

Dear gang,

This is going to be somewhat disorganized; I don't know exactly what the point or the question is supposed to be, although there are many little points and little questions scattered through this. Perhaps the big point is this: an early goal of the Atlas unitarity project should be to rewrite the Knapp/Baldoni-Silva JFA paper mentioned below, in a manner consistent with our general strategy.

I have been trying to think about the big induction step in our program to describe the unitary dual of a reductive group G : in which you look at one K -type μ , and try to describe

$$\Pi_u(G)(\mu),$$

the set of all irreducible unitary representations of G with lowest K -type μ . Very roughly speaking, this problem is supposed to be proven equivalent to some (nearly?) spherical problem ($\mu = \text{trivial}$) on some smaller group.

So I was waxing poetic on this subject to Tony Knapp, and saying that the Shimura correspondence paper that Jeff, Dan, Annegret, Peter, and I just finished was an illustration of the idea. Tony's response was this: what about my paper with Welleda?

Here's the subject of Tony's paper with Welleda (Journal of Functional Analysis, 1986). Suppose $P = MAN$ is a parabolic subgroup of G , with $\dim A = 1$, and suppose δ is a non-degenerate limit of discrete series representation for M . For any $\nu \in \hat{A}$, write $I(\delta \otimes \nu)$ for the principal series, and $J(\delta \otimes \nu)$ (which is irreducible except perhaps at $\nu = 0$, when it may split into two pieces). The problem they solve is

$$\text{when is } J(\delta \otimes \nu) \text{ unitary?}$$

In keeping with our (terrible!) notation from the Shimura paper, one could call this set $\mathbf{CS}(\delta) \subset \hat{A}$.

The connection with our general reduction step problem is this. Let μ be a lowest K -type of $I(\delta \otimes \nu)$ (of course the K -spectrum is independent of ν). Then

$$\{J(\delta \otimes \nu) \mid \nu \in \mathbf{CS}(\delta)\} \subset \Pi_u(G)(\mu).$$

(Well, essentially. At $\nu = 0$, the J may be a sum of two pieces, one of which is in $\Pi_u(G)(\mu)$.) If δ is actually a discrete series representation, then $\subset$ becomes $=$. SO in the setting of Tony and Welleda's paper, what we hope to do is prove that $\mathbf{CS}(\delta)$ looks exactly like some (rank one?) spherical complementary series.

Tony's point is that this goal is more or less ridiculous: that the set of cases to which one ought to hope to reduce (even in his paper with Welleda, which is in some sense the simplest case of our big induction) are a much larger and more complicated set than the spherical ones. His proof is that the set of cases to which he and Welleda reduce matters is much larger and more complicated than the spherical ones. My response (which I haven't dared to make to him, in part because I don't know whether it's true) is that they didn't try hard enough to reduce.

So I started trying to look carefully at the paper, which I am sorry to say I had never done before. My goal was to see what cases *really* don't admit reduction.

The problem is enormously complicated. One of my first steps was to make it worse, by dropping the Tony/Welleda assumption that δ is non-degenerate. (We work usually with the notion of "final." If δ is *any* non-zero limit of discrete series with $\dim A = 1$, then $\delta \otimes \nu$ is final at least for non-zero ν .)

So I started looking at $U(p, q)$. If $P = MAN$ is a cuspidal parabolic subgroup with $\dim A = 1$, then necessarily

$$M = U(p-1, q-1) \times \text{circle},$$

the circle being the diagonal $U(1)$ in $U(1, 1)$. In this picture the group A is made of cosh and sinh inside this same $U(1, 1)$. I won't go through all the notational nonsense, but I got a machine going for identifying these sets $\mathbf{CS}(\delta)$ with actual spherical complementary series, usually for some smaller $U(m, 1)$. The machine cruised along just fine until it came to a certain case in $U(5, 3)$, corresponding to the lowest K -type

$$\mu = (0, 0, 0, 0, 0) \times (1, 0, -1)$$

for $U(5) \times U(3)$. So M is $U(4, 2) \times U(1)$. The Harish-Chandra parameter for δ is

$$\lambda = (1/2, 1/2, 0, -1/2, -1/2) \times (1/2, 0, -1/2).$$

Here I have put the Cartan for M inside the standard maximal torus $U(1)^5 \times U(1)^3$ for K . The two coordinates equal to zero correspond to the $U(1)$ factor, and the other six coordinates are the Harish-Chandra parameter for $U(4, 2)$. In order to specify a limit of discrete series, I should also tell you a positive root system for M making λ dominant. This I'll do by writing

$$\rho_M = (5/2, 1/2, 0, -1/2, -5/2) \times (3/2, 0, -3/2).$$

For a variety of purposes, it's useful to have a positive root system for G containing this positive system for M , and making λ dominant. The corresponding half sum is

$$\rho_G = (7/2, 3/2, 1/2, -3/2, -7/2) \times (5/2, 0, -5/2).$$

The corresponding lowest K -type is pretty easy to calculate from this information (at least for Susana...). It's the μ listed above.

This example is excluded by Tony and Welleda since δ is degenerate, but I don't think that's important. Here is the problem. For real values of ν , the representation $I(\delta \otimes \nu)$ is irreducible for $|\langle \nu, \beta^\vee \rangle| \leq 3/2$ (with β the unique real root). This suggests trying to identify the complementary series with that for the spherical series in $U(3, 1)$. The bottom layer theory provides a way to try to do that. There is a θ -stable Levi subgroup

$$L_1 = U(1, 0)^2 \times U(3, 1) \times U(0, 1)^2$$

coming from the positive root system mentioned above; the $U(3, 1)$ factor corresponds to the coordinates of ρ_G between $-3/2$ and $3/2$. For this L_1 , the corresponding $L_1 \cap K$ -type is trivial on $U(3) \times U(1)$ (so it looks like a spherical case). We get unitarity out to $\nu = 3/2$ from the $U(3, 1)$ complementary series (so far so good). The difficulty is with *non*-unitarity for $\nu > 3/2$. In $U(3, 1)$ this is certified by the $\mathfrak{p}$ representations of $U(3) \times U(1)$, which have highest weights

$$(1, 0, 0) \times (-1), \quad (0, 0, -1) \times (1).$$

Bottom layer theory says that the signature on those types is reflected in the representations of K of highest weights

$$(0, 1, 0, 0, 0) \times (1, -1, -1), \quad (0, 0, 0, -1, 0) \times (1, 1, -1).$$

The trouble is that those highest weights are not dominant, so the K -types don't exist, so we don't get the non-unitarity certificates we need.

To a first approximation, I'm stuck. So what's the second approximation? For various reasons, I suspect that good K -types to look at for non-unitarity certificates are those of highest weights

$$(0, 0, 0, 0, 0) \times (2, -1, -1), \quad (0, 0, 0, 0, 0) \times (1, 1, -2).$$

These are bottom layer K -types with respect to some more or less obvious θ -stable parabolic with Levi factor

$$L_2 = U(1, 0) \times U(3, 3) \times U(1, 0).$$

Maybe it comes down to something like this. If we study representations of $U(3, 3)$ with lowest K -type $(0, 0, 0) \times (1, 0, -1)$ (or rather the special one-parameter subfamily of them implicit in this discussion) then we find that they are unitary for $|\nu| \leq 3/2$, and for larger ν there are non-unitarity certificates provided by the K -types

$$(1, 0, 0) \times (1, -1, -1), \quad (0, 0, -1) \times (1, 1, -1).$$

(This is bottom layer theory from a $U(3, 1)$ Levi: the same thing that didn't work for $U(5, 3)$.) I would like to know that in this case we can *also* get non-unitarity certificates by looking instead at the K -types

$$(0, 0, 0) \times (2, -1, -1), \quad (0, 0, 0) \times (1, 1, -2).$$

If I know that, bottom layer gives the non-unitarity certificates I want for $U(5, 3)$. (Incidentally, my actual opinion is that this does *not* work: that these other K -types do *not* provide non-unitarity certificates in $U(3, 3)$. But I would like to be proven wrong.)

If this all works, then of course the question is how general it can be made: given some non-unitarity certificates for a family of representations of G , how can we describe *different* non-unitarity certificates, which may be better suited to some inductive argument where G is showing up as a Levi subgroup?