $\Delta^+(\mathfrak{h}, \mathfrak{t})$ of positive roots, and write

$$\mathfrak{b} = \mathfrak{t} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{h}_\alpha = \mathfrak{t} + \mathfrak{n}$$

for the corresponding Borel subalgebra.

Let $\langle , \rangle$ be an $H$-invariant positive definite inner product on $\mathfrak{h}_0$. A weight $\lambda$ in $\mathfrak{t}^*$ is called *dominant* (with respect to $\mathfrak{b}$ or $\Delta^+$) if

$$\langle \lambda, \alpha \rangle \geqq 0, \quad \text{all } \alpha \in \Delta^+ .$$

Write $T$ for the normalizer of $\mathfrak{b}$ in $H$; we call $T$ a *Cartan subgroup* of $H$. The identity component $T_0$ is then a maximal torus in $H_0$. A *dominant representation* of $T$ is one whose differential is a sum of dominant (with respect to the given Borel subalgebra) weights. Since we will often consider the same Cartan subgroup $T$ in various compact groups, we may say *$H$-dominant* for definiteness.

**Proposition 4.5** (Cartan-Weyl). *Suppose $H$ is a compact Lie group, and $T$ is the Cartan subgroup associated to the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{h}$. Write $\mathfrak{n}$ for the nilradical of $\mathfrak{b}$. Then passage to $\mathfrak{n}$-invariant vectors defines a bijection from the set of irreducible representations of $H$, onto the set of irreducible dominant representations of $T$.*

If $\mu$ in $\hat{H}$ corresponds to $\gamma$ in $\hat{T}$, we will say that $\mu$ has highest weight $\gamma$, even though $\gamma$ may not be a one dimensional character of $T$.

We will sketch a construction of the inverse map from $T$ to $\hat{H}$. It is by no means the easiest one, but it is the one we will need later. It is convenient to generalize things a little.

*Definition 4.6.* Suppose $H$ is a compact Lie group, and $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{h}$. Write $\mathfrak{u}$ for the unipotent radical of $\mathfrak{q}$. Write $L$ for the normalizer of $\mathfrak{q}$ in $H$, the *Levi subgroup* of $\mathfrak{q}$. (The complexified Lie algebra $\mathfrak{l}$ is easily seen to be a Levi subalgebra of $\mathfrak{q}$.) Recall from [36], Definition 6.3.1, the *cohomological parabolic induction functors*, taking locally finite representations of $L$ to locally finite representations of $H$. They are defined by twisting by a fixed one dimensional representation $\tilde{\tau}$ of $L$; extending to a representation of $\mathfrak{q}$ trivial on $\mathfrak{u}$; algebraically producing to an $L$-finite representation of $\mathfrak{h}$; and applying Zuckerman's derived functors to get a representation of $H$. Briefly,

$$(\mathscr{R}^H)^i = \Gamma^i((\mathfrak{h}, L) \uparrow (\mathfrak{h}, H)) \circ \operatorname{pro}((\mathfrak{q}, L) \uparrow (\mathfrak{h}, L))(* \otimes_{\mathbb{C}} \tau^{\sim}).$$

The notation is explained in [36]. The twist $\tau^{\sim}$ is introduced for later convenience only. For not-so-distant convenience, we write it as

$$\tau^{\sim} = \tau \otimes (\Lambda^S \mathfrak{u});$$

here $S$ is the dimension of $\mathfrak{u}$.

We will also need the dual construction. Write $\mathfrak{q}^{op}$ for the complex conjugate of $\mathfrak{q}$ (which still has Levi subgroup $L$). Define

$$(\mathscr{L}^H)^j = \Gamma^j \circ \operatorname{ind}((\mathfrak{q}^{op}, L) \uparrow (\mathfrak{h}, L))(* \otimes_{\mathbb{C}} \tau^{\sim}).$$

This has formal properties closely related to those of $\mathscr{R}^H$ (cf. [38], Sect. 5).

**Proposition 4.7.** (see [11]) *In the setting of Definition 4.6, the functors $(\mathscr{R}^H)^i$ and $(\mathscr{L}^H)^j$ are zero for $i$ greater than $S$, and $j$ less than $S$. In degree $S$ the functors are isomorphic, and may be computed as follows. Fix a Borel subalgebra $\mathfrak{b}$ of $H$, contained in $\mathfrak{q}$. Write $T$ for its normalizer in $H$, and $T_L$ for its normalizer in $L$. Let $(\mu_L, V)$ be an irreducible representation of $L$, corresponding to the irreducible $L$-dominant representation $\gamma_L$ of $T_L$* (Proposition 4.5). *Set*

$$\gamma = \operatorname{Ind}(T_L \uparrow T)(\gamma_L \otimes \tau)$$

(with $\tau$ as in Definition 4.6). *If $\gamma$ is $H$-dominant, then $(\mathscr{L}^H)^S(V)$ is the sum of the irreducible representations of $H$ corresponding to the constituents of $\gamma$* (Proposition 4.5). *If $\gamma$ is not $H$-dominant, then $(\mathscr{L}^H)^S(V)$ is zero.*

In the setting of the proposition, it is easy to check that if $\gamma$ is not $H$-dominant, then no constituent of it is either.

Taking $\mathfrak{q}$ to be a Borel subalgebra, and $\tau$ trivial, we get a construction of the inverse of the bijection of Proposition 4.5.

We conclude this section by recalling the definition of lowest $K$-type. We will confine ourselves to the simple definition in [35], despite the technical merits of the more complicated (but equivalent) version in [36].

*Definition 4.8.* Suppose $G$ is a reductive Lie group, and $K$ is a maximal compact subgroup. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{k}$, and $T$ the corresponding Cartan subgroup. Write $2\rho_c$ for the sum of the roots of $\mathfrak{t}$ in $\mathfrak{b}$. Fix an irreducible representation $\mu$ of $K$, of highest weight $\gamma$ in $\hat{T}$. Let $\gamma_0$ in $\mathfrak{t}^*$ be a weight of $\gamma$. Define the *norm* of $\mu$ to be

$$\|\mu\| = \langle \mu + 2\rho_c, \mu + 2\rho_c \rangle.$$

If $X$ is any $(\mathfrak{g}, K)$-module, we say that $\mu$ is a *lowest $K$-type* of $X$ if

a) $\mu$ occurs in the restriction of $X$ to $K$; and

b) $\|\mu\|$ is minimal subject to (a).

As was remarked earlier, representations of general reductive groups may have several lowest $K$-types. The situation is completely described in 6.5 of [36], however. Specializing those results appropriately, we find

**Theorem 4.9.** *Suppose $G = \mathrm{GL}(n, \mathbb{F})$, and $X$ is an irreducible $(\mathfrak{g}, K)$-module. Then $X$ has a unique lowest $K$-type. It occurs with multiplicity one in $X$.*

## 5. Parametrization of $\hat{K}$

The program outlined at the beginning of Sect. 4 requires us to associate to each $\mu$ in $\hat{K}$ a subgroup $L$ of $G$, and a representation $\mu_L$ of $L \cap K$. We will do this on a case by case basis. Always we proceed in two steps. First, we assign to $\mu$ a weight $\lambda = \lambda(\mu)$ in the dual $\mathfrak{t}^*$ of a Cartan in $\mathfrak{k}$. We will define

(5.1)
$$L_\theta = L_\theta(\mu) \\ = \text{centralizer of } \lambda \text{ in } G.$$

(Except for the case of $\mathbb{H}$, the weight $\lambda$ will be the one constructed in Sect. 5.3 of [36].) We will also define $\mu_{L_\theta}$. Next, we will define $L$ to be the Levi subgroup of a real parabolic subgroup of $L_\theta$. (Except for the case of $\mathbb{R}$, $L$ will be all of $L_\theta$.) Finally, we will define $\mu_L$ to be a certain almost spherical representation of $L \cap K$.

Suppose first that $G$ is $\mathrm{GL}(n, \mathbb{C})$. Recall that $K$ is the group $\mathrm{U}(n)$. Define

(5.2)(a)
$$T = \text{group of diagonal unitary matrices} \\ = \mathrm{U}(1) \times \ldots \times \mathrm{U}(1) \quad (n \text{ copies}).$$

As usual, we identify

(5.2)(b)
$$\hat{T} = \mathbb{Z}^n.$$

Write $e_i$ for the usual basis elements of $\mathbb{Z}^n$. $T$ has the same weights on $\mathfrak{k}$ as on $\mathfrak{p}$; the non-zero ones are the weights of the form $e_i - e_j$, for $i$ different from $j$. We choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{k}$, corresponding to the positive roots

(5.3)
$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = \{e_i - e_j \mid i < j\}.$$

The dominant weights (and therefore the representations of $K$) correspond to decreasing sequences of integers:

(5.4)
$$\hat{K} \cong \{\gamma = (\gamma_1, \ldots, \gamma_n) \mid \gamma_1 \geqq \ldots \geqq \gamma_n\}.$$

The one dimensional representations of $K$ are all special (Definition 3.3); they correspond to the constant sequences. (In particular, they are naturally parametrized by $\mathbb{Z}$ for any $n$.)

Fix an irreducible representation $\mu$ of $K$, of highest weight $\gamma$. Define

(5.5)
$$\lambda(\mu) = \gamma.$$

Write $\pi = \pi(\mu)$ for the coarsest ordered partition of $n$ such that $\gamma$ is constant on the parts. Then

(5.6)
$$L_\theta = \mathrm{GL}(\pi, \mathbb{C})$$

(Definition 3.6), a product of copies of $\mathrm{GL}(p_j, \mathbb{C})$. We take $L = L_\theta$, and let $\mu_L$ be the representation of $L \cap K$ of highest weight $\gamma$. If we write $\gamma(j)$ for the constant value of $\gamma$ on the $j$th block of $\pi$, then

$$\mu_L = \otimes (\det)^{\gamma(j)}. \tag{5.7}$$

Clearly this is an almost spherical representation of $L \cap K$.

Next, suppose $G$ is $\mathrm{GL}(n, \mathbb{R})$. Recall that $K$ is $\mathrm{O}(n)$. Define $m = [n/2]$, and $\varepsilon = n - 2m$; thus

$$n = 2m + \varepsilon. \tag{5.8}$$

Recall that the group $\mathrm{SO}(2)$ (the identity component of $\mathrm{O}(2)$) is naturally isomorphic to the circle, by the identification of $\mathbb{R}^2$ (on which $\mathrm{SO}(2)$ acts) with $\mathbb{C}$ (on which the circle acts). Explicitly, the matrix

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5.9}$$

corresponds to $\exp(i\theta)$. Set

$$T_0 = \mathrm{SO}(2) \times \ldots \times \mathrm{SO}(2) \quad (m \text{ copies}), \tag{5.10}$$

embedded in $\mathrm{O}(n)$ in the obvious way. Using the coordinates (7.2) to identify $T_0$ with the product of $m$ circles, we get

$$(T_0)^\wedge \cong \mathbb{Z}^m. \tag{5.11}$$

Consider the following three sets of weights in $(T_0)^\wedge$:

$$\begin{aligned} A &= \{e_i \pm e_j \mid i<j\} \\ B &= \{2e_i\} \\ C &= \{e_i\} \end{aligned} \tag{5.12}$$

We choose as positive roots of $T_0$ in $\mathfrak{k}$

$$\Delta^+(\mathfrak{k}, \mathfrak{t}) = A \cup \varepsilon C; \tag{5.13}$$

this rather loose notation is intended to mean $A$ alone if $n$ is even, and the union of $A$ and $C$ if $n$ is odd. The roots of $T_0$ in $\mathfrak{p}$ are

$$\Delta(\mathfrak{p}, \mathfrak{t}) = \pm (A \cup B \cup \varepsilon C). \tag{5.14}$$

Let $r_n$ denote the matrix

$$r_n = \mathrm{diag}(1, \ldots, 1, -1). \tag{5.15}$$

Then $r_n$ normalizes $T_0$, and permutes the positive roots of $T_0$ in $\mathfrak{k}$. It acts trivially on the first $m-1$ $\mathrm{SO}(2)$ factors of $T_0$, and acts on the last by $(-1)^{(1+\varepsilon)}$ – that is, trivially if $n$ is odd and by inversion if $n$ is even. In any case, the Cartan subgroup of $K$ (Definition 4.4) is

$$T = T_0 \rtimes (1, r_n\} \tag{5.16}$$

If $n$ is odd, then $T$ is a direct product, and its irreducible representations are one dimensional. They are parametrized by pairs $(\gamma, \eta)$, with $\gamma$ (the weight of $T_0$) a sequence of $m$ integers, and $\eta$ (the weight of $r_n$) equal to 0 or 1. If $n$ is even, the irreducible representations of $T$ are of two types. First, there are the representations induced from $T_0$. These are two dimensional, and are parametrized by the inducing weight $\gamma$, a sequence of $m$ integers with $\gamma_m$ positive. Second, there are the one dimensional representations. They are parametrized by pairs $(\gamma, \eta)$, with $\gamma$ (the weight of $T_0$) a sequence of $m$ integers ending in zero, and $\eta$ (the weight of $r_n$) equal to 0 or 1.

Taking into account Proposition 4.5, we obtain a description of the representations of $\mathrm{O}(n)$.

**Proposition 5.17.** *The representations of* $\mathrm{O}(n)$ *are parametrized by pairs* $(\gamma, \eta)$, *subject to the following conditions. Write* $n = 2m + \varepsilon$ *as in* (5.8).

a) $\gamma$ *is a decreasing sequence of* $m$ non-negative integers.

b) *If* $n$ *is even and* $\gamma_m$ *is not zero, then* $\eta$ *is* $\frac{1}{2}$; *otherwise* $\eta$ *is* 0 *or* 1.

*Let* $\mu$ *be the representation of highest weight* $(\gamma, \eta)$. *If* $\eta$ *is* 0 *or* 1, *the restriction of* $\mu$ *to* $\mathrm{SO}(n)$ *is the irreducible representation of highest weight* $\gamma$. *If* $\eta$ *is* $\frac{1}{2}$, *the restriction is the sum of the representations of highest weights* $\gamma$ *and* $(\gamma_1, \ldots, -\gamma_m)$.

The two one dimensional representations of $\mathrm{O}(n)$ are those parametrized by $((0, \ldots, 0), 0)$ and $((0, \ldots, 0), 1)$. They are both special (Definition 3.3); they are the restrictions to $K$ of the trivial and determinant characters of $G$, respectively.

Let $\mu$ be an irreducible representation of $\mathrm{O}(n)$, of highest weight $(\gamma, \eta)$ as in Proposition 5.17. Let $p$ be the largest integer such that $\gamma_p$ is at least 2. Define

(5.18)
$$\begin{aligned}\lambda(\mu) &= (\max(\gamma_i - 1, 0)) \\ &= (\gamma_1 - 1, \ldots, \gamma_0 - 1, 0, \ldots, 0).\end{aligned}$$

Let $\pi$ be the coarsest ordered partition of $p$ such that $\gamma$ is constant on the parts of $\pi$. Then an easy calculation shows that

(5.19)
$$L_\theta = \mathrm{GL}(\pi, \mathbb{C}) \times \mathrm{GL}(n - 2p, \mathbb{R}).$$

(Here $\mathrm{GL}(\pi, \mathbb{C})$ denotes the obvious product of copies of $\mathrm{GL}(p_j, \mathbb{C})$. We let $\mu_{L_\theta}$ be the representation of $L_\theta \cap K$ of highest weight

$$((\gamma_1 - 1, \ldots, \gamma_p - 1, \gamma_{p+1}, \ldots, \gamma_m), \eta).$$

Write $\mu_f$ for the representation of $\mathrm{O}(n-2p)$ parametrized by the last $m-p$ coordinates of $\gamma$, and $\eta$. Let $\gamma(j)$ denote the constant value of $\gamma$ on the $j$-th part of $\pi$. Then

(5.20)
$$\mu_{L_\theta} = [\otimes \det{}^{\gamma(j)-1}] \otimes \mu_f.$$

Next, we turn to the description of $L$ itself. Write $q = n - 2p$. The last $[n/2] - p$ terms of $\gamma$ are zeros and ones; say there are $q'$ ones. Define $q_0$ and $q_1$ as follows:

(5.21)(a) if $\eta$ is 0 or $\frac{1}{2}$, then $q_1 = q'$, and $q_0 = q - q_1$;

and

(5.21)(b) if $\eta$ is 1 or $\frac{1}{2}$, then $q_0 = q'$, and $q_1 = q - q_0$.

Let

(5.22)
$$L = \mathrm{GL}(\pi, \mathbb{C}) \times \mathrm{GL}(q_0, \mathbb{R}) \times \mathrm{GL}(q_1, \mathbb{R}).$$

Set

(5.23) $$\mu_L = [\otimes \det^{\gamma(j)-1}] \otimes 1 \otimes \det .$$

Obviously this is an almost spherical representation of $L \cap K$, but it may appear to have been pulled from a hat. To allay the reader's fears on this point, we include here a lemma which will not be needed (or proved) until section 8.

**Lemma 5.24.** *Suppose $q_0$ and $q_1$ are non-negative integers, and $q = q_0 + q_1$. Write $q = 2r + \varepsilon$, in analogy with* (5.8). *Then there is a unique decreasing sequence $\gamma$ of $r$ ones and zeros: and an $\eta$ equal to* 0, $\frac{1}{2}$, *or* 1, *with the following properties. Write $q'$ for the number of ones in $\gamma$. First, $\eta$ is $\frac{1}{2}$ if and only if $q$ is even and $\gamma_r$ is* 1. *Second,* (5.21)(a) *and* (b) *hold.*

*Write $\mu_f = \mu_f(q_0, q_1)$ for the irreducible representation of* O$(q)$ *of highest weight* $(\gamma, \eta)$ (Proposition 5.17). *Then $\mu_f$ is the lowest* O$(q)$*-type of*

$$\mathrm{Ind}(\mathrm{O}(q_0) \times \mathrm{O}(q_1) \uparrow \mathrm{O}(q))\,(1 \otimes \det).$$

We defined lowest $K$-type only for $(\mathfrak{g}, K)$-modules, but obviously the definition requires only the $K$-structure. In any case, the induced representation in question is the restriction to $K$ of a $(\mathfrak{g}, K)$-module.

Next, suppose $G$ is $\mathrm{GL}(n, \mathbb{H})$, so that $K$ is the (connected) compact symplectic group $\mathrm{Sp}(n)$. The group $\mathrm{Sp}(1)$ is the group of unit quaternions. This obviously contains a copy of the unit circle $\mathrm{U}(1)$, the complex numbers of absolute value 1. The natural embedding of $\mathrm{Sp}(1)^n$ into $\mathrm{Sp}(n)$ then defined a subgroup

(5.25)(a) $$T = \mathrm{U}(1) \times \ldots \times \mathrm{U}(1) \quad (n \text{ copies})$$

of $K$. It is a maximal torus and a Cartan subgroup. Clearly

(5.25)(b) $$\hat{T} \cong \mathbb{Z}^n.$$

Consider the weights

(5.26)(a) $$A = \{e_i \pm e_j \mid i < j\}, \quad B = \{2e_i\}.$$

We can choose as positive roots of $T$ in $\mathfrak{k}$

(5.26)(b) $$\Delta^+(\mathfrak{k}, \mathfrak{t}) = A \cup B.$$

The roots of $T$ in $\mathfrak{p}$ are

(5.27) $$\Delta(\mathfrak{p}, \mathfrak{t}) = \pm A.$$

The dominant weights are decreasing sequences of $n$ positive integers:

(5.28) $$\hat{K} \cong \{\gamma = (\gamma_1, \ldots, \gamma_n) \mid \gamma_1 \geqq \ldots \geqq \gamma_n \geqq 0\}.$$

Since $K$ is connected and semisimple, the only one dimensional representation is the trivial one; it is special (Definition 3.3).

Fix a representation $\mu$ of $K$, associated to $\gamma$ by (5.28). Define

(5.29) $$\lambda(\mu)_i = \gamma_i + 1, \text{ if } \gamma_i \text{ is positive; or}$$
$$= 0, \text{ if } \gamma_i \text{ is zero.}$$

We will let $L$ be equal to the centralizer $L_\theta$ of $\lambda$. To describe it, write $q$ for the number of zeros in the sequence $\gamma$, and $p=n-q$. Let $\pi$ be the coarsest ordered partition of $p$ so that $\gamma$ (or rather the first $p$ coordinates thereof) is constant on the parts of $\pi$. Then

$$L=\mathrm{GL}(\pi,\mathbb{C})\times \mathrm{GL}(q,\mathbb{H}). \tag{5.30}$$

We let $\mu_L$ be the representation of $L\cap K$ with highest weight $\gamma$. With notation analogous to that in (5.7), this is

$$\mu_L=[\otimes \det{}^{\gamma(j)+1}]\otimes 1. \tag{5.31}$$

Clearly it is special (Definition 3.3).

## 6. The induction functors

In this section, we will define the functors which will exhibit the bijection between unitary representations of $G$ with lowest $K$-type $\mu$, and (almost spherical) representations of $L$ containing the $L\cap K$-type $\mu_L$. Once again this construction will be in two steps, corresponding to the two steps in the definition of $L$. The functors from representations of $L$ to representations of $L_\theta$ will be ordinary (normalized) parabolic induction. To go from $L_\theta$ to $G$, we will use cohomological parabolic induction as in [36] (but with a twist in the normalization). We therefore begin by recalling its definition.

*Definition 6.1* (cf. [36], Definition 5.2.1). A *$\theta$-stable parabolic subalgebra* of $\mathfrak{g}$ is a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, stable under the (complexified) Cartan involution $\theta$. Necessarily such a $\mathfrak{q}$ meets its complex conjugate in a Levi factor $\mathfrak{l}$ of $\mathfrak{q}$. Write

$$\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$$

for the corresponding Levi decomposition of $\mathfrak{q}$, and $L$ for the normalizer of $\mathfrak{q}$ in $G$ (the *Levi subgroup* of $\mathfrak{q}$). Recall from [34], Definition 6.3.1, the cohomological parabolic induction functors (from $(\mathfrak{l}, L\cap K)$-modules to $(\mathfrak{g}, K)$-modules)

$$\begin{aligned}\mathscr{R}^i&=\mathscr{R}^i((\mathfrak{q},L\cap K)\uparrow(\mathfrak{g},K))\\&=\Gamma^i((\mathfrak{g},L\cap K)\uparrow(\mathfrak{g},K))\circ \mathit{pro}((\mathfrak{q},L\cap K)\uparrow(\mathfrak{g},L\cap K))(*\otimes_{\mathbb{C}}\tau^{\sim}).\end{aligned}$$

(The symbols are explained in [36].) Here we have allowed a twist by a one dimensional character $\tau^{\sim}$ of $L$. (In [36], $\tau^{\sim}$ was fixed as the top exterior power of $\mathfrak{u}$. This normalization has not aged well. Fixing it properly in general would lead us too far astray, but we cannot resist the temptation to improve matters for GL $(n)$.) We will write $(\mathscr{R}^K)^i$ for the analogous functors with $\mathfrak{q}$ and $\mathfrak{g}$ replaced by $\mathfrak{q}\cap\mathfrak{k}$ and $\mathfrak{k}$ (defined in Definition 4.6); these take representations of $L\cap K$ to representations of $K$. As in Sect. 5 of [38], it is actually more convenient to consider the functors

$$\mathscr{L}^j=\Gamma^j\circ \mathrm{ind}((\mathfrak{q}^{op},L\cap K)\uparrow(\mathfrak{g},L\cap K))(*\otimes_{\mathbb{C}}\tau^{\sim})$$

and $(\mathscr{L}^K)^j$. We will mention properties of $\mathscr{R}$ only because it is $\mathscr{R}$ rather than $\mathscr{L}$ which is discussed in [36].

If $Y$ is any $(\mathfrak{l}, L\cap K)$-module, the inclusion of $U(\mathfrak{k})$ in $U(\mathfrak{g})$ induces a natural homomorphism of $(\mathfrak{k}, L\cap K)$-modules

$$\begin{aligned}&\operatorname{ind}((\mathfrak{q}^{op}\cap\mathfrak{k}, L\cap K)\uparrow(\mathfrak{k}, L\cap K))(Y\otimes\tau^{\sim})\\ &\quad\hookrightarrow\operatorname{ind}((\mathfrak{q}^{op}, L\cap K)\uparrow(\mathfrak{g}, L\cap K))(Y\otimes\tau^{\sim}),\end{aligned}$$

and so a family of homomorphisms

$$\psi^j\colon(\mathscr{L}^K)^j Y\to\mathscr{L}^j Y \tag{6.2)(a}$$

of $K$ modules. Similarly, we have homomorphisms

$$\phi^i\colon\mathscr{R}^i Y\to(\mathscr{R}^K)^i Y. \tag{6.2)(b}$$

Here are some of the less obvious formal properties of these functors which we will use.

**Proposition 6.3.** *In the setting of Definition 6.1, let $Y$ be an $(\mathfrak{l}, L\cap K)$-module, and let $S$ be the dimension of $\mathfrak{u}\cap\mathfrak{k}$.*

a) *$\mathscr{R}^i Y$ and $(\mathscr{R}^K)^i(Y)$ are zero for $i$ greater than $S$.*

b) *$\mathscr{L}^j$ and $(\mathscr{L}^K)^j$ are zero for $j$ less than $S$.*

c) *The homomorphism $\phi^S$ of (6.2) is surjective, and $\psi^S$ is injective.*

Part (a) of this proposition, due to Zuckerman, is Corollary 6.3.21 in [36]; and (b) is analogous ([38], Lemma 5.2(b)). Part (c) is discussed in Definition 6.9 of [36].

As background and motivation for what is to come, we recall now a few more pieces of the Langlands classification of irreducible $(\mathfrak{g}, K)$-modules (for general reductive $G$, for the moment). One of the key ideas in that classification is

**Theorem 6.4** (Langlands; cf. [36], Sect. 6.5). *Suppose $G$ is a reductive Lie group, with maximal compact subgroup $K$. Any irreducible $(\mathfrak{g}, K)$-module $X$ is the unique irreducible quotient of a certain "standard" $(\mathfrak{g}, K)$-module $C(X)$. $C(X)$ has the following properties, which characterize it uniquely;*

a) *The lowest $K$-types of $C(X)$ are precisely the same as those of $X$.*

a)′ *The leading exponents in the asymptotic expansions of matrix coefficients of $C(X)$ are the same as those for $X$.*

b) *If $X'$ is any $(\mathfrak{g}, K)$-module of finite length satisfying* (a) *or* (a)′, *and $X$ is a composition factor of $X'$, then*

$$\operatorname{Hom}_{\mathfrak{g},K}(C(X), X')\neq 0.$$

(We have included (a)′ only as motivation for those to whom the words have a meaning already; others may safely ignore it.) The point is that $C(X)$ is some sort of canonical covering of $X$; not precisely a projective cover, but something with that general flavor.

Here is a more precise version of the reduction step alluded to at the beginning of Sect. 4.

**Theorem 6.5** ([36], Theorem 6.5.12). *Suppose $G$ is a reductive Lie group, with maximal compact subgroup $K$. Fix a representation $\mu$ of $K$. Then there is attached to $\mu$ a $\theta$-stable parabolic subalgebra*

$$\mathfrak{q}=\mathfrak{q}_{cl}(\mu),$$

*and a representation $\mu_L$ of $L\cap K$ (notation as in Definition 6.1), with the following properties;*

a) *There is a bijection $X_L \to X$ from irreducible $(\mathfrak{l}, L\cap K)$-modules with lowest $L\cap K$-type $\mu_L$, onto irreducible $(\mathfrak{g}, K)$-modules with lowest $K$-type $\mu$.*

b) *Suppose $X_L$ has lowest $L\cap K$-type $\mu_L$. Then (with $S$ equal to the dimension of $\mathfrak{u}\cap\mathfrak{k}$),*

$$\mathscr{L}^S(C(X_L)) = C(X): \quad \mathscr{L}^j(C(X_L)) = 0,\ j \neq S.$$

c) *$X$ is the unique irreducible quotient of $\mathscr{L}^S(C(X_L))$: and $X$ is a quotient of $\mathscr{L}^S(X_L)$.*

d) *$(\mathscr{L}^K)^S$ defines a bijection from the set of lowest $L\cap K$-types of $X_L$ onto the set of lowest $K$-types of $X$. In particular,*

$$(\mathscr{L}^K)^S(\mu_L) = \mu.$$

Using a simple induction by stages argument ([36], Proposition 6.3.6), we deduce

**Corollary 6.6.** *The conclusions of Theorem 6.5 remain valid if $\mathfrak{q}_{cl}$ is replaced by any $\theta$-stable $\mathfrak{q}$ containing it.*

We now return to the special case of $G = \mathrm{GL}(n, \mathbb{F})$. Fix $\mu$ in $\hat{K}$, and define $\lambda$ as in (5.5), (5.18), and (5.29). Recall that $\lambda$ belongs to a fixed Cartan subalgebra $\mathfrak{l}$ of $\mathfrak{k}$. We want to define a $\theta$-stable parabolic subalgebra

(6.7)(a) $$\mathfrak{q}_\theta = \mathfrak{l}_\theta + \mathfrak{u}_\theta$$

of $\mathfrak{g}$; recall that $L_\theta$ has already been defined in (5.1). It is enough to specify the set of weights of $\mathfrak{t}$ in $\mathfrak{u}_\theta$. These are given by

(6.7)(b) $$\Delta(\mathfrak{u}_\theta, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \alpha, \lambda\rangle > 0\}.$$

For comparison, note that

(6.7)(c) $$\Delta(\mathfrak{l}_\theta, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \alpha, \lambda\rangle = 0\}.$$

**Lemma 6.8.** *In the setting of (6.7), the parabolic $\mathfrak{q}_\theta$ contains the one $\mathfrak{q}_{cl}$ of Theorem 6.5. In particular, all the conclusions of that theorem are available for $\mathfrak{q}_\theta$.*

*Sketch of proof.* The definition of $\mathfrak{q}_{cl}$ is contained in Definition 5.3.22 (and hence in Proposition 5.3.3) of [36]. It is analogous to (6.7), but uses a different weight $\lambda_{cl}$. Carrying out the calculation in Proposition 5.3.3 of [36] shows that $\lambda_{cl}$ is equal to ous $\lambda$ for $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{R})$; so the two parabolics coincide in those cases. For $\mathbb{H}$, in the notation of (5.29),

$$(\lambda_{cl})_i = \gamma_i + 1, \text{ all } i.$$

Once this is verified, the claim in the lemma is very easy to check. We leave the details to the reader.

We now choose the shift $\tau^{\sim}$ appearing in the definition of cohomological induction.

**Lemma 6.9.** *In the setting of (6.7), write*

$$L_\theta = \mathrm{GL}(\pi, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{F})$$

*as in* (5.6), (5.19), *and* (5.30). *Let* $\delta$ *be* 0 *if* $\mathbb{F}=\mathbb{C}$, 1 *if* $\mathbb{F}=\mathbb{R}$, *and* $-1$ *if* $\mathbb{F}=\mathbb{H}$. *Consider the one dimensional character*

$$\tau = \det{}^{\delta} \otimes 1$$

*of* $L_\theta \cap K$. *Define*

$$\tau^{\sim} = \tau \otimes (\Lambda^S \mathfrak{u}_\theta \cap \mathfrak{k})$$

(*cf. Definition 4.6*). *Then the differential of* $\tau^{\sim}$ *is* $\rho(\mathfrak{u}_\theta)$, *half the weight of* $\mathfrak{l}_\theta \cap \mathfrak{k}$ *on the top exterior power of* $\mathfrak{u}_\theta$. *Consequently* $\tau^{\sim}$ *extends uniquely to a character* (*still denoted* $\tau^{\sim}$) *of* $L_\theta$.

*Sketch of proof.* Write $R$ for the dimension of $\mathfrak{u}_\theta \cap \mathfrak{p}$. What must be shown is that

$$\tau^2 \otimes (\Lambda^S(\mathfrak{u}_\theta \cap \mathfrak{k}))^2 = \Lambda^{R+S}(\mathfrak{u}_\theta).$$

This is equivalent to

$$\tau^2 \otimes \Lambda^S(\mathfrak{u}_\theta \cap \mathfrak{k}) = \Lambda^R(\mathfrak{u}_\theta \cap \mathfrak{p}). \tag{6.10}$$

Over $\mathbb{C}$, $\mathfrak{k}$ and $\mathfrak{p}$ are isomorphic as representations of $K$, and (6.10) is clear. Over $\mathbb{R}$ or $\mathbb{H}$, the calculations in Sect. 5 ((5.13) and (5.14), and (5.26) and (5.27)) show that the weights of $\mathfrak{t}$ in $\mathfrak{k}$ and $\mathfrak{p}$ are nearly the same; they differ only by various $2e_i$. Careful inspection of this claim gives (6.10); details are left to the reader. The point of the final assertion of the lemma is that $\rho(\mathfrak{u}_\theta)$ automatically defines a Lie algebra character. The question of whether such a character exponentiates is always settled on a maximal compact subgroup. Q.E.D.

*Definition 6.11.* In the setting of (6.7) (which depends on a choice of a representation $\mu$ of $K$), define

$$\mathscr{I}_\theta = \mathscr{L}^S((\mathfrak{q}_\theta, L_\theta \cap K) \uparrow (\mathfrak{g}, K)),$$

a functor from $(\mathfrak{l}_\theta, L_\theta \cap K)$-modules to $(\mathfrak{g}, K)$-modules (Definition 6.1). We use the twist $\tau^{\sim}$ of Lemma 6.9. Put

$$(\mathscr{I}^K)_\theta = (\mathscr{L}^K)^S,$$

a functor from representations of $L_\theta \cap K$ to representations of $K$.

There are at least three simple but important observations to make about this definition. First, Corollary 6.6 applies (by Lemma 6.8): these functors relate representation theory for $G$ and for $L_\theta$ in the strong way outlined in Theorem 6.5. Second, $(\mathscr{I}^K)_\theta$ has been computed in Proposition 4.7; it essentially twists highest weights by the factor $\tau$ appearing in Lemma 6.9. In particular, inspection of (5.7), (5.20), and (5.31) shows that

$$(\mathscr{I}^K)_\theta(\mu_{L\theta}) = \mu. \tag{6.12}$$

This makes the notation of Sect. 5 consistent with that in Theorem 6.5. Third, suppose (following Harish-Chandra) that we identify infinitesimal characters with Weyl group orbits in duals of Cartan subalgebras. Then $\mathscr{I}_\theta$ preserves infinitesimal characters precisely; the $\rho$ shift appearing in (say) Proposition 6.3.11 of [36] is cancelled by our new choice of $\tau^{\sim}$.

As a final observation, we note that we can drop some of the occurrences of the phrase "lowest $K$-type".

**Lemma 6.13.** *Suppose $G = \mathrm{GL}(n, \mathbb{F})$, $\mu$ is an irreducible representation of $K$, and $X$ is an irreducible $(\mathfrak{g}, K)$-module containing the $K$-type $\mu$. Assume that either*

a) *$\mu$ is special (Definition 3.3): or*

b) *$\mathbb{F} = \mathbb{R}$, and $\mu$ is one of the representations $\mu_f$ described in Lemma 5.24.*

*Then $\mu$ is the unique lowest $K$-type of $X$.*

*Sketch of proof.* In case (a), this is an elementary consequence of the definition of lowest. The main point is that the differential of a special representation of $K$ lives on the center of $\mathfrak{g}$. Case (b) is a consequence of the fact that that $\mu_f$ is fine, and the general theory of fine $K$-types (cf. [35]). It can also be deduced directly from the subquotient theorem and the definition of lowest.

**Corollary 6.14.** *In the setting of Definition 6.11, any irreducible $(\mathfrak{l}_\theta, L_\theta \cap K)$-module containing $\mu_{L_\theta}$ has that as its unique lowest $L_\theta \cap K$-type.*

*Definition 6.15.* In the setting of (6.7), let $L$ be the subgroup of $L_\theta$ defined in Sect. 5; recall that it is equal to $L_\theta$ unless $\mathbb{F} = \mathbb{R}$, in which case it is given by (5.22). Fix a real parabolic subgroup $P$ of $L_\theta$, with Levi factor $L$; write

$$P = LN$$

for the Levi decomposition. Of course $P = L_\theta$ unless $\mathbb{F} = \mathbb{R}$; and in that case, we can take $P$ to be the block upper triangular matrices in the $\mathrm{GL}(q, \mathbb{R})$ factor of $L_\theta$ (together with the entire $\mathrm{GL}(\pi, \mathbb{C})$ factor). Define

$$\mathscr{I}_{\mathbb{R}} = \mathrm{Ind}(P \uparrow L_\theta)$$

(normalized parabolic induction), a functor from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{l}_\theta, L_\theta \cap K)$-modules. Put

$$(\mathscr{I}^K)_{\mathbb{R}} = \mathrm{Ind}(L \cap K \uparrow L_\theta \cap K),$$

a functor from representations of $L \cap K$ to representations of $L_\theta \cap K$.

Obviously this is a much more familiar kind of object than $\mathscr{I}_\theta$; everything is really going on inside $\mathrm{GL}(q, \mathbb{R})$. The functor $(\mathscr{I}^K)_{\mathbb{R}}$, for example, is essentially induction from $\mathrm{O}(q_0) \times \mathrm{O}(q_1)$ to $\mathrm{O}(q)$. For now, the most serious observation we want to make about $(\mathscr{I}^K)_{\mathbb{R}}$ is Lemma 5.24. Although it is not strictly necessary, we should also mention the weak version of Theorem 6.5 available in this context.

**Theorem 6.16.** *Suppose $q_0$ and $q_1$ are non-negative integers, and $q = q_0 + q_1$. Put $G = \mathrm{GL}(q, \mathbb{R})$, and $\mu = \mu_f$ as in Lemma 5.25. Define $L$ to be $\mathrm{GL}(q_0, \mathbb{R}) \times \mathrm{GL}(q_1, \mathbb{R})$, and $\mu_L$ to be the representation $1 \otimes \det$ of $\mathrm{O}(q_0) \times \mathrm{O}(q_1)$. Define $\mathscr{I}_{\mathbb{R}}$ and $(\mathscr{I}^K)_{\mathbb{R}}$ as in Definition 6.15.*

a) *There is a bijection $X_L \to X$ from irreducible $(\mathfrak{l}, L \cap K)$-modules, almost spherical of type $\mu_L$, onto irreducible $(\mathfrak{g}, K)$-modules containing the $K$-type $\mu$ (cf. Lemma 6.13(b)).*

b) *Suppose $X_L$ contains $\mu_L$. Then $\mathscr{I}_{\mathbb{R}}(C(X_L))$ and $C(X)$ have the same irreducible composition factors and multiplicities.*

c) *X is the unique irreducible subquotient of either* $\mathscr{I}_{\mathbb{R}}(C(X_L))$ *or of* $\mathscr{I}_{\mathbb{R}}(X_L)$ *containing* $\mu$.

d) $\mu$ *is the lowest K-type of* $(\mathscr{I}^K)_{\mathbb{R}}(\mu_L)$.

Part (a) follows essentially from the subquotient theorem (compare Chap. 4 of [36], especially Theorem 4.4.8.) Since the standard modules here are ordinary principal series for $\mathrm{GL}(q, \mathbb{R})$, part (b) is just induction by stages. Part (c) follows from (b), and (d) is Lemma 5.24.

*Definition 6.17.* Fix an irreducible representation $\mu$ of $K$, and define $L$ as in (5.6), (5.22), and (5.30). Set

$$\mathscr{I} = \mathscr{I}_\theta \circ \mathscr{I}_{\mathbb{R}}$$

(Definitions 6.11 and 6.15), a functor from $(\mathfrak{l}, L \cup K)$-modules to $(\mathfrak{g}, K)$-modules. Put

$$\mathscr{I}^K = (\mathscr{I}^K)_\theta \circ (\mathscr{I}^K)_{\mathbb{R}}$$

a functor from representations of $L \cap K$ to representations of $K$.

Here is the main theorem.

**Theorem 6.18.** *Suppose* $\mathbb{F}$ *is* $\mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$, $G = \mathrm{GL}(n, \mathbb{F})$, *and* $\mu$ *is an irreducible representation of the standard maximal compact subgroup K of G (Definition 4.2). Define* $(L, \mu_L)$ *as in Sect. 5 ((5.6) and (5.7); (5.22) and (5.23); and (5.30) and (5.31)). Then L is a product of various* $\mathrm{GL}(m_i, \mathbb{F}_i)$, *and* $\mu_L$ *is a special one dimensional representation of* $L \cap K$ *(Definition 3.3). The functor* $\mathscr{I}$ *of Definition 6.17 defines a bijection from the set of irreducible unitary representations of L, almost spherical of type* $\mu_L$, *onto the set of irreducible unitary representations of G of lowest K-type* $\mu$. *In particular,* $\mathscr{I}$ *has the following three properties.*

a) *If Y is a basic almost spherical representation of L of type* $\mu_L$, *then* $\mathscr{I}Y$ *is unitary.*

b) *If Y is a basic almost spherical representation of L of type* $\mu_L$, *then* $\mathscr{I}Y$ *is irreducible.*

c) *If X is any irreducible unitary representation of G of lowest K-type* $\mu$, *then there is a unitary almost spherical representation Y of L, such that X is a subquotient of* $\mathscr{I}Y$.

Together with Theorem 3.8, this parametrizes the unitary dual of $G$. The reader may be unhappy with the pervasive use of cohomological induction in the statement, even over $\mathbb{C}$ where it is not needed. This is done partly for philosophical reasons, but mostly for convenience in the proof. In those cases (for example always over $\mathbb{C}$) when $L$ is the Levi factor of a real parabolic subgroup, we can (using known "independence of polarization" results) replace $\mathscr{I}$ by ordinary parabolic induction; and in general, $\mathscr{I}$ may be built mostly from ordinary induction. We will return to this point in Sect. 17. In any case, the functors $\mathscr{I}$ are quite computable (for example on the level of distribution characters).

We conclude this section with the $(\mathfrak{g}, K)$-module analogue of Theorem 6.18.

**Theorem 6.19.** *Suppose we are in the setting of Theorem 6.18.*

a) *There is a bijection* $X_L \rightarrow X$ *from irreducible* $(\mathfrak{l}, L \cap K)$*-modules, almost spherical of type* $\mu_L$, *onto irreducible* $(\mathfrak{g}, K)$*-modules with lowest K-type* $\mu$.

b) *$X$ is the unique irreducible subquotient of either $\mathscr{I}(C(X_L))$ or of $\mathscr{I}(X_L)$ containing $\mu$.*

c) *$\mu$ is the lowest $K$-type of $\mathscr{I}^K(\mu_L)$.*

This is immediate from Corollary 6.6, the first remark after Definition 6.11, and Theorem 6.16. We lose (b) of Theorem 6.16 because $\mathscr{I}$ is not exact; this could be remedied by replacing $\mathscr{I}_\theta$ by an alternating sum of cohomological induction functors.

Because of this theorem, parts (a), (b), and (c) of Theorem 6.18 imply the rest of that theorem.

## 7. Small representations of $K$

In this section, we will begin to introduce some of the ideas needed to explain the proof of Theorem 6.18. We must begin in the context of Theorem 3.8.

*Definition 7.1.* Fix a collection (one for each $m$) of special one dimensional representations $\mu_m$ of $K(m, \mathbb{F})$, the restrictions to $K$ of some characters $j_m$ of $G$ (Definition 3.3). Fix $n$, and an integer $q$ between 0 and $n$. Define

$$Z(q; n) = \operatorname{Ind}(K(q) \times K(n-q) \uparrow K(n))(\mu_q \otimes \mu_{n-q}),$$

an infinite dimensional representation of $K(n)$. (We will write $Z(q)$ when no confusion will result; in general we want to consider these representations for fixed $n$ and varying $q$.) It will often be convenient to write

$$Z(-1; n) = 0.$$

Although we could proceed immediately with some abstract definitions, it is perhaps more helpful to recall the (well known) decomposition of $Z(q)$ into irreducibles. Set

$$\begin{aligned} X(q; n) &= K(n)/(K(q) \times K(n-q)) \\ &= \text{Grassmanian of } q\text{-planes in } \mathbb{F}^n. \end{aligned} \tag{7.2}$$

For example, $X(1; n)$ is $n-1$ dimensional projective space over $\mathbb{F}$. Obviously

$$Z(q; n) = \mu_n \otimes (\text{functions on } X(q; n)). \tag{7.3}$$

The space $X(q; n)$ is a symmetric homogeneous space for $K(n)$, so the space of functions on $X(q; n)$ is computed by Helgason's theorem ([17], Corollary V.4.2). Making explicit the definitions there, and using the parametrizations of $\hat{K}$ in (5.4), Proposition 5.17, and (5.28), we arrive at the following lemma.

**Lemma 7.4.** *In the setting of Definition 7.1, suppose $q \leqq [n/2]$. Then the $K$-types of $Z(q)$ all occur with multiplicity one. They are parametrized by* (weakly) *decreasing sequences $(a_1, \ldots, a_q)$ of $q$ non-negative integers, as follows.*

a) *Suppose $\mathbb{F} = \mathbb{C}$, and $\mu$ is* $\det^m$ *(of highest weight $(m, \ldots, m)$). Then the constituents of $Z(q)$ have highest weights*

$$(m + a_1, m + a_2, \ldots, m + a_q, m, \ldots, m, m - a_q, \ldots, m - a_1).$$

b) *Suppose* $\mathbb{F}=\mathbb{R}$, *and* $\mu$ *is* $\det^\eta$, *with* $\eta$ *equal to zero or one (with highest weight* $((0, \ldots, 0), \eta)$. *Then the constituents of* $Z(q)$ *have highest weights*

$$((2a_1, \ldots, 2a_q, 0, \ldots, 0), \eta'),$$

*with* $\eta'$ *equal to* $\eta$ *or* $\frac{1}{2}$.

c) *Suppose* $\mathbb{F}=\mathbb{H}$, *so that* $\mu$ *must be trivial. Then the constituents of* $Z(q)$ *have highest weights*

$$(a_1, a_1, a_2, \ldots, a_q, a_q, 0, \ldots, 0).$$

Since obviously $Z(q)$ is isomorphic to $Z(n-q)$, this lemma computes $Z(q)$ in all cases.

**Corollary 7.5.** *In the setting of Definition 7.1, suppose* $q' \leqq q \leqq [n/2]$. *Then there is an inclusion*

$$Z(q') \hookrightarrow Z(q).$$

This inclusion is analytically quite interesting, and may be obtained more explicitly in various ways. For example, Lemma 2.4(d) exhibits an inclusion of two $(\mathfrak{g}, K)$-modules. Restricted to $K$, they are just $Z(m-1; 2m)$ and $Z(m; 2m)$. The proof of that lemma (which relies on fairly serious results still to come) extends easily to yield all the inclusions of Corollary 7.5.

Alternatively (at least as a point of view; the mathematics is in some sense the same), there is a Radon transorm from functions on $X(q')$ to functions on $X(q)$, obtained by integrating over $q'$-planes contained in a fixed $q$-plane. From this point of view, there is some analysis to do to prove the injectivity of the transform.

*Definition 7.6.* In the setting of Definition 7.1, define (for $0 \leqq q \leqq [n/2]$)

$$\omega_q(\mu; n) = \text{lowest } K\text{-type of } Z(q; n)/Z(q-1; n).$$

This makes sense because of Corollary 7.5. In the parametrization of Lemma 7.4, $\omega_q$ corresponds to the sequence

$$(a_i) = (1, \ldots, 1, 0, \ldots, 0) \quad (q \text{ ones}).$$

In particular, $\omega_0(\mu)$ is $\mu$ itself. The set of *small representations of K associated to* $\mu$ is

$$S(\mu) = \{\omega_q(\mu) \mid 1 \leqq q \leqq [n/2]\}.$$

More generally, suppose $G$ is a product of various $GL(m_i, \mathbb{F}_i)$, and $\mu$ is a special one dimensional representation of $K$; say $\mu = (\otimes \mu_i)$. Then

$$S(\mu) = \{\otimes \delta_i \mid \delta_i = \mu_i \text{ for } i \neq i_0, \text{ and } \delta_{i_0} \in S(\mu_{i_0})\}.$$

*Definition 7.7.* Let $Y$ be an irreducible almost spherical $(\mathfrak{g}, K)$-module of type $\mu$. We say that *Y satisfies Hypothesis 7.7* if it falls in one of the following three cases.

Case 1. $Y$ does not admit an invariant Hermitian form $\langle , \rangle$.

Case 2. $Y$ admits an invariant Hermitian form which is positive on $\mu$, but not on every $K$-type in $S(\mu)$ (Definition 7.6).

Case 3. $Y$ is basic (Definition 3.7).

Obviously Theorem 3.8(d) follows immediately from

**Theorem 7.8.** *Suppose $G = \mathrm{GL}(n, \mathbb{F})$, and $Y$ is an irreducible almost spherical $(\mathfrak{g}, K)$-module of type $\mu$ (Definition 3.3). Then $Y$ satisfies Hypothesis 7.7.*

This theorem says that $Y$ can fail to be unitary only if it admits no invariant Hermitian form, or if its form has different signs on the $K$-type $\mu$ and some small $K$-type $\omega$. Its advantage over Theorem 3.8 is not that it is intrinsically interesting – after all, it says nothing about the unitary representations – but that it is ideally suited to proving Theorem 6.18. Its proof is in Sects. 12–14.

## 8. Relatively small representations of $K$: Fine case

In this section, we will formulate an extension of Definition 7.6 to general representations of $K$. The most difficult case is that of the $K$-types considered in Lemma 5.24, and this section will be devoted to it. It is rather technical, and the reader may wish to omit it. We need from it only Corollary 8.12.

We begin with a very useful general fact.

**Proposition 8.1.** *Suppose $G$ is a real reductive group, $K$ is a maximal compact subgroup, and $P$ is a parabolic subgroup. Fix a Levi decomposition $P = LN$ with $L$ stable under the Cartan involution. Write $M = L \cap K$.* Let $Y$ be an irreducible $(\mathfrak{l}, M)$-*module, and $Z$ an irreducible representation of $M$ occuring in $Y$. Put*

$$X = \mathrm{Ind}(P \uparrow G)(Y)$$

*If $Z$ is not a lowest $M$-type of $Y$, then* Ind $(M \uparrow K)(Z)$ *contains no lowest $K$-type of $X$. That is, the lowest $K$-types of the induced are contained in the induced from the lowest.*

*Proof.* Recall the parameter $\lambda(\mu)$ attached to a representation of $K$ by Proposition 5.3.1 of [36]. (It was referred to as $\lambda_{cl}$ in the proof of Lemma 6.8, to avoid confusion with the $\lambda$ defined only for $GL(n)$ in section 5. Since it is used now only within this proof, no confusion should result.) We will use the fact ([35], Lemma 8.8) that lowest can be defined by minimizing the length of this parameter. Let $Z_0$ be a lowest $M$-type of $Y$. Then we conclude that

$$|\lambda(Z_0)| < |\lambda(Z)|. \tag{8.2}$$

Recall from [35] that $\lambda(Z_0)$ is the Harish-Chandra parameter for the discrete series representation $\pi$ figuring in the Langlands classification for $Y$. Let $\mu$ be a lowest $K$-type of $X$, and $X_0$ an irreducible subquotient of $X$ containing $\mu$. Then $\pi$ is also the Langlands discrete series for $X_0$, so $\lambda(\mu)$ is actually conjugate to $\lambda(Z_0)$. In particular,

$$|\lambda(Z_0)| = |\lambda(\mu)|. \tag{8.3}$$

Choose a representation $Y'$ of $L$ with lowest $M$-type $Z$ (as is always possible), and let $\mu'$ be the lowest $K$-type of $X' = \mathrm{Ind}(Y')$. Applying (8.3) to these representations gives

$$|\lambda(Z)| = |\lambda(\mu')|. \tag{8.4}$$

If $\mu''$ is any $K$-type of $\mathrm{Ind}(Z)$, then $\mu''$ occurs in $X'$: so (using the analogue of (8.2) for $G$)

$$|\lambda(\mu')| \leqq |\lambda(\mu'')|. \tag{8.5}$$

Combining (8.2)–(8.5) gives

$$|\lambda(\mu)| < |\lambda(\mu'')|,$$

which is the conclusion of the proposition. Q.E.D.

Here are the standing hypotheses for the rest of this section. $G$ will be $\mathrm{GL}(n, \mathbb{R})$. Fix non-negative integers $q_0$ and $q_1$, such that

(8.6)(a) $$q_0 + q_1 = n.$$

Recall from Lemma 5.24 the representation

(8.6)(b) $$\mu = \mu_f(q_0, q_1).$$

The representation $\mu$ is defined by specifying its highest weight. It is an easy exercise to calculate the highest weights of the exterior algebra representations, and deduce that

(8.6)(c) $$\mu = [\Lambda^{q_1}(\mathbb{R}^n)] \otimes_{\mathbb{R}} \mathbb{C}.$$

It will be convenient to write

(8.6)(d) $$M = \mathrm{O}(q_0) \times \mathrm{O}(q_1).$$

*Proof of Lemma 5.24.* We want to show that $\mu$ is the lowest $K$-type of $\mathrm{Ind}(1 \otimes \det)$. We will apply Proposition 8.1, with $P = P(q_0, q_1)$ (Definition 3.6). The conclusion is that we may replace $1 \otimes \det$ by (the restriction to $M$ of) any irreducible representation $Y$ with unique lowest $M$-type $1 \otimes \det$. We choose for $Y$ a full principal series representation of $\mathrm{GL}(q_0) \times \mathrm{GL}(q_1)$. Then $X = \mathrm{Ind}(Y)$ is a full principal series for $G$. We need to show that $\mu$ is the lowest $K$-type of $X$. This follows from Sect. 5.3 of [36], because $\mu$ is fine (as was first observed in [5]). Q.E.D.

*Definition 8.7.* In the setting (8.6), and using the notation of Definition 7.6, define

$\omega_{q,0}(\mu)$ = lowest $K$-type of $\mathrm{Ind}(M \uparrow K)(\omega_q(1) \otimes \det)$ $(0 \leqq q \leqq [q_0/2])$

$\omega_{q,1}(\mu)$ = lowest $K$-type of $\mathrm{Ind}(M \uparrow K)(1 \otimes \omega_q(\det))$ $(0 \leqq q \leqq [q_1/2])$.

The set of *relatively small representations of $K$ associated to $\mu$* is

$$S(\mu) = \{\omega_{q,0}(\mu) \mid 0 < q \leqq [q_0/2]\} \cup \{\omega_{q,1}(\mu) \mid 0 < q \leqq [q_1/2]\}.$$

Because of the last part of Definition 7.6, this is precisely the set of lowest $K$-types of $\mathrm{Ind}(M \uparrow K)(\omega)$, as $\omega$ runs over $S(1 \otimes \det)$.

We want to compute $S(\mu)$ explicitly.

**Lemma 8.8.** *In the notation of Definition 8.7, the highest weight of $\omega_{q,0}(\mu)$ may be computed as follows. Recall $m$ and $\varepsilon$ from (5.8). Write $(\gamma, \eta)$ for the highest weight* *of the representation $\mu_f(q_0-2q, q_1)$ of* O$(n-2q)$; *this is defined in Lemma 5.24. (In particular, $\gamma$ is a string of ones and zeros.) Then the highest weight of $\omega_{q,0}(\mu)$ is*

$$((2, \ldots, 2, \gamma), \eta) \quad (q\, 2's).$$

*An analogous formula applies to $\omega_{q,1}(\mu)$.*

*Proof.* The representation $\omega_q(1; 2q)$ of O$(2q)$ has highest weight $((2, \ldots, 2), \frac{1}{2})$. It is very easy to check from the definition that $\omega_q(1; q_0)$ is the lowest $K$-type of

$$\text{(8.9)} \qquad \operatorname{Ind}(\mathrm{O}(2q) \times \mathrm{O}(q_0-2q))(\omega_q \otimes 1).$$

(Part of the reason this is easy is that the inducing subgroup contains a Cartan subgroup.) It therefore follows from Proposition 8.1 and Definition 8.7 that $\omega_{q,0}(\mu)$ is the lowest $K$-type of

$$\text{(8.10)} \qquad \operatorname{Ind}(\mathrm{O}(2q) \times \mathrm{O}(q_0-2q) \times \mathrm{O}(q_1))(\omega_q \otimes 1 \otimes \det).$$

Now we reverse the reasoning, and compute the lowest $K$-type by induction by stages with the last two factors grouped together. Lemma 5.24 and Proposition 8.1 imply that $\omega_{q,0}(\mu)$ is the lowest $K$-type of

$$\text{(8.11)} \qquad \operatorname{Ind}(\mathrm{O}(2q) \times \mathrm{O}(n-2q))(\omega_q \otimes \mu_f(q_0-2q, q_1)).$$

Again because the inducing subgroup contains a Cartan for $K$, this lowest $K$-type is easy to compute; it is the one given in the conclusion of the lemma. Q.E.D.

**Corollary 8.12.** *In the setting of Definition 8.7, taking the lowest K-type of* Ind$(M \uparrow K)$ *defines a bijection from $S(1 \otimes \det)$ onto $S(\mu)$. Fix $\omega$ in $S(\mu)$, and a standard* SO(2) *inside K. Then the weights of the* SO(2) *in $\omega$ are integers between* $-2$ *and* 2.

It is only the latter assertion of this corollary which we will really need, and that only in the following weak form: if $\delta$ belongs to $S(1 \otimes \det)$, then there is a representation $\omega$ of $K$ which contains $\delta$, and has SO(2) weights in between $-2$ and 2. This sounds like it should be very easy (since the SO(2) weights of $\delta$ are between $-2$ and 2). To see that there was really something to check, consider the analogous problem when $\delta$ is $1 \otimes \det$. This has all its SO(2) weights equal to zero; but any representation of $K$ containing it must have some non-zero SO(2) weights.

## 9. Relatively small representations of $K$: General case

We have up until now defined $S(\mu)$ when $\mu$ is special one dimensional, and when $\mu$ is one of the representations of O$(n)$ in Lemma 5.24. We extend this definition to product groups as at the end of Definition 7.6. This covers the case of the representations $\mu_{L_\theta}$ of $L_\theta \cap K$, introduced in Sect. 5. The following easy lemma is one of the main steps in the reduction of Theorem 6.18 to the almost spherical case. It will guarantee the existence of representations of $K$ on which it is possible to study Hermitian forms.

**Lemma 9.1.** *Suppose $G = \mathrm{GL}(n, \mathbb{F})$, and $\mu$ is an irreducible representation of K. Define $L_\theta$ and $\mu_{L_\theta}$ as in Sect. 5 (cf.* (5.6) *and* (5.7); (5.19) *and* (5.20); *and* (5.30) *and*

(5.31)). *Recall the character* $\tau$ *of* $L_\theta \cap K$ *defined in Lemma 6.9. Suppose* $\omega_{L_\theta}$ *belongs to* $S(\mu_{L_\theta})$. *Then the highest weight of* $\omega_{L_\theta} \otimes \tau$ *is K-dominant: so*

$$\omega = (\mathscr{I}^K)_\theta(\omega_{L_\theta} \otimes \tau)$$

(*cf. Definition 6.11 and Proposition 4.7*) *is a non-zero representation of K.*

*Proof.* We have computed all the highest weights and positive roots explicitly, so we only need to inspect the results in each case. Consider for example the case $\mathbb{F} = \mathbb{R}$, with $\omega_{L_\theta}$ differing from $\mu_{L_\theta}$ only on the $GL(n-2p, \mathbb{R})$ factor. Using the notation around (5.19), we see from Lemma 8.8 that $\omega_{L_\theta}$ has highest weight of the form

$$(9.2) \quad ((\gamma_1 - 1, \ldots, \gamma_p - 1, 2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0), \eta').$$

Tensoring with $\tau$ adds 1 to the first $p$ coordinates. The resulting weight will be dominant if the sequence is decreasing and non-negative. Since $\gamma$ already has this property, we only need to check that $\gamma_p$ is at least 2. But $p$ was defined exactly to make this true, proving the lemma in this case. The other cases are similar, slightly easier, and notationally more complicated; so we leave them to the reader. Q.E.D.

*Definition 9.3.* In the setting of Lemma 9.1, define the set $S(\mu)$ of *relatively small representations of K associated to* $\mu$ to be the set of all irreducible constituents of the various $(\mathscr{I}^K)_\theta(\omega_{L_\theta})$, as $\omega_{L_\theta}$ runs over $S(\mu_{L_\theta})$. This latter set is defined in Definitions 7.6 and 8.7, and the remarks preceding Lemma 9.1.

Even though we need no more than is contained in Lemma 9.1, careful inspection of the calculations in it actually proves a bit more.

**Corollary 9.4.** *In the setting of Lemma 9.1, we have*

a) $(\mathscr{I}^K)_\theta$ *defines a bijection from* $S(\mu_{L_\theta})$ *onto* $S(\mu)$: *and*

b) *taking the lowest K-type of* $\mathscr{I}^K(\omega_{L_\theta})$ *defines a bijection from* $S(\mu_L)$ *onto* $S(\mu)$. *Here we use the notation of Definition 6.17.*

We record here the part of Theorem 6.18 which will follow from Theorem 7.8.

**Theorem 9.5.** *Suppose we are in the setting of Theorem 6.18, and X is an irreducible* $(\mathfrak{g}, K)$*-module with lowest K-type* $\mu$. *Assume that X admits an invariant Hermitian form* $\langle , \rangle$, *which we may assume to be positive on* $\mu$. *Let* $X_L$ *be the irreducible* $(\mathfrak{l}, L \cap K)$*-module corresponding to X (Theorem 6.19). Then either*

a) $X_L$ *is basic: or*

b) *there is a K-type* $\omega$ *in* $S(\mu)$ *(Definition 9.3) on which* $\langle , \rangle$ *is not positive.*

This includes (the contrapositive of) Theorem 6.18(c). It follows from Theorem 7.8 and Proposition 10.2 below.

## 10. Induction and Hermitian forms

In this section, we will recall some known results about the effect of various kinds of induction on Hermitian forms. These will be the basic tools used in proving Theorems 7.8 and 9.5. We begin with a definition from [38]. Recall that a $(\mathfrak{g}, K)$-module is called *admissible* if each $K$-type has finite multiplicity.

*Definition 10.1.* Let $Y$ be an admissible representation of $G$, with an invariant Hermitian form $\langle,\rangle$. The *signature of* $\langle,\rangle$ is a triple $(p, q, z)$ of three functions from $\hat{K}$ to $\mathbb{N}$, defined as follows. Fix an irreducible representation $(\delta, V_\delta)$ of $K$, and a positive invariant Hermitian form on $V_\delta$. Then

$$Y_\delta = \mathrm{Hom}_K(V_\delta, Y) \cong (V_\delta)^* \otimes_K Y$$

acquires an invariant Hermitian form $\langle,\rangle_\delta$. We define $z(\delta)$ to be the dimension of the radical of $\langle,\rangle_\delta$, and $(p(\delta), q(\delta))$ to be the signature of the induced non-degenerate form on $Y_\delta/(\mathrm{rad}\langle,\rangle_\delta)$. Thus the multiplicity of $\delta$ in $Y$ is given by

$$m(\delta) = p(\delta) + q(\delta) + z(\delta).$$

For definiteness, we may sometimes write $p(\delta, Y)$, etc.

In terms of signatures, we can formulate the reduction theorem which gives Theorem 9.5 from Theorem 7.8.

**Proposition 10.2.** *Suppose $G$ is* $\mathrm{GL}(n, \mathbb{F})$. *Let $X$ be an irreducible representation of $G$ admitting a non-zero invariant Hermitian form $\langle,\rangle$. Let $\mu$ be the lowest $K$-type of $X$, and define $(L, \mu_L)$ as in section 5. Let $X_L$ be the irreducible almost spherical representation of $L$ of type $\mu_L$ corresponding to $X$ (Theorem 6.19).*

a) *$X_L$ admits a non-zero invariant Hermitian form $\langle,\rangle_L$. Fix a $K$-type*

$$\omega \in \{\mu\} \cup S(\mu)$$

*(Definition 9.3), and let*

$$\omega_L \in \{\mu_L) \cup S(\mu_L)$$

*be the corresponding $L \cap K$ type (Corollary 9.4). Assume that $\langle,\rangle$ is normalized to be positive on $\mu$, and $\langle,\rangle_L$ to be positive on $\mu_L$. Then*

b) $$p(\omega, X) \geqq p(\omega_L, X_L): \quad q(\omega, X) \geqq q(\omega_L, X_L)$$

*(cf. Definition 10.1).*

We will prove this result at the end of this section.

We consider now the behavior of Hermitian forms under ordinary parabolic induction. We may as well work in the context of a general reductive Lie group $G$, with maximal compact subgroup $K$. Suppose $Q$ is a parabolic subgroup of $G$. Then necessarily

(10.3)(a) $$L = Q \cap \theta Q$$

is a Levi factor of $Q$. Write $U$ for the unipotent radical of $Q$. Then there is a Levi decomposition

(10.3)(b) $$Q = LU,$$

a semidirect product with $U$ normal.

Each element of $G$ can be written as a product of an element of $K$ and an element of $Q$:

(10.4) $$G = KQ.$$

**Proposition 10.5.** *Let $Q = LU$ be a parabolic subgroup of $G$ as in (10.3), and $Y$ an $(\mathfrak{l}, L \cap K)$-module.*

a) *As representations of $K$,*

$$\operatorname{Ind}(Q \uparrow G)(Y) \cong \operatorname{Ind}(L \cap K \uparrow K)(Y).$$

*This isomorphism is defined by restricting functions in the induced representation to $K$. In particular (writing $m(\tau, \sigma)$ for the multiplicity of a representation $\tau$ in an appropriate restriction of $\sigma$), we have for any $\mu$ in $\hat{K}$,*

$$m(\mu, \operatorname{Ind}(Q \uparrow G)(Y)) = \sum_{\tau \in (L \cap K)^{\wedge}} m(\tau, \mu)\, m(\tau, Y).$$

b) *Suppose $Y$ admits an invariant Hermitian form $\langle , \rangle_L$. If we regard elements of the induced representation as functions on $K$ with values in $Y$ (as is possible by* (a)), *then*

$$\langle v, w \rangle = \int_K \langle v,(k), w(k) \rangle_L\, dk$$

*defines an invariant Hermitian form on the induced representation. Its signature (Definition 10.1) is given by the formula in* (a), *with the first and third $m$'s replaced by $p$, $q$, or $z$. In particular, the induced form is non-degenerate (respectively, positive definite) if and only if $\langle , \rangle_L$ is.*

This is standard and easy. In (b), the fact that induction preserves unitarity goes back at least to Wigner. The importance of the (equally simple) fact that it also preserves failure to be unitary was noticed more recently; I learned of it in the dissertation of Birgit Speh [28].

We turn now to the case of cohomological parabolic induction.

**Proposition 10.6.** *In the setting of Definition 6.1, let $Y$ be an $(\mathfrak{l}, L \cap K)$-module, endowed with an invariant Hermitian form $\langle , \rangle_L$ (which may be degenerate). Let $S$ be the dimension of $\mathfrak{u} \cap \mathfrak{k}$.*

a) *$\mathscr{L}^S Y$ and $(\mathscr{L}^K)^S Y$ carry induced Hermitian forms*

$$\mathscr{L}^S(\langle , \rangle_L) = \langle , \rangle_G = \langle , \rangle, \quad (\mathscr{L}^K)^S(\langle , \rangle_L) = \langle , \rangle_K.$$

b) *The restriction (that is, pullback via $\psi^S$; cf.* (6.2)a)) of *$\langle , \rangle_G$ to $(\mathscr{L}^K)^S$ is $\langle , \rangle_K$.*

c) *$(\mathscr{L}^K)^S$ carries positive definite forms to positive definite forms.*

This is closely related to the theorem of Enright and Wallach in [11]. Part (c) is due to Enright; the entire proposition may be found in Corollary 5.5 and Proposition 6.10 of [36].

One should keep in mind the fact (Proposition 6.3(c)) that $\psi^S$ is an inclusion. It is helpful to drop the map from the notation, writing

$$\mathscr{L}^K Y \subset \mathscr{L} Y. \tag{10.7}$$

This inclusion plays some of the role of the isomorphism in Proposition 10.5(a).

We have most of the machinery needed to prove the following lemma, but some otherwise useless definitions would be required first. We will therefore regard it as a formal consequence of the fact that the set of irreducible $(\mathfrak{g}, K)$-modules admitting non-zero invariant Hermitian forms is explicitly known (by the work of Knapp-Zuckerman and others; see for example [37] or [38]).

**Lemma 10.8.** *In the setting either of Theorem 6.5 or of Theorem 6.16, $X$ admits a non-zero invariant Hermitian form if and only if $X_L$ does.*

**Proposition 10.9.** *In the setting of Theorem 6.5, suppose $X$ admits a non-zero invariant Hermitian form $\langle,\rangle$. Let $\mu$ be a lowest $K$-type of $X$, and $\mu_L$ the corresponding lowest $L\cap K$ type of $X_L$. Let $\langle,\rangle_L$ be a non-zero invariant Hermitian form on $X_L$ (Lemma 10.8). Assume that $\langle,\rangle$ is normalized to be positive on $\mu$, and $\langle,\rangle_L$ to be positive on $\mu_L$. Fix an irreducible representation $\omega_L$ of $L\cap K$. Assume that $(\mathscr{L}^K)^S(\omega_L)$ is non-zero: fix an irreducible constituent $\omega$ of it. Then*

$$p(\omega, X) \geqq p(\omega_L, X_L)$$
$$q(\omega, X) \geqq q(\omega_L, X_L)$$

*(cf. Definition 10.1).*

*Proof.* Since $X_L$ is a quotient of $C(X_L)$ by some $Q_L$ (Theorem 6.4), we can regard $\langle,\rangle_L$ as a form on $C(X_L)$ with radical $Q_L$. Write $\langle,\rangle_G$ for the induced form on

$$C(X) = \mathscr{L}^S(C(X_L))$$

(Theorem 6.5(b) and Proposition 10.6). By Proposition 10.6(b) and (c), $\langle,\rangle_G$ is positive on $\mu$. Because $X$ is the unique irreducible quotient of $C(X)$ (say by $Q$), and occurs only once as a composition factor, a formal argument shows that the radical of $\langle,\rangle_G$ is $Q$. So $\langle,\rangle_G$ is just $\langle,\rangle$, up to a positive constant multiple. Write $\langle,\rangle_K$ for the form on $(\mathscr{L}^K)^S(C(X_L))$ induced by $\langle,\rangle_L$. Proposition 10.6(c) gives

$$p(\omega, \langle,\rangle_K) \geqq p(\omega_L, X_L)$$
$$q(\omega, \langle,\rangle_K) \geqq q(\omega_L, X_L).$$

Proposition 10.6(b) gives

$$p(\omega, \langle,\rangle) \geqq p(\omega, \langle,\rangle_K)$$
$$q(\omega, \langle,\rangle) \geqq q(\omega, \langle,\rangle_K).$$

The proposition follows. Q.E.D.

This argument used Theorem 6.5(b) in an essential way. The analogous statement fails in the context of Theorem 6.16. To replace it, we need the following lemma.

**Lemma 10.10.** *In the setting of Theorem 6.16, fix an irreducible representation $\omega$ of $K$. Assume that in the restriction of $\omega$ to a standard* SO(2), *only the weights between $-2$ and $2$ appear. Then the multiplicity of $\omega$ in $X$ is the same as its multiplicity in $\mathscr{I}_{\mathbb{R}}(X_L)$.*

This is proved in precisely the same way as Lemma 13.3 below, so we leave the details to the reader.

Here is an analogue of Proposition 10.9.

**Proposition 10.11.** *In the setting of Theorem 6.16, suppose $X$ admits a non-zero invariant Hermitian form $\langle,\rangle$. Let $\langle,\rangle_L$ be a non-zero invariant Hermitian form on $X_L$ (Lemma 10.8). Assume that $\langle,\rangle$ is normalized to be positive on $\mu$, and $\langle,\rangle_L$ to be positive on $\mu_L$. Fix an irreducible representation $\omega_L$ of $L\cap K$, and an irreducible* *constituent $\omega$ of $(\mathscr{I}^K)_{\mathbb{R}}(\omega_L)$. Assume that in the restriction of $\omega$ to a standard* SO (2), *only the weights between* $-2$ and 2 appear. Then

$$p(\omega, X) \geqq p(\omega_L, X_L)$$
$$q(\omega, X) \geqq q(\omega_L, X_L)$$

*(cf. Definition 10.1).*

*Proof.* One essentially imitates the proof of Proposition 10.9, using Proposition 10.5(b) in place of Proposition 10.6. One has to find some relationship between $\langle , \rangle_G$ and $\langle , \rangle$ on $\omega$; this is accomplished by Lemma 10.10. Details are left to the reader. Q.E.D.

In light of the defining properties of $S(\mu)$ established in Sect. 8 and 9, we see that Proposition 10.2 follows from Propositions 10.9 and 10.11.

## 11. Spherical representations: General results

At last it is time to pass beyond formal preliminaries to the substance of the argument. The first item on the agenda is the proof of Theorem 7.8. In Sect. 12 through 14, we will establish it for each of the three fields in question. First, however, we need some notation and results common to the three cases. Write

(11.1)(a) $$A = \text{group of } n \times n \text{ diagonal matrices with positive real entries}$$

(11.1)(b) $$M = \text{centralizer of } A \text{ in } K \cong K(1, \mathbb{F}) \times \ldots \times K(1, \mathbb{F}) \quad (n \text{ copies}).$$

Here $K(1, \mathbb{F})$ (notation (3.2)) is the maximal compact subgroup of the multiplicative group of $\mathbb{F}$; it is U (1), $\mathbb{Z}/2\mathbb{Z}$, or Sp (1) (which is SU (2)) according as $\mathbb{F}$ is $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{H}$. The group $A$ may be taken as the one figuring in an Iwasawa decomposition of $G$. The Lie algebra of $A$ is

(11.2)(a) $$\mathfrak{a}_0 \cong \mathbb{R}^n,$$

the isomorphism identifying the exponential map for $A$ with the usual exponential map on each diagonal entry. Consequently

(11.2)(b) $$\hat{A} \cong \mathfrak{a}^* \cong \mathbb{C}^n.$$

The set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$ is

(11.2)(c) $$\Delta(\mathfrak{g}, \mathfrak{a}) = \{e_i - e_j\};$$

each has multiplicity $d$ (the dimension of $\mathbb{F}$ over $\mathbb{R}$). The little Weyl group is

(11.2)(d) $$W(A) = S_n,$$

the group of permutations of the coordinates in $A$. Taking the obvious choice for $\Delta^+$ (cf. 5.3)), we get

$$(11.3) \qquad N = \text{upper triangular matrices with ones on the diagonal}$$

for the Iwasawa N. Notice that the group $B$ of (3.5) is just $MAN$.

*Definition 11.4.* Suppose $\nu$ is a character of $A$ (which, by (11.2)(b), we may regard as an element of $\mathbb{C}^n$). The *spherical principal series representation with parameter* $\nu$, $I(\nu)$, is the Harish-Chandra module of

$$\mathrm{Ind}(B \uparrow G)(\nu).$$

Here we regard $\nu$ as a character of $B = MAN$ by making $M$ and $N$ act trivially, and we use normalized induction. (With this parametrization, $I(\nu)$ is naturally unitary whenever $\nu$ lies in $i\mathbb{R}^n$.) As is well known,

$$I(\nu)|_K \cong \mathrm{Ind}(M \uparrow K)(\mathbb{C}).$$

In particular, the trivial representation of $K$ occurs exactly once in $I(\nu)$. Write

$$J(\nu) = \text{unique irreducible subquotient of } I(\nu) \text{ containing the trivial representation of } K.$$

Here are some of the basic results about spherical representations. They are by now partly "classical", but some proofs and references may be found in [22] and [24].

**Theorem 11.5.** *Fix notation as above.*

a) *If* $w \in W(A)$ *and* $\nu \in \mathfrak{a}^*$, *then* $J(\nu)$ *is isomorphic to* $J(w\nu)$.

b) *If* $\nu$ *and* $\nu'$ *belong to* $\mathfrak{a}^*$, *and* $J(\nu)$ *is isomorphic to* $J(\nu')$, *then there is a* $w \in W(A)$ *with* $w\nu = \nu'$.

*In the remaining statements,* $\nu$ *is an element of* $\mathfrak{a}^*$, *which is sometimes identified with* $\mathbb{C}^n$.

c) $J(\nu)$ *admits a non-degenerate invariant Hermitian form if and only if* $\nu$ *is conjugate under* $W(A)$ *to* $-\bar{\nu}$: *that is, if and only if the sequence* $(\nu_i)$ *is a permutation of* $(-\bar{\nu}_i)$.

d) *Assume that the sequence* $(\mathrm{Re}\,\nu_i)$ *is decreasing. Then* $J(\nu)$ *is the unique irreducible quotient of* $I(\nu)$.

e) *Assume that the sequence* $(\mathrm{Re}\,\nu_i)$ *is increasing. Then* $J(\nu)$ *is the unique irreducible subrepresentation of* $I(\nu)$.

*Fix* $m < n$. *Let* $s$ *be the permutation transposing* $m$ *and* $m+1$, *and let* $\nu' = s\nu$.

f) *There is a natural intertwining operator* $A(s)$ *from* $I(\nu)$ *to* $I(\nu')$: *it is induced from the corresponding operator on the minimal parabolic subgroup corresponding to* $s$.

g) *If* $(\mathrm{Re}\,\nu_m) \geqq (\mathrm{Re}\,\nu_{m+1})$, *then* $A(s)$ *is non-zero on the trivial representation of* $K$.

h) $A(s)$ *is an isomorphism unless* $\nu_m - \nu_{m+1}$ *is of the form* $\pm(d+2k)$, *with* $d$ *the real dimension of* $\mathbb{F}$, *and* $k$ *in* $\mathbb{N}$.

The following corollary, although not decisive, indicates the general way in which we will use this result.

**Corollary 11.6.** *Fix* $\nu$ *in* $\mathfrak{a}^*$. *Partition the various* $\nu_i$ *into subsets in such a way that if* $\nu_i$ *and* $\nu_j$ *belong to different subsets, then* $\nu_i - \nu_j$ *is not an even integer (if* $\mathbb{F}$ *is* $\mathbb{C}$ *or* $\mathbb{H}$), *or* *not an integer (if* $\mathbb{F}$ *is* $\mathbb{R}$*). Permute* $v$ *to a new sequence* $v'$*, in which each subset is an interval: that is, of the form*

$$\{(v')_p, (v')_{p+1}, \ldots, (v')_q\}.$$

*The subsets then define an ordered partition* $\pi$ *of* $n$ *(Definition 3.6). For each part* $p_j$ *of* $\pi$*, let* $(v')^j$ *be the corresponding interval of the* $(v')_i$*'s, and let* $J_j$ *be the representation* $J((v')^j)$ *of* $\mathrm{GL}(p_j, \mathbb{F})$*. Then*

$$J(v) \cong \mathrm{Ind}(P(\pi)\uparrow G)(\otimes J_j).$$

*Proof.* We may assume without loss of generality that the $v_i$ are ordered with decreasing real parts, and that the permutation to $v'$ does not disturb the ordering within each subset. That permutation may clearly be written as a product of transpositions satisfying the condition in Theorem 11.5(h). Consequently $I(v)$ is isomorphic to $I(v')$. Write $w_0$ for the permutation which reverses the order of all the coordinates, and $w_1$ for the one reversing the order within each part of $\pi$. The same argument shows that $I(w_0 v)$ is isomorphic to $I(w_1 v')$.

On the other hand, parts (d) and (e) of Theorem 11.5 guarantee that there is a unique (up to multiple) non-zero intertwining map from $I(v)$ *to* $I(w_0 v)$, and that its image is $J(v)$. By the first part of the proof, it follows that any non-zero map from $I(v')$ to $I(w_1 v')$ has image $J(v)$.

Finally, define representations $I_j(v')$ and $I_j(w_1 v')$ in analogy with $J_j$; then $I(v')$ is induced from the product of the $I_j(v')$ on $P(\pi)$. By Theorem 11.5 applied to $\mathrm{GL}(p_j)$, there is a map from $I_j(v')$ to $I_j(w_1 v')$, with image $J_j$. The induced map from $I(v')$ to $I(w_1 v')$ has image $\mathrm{Ind}(\otimes J_j)$. By the preceding paragraph, this completes the proof. Q.E.D.

**Corollary 11.7.** *Theorem 7.8 may be reduced to the case* $Y = J(v)$, with all coordinates of $v$ real.

The argument for this is based on the following simple lemmas, which we will use repeatedly.

**Lemma 11.8.** *In the setting of Definition 3.6, fix a part* $p_j$ *of* $\pi$*, and a small representation* $\omega_q(1; p_j)$ *of* $K(p_j)$*. Let* $\delta_\pi$ *be the representation of* $K(\pi)$ *which is* $\omega_q$ *on the* $K(p_j)$ *factor, and trivial on all the other factors. Then*

$$\omega_q(1; n) \subset \mathrm{Ind}(K(\pi)\uparrow K)(\delta_\pi).$$

*Proof.* Perhaps the easiest method is to show that $\omega_q(1; n)$ is actually the lowest $K$-type of the right side. To prove that, Proposition 8.1 allows us to replace $\delta_\pi$ by something of which it is the lowest $K(\pi)$-type. Write $\pi'$ for the refinement of $\pi$ in which $p_j$ is replaced by $2q$ and $p_j - 2q$. Define $\delta_{\pi'}$ to be the representation of $K(\pi')$ which is $\omega_q(1; 2q)$ on $K(2q)$, and trivial on the other factors. Then (8.9) says that we may replace $\delta_\pi$ by

$$\mathrm{Ind}(K(\pi')\uparrow K(\pi))(\delta_{\pi'}).$$

By induction by stages, we can therefore replace $\pi$ by $\pi'$ throughout. Now compute the lowest $K$-type by induction by stages through $K(2q) \times K(n-2q)$, using (8.9) again. Q.E.D.

**Lemma 11.9.** *In the setting of Definition 3.6, let $J_\pi = \otimes J_j$ be an irreducible spherical representation of $G(\pi)$. Assume that $J_\pi$ admits an invariant Hermitian form, and that $J = \mathrm{Ind}(J_\pi)$ is irreducible. If each $J_j$ satisfies Hypothesis 7.7, then so does $J$.*

*Proof.* Assume that all Hermitian forms are positive on the spherical vectors. We use Proposition 10.5. First of all, it says that $J$ admits an invariant Hermitian form. If $J$ is basic, there is nothing to prove; so assume it is not. The set of basic representations is closed under induction, so some $J_j$ must not be basic. By Hypothesis 7.7, there is a small representation $\delta_j$ of $K(p_j)$ such that the form on $J_j$ is partly negative on $\delta_j$. Define $\delta_\pi$ to be the product of $\delta_j$ with trivial representations on the other factors. By Lemma 11.8,

$$\mathrm{Ind}(K(\pi)\uparrow K)(\delta_\pi)$$

contains a small representation $\omega$ of $K$. By Proposition 10.5, the form on $J$ is partly negative on $\omega$. Q.E.D.

*Proof of Corollary 11.7.* We may as well assume that $Y$ has an invariant Hermitian form. Partition the $v_i$ by putting two of them in the same class whenever they have the same imaginary part. This partition satisfies the hypotheses of Corollary 11.6. The condition in Theorem 11.5(c) for having an invariant Hermitian form must be satisfied by each subset of the $v_i$'s separately; so the representations $J_j$ of Corollary 11.6 all admit invariant Hermitian forms. By Lemma 11.9, it is enough to prove Theorem 7.8 for each $J_j$ separately. This reduces us to the case when all $v_i$ have the same imaginary part $c$. Since

$$J(v)\otimes(\delta_n)^t \cong J((v_i + dt)) \tag{11.10}$$

(with $d$ the dimension of $\mathbb{F}$, and $\delta_n$ as in (2.2)), we can change the imaginary part of $v$ without changing unitarity. Thus finally we are reduced to the case of imaginary part zero. Q.E.D.

A reduction argument like Corollary 11.7 is available for all reductive groups (cf. [37], Corollary 3.6).

We will often use the following lemma.

**Lemma 11.11.** *Suppose the sequence* $\mathrm{Re}(v_i)$ *is decreasing. Then $J(v)$ is finite dimensional if and only if for every $i$, $v_i - v_{i+1}$ is of the form $d + 2k$, with $d$ the dimension of $\mathbb{F}$ and $k$ in $\mathbb{N}$. It is one dimensional if and only if $v_i - v_{i+1}$ is always equal to $d$.*

This result has the same status as Theorem 11.5. Using it, we can identify the Stein complementary series in these parameters. Put

(11.12)(a) $$\rho(t) = d((n-1)/2 + t, (n-3)/2 + t, \ldots, -(n-1)/2 + t).$$

(It will sometimes be convenient to write $\rho(t; n)$ instead of $\rho(t)$.) Then

(11.12)(b) $$\sigma_{2m}(1, t) \supset J(\rho(t), \rho(-t)).$$

Two coordinates of the $v$ parameter here which come from different blocks differ by $d(m+2t)$, with $m$ an integer. Corollary 11.6 now implies that equality holds in (11.12)(b) whenever $t \notin \frac{1}{2}\mathbb{Z}$ (over $\mathbb{R}$ or $\mathbb{C}$), or $t \notin \frac{1}{4}\mathbb{Z}$ (over $\mathbb{H}$). This proves all the irreducibility assertions in Lemma 2.4 except at $t = 0$ or (if $\mathbb{F} = \mathbb{H}$) $t = \frac{1}{4}$.

## 12. Proof of Theorem 7.8: Complex case

In this section, $G$ will be $\mathrm{GL}(n, \mathbb{C})$. We use other notation as in Sect. 5 and 11. Fix $\nu$ in $\mathfrak{a}^*$; we will generally regard $\nu$ as a sequence of $n$ complex numbers. Rearrange the sequence in such a way that the following properties hold: there is an ordered partition

(12.1)(a) $$\pi = (p_1, \ldots, p_r)$$

of $n$, such that

(12.1)(b) $$\nu_i - \nu_{i+1} \in 2\mathbb{N} - \{0\}$$

when $i$ and $i+1$ belong to the same block of $\pi$. We also assume that $\pi$ is maximal (in the usual partial order on partitions) with respect to this property. Write $\nu^j$ for the restriction of $\nu$ to the $p_j$ block of $\pi$; it is a sequence of complex numbers, strictly decreasing by positive even integers. (A little thought should convince the reader that the set of pairs $(p_j, \nu^j)$ is determined uniquely up to permutation.) By Lemma 11.11, the corresponding spherical representation $J_j$ of $\mathrm{GL}(p_j, \mathbb{C})$ is finite dimensional.

**Proposition 12.2.** *Let $J(\nu)$ be any irreducible spherical representation of* $\mathrm{GL}(n, \mathbb{C})$. *Define $\pi$ and the various $J_j$ as above. Then*

$$J(\nu) \cong \mathrm{Ind}(P(\pi)\uparrow G)(\otimes J_j).$$

Let me emphasize again that this result was conjectured by Enright and proved with Barbasch. It includes all the irreducibility assertions in Lemma 2.4 and its proof over $\mathbb{C}$.

*Proof.* Using Corollary 11.6, we can reduce to the case when all the $\nu_i$ are congruent mod $2\mathbb{Z}$. (The reduction serves only to simplify the notation somewhat.) Write $\nu'$ for the sequence rearranged in decreasing order. The multiplicities in this sequence give a second ordered partition $\xi = (q_k)$ of $n$, with $s$ parts. The largest part $p_m$ of $\pi$ is the length of the longest possible sequence of distinct $\nu_i$'s. This is nothing but the number $s$ of parts of $\xi$. (It is also the largest block of the transpose of $\xi$.) The corresponding sequence $\nu^m$ consists of one representative from each block of $\nu'$. Continuing in this way, one finds that $\pi$ and $\xi$ are transpose partitions.

The next step in the argument works for any complex reductive group.

**Proposition 12.3.** *Let $G$ be a complex connected reductive Lie group, and $J$ an irreducible spherical representation of $G$. Write $\mathscr{I}$ for the annihilator of $J$ in $U(\mathfrak{g})$.*

a) *$\mathscr{I}$ is the unique maximal ideal in $U(\mathfrak{g})$ of the same infinitesimal character as $J$. Any other primitive ideal of this infinitesimal character has strictly larger Gelfand-Kirillov dimension.*

*Write $S$ for the set of irreducible representations of $G$ having the same infinitesimal character, Gelfand-Kirillov dimension, and central character as $J$. Write $V$ for the left cell representation of the integral Weyl group $W(J)$ attached to $\mathscr{I}$ by Joseph and Lusztig (cf.* [20]*).*

b) *The cardinality of $S$ is the dimension of* $\mathrm{Hom}_{W(J)}(V, V)$.

*Proof.* Part (a) follows from Duflo's analysis in [8]. Part (b) may be found in [2], Proposition 5.25. Q.E.D.

**Corollary 12.4.** *No other representation of* GL$(n, \mathbb{C})$ *has the same infinitesimal character, central character, and Gelfand-Kirillov dimension as* $J(\nu)$.

*Proof.* The left cell representations of the symmetric group are all irreducible (see [21]), so the Hom in Proposition 12.3(b) is one dimensional. Q.E.D.

It is possible to argue a little more directly from Joseph's results on GL$(n)$, but Proposition 12.3 seemed to be worth stating in general.

To finish the proof, we observe that Joseph calculates the Gelfand-Kirillov dimension of $J(\nu)$ in [19]. It is

$$2n(n-1) - 2\Sigma p_j(p_j - 1).$$

But this is exactly the Gelfand-Kirillov dimension of the induced representation in Proposition 12.2. By Corollary 12.4, $J(\nu)$ is the only possible composition factor of the induced representation. Q.E.D.

The reader who is unhappy with the invocation of soft-core non-commutative algebra to prove what amounts to a result about intertwining operators will find that the argument given in the next section for $\mathbb{R}$ can be adapted to this case as well.

*Proof of Theorem 7.8 over* $\mathbb{C}$. We proceed by induction on $n$. We may assume that $Y = J(\nu)$ is not basic, but that it does admit an invariant Hermitian form. We want to show that $Y$ is in Case 2 of Hypothesis 7.7. By Corollary 11.7, we may assume $\nu$ is real. Theorem 11.5(c) says that $\nu$ must be a permutation of $-\nu$. Choose notation as in (12.1). Write $p = p_1$, and consider the sequence

$$\nu^1 = (\nu_1, \ldots, \nu_p), \tag{12.5}$$

which decreases by positive even integers. There are two cases. First, assume that $\nu^1$ is a permutation of $-\nu^1$. Then the same is true for the remaining $n-p$ coordinates $\nu'$ of $\nu$. This means that the finite dimensional representation $J(\nu^1)$, and also $J(\nu')$, admit invariant Hermitian forms. We are therefore in the setting of Lemma 11.9. By inductive hypothesis, that lemma now reduces us to the case when $p = n$, so that $J(\nu)$ is finite dimensional. That case is treated by the following lemma.

**Lemma 12.6.** *Let* $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ *be a Cartan decomposition of a semisimple Lie algebra. Let* $F$ *be an irreducible finite dimensional spherical representation of* $\mathfrak{g}_0$, *admitting an invariant Hermitian form positive on the* $\mathfrak{k}_0$*-fixed line* $F_0$. *If* $F$ *is not one dimensional, then* $F$ *has a* $\mathfrak{k}_0$*-invariant subspace* $F_1$, *isomorphic to a subspace of* $\mathfrak{p}$, *on which the form is negative.*

We defer the proof for a moment. Lemma 7.4 and Definition 7.6 show that the $\mathfrak{p}$ representation is $\omega_1$, which is a small representation attached to the trivial representation of $K$; so $J(\nu)$ falls in Case 2 of Hypothesis 7.7, as we wished to show.

Next, suppose that $\nu^1$ is not a permutation of $-\nu^1$. Since $\nu$ is a permutation of $-\nu$, it follows that $-\nu^1$ must be one of the other $\nu^j$; say $\nu^2$, without loss of generality. Once again an application of Lemma 11.9 and the inductive hypothesis reduces us to the case when $\nu = (\nu^1, \nu^2)$. Say $\nu$ is congruent to $\varepsilon$ mod $\mathbb{Z}$, with $\varepsilon$ strictly between $-1$ and 1. After interchanging the roles of $\nu^1$ and $\nu^2$, we may assume $\varepsilon$ is between 0 and 1. Define

$$(12.7)\qquad \begin{aligned} (\nu^1)^- &= (\nu^1) - (\varepsilon, \ldots, \varepsilon) \\ (\nu^1)^+ &= (\nu^1) + ((1-\varepsilon), \ldots, (1-\varepsilon)) \\ \nu^+ &= ((\nu^1)^+, \ -(\nu^1)^+) \\ \nu^- &= ((\nu^1)^-, \ -(\nu^1)^-) \\ Y^\pm &= J(\nu^\pm). \end{aligned}$$

A standard continuity argument shows that the signatures of the forms on $Y$ and $Y^\pm$ (Definition 10.1) satisfy

$$(12.8)\qquad \begin{aligned} p^\pm(\delta) &\leqq p(\delta) \\ q^\pm(\delta) &\leqq q(\delta), \end{aligned}$$

with obvious notation.

The terms of $\nu^\pm$ are all congruent mod $2\mathbb{Z}$, and have multiplicity one or two. By Proposition 12.2, each is induced from a finite dimensional Hermitian representation $E^\pm$ *of* $\mathrm{GL}(r^\pm) \times \mathrm{GL}(s^\pm)$. If either of these representations has dimension greater than one, Lemma 12.6 and Lemma 11.9 say that the corresponding $Y^\pm$ satisfies Case 2 of Hypothesis 7.7. By (12.8), $Y$ does as well. So we may assume that $E^\pm$ are one dimensional. This means that, up to permutation,

$$(12.9)\qquad \nu^\pm = (\rho(0; r^\pm), \ \rho(0; s^\pm))$$

(notation (11.12)(a)). A simple argument shows that this is impossible unless the coordinates of $\nu^1$ decrease by exactly 2. That is,

$$(12.10)\qquad \nu^1 = \rho(t; p)$$

for some real number $t$ not in $\frac{1}{2}\mathbb{Z}$. Possibly interchanging the roles of $\nu^1$ and $\nu^2$ again, we may assume that $t$ is positive. By (12.9), $t$ is at most $n/2$. We are assuming that $Y$ is not basic. By (11.12)(b), this means that $t$ is greater than $\frac{1}{2}$.

Write $I(s)$ for $\sigma_{2p}(1, s)$ (Definition 2.3), and $J(s)$ for its irreducible spherical subquotient; these coincide unless $s \in \frac{1}{2}\mathbb{Z}$. Recall from Definition 7.6 that the set of small representations attached to the trivial representation consists of $p$ representations $\omega_k$, for $k$ running from 1 to $p$; and that $\omega_0$ is the trivial representation of $K$. By Lemma 7.4, all of these occur in $I(s)$, with multiplicity 1. Fix a positive integer $m$ less than or equal to $p$. Lemma 7.4 and Proposition 12.2 show that

$$(12.11)\qquad \omega_k \text{ occurs in } J(m/2) \text{ if and only if } k \leqq p - m.$$

In particular, $\omega_{p-m}$ occurs in $J(m/2)$, and $\omega_{p-m+1}$ does not. This suggests

**Lemma 12.12.** *With notation as above, suppose $s$ is between $m/2$ and $(m+1)/2$. Then the Hermitian form on $J(s)$ which is positive on the spherical vector is negative on $\omega_{p-m+1}$.*

*Proof.* The representation $\mu = \omega_{p-m}$ occurs in $J(z)$ for all $z$ between 0 and $s$ (by (12.11); so a continuity argument shows that the form on $J(s)$ is positive on $\mu$. Let $Z$ denote the unique irreducible subquotient of $I((m+1)/2)$ containing $\mu$. We may endow $Z$ with a Hermitian form positive on $\mu$. By Theorem 3.8 of [38] (that is, "obviously"), it suffices to show that the form on $Z$ is negative on $\omega = \omega_{p-m+1}$. To do this, we need to identify $Z$ in the parametrization of irreducible $(\mathfrak{g}, K)$-modules with lowest $K$-type $\mu$, given by Theorem 6.19. First, Lemma 7.4 guarantees that $\mu$ is the lowest $K$-type of $Z$. The pair $(L, \mu_L)$ attached to $\mu$ is computed by (5.6) and (5.7). $L$ is $\mathrm{GL}(p-m) \times \mathrm{GL}(2m) \times \mathrm{GL}(p-m)$, and $\mu_L$ is $\det \otimes 1 \otimes \det^{-1}$. The $K$-type $\omega$ belongs to $S(\mu)$ (Definition 9.3); in fact it corresponds to the $\mathfrak{p}$ representation of the $\mathrm{U}(2m)$ factor. Write

$$Z_L = U \otimes V \otimes W$$

for the representation of $L$ corresponding to $Z$ (Theorem 6.19). Here $V = J(\phi)$ is a spherical representation of $\mathrm{GL}(2m)$; and $U$ is of the form

$$U = J(\psi) \otimes (\det/|\det|).$$

with $J(\psi)$ spherical for $\mathrm{GL}(p-m)$. By symmetry (more precisely, using the inverse transpose automorphism of $\mathrm{GL}(n)$), we see that

$$W = J(\psi) \otimes (\det/|\det|)^{-1}.$$

By Proposition 10.2, we must show that the form on $J(\phi)$ is negative on the $\mathfrak{p}$ representation of $U(2m)$.

To do this, we must say something about $\phi$. We compute the infinitesimal character of $Z$, first in terms of $Z_L$, and then in terms of the induced representation $I((m+1)/2)$. The conclusion is that

$$((\psi_i + 1), \phi, (\psi_i - 1)) = w(\rho((m+1)/2; p), \rho(-(m+1)/2; p)) \tag{12.13}$$

for some permutation $w$ of $2p$. More explicitly, the term on the right is (up to permutation)

$$(p+m, p+m-2, \ldots, -p+m+2, p-m-2, p-m-4, \ldots, -p-m)$$

In particular, all the terms in $\phi$ are congruent mod $2\mathbb{Z}$, and they have multiplicity at most two. By Proposition 12.2, $J(\phi)$ is induced from a product of two finite dimensional Hermitian representations. By Lemma 12.6, it is enough to show that these are not both trivial. Suppose they are; then $\phi$ is a permutation of $(\rho(0; r), \rho(0; 2m-r)$ for some $r$ between 0 and $2m$.

It will be convenient to assume that $\psi$ is decreasing. The largest term of $\phi$ is at most $2m-1$. Since the largest term on the right in (12.13) is $p+m$ (which is at least $2m$), it follows that $\psi_1$ is $p+m-1$. This is already a contradiction if $p$ is equal to $m$ (when $\psi_1$ is undefined) or $m+1$ (when $\psi_1$ must be zero by the symmetry condition). Assume then that $p$ is at least $m+2$. By symmetry, $\psi_{p-m}$ is $1-p-m$. Eliminating the four terms corresponding to $\psi_1$ and $\psi_{p-m}$ from both sides of (12.13), we get another equation of the same form, with $p$ replaced by $p-2$, and $m$ unchanged. Continuing in this way, we eventually arrive at a contradiction. Q.E.D.

Lemma 12.12 is an explicit version of Theorem 7.8 in the case to which we had reduced it. This therefore completes the proof of Theorem 7.8 over $\mathbb{C}$.

*Proof of Lemma 12.6.* Let $\{X_i\}$ be an orthogonal basis of $\mathfrak{k}_0$ for the Killing form, with elements of length $-1$, and $\{Y_j\}$ an orthonormal basis of $\mathfrak{p}_0$. Pick a non-zero vector $v$ in $F_0$. Since $F$ is non-trivial, the Casimir operator $\Omega$ has a positive eigenvalue on $F$. Hence

$$0 < \langle \Omega v, v\rangle = -\Sigma\langle (X_i)^2 v, v\rangle + \Sigma\langle (Y_j)^2 v, v\rangle .$$

The first term is 0 since $v$ is spherical. Since $Y_j$ is a skew-Hermitian operator, the second is

$$-\Sigma\langle Y_j v, Y_j v\rangle .$$

So some of the $Y_j v$ have negative length. On the other hand, they obviously span a $\mathfrak{k}$-stable subspace of $F$, which is a homomorphic image of $\mathfrak{p}$. Q.E.D.

## 13. Proof of Theorem 7.8: Real case

Our treatment of the real and quaternionic cases is complicated by the fact that nothing so simple as Proposition 12.2 is true. A simple example is the representation $J(2,1,-1,-2)$ of $\mathrm{GL}(4,\mathbb{R})$. It is contained in several different induced from finite dimensional representations, but is equal to none of them. We will prove a much weaker version of Proposition 12.2 (Proposition 13.4), which suffices to treat reducibility questions for basic representations and a few others. We need a different kind of result to show that the remaining representations are of no interest for the unitary theory. Here it is. (Throughout this section, $G$ will be $\mathrm{GL}(n,\mathbb{R})$, and $v$ will be a more or less fixed element of $\mathfrak{a}^*$.)

**Proposition 13.1.** *Suppose $v$ is divided into two subsequences $v^1$ and $v^2$, with $p$ and $q$ terms respectively. Assume that*

*i) if $i$ and $j$ belong to different blocks, then $v_i - v_j \neq \pm 1$. Write $P$ for the parabolic $P(p,q)$ (Definition 3.6), and*

$$I(p,q) = \mathrm{Ind}(P \uparrow G)(J(v^1) \otimes J(v^2)) .$$

*Then any $K$-type of $I(p,q)/J(v)$ must have highest weight $(\gamma,\eta)$ (Proposition 5.17) with $\gamma_1$ at least 3. In particular, any small $K$-type $\delta$ attached to the trivial representation of $K$ (Definition 4.7) occurs in $I(p,q)$ exactly as often as in $J(v)$.*

We will prove this in a moment. What matters most about it is

**Corollary 13.2.** *In the setting of Theorem 13.1, assume that $J(v^1)$ and $J(v^2)$ are Hermitian. If they both satisfy Hypothesis 7.7, then $J(v)$ does as well.*

The point is that Proposition 13.1 allows us to compute the form on small $K$-types in $I(p,q)$ instead of $J(v)$. One can therefore argue as in the proof of Lemma 11.9.

*Proof of Proposition 13.1.* The argument is modelled on the proof of Corollary 11.6. Instead of examining the intertwining operator on full induced representations $I(*)$, however, we confine attention to a single $K$-primary subspace $I(*)_\delta$. The role of Theorem 11.5(h) is played by

**Lemma 13.3.** *In the setting of Theorem 11.5(f), suppose $\delta$ is a representation of $K$ with highest weight $(\gamma,\eta)$, and that $\gamma_1$ is at most 2. Assume that the real part of $\phi = v_m - v_{m+1}$ is non-negative. If $\phi$ is not 1, then $A(s)$ is an isomorphism from $I(v)_\delta$ to $I(v')_\delta$.*

*Proof.* By Theorem 11.5(h), we may assume that $\phi = 2k+1$, with $k$ a positive integer. By Theorem 11.5(f), the kernel of $A(s)$ is induced from the corresponding kernel $Z$ on $GL(2,\mathbb{R}) \times GL(1,\mathbb{R})^{n-2}$. As is well known, $Z$ is a discrete series representation, with weights $\pm(2k+2)$, $\pm(2k+4)$, ... on SO(2). Consequently, any representation of O$(n)$ occurring in the kernel of $A(s)$ must contain an SO(2) weight of the form $2m$, for some $m$ greater than $k$. Our assumption on $\delta$ forces all its SO(2) weights to be $\pm 2$, $\pm 1$, or 0; so $I(v)_\delta$ does not meet the kernel of $A(s)$. Q.E.D.

**Proposition 13.4.** *Assume that $v$ contains a subsequence of the form*

$$v^1 = (p+z, p-1+z, \ldots, 1+z),$$

*so that $J(v^1)$ is one dimensional (Lemma 11.11). Write $v^2$ (say with $q$ terms) for the remaining $n-p$ coordinates of $v$. Assume that*

i) *if $v_j$ is any coordinate of $v^2$ which is congruent to $z$ mod $\mathbb{Z}$, then $v_j$ is equal to some coordinate of $v^1$.*

*Write $P$ for the parabolic $P(p,q)$ (Definition 3.6). Then*

$$J(v) \cong \mathrm{Ind}(P\uparrow G)(J(v^1) \otimes J(v^2)).$$

(Together with Corollary 11.6, this contains all the irreducibility assertions in Lemma 2.4 and its proof over $\mathbb{R}$.)

The non-formal part of the proof is contained in the following result, which we prove first.

**Lemma 13.5.** *With notation roughly following Proposition 13.4,*

$$J(r, r, r-1, r-2, \ldots, 1) \cong \mathrm{Ind}(P\uparrow G)(J(r) \otimes J(r, r-1, \ldots, 1)).$$

*Proof.* Write $Y$ for the right side; we must show that $Y$ is irreducible. By Lemma 11.11, the inducing representation is one dimensional. By Lemma 7.4, the $K$-types of $Y$ form a one parameter family. It is easy to see that the action of $\mathfrak{p}$ can move at most one step up or down in this family. It follows that if $Y$ is reducible, then its spherical composition factor is finite dimensional. This is impossible by Lemma 11.11. Q.E.D.

**Corollary 13.6.**

$$\mathrm{Ind}(P(1,r)\uparrow G)(J(r) \otimes J(r, \ldots, 1)) \cong \mathrm{Ind}(P(r,1)\uparrow G)(J(r, \ldots, 1) \otimes J(r)).$$

*Proof of Proposition 13.4.* Using Corollary 11.6, we can reduce to the case when all $v_j$ are congruent mod $\mathbb{Z}$. Using (11.10), we may assume that $z=0$; so we can write

$$v = (p, \ldots, p, p-1, \ldots, p-1, \ldots, 1, \ldots, 1). \tag{13.7}$$

We also take $v^1$ and $v^2$ to be in decreasing order. Define

$$v' = (v^1, v^2). \tag{13.8}$$

Write $w^0$ for the longest element of the symmetric group $S_n$ (which reverses the order of the $n$ coordinates), and $w^1$ and $w^2$ for the corresponding elements of $S_p$ and $S_q$, respectively. There is a commutative diagram of intertwining operators

$$
\begin{tikzcd}
I(v') \arrow[r, dash, "A(w^1)"] \arrow[d, dash, "A(w^2)"'] & I(w^1 v') \arrow[d, dash, "A(w^2)"] \\
I(w^2 v') \arrow[r, dash, "A(w^1)"'] & I(w^1 w^2 v')
\end{tikzcd} \tag{13.9}
$$

Each of these operators is induced from an operator on GL$(p)$ or GL$(q)$; all are non-zero on the spherical vectors. The induced representation in the proposition is clearly the image of $A(w^1)A(w^2)$; we must show that it is irreducible. This will follow from the following two facts:

(13.10)(a) $\qquad I(v')/\ker A(w^1)$ has $J(v)$ as its unique irreducible quotient;

and

(13.10)(b) $\qquad$ the image of $A(w^1)$ in $I(w^1 w^2 v')$ has $J(v)$ as its unique irreducible submodule.

(The point is that these two statements force $J(v)$ to be the unique irreducible submodule and quotient of the image of $A(w^1)A(w^2)$.) Because they are formally identical, we will prove only (a).

Because of Theorem 11.5(d), it is enough to prove that

$$\operatorname{Ind}(P(p,q)\uparrow G)(J(p,p-1,\ldots,1)\otimes I(v^2))$$

is a homomorphic image of $I(v)$. We proceed by induction on $n$. By induction by stages, this induced representation is isomorphic to

$$\operatorname{Ind}(J(p,p-1,\ldots,1)\otimes I(p,\ldots,p)\otimes I(p-1,p-1,\ldots,1)),$$

say with $r-1$ $p$'s in the middle term. By Corollary 13.6, this is isomorphic to

$$\operatorname{Ind}(I(p,\ldots,p)\otimes J(p,p-1,\ldots,1)\otimes I(p-1,p-1,\ldots,1)).$$

This representation is clearly a homomorphic image of

$$Y=\operatorname{Ind}(I(p,\ldots,p,p)\otimes J(p-1,p-2,\ldots,1)\otimes I(p-1,p-1,\ldots,1)),$$

with $r$ $p$'s. By inductive hypothesis,

$$\operatorname{Ind}(P(p-1,q-r+1)\uparrow \operatorname{GL}(n-r))(J(p-1,\ldots,1)\otimes I(p-1,p-1,\ldots,1)$$

is a homomorphic image of $I(p-1,p-1,\ldots,1,1)$. By induction by stages, it follows that $Y$ is a homomorphic image of $I(v)$. This proves (13.10)(a), and hence the proposition. Q.E.D.

The argument from here on parallels the complex case; but since it is more complicated in detail, we will explicitly isolate the first main step.

**Proposition 13.11.** *Let $J(v)$ be a Hermitian spherical representation of $GL(n,\mathbb{R})$: we take the form positive on the $K$-fixed vector. Assume that all the coordinates of $v$ are real and congruent* mod $\mathbb{Z}$. *There are two mutually exclusive alternatives. The first is*

a) *$J(\nu)$ is not basic. In that case, the form is not positive on the $\mathfrak{p}$ representation of $K$.*

*The second is*

b) *$J(\nu)$ is basic. In that case, $\nu$ (after permutation) is of the form*

$$\nu = (j, \ldots, j, j-1, \ldots, j-1, \ldots, -j, \ldots, -j).$$

*Here $2j$ is a non-negative integer: and the term $j-k$ occurs $p_k$ times. We have $p_k = p_{-k}$, and $p_k$ increases for $k \leqq j$.*

Because the $\mathfrak{p}$ representation is small (Definition 7.6), this proposition includes Theorem 7.8 for integral $\nu$.

*Proof.* We proceed by induction on $n$. Assume that $\nu$ is decreasing. By Theorem 11.5(c), $\nu$ must be a permutation of $-\nu$. Write $j = \nu_1$; then $\nu$ must be of the form in (b), but with no conditions on the $p_k$. Since $\nu$ is a permutation of $-\nu$, $p_k = p_{-k}$. It is easy to see that $J(\nu)$ is basic exactly when the $p_k$ increase for $k \leqq j$. So suppose they do not; we want to establish the conclusion of (a).

Assume first that $p_k$ is actually zero, for some $k$ with $0 < k \leqq j$. Write $\nu^1$ for the coordinates which exceed $j-k$ in absolute value, and $\nu^2$ for the rest. This partition satisfies the hypotheses of Proposition 13.1. That proposition allows us to reduce to the case $\nu = \nu^1$; that is, to

$$\nu = (j, j, \ldots, j-k+1, -(j-k+1), \ldots, -j, -j).$$

Define

$$\nu(t) = \nu + (t, \ldots, t, -t, \ldots, -t).$$

Proposition 13.1 applies to $\nu(t)$ and the parabolic $P(n/2, n/2)$, for all positive $t$. It shows that the multiplicity of the $\mathfrak{p}$ representation (or any small representation of $K$) in $J(\nu(t))$ is independent of $t$. We want to show that the form on the $\mathfrak{p}$ representation is not positive. By a continuity argument, it suffices to do this for large positive $t$. For such $t$, the eigenvalue of the Casimir operator is positive. The proof of Lemma 14.6 now gives the desired conclusion.

Next, assume that all the terms $j-k$ actually occur in $\nu$. Set

$$\nu^1 = (j, j-1, \ldots, -(j-1), -j),$$

and write $\nu^2$ for the rest of the coordinates. Proposition 13.4 shows that

$$J(\nu) \cong \operatorname{Ind}(\mathbb{C} \otimes J(\nu^2)).$$

Because $\nu$ is assumed to fall in case (a) of the proposition, $\nu^2$ does as well. By inductive hypothesis, its form is not positive on $\mathfrak{p}$; so that for the induced representation $J(\nu)$ cannot be either. Q.E.D.

*Proof of Theorem 7.8 over* $\mathbb{R}$. We proceed by induction on $n$. Suppose $Y = J(\nu)$ admits a Hermitian form, but is not basic. By Corollary 11.7, we may assume $\nu$ is real. After permutation, we may assume that $\nu_1$ is the largest of the coordinates, and that

$$\text{(13.12)(a)} \qquad \nu^1 = (\nu_1, \ldots, \nu_p)$$

consists of all the coordinates congruent to $\nu_1$ mod $\mathbb{Z}$, in decreasing order. By Theorem 11.5(c), $-\nu_1$ is among the $\nu_i$. If it occurs in $\nu^1$, then Corollary 11.6, Lemma 11.9, and the inductive hypothesis quickly reduce matters to the case $\nu = \nu^1$. That case is treated by Proposition 13.11. So we may assume that $-\nu_1$ is not congruent to $\nu_1$. Then

(13.12)(b) $$\nu^2 = (-\nu_p, \ldots, -\nu_1)$$

is (up to permutation) a subsequence of $\nu$. Corollary 11.6, Lemma 11.9, and the inductive hypothesis allow us to assume

(13.13)(c) $$\nu = (\nu^1, \nu^2).$$

Assume next that there is a term $\nu_1 - k$, lying between $\nu_1$ and $\nu_p$, which does not appear as a coordinate of $\nu^1$. Set

$$\lambda^1 = (\nu_1, \nu_2, \ldots, \nu - k + 1)$$
$$\rho^1 = \text{remaining coordinates of } \nu^1$$
$$\lambda = (\lambda^1, -\lambda^1),\ \rho = (\rho^1, -\rho^1).$$

The partition $(\lambda, \rho)$ of the coordinates of $\nu$ satisfies the hypotheses of Proposition 13.1, and so allows us to reduce to a smaller $n$. (Since $\nu$ is not basic, one of the two pieces must also fail to be.)

We may therefore assume that $\nu^1$ contains the subsequence

(13.14) $$\lambda^1 = (\nu_1, \nu_1 - 1, \ldots, \nu_p).$$

Proposition 13.4 guarantees that $J(\nu)$ is induced from $J(\lambda^1, -\lambda^1)$ and another Hermitian representation. By inductive hypothesis, we are finally reduced to the case $\nu^1 = \lambda^1$. The analysis of this case is exactly parallel to that given for the complex case, and we leave it to the reader. Q.E.D.

## 14. Proof of Theorem 7.8: Quaternionic case

In this section, $G$ is $GL(n, \mathbb{H})$.

**Proposition 14.1.** *Suppose $\nu \in \mathfrak{a}^*$ is divided into two subsequences $\nu^1$ and $\nu^2$*, with $p$ *and $q$ terms respectively. Assume that*

i) *if $i$ and $j$ belong to different blocks, then $\nu_i - \nu_j$ is not equal to $\pm 4$.*
*Write $P$ for the parabolic $P(p, q)$ (Definition 3.6), and*

$$I(p, q) = \mathrm{Ind}(P \uparrow G)(J(\nu^1 \otimes J(\nu^2)).$$

*Then any $K$-type of $I(p, q)/J(\nu)$ must have a highest weight $\gamma$* (cf. (5.28)) *with $\gamma_1$ at least* 2. *In particular, any small representation of $K$ (Definition 7.6) occurs in $I(p, q)$ exactly as often as in $J(\nu)$.*

*Outline of proof.* The argument is exactly as in the real case. The key to the calculation is the fact that if $J$ is the quotient of the principal series $I(4+2k)$ for $GL(2, \mathbb{H})$, then the $Sp(2)$-types of $I(4+2k)/J$ are of the form $(k+1, k+1)$,

$(k+2, k+2)$, and so on. Because GL$(2, \mathbb{H})$ is locally isomorphic to SO$(5,1) \times \mathbb{R}$, this fact is well known.

**Proposition 14.2.** *Assume that $v$ contains a subsequence of the form*

$$v^1 = (4p+z,\ 4(p-1)+z,\ \ldots,\ 4+z),$$

*so that $J(v^1)$ is one dimensional (Lemma 11.11). Write $v^2$ (say with $q$ terms) for the rest of the coordinates of $v$. Assume that*

i) *if $v_j$ is any coordinate of $v^2$ which is congruent to $z$ mod $2\mathbb{Z}$, then $v_j - z$ is an even integer between 2 and $4p+2$ (inclusive).*

*Write $P$ for the parabolic $P(p,q)$. Then*

$$J(v) \cong \mathrm{Ind}(P \uparrow G)(J(v^1) \otimes J(v^2)).$$

This result will complete the proofs of the irreducibility assertions in Lemma 2.4 and its proof.

**Lemma 14.3.** *If $s$ is any real number in the open internal $(4r+4, 0)$, then*

$$J(s,\ 4r,\ 4r-4,\ \ldots,\ 4) \cong \mathrm{Ind}(P(1,r) \uparrow G)(J(s) \otimes J(4r,\ 4r-4,\ \ldots,4)).$$

This is proved in exactly the same way as Lemma 13.5. Using this lemma, one can prove Proposition 14.2 by following the argument for Proposition 13.4. Details are left to the reader.

Just as for $\mathbb{R}$, we begin with a special case of Theorem 7.8.

**Proposition 14.4.** *Let $J(v)$ be a Hermitian spherical representation of* GL$(n, \mathbb{H})$; *we take the form positive on the $K$-fixed vector. Assume that all the coordinates of $v$ are even integers. There are two mutually exclusive alternatives. The first is*

a) *$J(v)$ is not basic. In that case, the form is not positive on the $\mathfrak{p}$ representation of $K$.*

*The second alternative is*

b) *$J(v)$ is basic. In that case, $v$ (after permutation) is of the form*

$$v = (v^1, v^2),$$

*with each subsequence of the form*

$$v^r = (4j,\ \ldots,\ 4j,\ 4j-4,\ \ldots,\ 4j-4,\ \ldots,\ -(4j-4),\ -4j,\ \ldots,\ -4j).$$

*Here $2j$ is a non-negative integer: and the term $4j-4k$ occurs $p_k$ times. We have $p_k = p_{-k}$, and $p_k$ increases with $k$ for $k \leqq j$.*

*Proof.* We proceed by induction on $n$. By Theorem 11.5(c), $v$ must be a permutation of $-v$. By hypothesis, each coordinate is congruent to its negative mod 4. We partition $v$ into the two congruence classes $v^1$ and $v^2$ mod 4. This partition satisfies the hypotheses of Proposition 14.1, and the two representations $J(v^r)$ carry Hermitian forms. Since the $\mathfrak{p}$ representation is small (cf. (8.11)), Proposition 14.1 and the inductive hypothesis reduce us to the case $v = v^1$. The rest of the argument is exactly like that for Proposition 13.13. Q.E.D.

The rest of the proof of Theorem 7.8 is quite similar to the real case; the argument needs to be modified just as the proof of Proposition 13.11 was modified to give Proposition 14.4. Details are left to the reader.

## 15. Reduction to the spherical case: Unitarity

In this section, we recall some deeper results about the cohomological parabolic induction functors, and use them to prove Theorem 6.18(a). This is based on the ideas in Sect. 7 and 8 of [38]. Unfortunately, none of the results stated there is entirely adequate for the present situation. Here is an extension which meets our needs. We begin by recalling the notation in [38], suitably modified to take into account the more general twisting $\tau^{\sim}$ allowed in Definition 6.1.

So suppose $G$ is a reductive Lie group, and

(15.1)(a) $$\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$$

is a $\theta$-stable parabolic subalgebra. Fix a Cartan subalgebra

(15.1)(b) $$\mathfrak{h} \subset \mathfrak{l}.$$

Define

(15.1)(c) $$\rho(\mathfrak{u}) = \text{half the sum of the roots of } \mathfrak{h} \text{ in } \mathfrak{u}$$

(cf. Lemma 6.9). Let

(15.1)(d) $$Y = \text{an irreducible } (\mathfrak{l}, L \cap K)\text{-module}.$$

Recall that Harish-Chandra parametrizes the infinitesimal character of a representation by a weight in a Cartan subalgebra, defined up to the Weyl group. Fix a weight

(15.1)(e) $$\lambda \in \mathfrak{h}^*$$

with the property that the infinitesimal character of $Y$ is defined by the weight

(15.1)(f) $$\lambda + \rho(\mathfrak{u}) - \tau^{\sim}.$$

Here we have used the same letter $\tau^{\sim}$ to denote the restriction to $\mathfrak{h}$ of the differential of the twist $\tau^{\sim}$ appearing in Definition 6.1. For the case of the functors $\mathscr{I}_\theta$ of Definition 6.11, Lemma 6.9 guarantees that the two twists cancel:

(15.1)(g) $$\lambda = \text{infinitesimal character of } Y \text{ (case of } \mathrm{GL}(n)).$$

The reason for the twists is Proposition 6.3.11 of [36], which says that

(15.1)(h) $$\lambda = \text{infinitesimal character of } \mathscr{L}^j Y \text{ (general case)}.$$

**Proposition 15.2.** *In the setting* (15.1), *assume that the Cartan subalgebra* $\mathfrak{h}$ *has an orthogonal decomposition*
$$\mathfrak{h} = \mathfrak{s} + \mathfrak{c}$$
*with the following properties*;

i) $\mathfrak{s} \subset [\mathfrak{l}, \mathfrak{l}]$;

ii) $\mathfrak{c} \supset$ *center of* $\mathfrak{l}$;

iii) $\mathrm{Re}\langle\alpha^\vee, \lambda|_{\mathfrak{c}}\rangle > -1$, *all* $\alpha \in \Delta(\mathfrak{u}, \mathfrak{h})$; *and*

iv) *if $F$ is any finite dimensional representation of* $\mathfrak{l}$, *and* $\lambda' + \rho(\mathfrak{u}) - \tau^{\sim}$ *is an infinitesimal character occurring in* $Y \otimes F$, *then (perhaps after replacing* $\lambda'$ *by a* $W(\mathfrak{l})$

*conjugate), we may assume that*

$$\lambda'|_{\mathfrak{c}} = \lambda|_{\mathfrak{c}} + (\textit{weight of } \mathfrak{c} \textit{ in } F)$$
$$\operatorname{Re}\langle \lambda|_{\mathfrak{s}}, \lambda|_{\mathfrak{s}} \rangle \leqq \operatorname{Re}\langle \lambda'|_{\mathfrak{s}}, \lambda'|_{\mathfrak{s}} \rangle .$$

*Then*

a) *the cohomologically induced representation* $\mathscr{L}^j Y$ *vanishes except for* $j = S$: *and*

b) *if* $Y$ *is a unitary* $(\mathfrak{l}, L \cap K)$*-module, then* $\mathscr{L}^S Y$ *is a unitary* $(\mathfrak{g}, K)$*-module.*

We take this opportunity to point out a mistake in [38]. The statement of Proposition 8.18 there omits the first condition in (iv) above; but it is certainly used in the proof.

*Proof.* By Theorem 7.1 of [38], it suffices to show that the map $\psi_t$ of (8.3)(c) in [38] is injective for $t \geqq 0$. The proof of this fact parallels that of Proposition 8.18 in [38] exactly. The only difference is the derivation of (8.15) in [38]. Just as before, one gets a formula (in which the restrictions are indicated by subscripts)

$$(\lambda')_{\mathfrak{c}} = \lambda_{\mathfrak{c}} + (\Sigma n_\alpha \alpha)|_{\mathfrak{c}} . \tag{15.3}$$

Here the second term is a sum of roots in $\mathfrak{u}$. We want to show that $\lambda_{\mathfrak{c}}$ is strictly shorter than $(\lambda')_{\mathfrak{c}}$ unless the second term is zero. To do this, first choose the $n_\alpha$ in (15.3) so that their sum is as small as possible. If two of the roots which appear have a negative inner product, they may be replaced by their sum (which is a root). Consequently, we may assume that all the roots appearing have non-negative inner products with each other. Now we compute

$$\begin{aligned}\langle (\lambda')_{\mathfrak{c}}, (\lambda')_{\mathfrak{c}} \rangle - \langle \lambda_{\mathfrak{c}}, \lambda_{\mathfrak{c}} \rangle &= \langle \Sigma n_\alpha \alpha, 2\lambda_{\mathfrak{c}} + \Sigma n_\alpha \alpha \rangle \\ &\geqq \Sigma \left[ (n_\alpha)^2 \langle \alpha, \alpha \rangle + 2 n_\alpha \langle \alpha, \lambda_{\mathfrak{c}} \rangle \right] \\ &= \Sigma n_\alpha \langle \alpha, \alpha \rangle (n_\alpha + \langle \alpha^\vee, \lambda_{\mathfrak{c}} \rangle) .\end{aligned}$$

By hypothesis (iii), each of the terms in parentheses at the end has positive real part. The sum therefore has positive real part unless it is empty, as we wished to show. Now proceed as in [38]. Q.E.D.

*Proof of Theorem 6.18 (a).* We need to show that the correspondence of Theorem 6.19 takes unitary (basic almost spherical) representations to unitary representations. Recall (Definition 6.17) that the correspondence is contructed in two steps. The first of these is unitary induction, which preserves unitarity (Proposition 10.5). We will treat the second step $\mathscr{I}_\theta$ using the criterion of Proposition 15.2. Assume therefore that $\mathfrak{q}_\theta = \mathfrak{l}_\theta + \mathfrak{u}_\theta$ is as in Definition 6.11, and use the notation there. Let $Y$ be an irreducible $(\mathfrak{l}_\theta, L_\theta \cap K)$-module of lowest $L_\theta \cap K$-type $\mu_{L_\theta}$. We are assuming that $Y$ is of the form $\mathscr{I}_{\mathbb{R}} Z$ for some basic almost spherical representation $Z$ of $L$. Since the functor $\mathscr{I}_{\mathbb{R}}$ is real parabolic induction, we conclude (from the definition of basic) that $Y$ must be induced from a one dimensional representation $\phi$ of a parabolic subgroup $P = MAN$ of $L_\theta$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{m} + \mathfrak{a}$; write $\mathfrak{c}$ for the center of $\mathfrak{m}$, and $\mathfrak{s}$ for $\mathfrak{h} \cap [\mathfrak{m}, \mathfrak{m}]$. This will be the decomposition required in Proposition 15.2. The infinitesimal character of the inducing representation $\phi$ (and hence of $Y$ itself) may be identified with the weight

$$\lambda = (\lambda_c, \lambda_s) \in \mathfrak{h}^*,$$

defined by

$$\lambda_c = \text{differential of } \phi$$
$$\lambda_s = \rho(\mathfrak{m}).$$

(The last term is half the sum of any set of positive roots for $\mathfrak{m}$.)

With this notation established, the first two hypotheses of Proposition 15.2 are clear. The third is this: for each root $\alpha$ of $\mathfrak{h}$ in $\mathfrak{u}$,

$$\operatorname{Re}\langle \alpha^\vee, \lambda_c \rangle > -1. \tag{15.4}$$

It is convenient to prove this for each of the three cases separately. In each case, the weight $\lambda_c$ is a sum of three terms,

$$\lambda_c = \lambda_\gamma + \lambda_v + \lambda_t. \tag{15.5)(a}$$

The first, $\lambda_\gamma$, is the restriction of $\lambda$ to the compact part of the center of $\mathfrak{l}_\theta$. Obviously it coincides with the corresponding restriction of the highest weight of $\mu_{L_\theta}$. By inspection of the definitions of $\mu_{L_\theta}$ and $\lambda(\mu)$ in Sect. 5, one sees that

$$\lambda_\gamma = \lambda(\mu). \tag{15.5)(b}$$

Because of (6.7), it follows that

$$\langle \alpha, \lambda_\gamma \rangle > 0. \tag{15.5)(c}$$

The second term comes from the unitary part of the character $\phi$ on the vector subgroup $A$. It does not contribute to the real part of the inner product (15.4), and we will ignore it. The last, $\lambda_t$, comes from the real part of the character on $A$. It arises from the Stein complementary series involved in $Y$. It is restricted to be small, but can be negative on $\alpha$. Over $\mathbb{R}$ and $\mathbb{C}$, we will show that

$$\langle \alpha^\vee, \lambda_t \rangle > -1; \tag{15.5)(d}$$

together with (15.5)(c), this will certainly prove (15.4). Over $\mathbb{H}$, we will find that

$$\langle \alpha^\vee, \lambda_t \rangle > -(3/2); \tag{15.5)(e}$$

but that

$$\langle \alpha^\vee, \lambda_\gamma \rangle \geqq \tfrac{1}{2}. \tag{15.5)(f}$$

Adding these gives (15.4).

So suppose first that $\mathbb{F} = \mathbb{C}$. We may identify $\mathfrak{h}^*$ with $\mathbb{C}^n \times \mathbb{C}^n$. The roots are of the form $e_i - e_j$. The precise form of the weight $\lambda_t$ is not important; what matters is that all its coordinates are less than $\frac{1}{2}$ in absolute value. Now (15.5)(d) is clear over $\mathbb{C}$.

Next, suppose $\mathbb{F} = \mathbb{R}$. We may identify $\mathfrak{h}^*$ with $\mathbb{C}^n$, with roots $e_i - e_j$. Again all coordinates of $\lambda_t$ are less than $\frac{1}{2}$ in absolute value, and (15.5)(d) follows.

Over $\mathbb{H}$, we can identify $\mathfrak{h}^*$ with $\mathbb{C}^{2n}$, with roots $e_i - e_j$. Write $\gamma$ for the highest weight of the lowest $K$-type; say the first $p$ coordinates are non-zero. Because of

(15.5)(b) and (5.29), $\lambda_\gamma$ is

$$\tfrac{1}{2}(\gamma_1+1, \ldots, \gamma_p+1, 0, \ldots, 0, -(\gamma_p+1), \ldots, -(\gamma_1+1)).$$

Because the $\gamma_i$ are integers, (15.5)(f) follows from (15.5)(c). The first and last $p$ coordinates of $\lambda_t$ are less than $\frac{1}{2}$ in absolute value; the middle $2n-2p$ are less than 1. The inequality (15.5)(e) follows.

Finally, we must verify hypothesis (iv) of Proposition 15.2. Any constituent of $Y \otimes F$ is induced from a constituent of $\phi \otimes (F|_{MA})$. If we write $\lambda'$ for the infinitesimal character of that constituent, then $\lambda'$ satisfies the two conditions in question. Q.E.D.

## 16. Irreducibility results

The only result still unproved is Theorem 6.18(b): that the functor $\mathscr{I}$ takes irreducible (basic almost spherical) representations of $L$ to irreducible representations of $G$. We first dispose of (or rather ignore) the ordinary induction aspect of this problem.

**Lemma 16.1.** *Suppose $n=p+q$, and $Y^0$ and $Y^1$ are basic spherical representations of* GL$(p, \mathbb{R})$ *and* GL$(q, \mathbb{R})$, *respectively. Then*

$$\mathrm{Ind}(P(p,q)\uparrow G)(Y^0 \times [Y^1 \otimes sgn(\det)])$$

*is irreducible.*

This follows from a slight generalization of Proposition 13.4, the formulation and proof of which we leave to the reader.

We must therefore prove a result about irreducibility of cohomologically induced representations. To state it, we need some language from [7] and [8].

*Definition 16.2.* Let $\mathfrak{g}$ be a complex reductive Lie algebra, and $G_{\mathbb{C}}$ a corresponding algebraic group. Fix a Cartan involution $\theta$ of $G_{\mathbb{C}}$, and a commuting Chevalley antiautomorphism $g \to {}^t g$. Define $\sigma g = {}^t(\theta g)^{-1}$. (If $G_{\mathbb{C}}$ is GL$(n, \mathbb{C})$, we can take $\theta$ to be conjugate transpose inverse, and the Chevalley antiautomorphism to be transpose. Then $\sigma$ is just complex conjugation of matrices.) Regard $\mathfrak{g}$ as a real Lie algebra, and write $\mathfrak{g}_{\mathbb{C}}$ for its complexification; this has a new complex structure $j$, in addition to the multiplication by $i$ from $\mathfrak{g}$. Define

$$\mathfrak{g}^L = \{\tfrac{1}{2}(X - jiX) \mid X \in \mathfrak{g}\} \tag{16.3}$$
$$\mathfrak{g}^R = \{\tfrac{1}{2}(\sigma X + ji(\sigma X)) \mid X \in \mathfrak{g}\},$$

the holomorphic and anti-holomorphic tangent spaces to $G_{\mathbb{C}}$ at the identity. The indicated parametrizations define complex-linear isomorphisms $\phi^L$ and $\phi^R$ of $\mathfrak{g}$ with $\mathfrak{g}^L$ and $\mathfrak{g}^R$, respectively. We will therefore sometimes write

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \times \mathfrak{g} = \mathfrak{g}^L \times \mathfrak{g}^R. \tag{16.4}$$

Write $D$ for the fixed points of $\theta$, and $\mathfrak{d}_0$ for its Lie algebra. In the identification (16.4), its complexification is

$$\mathfrak{d} = \{(X, -{}^tX) \mid X \in \mathfrak{g}\}. \tag{16.5}$$

This provides an isomorphism $\phi^{\mathfrak{d}}$ of $\mathfrak{g}$ with $\mathfrak{d}$.

Suppose $M$ is a $(\mathfrak{g}, \mathfrak{g})$-bimodule. (This means that $M$ is both a left and a right module for $\mathfrak{g}$, and that the two structures commute.) We make $M$ into a left module for $\mathfrak{g}_{\mathbb{C}}$ using (16.14):

(16.6)(a) $\qquad \phi^L(X)m = Xm, \quad \phi^R(Y)m = m({}^tY).$

The action of $\mathfrak{d}$ is then

(16.6)(b) $$\phi^{\mathfrak{d}}(X)m = Xm - mX.$$

A *Harish-Chandra bimodule* for $G_{\mathbb{C}}$ is a bimodule $M$, endowed with an algebraic (that is, holomorphic and locally finite) action

$$\mathrm{Ad}\colon G_{\mathbb{C}} \to \mathrm{End}(M),$$

the differential of which (written ad) is

$$\mathrm{ad}(X)m = Xm - mX.$$

The point of the definitions above is to identify the category of Harish-Chandra bimodules with the category of Harish-Chandra modules (or rather $(\mathfrak{g}_{\mathbb{C}}, D)$-modules) for $G_{\mathbb{C}}$ (regarded as a real group).

*Example 16.7.* If $I$ is a two-sided ideal in $U(\mathfrak{g})$, then $M_I = U(\mathfrak{g})/I$ is a Harish-Chandra bimodule. If $I$ is primitive, then $M_I$ has finite length, and one can apply to it the theory of finite length Harish-Chandra modules for $G_{\mathbb{C}}$. This is the main idea in [7] and [8].

Here is the irreducibility result that we need. Although we will not explicitly invoke it, the Beilinson-Bernstein theory of [4] is obviously an important motivation for the proof.

**Proposition 16.8** (J. Bernstein). *Suppose we are in the setting of Proposition 15.2, with the same hypotheses on $Y$. Let $R_L$ be an algebra of endomorphisms of $Y \otimes \tau^{\sim} \otimes \mathbb{C}_{-\rho(\mathfrak{u})}$, containing the image of $U(\mathfrak{l})$. This makes $R_L$ into an $(\mathfrak{l}, \mathfrak{l})$-bimodule: assume it is a Harish-Chandra bimodule of finite length. Let $Q_{\mathbb{C}}$ be the real parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. Then*

$$R = \mathrm{Ind}(Q_{\mathbb{C}} \uparrow G_{\mathbb{C}})(R_L)$$

*(which is a Harish-Chandra module for $G_{\mathbb{C}}$) may be endowed with the structure of an algebra. We have*

a) *$R$ acts naturally on $X = \mathscr{L}^S Y$, and $X$ is an irreducible $(R, K)$-module (or zero).*

b) *The action of $U(\mathfrak{g})$ on $X$ is induced by the homomorphism*

$$U(\mathfrak{g}) \to R$$

*which (in terms of the $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ action on $R$) sends $u$ to $(u \otimes 1)1_R$.*

*In particular, $X$ is irreducible if the $\mathfrak{d}$-fixed vector of the induced representation $R$ is cyclic.*

Probably this is true if one assumes only that the produced module

$$\mathrm{Hom}_{\mathfrak{q}}(U(\mathfrak{g}), Y \otimes \tau^{\sim})$$

(or rather its $L \cap K$-finite part) is irreducible; this at any rate is enough to make $R$ an algebra (cf. Proposition 16.9 below).

We will postpone most of the proof of Proposition 16.8 for a moment, but one of the main ideas will be helpful in the application of it.

**Proposition 16.9** (Conze-Berline, Duflo [7], Proposition 5.5). *Suppose $Q_{\mathbb{C}} = L_{\mathbb{C}} U_{\mathbb{C}}$ is a parabolic subgroup of the complex reductive group $G_{\mathbb{C}}$. Let $Z$ be a Harish-Chandra module for $L_{\mathbb{C}}$. Assume that $Z$ is of the form*

$$L_{\mathbb{C}} \cap D\text{-finite part of } \operatorname{Hom}(Z^1, Z^2),$$

*for two $\mathfrak{l}$-modules $Z^1$ and $Z^2$. Define*

$$X^1 = \operatorname{ind}(\mathfrak{q}^{op} \uparrow \mathfrak{g})(Z^1 \otimes \mathbb{C}_{\rho(\mathfrak{u})})$$
$$X^2 = \operatorname{pro}(\mathfrak{q} \uparrow \mathfrak{g})(Z^2 \otimes \mathbb{C}_{\rho(\mathfrak{u})}).$$

*Then $X = \operatorname{Ind}(Q_{\mathbb{C}} \uparrow G_{\mathbb{C}})(Z)$ is isomorphic to*

$$D\text{-finite part of } \operatorname{Hom}(X^1, X^2).$$

*In particular, if $Z^1 = Z^2$ (so that the inducing representation has an algebra structure), and the natural map from $X^1$ to $X^2$* ([38], *Lemma 5.15*) *is injective, then the induced representation has an algebra structure.*

This is actually proved in [7] only under slightly more restrictive hypotheses; but the argument there gives this result.

*Proof of Theorem 6.18 (b).* Because we have already dealt with $\mathscr{I}_{\mathbb{R}}$, we can consider only $\mathscr{I}_\theta$. To be consistent with the notation in Sect. 6, we should write $\mathfrak{q}_\theta$ for our parabolic subalgebra; but we drop the $\theta$ for simplicity. In that setting, $Y$ is induced from a one dimensional character $\phi$ of a parabolic subgroup $P(L)$ of $L$. That is, $Y$ is the $L \cap K$-finite part of the module

$$\operatorname{Hom}_{\mathfrak{p}(L)}(U(\mathfrak{l}), \mathbb{C}_{\phi+\rho}); \tag{16.10}$$

here $\rho$ is $\rho$ for the parabolic $P(L)$. We may assume that $P(L)$ is chosen so that

$$Z^1 = \operatorname{ind}(\mathfrak{p}(L) \uparrow \mathfrak{l})(\mathbb{C}_{\phi+\rho})$$

is irreducible. We set

$$R_L = L_{\mathbb{C}} \cap D\text{-finite part of } \operatorname{Hom}(Z^1, Z^1) \cong \operatorname{Ind}(P(L)_{\mathbb{C}} \uparrow L_{\mathbb{C}})(\Phi). \tag{16.11)(a}$$

Here $\Phi$ denotes the (one dimensional) endomorphism ring of the module $\phi$ for the Levi factor of $\mathfrak{p}(L)$, regarded as a Harish-Chandra module; the isomorphism in (16.11)(a) is a consequence of Proposition 16.9.

Now define

$$P_{\mathbb{C}} = P(L)_{\mathbb{C}} U_{\mathbb{C}}, \tag{16.11)(b}$$

a parabolic in $G_{\mathbb{C}}$ contained in $Q_{\mathbb{C}}$. The ring $R$ of Proposition 16.8 is

$$R = \operatorname{Ind}(P_{\mathbb{C}} \uparrow G_{\mathbb{C}})(\Phi). \tag{16.11)(c}$$

We must show that the $D$-fixed vector in this induced representation is cyclic. We will sketch two proofs of this fact.

For the first argument, we need to describe $\Phi$ a little more precisely. The complexified Iwasawa $\mathfrak{a}$ in $G_{\mathbb{C}}$ may be naturally identified (up to the Weyl group) with a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We may assume that $\mathfrak{h}$ lies inside $\mathfrak{p}(L)$, so the one dimensional character $\phi$ gives a weight in $\mathfrak{h}^*$; this weight was called $\lambda_c$ in Sect. 15. Inspecting the definitions, one finds

(16.12) $\qquad$ the differential of $\Phi$ on $\mathfrak{a}$ is $2\lambda_c$.

Because of (15.4), the cyclicity of the $D$-fixed vector in $R$ is now a consequence of the following proposition.

**Proposition 16.13.** *Suppose $P=P(\pi)$ is a standard parabolic subgroup of $G=\mathrm{GL}(n,\mathbb{C})$, and $\Phi$ is a one dimensional character of $P$ which is trivial on $P\cap K$. Write $v\in\mathbb{C}^n$ for the differential of $\phi$ restricted to $A$ (cf. (9.2)). Assume that if $i<j$, and $i$ and $j$ belong to different blocks of $\pi$, then the real part of $v_i-v_j$ is greater than $+2$. Then the $K$-fixed vector is cyclic in* $\mathrm{Ind}(P\uparrow G)(\Phi)$.

This can be proved by the argument given for Proposition 13.4. Details are left to the reader.

The second proof is due to Borho and others (cf. [6], [20]). Notice first that Proposition 16.9 exhibits $R$ as a ring of endomorphisms of a highest weight module $V$. Write $I$ for the annihilator of $V$ in $U(\mathfrak{g})$; then we are investigating the inclusion

(16.14) $$U(\mathfrak{g})/I\to R.$$

As a representation of the maximal compact subgroup $D$, the right side is

(16.15) $$\mathrm{Ind}(P_{\mathbb{C}}\cap D\uparrow D)(\mathbb{C}).$$

To compute the left side, we pass to the associated graded ring

(16.16)(a) $$M=S(\mathfrak{g})/gr\,I.$$

Now $I$ is the annihilator of a highest weight module induced from a finite dimensional representation of $\mathfrak{p}$. Therefore [6] implies that the associated variety of $M$ (inside $\mathfrak{g}^*=\mathrm{Spec}(S(\mathfrak{g}))$) is equal to the closure $\mathscr{V}$ of the $G_{\mathbb{C}}$ orbit of the nil radical of $\mathfrak{p}$. This gives rise to a surjective map of representations of $D$

(16.16)(b) $$M\to \text{algebraic functions on } \mathscr{V}.$$

But the right side of (16.16)(b) is the same as (16.15), by the (deep) theorem of Kraft and Procesi in [25]. So the left side of (16.14) is as large as the right as representations of $D$; so they coincide. Q.E.D.

*Proof of Proposition 16.8.* By Proposition 16.9, $R$ may be regarded as an algebra of endomorphisms of

(16.17) $$Z=\mathrm{ind}((q^{op},L\cap K)\uparrow(\mathfrak{g},K))(Y\otimes\tau^{\sim}).$$

The techniques of [11] show how to make $R$ act on the derived functor modules $\Gamma^i Z$, in such a way as to recover the action of $U(\mathfrak{g})$ on its image in $R$. What we want to show is that, under the hypotheses of the proposition, this action is actually irreducible.

Fix a large integer $n$, and let $F_L$ be the $n$th power of the one dimensional representation of $L$ on $\Lambda^{\dim \mathfrak{u}}\mathfrak{u}$. Write $Y' = Y \otimes F_L$; this is still an irreducible unitary representation of $L$, satisfying the conditions of Proposition 13.4. Write $E_L$ for the (one dimensional) ring of endomorphisms of $F_L$, regarded as a Harish-Chandra module for $L_{\mathbb{C}}$. Then $(R_L)' = R_L \otimes E_L$ is a finite length Harish-Chandra bimodule of endomorphisms of $Y' \otimes \otimes \tau^{\sim} \mathbb{C}_{-\rho(\mathfrak{u})}$, containing the image of $U(\mathfrak{l})$. Consequently, $R' = \operatorname{Ind}(Q_{\mathbb{C}} \uparrow G_{\mathbb{C}})((R_L)')$ is an algebra of endomorphisms of $X' = \mathscr{L}^S Y'$. What we have gained by all of this is that if $n$ is large enough, $X'$ is actually irreducible as a $(\mathfrak{g}, K)$-module ([38], Proposition 4.18); so it is certainly irreducible as an $(R', K)$ module.

We want to relate $X'$ and $R'$ to $X$ and $R$. Let $F^*$ be the finite dimensional representation of $G_{\mathbb{C}}$ of lowest weight $(F_L)^*$, and let $E = F \otimes F^*$ be its endomorphism ring. By the Jacobson density theorem, the algebra $R' \otimes E$ acts irreducibly on $X' \otimes F^*$. List the distinct infinitesimal characters occuring in $X' \otimes F^*$ as $\chi_1, \ldots, \chi_r$. Write $P_i$ for the functor which takes a $\mathfrak{Z}(\mathfrak{g})$-finite $\mathfrak{g}$-module to its summand of generalized infinitesimal character $\chi_i$. Then

$$X' \otimes F^* = \Sigma P_i(X' \otimes F^*) = \Sigma X_i .$$

Similarly, $P_i$ can be made to act on Harish-Chandra modules for $G_{\mathbb{C}}$, regarded as $\mathfrak{g}^L$ or (after twisting by the Chevalley automorphism) $\mathfrak{g}^R$ modules; we write these two functors as $(P_i)^L$ and $(P_i)^R$, respectively. Put

$$R_{ij} = [(P_i)^L (P_j)^R](R') .$$

Then $R_{ij}$ maps $X_j$ to $X_i$; by the density theorem again, it is dense in the space of all such linear transformations. In particular, $X_i$ is an irreducible module for $R_{ii}$.

To finish the proof, it suffices to show that our original $X$ is one of the $X_i$ (say $X_1$), and that $R_{11}$ is $R$. These two assertions are proved in exactly the same way (indeed the second is in a certain sense a special case of the first), so we concentrate on the first. We have

$$\text{(16.18)(a)} \qquad X' \otimes F^* \cong \Gamma^S(Z \otimes F^*)$$

(cf. Definition 5.8 and (16.17)); and

$$\text{(16.18)(b)} \qquad Z \otimes F^* \cong \operatorname{ind}((Y' \otimes \tau^{\sim}) \otimes F^*) .$$

**Lemma 16.19.** *In the setting above, let $F_1$ be any irreducible representation of $L$ occurring in $F$, and let $\lambda_1 + \tau^{\sim} - \rho(\mathfrak{u})$ be an infinitesimal character for $L$ occuring in $(Y')_L \otimes (F_1)^*$. Assume that $\lambda_1$ is in the $\mathfrak{g}$ Weyl group orbit of $\lambda$. Then $F_1$ is $F_L$.*

*Proof.* The infinitesimal character for $L$ of $Y'$ is $\lambda + 2n\rho(\mathfrak{u}) + \tau^{\sim} - \rho(\mathfrak{u})$; this is the $n$ introduced at the beginning of the proof of Proposition 16.8. As a subrepresentation of $F$, $F_1$ has weights of the form

$$2n\rho(\mathfrak{u}) - \Sigma n_\alpha \alpha ,$$

the sum extending over roots in $\mathfrak{u}$. This leads to an equation

$$\lambda_1 + \tau^{\sim} - \rho(\mathfrak{u}) = [\lambda + 2n\rho(\mathfrak{u}) + \tau^{\sim} - \rho(\mathfrak{u})] - [2n\rho(\mathfrak{u}) - \Sigma n_\alpha \alpha] .$$

Consequently

$$w\lambda = \lambda + \Sigma n_\alpha \alpha .$$

In particular, the two sides of this last equation have the same length. By the proof of Proposition 15.2, this implies that all the $n_\alpha$ are zero. It follows that the weight $2n\rho(\mathfrak{u})$ occurs in $F_1$, which proves the lemma. Q.E.D.

By standard arguments (cf. [36], Lemma 7.2.3 and Proposition 7.4.1), Lemma 16.19 implies that $Y$ is one of the $Y_i$, as we wished to show. This completes the proof of Proposition 16.8, and that's the ball game.

## 17. Other constructions and parametrizations

In this section, we consider other ways of organizing the classification of unitary representations. The purpose of this exercise is to provide a more flexible toolkit for possible harmonic analysis applications, and to gain a little insight into possible generalizations of these results to other groups. Neither of these purposes seems to demand proofs, so we omit them. On the other hand, it is perhaps worthwhile to keep the discussion in the framework of general reductive Lie groups for as long as possible.

*Definition 17.1.* Suppose $G$ is a reductive Lie group. A *Levi subgroup* $L$ of $G$ is the centralizer in $G$ of a reductive abelian subalgebra of $\mathfrak{g}_0$. Any such $L$ is conjugate by $G$ to one which is stable under the Cartan involution $\theta$; so we assume that is the case. Write

(a) $$\mathfrak{c} = \text{center of } \mathfrak{l} = \mathfrak{c}\cap\mathfrak{k} + \mathfrak{c}\cap\mathfrak{p}.$$

Define

(b) $$L_\theta = \text{centralizer of } \mathfrak{c}\cap\mathfrak{k} \text{ in } G.$$

Then $L$ is the Levi subgroup of a real parabolic subgroup

(c) $$P = LN$$

of $L_\theta$. There is a $\theta$-stable parabolic subalgebra

(d) $$\mathfrak{q}_\theta = \mathfrak{l}_\theta + \mathfrak{u}_\theta$$

of $\mathfrak{g}$, with Levi subgroup $L_\theta$. As usual, define

(e) $$\rho(\mathfrak{u}_\theta) = \text{half the trace of } \mathfrak{l}_\theta \text{ on } \mathfrak{u}_\theta.$$

To simplify notation, we will assume that $L_\theta$ has a character

(f) $$\tau^{\sim} : L_\theta \to \mathbb{C}$$

with differential $\rho(\mathfrak{u}_\theta)$. This is automatic for $GL(n)$, by a generalization of Lemma 6.9. (In general, following Duflo, one shoud introduce a two-fold cover of $L_\theta$ on which $\tau^{\sim}$ exists, and work with representations of it.)

Fix now an irreducible $(\mathfrak{l}, L\cap K)$-module $Y$. Define

(g) $$\lambda_0(Y) = \text{weight by which } \mathfrak{c}\cap\mathfrak{k} \text{ acts in } Y \in (\mathfrak{c}\cap\mathfrak{k})^*.$$

We say that $\mathfrak{q}_\theta$ is *weakly non-negative for* $Y$ if for every weight $\alpha$ of $\mathfrak{c}\cap\mathfrak{k}$ in $\mathfrak{u}_\theta$,

$$\text{(h)} \qquad \langle\lambda_0(Y),\alpha\rangle \geqq 0 .$$

Obviously such $\mathfrak{q}_\theta$ exist.

Using $P$ and $\mathfrak{q}_\theta$, one can proceed exactly as in Definition 6.15 to construct a functor $\mathscr{I}$ from $(\mathfrak{l}, L\cap K)$-modules to $(\mathfrak{g}, K)$-modules. We call $\mathscr{I}$ *weakly non-negative for* $Y$ if $\mathfrak{q}_\theta$ is.

In this generality, one can expect essentially nothing good to be true about $\mathscr{I}Y$. One needs to restrict $Y$ to be small in some sense; exactly what sense is an excellent and interesting question. Once this is done, $\mathscr{I}Y$ should be unitary (though not necessarily irreducible); it will depend heavily on the choice of $\mathfrak{q}_\theta$. For GL$(n)$, the situation is rather good.

**Theorem 17.2.** *Suppose* $G=\mathrm{GL}(n,\mathbb{F})$, *and* $L$ *is a* $\theta$*-stable Levi factor of* $G$. *Use the notation of Definition 17.1. Then* $L$ *is of the form*

$$L=\mathrm{GL}(\pi,\mathbb{C})\times\mathrm{GL}(\xi,\mathbb{F}).$$

*Here* $\pi$ *is an ordered partition of* $p$, *and* $\xi$ *is an ordered partition of* $q$. *In the case of* $\mathbb{C}$, $p=n$ *and* $q=0$. *For* $\mathbb{R}$, $n=2p+q$. *For* $\mathbb{H}$, $n=p+q$. *In any case,*

$$L_\theta=\mathrm{GL}(\pi,\mathbb{C})\times\mathrm{GL}(q,\mathbb{F}).$$

*Let* $Y$ *be an irreducible unitary* $(\mathfrak{l}, L\cap K)$-module. Assume *that on each* GL *factor of* $L$, $Y$ *is either a one dimensional character, or a Stein complementary series representation. Fix* $\mathscr{I}$ *weakly non-negative for* $Y$ *(Definition 17.1).*

a) $\mathscr{I}Y$ *is unitary.*

b) $\mathscr{I}Y$ *is irreducible or zero.*

c) *If* $\mathscr{I}'$ *is also weakly non-negative for* $Y$, *then* $\mathscr{I}'Y\cong\mathscr{I}Y$.

*Definition 17.3.* Suppose $G$ is GL$(n,\mathbb{F})$. A *Levi datum* for $G$ is a pair $(L, Y)$ as in Theorem 17.2. (In particular, $Y$ is a tensor product of unitary characters and Stein complementary series.) Fix $\mathscr{I}$ as in Theorem 17.2, and define

$$\mathscr{I}(L\uparrow G)(Y)=\mathscr{I}Y,$$

a unitary $(\mathfrak{g}, K)$-module which is irreducible or zero.

Write

$$L=\mathrm{GL}(\pi,\mathbb{C})\times\mathrm{GL}(\xi,\mathbb{F}).$$

as in Theorem 17.2. The parameter $\lambda_0(Y)$ is specified by an integer $\lambda_0(i)$ (the compact part of the central character) on each factor GL$(p_i,\mathbb{C})$; it is necessarily zero on GL$(\xi,\mathbb{F})$ (which has no central compact torus). We say that $(L, Y)$ is *non-degenerate* if each $\lambda_0(i)$ is at least equal to $d_{\mathbb{F}}$ in absolute value. Here

$$d_{\mathbb{C}}=0,\quad d_{\mathbb{R}}=1,\quad d_{\mathbb{H}}=2.$$

The motivation for the definition of non-degenerate is the next result, which is just a rewording of Theorems 6.18 and 3.8.

**Theorem 17.4.** *Any unitary irreducible representation of* GL$(n,\mathbb{F})$ *is of the form*

$$\mathscr{I}(L\uparrow G)(Y)$$

*for some non-degenerate Levi datum $(L, Y)$ (Definition 17.3). The pair $(L, Y)$ is unique up to conjugacy under $K$. The parameter $\lambda(\mu)$ attached to the lowest $K$-type $\mu$ of $\mathscr{I}Y$ in section 5 is conjugate to $\lambda_0(Y)$.*

The last assertion of this theorem makes sense for degenerate data as well, but is false in that case if $\mathbb{F}=\mathbb{H}$.

Here are the two basic "independence of polarization" results, which give some flexibility in the construction of $\mathscr{I}Y$. In both cases, we begin with a chain of Levi subgroups

$$L \subset L' \subset G. \tag{17.5}$$

We also fix an irreducible unitary $(\mathfrak{l}, L\cap K)$-module $Y$, such that $(L, Y)$ is a Levi datum (Definition 17.3).

**Theorem 17.6.** *In the setting* (17.5), *assume that $L'$ is a Levi subgroup of a real parabolic subgroup $P'=L'N'$ of $G$. Then*

$$\mathscr{I}(L\uparrow G)(Y) \cong \operatorname{Ind}(P'\uparrow G)(\mathscr{I}(L\uparrow L')(Y)).$$

One immediate corollary of this theorem is that unitarily induced representations of $\mathrm{GL}(n, \mathbb{F})$ are all irreducible.

**Theorem 17.7.** *In the setting* (17.5), *assume that $L'$ is the Levi subgroup of a $\theta$-stable parabolic subalgebra $\mathfrak{q}'=\mathfrak{l}'+\mathfrak{u}'$ of $\mathfrak{g}$. Recall that $\mathfrak{c}\cap\mathfrak{k}$ denotes the compact part of the center of $\mathfrak{l}$, and that $\lambda_0(Y)$ belongs to the dual of this space. Assume that for every weight $\alpha$ of $\mathfrak{c}\cap\mathfrak{k}$ in $\mathfrak{u}'$, we have*

$$\langle\lambda_0(Y), \alpha\rangle \geqq 0 \tag{i}$$

*Then*

$$\mathscr{I}(L\uparrow G)(Y) \cong \mathscr{L}^S((\mathfrak{q}', L'\cap K)\uparrow(\mathfrak{g}, K))(\mathscr{I}(L\uparrow L')(Y)).$$

*The other $\mathscr{L}^j(\mathscr{I}(L\uparrow L')Y)$ are all zero.*

Even if $L'$ is the Levi subgroup of some $\theta$-stable parabolic, it may not be possible to find one satisfying the positivity condition (i). In that case, I know of no construction of $\mathscr{I}(L\uparrow G)(Y)$ from $\mathscr{I}(L\uparrow L')(Y)$.

*Definition 17.8.* Suppose we are in the setting of Definition 17.1; use the notation there. Set

$$L_{\mathbb{R}} = \text{centralizer of } \mathfrak{c}\cap\mathfrak{p} \text{ in } G \tag{a}$$

Then $L$ is the Levi factor of a $\theta$-stable parabolic subalgebra

$$\mathfrak{q}'=\mathfrak{l}+\mathfrak{u}' \tag{b}$$

of $\mathfrak{l}_{\mathbb{R}}$. There is a real parabolic subgroup

$$P'=L_{\mathbb{R}}N' \tag{c}$$

of $G$.

Using $\mathfrak{q}'$ and $P'$, one can proceed as in Definition 6.15 (but with the order of the steps reversed) to construct a functor $\mathscr{I}'$ from $(\mathfrak{l}, L\cap K)$-modules to $(\mathfrak{g}, K)$-modules.

Again there is a notion of weakly non-negative, still referring only to the cohomological induction step of the construction.

Using Theorems 17.6 and 17.7, one deduces immediately

**Corollary 17.9.** *In the setting of Definitions 17.1 and 17.8, suppose $G$ is* GL $(n, \mathbb{F})$. *In the notation of Theorem 17.2, we have*

$$L_{\mathbb{R}} = \mathrm{GL}\,(e_{\mathbb{F}}\pi, \mathbb{F}) \times \mathrm{GL}\,(\xi, \mathbb{F}).$$

*The notation means that each part of $\pi$ is multiplied by the constant $e_{\mathbb{F}}$, which is 2 for $\mathbb{R}$ and 1 otherwise. If $Y$ is an irreducible unitary $(\mathfrak{l}, L\cap K)$-module, such that $(L, Y)$ is a Levi datum (Definition 17.3), and $\mathfrak{q}'$ is chosen to be weakly non-negative for $Y$, then*

$$\mathscr{I}Y \cong \mathscr{I}'Y.$$

Using this theorem, one sees that the only cases when the cohomological induction functor is absolutely needed are to go from a unitary character or a Stein complementary series of GL $(n, \mathbb{C})$ to a representation of GL $(e_{\mathbb{F}}n, \mathbb{F})$. The resulting representations of GL $(e_{\mathbb{F}}n, \mathbb{F})$ are the "extra building blocks" mentioned in the introduction.

To complete the picture of induction we have developed, we only need to compute $\mathscr{I}Y$ in the case of degenerate data. Using induction by stages (Theorems 17.6 and 17.7), one is reduced to a very few cases. Theses are dealt with by the next results.

**Proposition 17.10.** *Suppose $G$ is* GL$(2n, \mathbb{R})$, *$L$ is* GL$(n, \mathbb{C})$, *and $Y$ is a unitary character or a Stein complementary series for $L$. Assume that the datum $(L, Y)$ is degenerate: that is, that $Y$ has an $L\cap K$-fixed vector (Definition 17.3). Let $L'$ denote the subgroup* GL$(n, \mathbb{R}) \times$ GL$(n, \mathbb{R})$, *which is a Levi subgroup of a real parabolic subgroup $P' = L'N'$ of $G$. Then there is a unitary character or Stein complementary series $Y'$ of $L'$, such that*

$$\mathscr{I}Y \cong \mathrm{Ind}\,(P' \uparrow G)\,(Y').$$

*Explicitly, $Y'$ is described as follows.*

a) *Suppose $Y$ is the unitary character $\delta^{iv}$ (cf. (2.2)) of $L$.*

$$Y' = (\delta^{iv}) \otimes (\mathrm{sgn}\,(\det) \cdot \delta^{iv}).$$

b) *Suppose $n = 2m$ is even, and $Y$ is a Stein complementary series*

$$\sigma_{2m}(\delta^{iv}, t)$$

*(cf. Definition 2.3) of $L$. Then*

$$Y' = \sigma_{2m}(\delta^{iv}, t) \otimes \sigma_{2m}(\mathrm{sgn}\,(\det) \cdot \delta^{iv}, t).$$

**Proposition 17.11.** *Suppose $G$ is* GL$(n, \mathbb{H})$, *$L$ is* GL$(n, \mathbb{C})$, *and $Y$ is a unitary character or a Stein complementary series for $L$. Assume that $(L, Y)$ is degenerate; that is, that the lowest* U$(n)$*-type of $Y$ is trivial,* det, *or* $(\det)^{-1}$.

a) *If the lowest* U$(n)$*-type of $Y$ is trivial, then $\mathscr{I}Y$ is zero. Suppose for the rest of the theorem that the lowest* U$(n)$*-type of $Y$ is* det. *(The* $(\det)^{-1}$ *case is identical.)*

*Twisting Y by* $\delta^{2iv}$ *twists* $\mathscr{I}Y$ *by* $\delta^{iv}$, *so we may as well assume that Y is trivial on the split part of the center of L. There are several cases.*

b) *Suppose* $n = 2m + 1$, *and Y is a character. Then* $\mathscr{I}Y$ *is unitarily induced from the trivial character of* $\mathrm{GL}(m+1, \mathbb{H}) \times \mathrm{GL}(m, \mathbb{H})$.

c) *Suppose* $n = 2m$, *and Y is a character. Then* $\mathscr{I}Y$ *is the Stein complementary series* $\sigma_{2m}(1, \frac{1}{4})$.

d) *Suppose* $n = 4m$, *and Y is the Stein complementary series with parameter t. Then* $\mathscr{I}Y$ *is induced from* $\mathrm{GL}(2m, \mathbb{H}) \times \mathrm{GL}(2m, \mathbb{H})$, *by the Stein complementary series*

$$\sigma_{2m}(1, \tfrac{1}{4} + t/2) \otimes \sigma_{2m}(1, \tfrac{1}{4} - t/2).$$

e) *Suppose* $n = 4m + 2$, *and Y is the Stein complementary series with parameter t. Then* $\mathscr{I}$ *is induced from* $\mathrm{GL}(2m+2, \mathbb{H}) \times \mathrm{GL}(2m, \mathbb{H})$, *by the Stein complementary series*

$$\sigma_{2m+2}(1, t/2) \otimes \sigma_{2m}(1, t/2).$$

The most striking feature of this proposition is perhaps (c): a representation which looks entirely "complementary" can in fact be realized in a "discrete" way.

*Acknowledgements.* This work was done during a visit to Université Paris VII. I would like to thank Michel Duflo and the UER de Mathématiques for their hospitality.

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Oblatum 30-IV-1985/2-IX-1985