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On the Structure of Kazhdan–Lusztig Cells for Branched Dynkin Diagrams

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1. INTRODUCTION

Suppose G is a reductive Lie group. A variety of objects related to the infinite-dimensional representations of G carry "coherent continuation representations" of an appropriate Weyl group or Hecke algebra. Perhaps the most interesting examples are certain spaces of invariant eigendistributions on G. Here the action of the Weyl group plays a central role in the Kazhdan–Lusztig conjectures [12], which identify the characters of irreducible representations among all invariant eigendistributions. The analogous construction for highest weight modules was first used systematically by Jantzen [3], and applied to the theory of primitive ideals in the enveloping algebra by Borho and Jantzen [1]. Joseph (in [4], for example) extended these ideas substantially, and the resolution of the Kazhdan–Lusztig conjectures finally permitted the complete description of primitive ideals (as an ordered set) in terms of Weyl groups.

Despite their great theoretical importance, the Kazhdan–Lusztig conjectures (because of their complexity) are of limited value for calculation. It is often valuable to have less information in a more explicit form. The simplest example of what we have in mind is the τ-invariant of Borho and

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Jantzen and Duflo [1, 2]. In all the examples above, it provides useful (but incomplete) information about the action of a simple reflection (or even a corresponding generator of a Hecke algebra) in a coherent continuation representation. Very roughly speaking, the information it provides comes from little $SL(2)$'s inside G. At the next higher level of sophistication is the generalized τ-invariant, which is essentially the subject of [8, 10]. Here one exploits some of the relations in a Weyl group arising from rank two Levi subgroups, and gets somewhat better information about coherent continuation representations.

It is natural to stop with rank two subgroups, since all the defining relations of a Weyl group appear at that level. However, coherent continuation representations have more structure than arbitrary Weyl group representations: Lusztig's theory of "special" Weyl group representations intervenes, for reasons that are still not entirely clear. This theory begins with the Weyl groups of types BC_2, D_4, and G_2, in the sense that each of these groups has a non-special representation whose restriction to any proper Levi subgroup is a sum of special representations. We may therefore expect the structure of coherent continuation representations (and so of irreducible characters and primitive ideals) to be restricted by the presence of a Levi factor of type D_4 in ways that are difficult to predict with more elementary considerations. (There was a period in the late 1970s when the enveloping algebra of $\mathfrak{so}(8)$ was known to have either 36 or 38 primitive ideals contained in the augmentation ideal. The correct answer is 36. Our purpose here is to explore an analogous situation for $(\mathfrak{g}, K)$-modules.)

In this paper we describe explicitly some restrictions on the nature of coherent continuation arising from Levi factors of type D_4. The results seem to us very appealing in their own right. They are used by the first author in her work on the parametrization of primitive ideals and the description of the annihilator of an irreducible Harish-Chandra module.

The paper is organized a follows. Section 2 recalls the definition of coherent continuation representations and presents the main results (Theorem 2.15 and its corollaries). Section 3 is devoted to the proof. Section 4 considers the problem of making explicit calculations; the basic result is Proposition 4.4.

2. Statement of the Main Theorem

Suppose G is a reductive Lie group in Harish-Chandra's class. (We do not require G to be linear.) Fix a Cartan subalgebra $\mathfrak{h}_a$ of the complexified Lie algebra $\mathfrak{g}$ of G. (The subscript *a stands for "abstract"; it indicates that* $\mathfrak{h}_a$ *will be used as a kind of reference point when several Cartan subalgebras are considered at the same time.* Write R_a for the root system of

$\mathfrak{h}_a$ in $\mathfrak{g}$, $(R_a)^+$ for a fixed positive root system, and W_a for the Weyl group. Recall that a weight λ in $(\mathfrak{h}_a)^*$ is called *dominant* if $\langle \phi^\vee, \lambda \rangle$ is not a negative integer for any ϕ in $(R_a)^+$, and *regular* if $\langle \phi^\vee, \lambda\rangle$ is not zero for any ϕ in R_a. The set of *integral roots* for λ is

$$R_a(\lambda) = \{\phi \in R_a \mid \langle \phi^\vee, \lambda \rangle \in \mathbb{Z}\}.$$

This is a root system, but it need not be the root system of a subalgebra of $\mathfrak{g}$. The set of simple roots for the positive system

$$(R_a(\lambda))^+ = R_a(\lambda) \cap (R_a)^+$$

is called $\Pi(\lambda)$. The Weyl group of $R_a(\lambda)$ is written $W_a(\lambda)$, the *integral Weyl group*. It may also be defined by

$$W_a(\lambda) = \{w \in W_a \mid w\lambda - \lambda \text{ is a sum of roots}\}.$$

$W_a(\lambda)$ is a Coxeter group with generating reflections

$$S_a(\lambda) = \{s_\phi \mid \phi \in \Pi(\lambda)\}.$$

The weight λ is called *integral* if $R_a(\lambda) = R_a$ (or, equivalently, if $W_a(\lambda) = W_a$). Occasionally we will also need the *singular roots* for λ

$$(R_a)^\lambda = \{\phi \in R_a \mid \langle \phi^\vee, \lambda \rangle = 0\}.$$

The Weyl group of this root system is the subgroup $(W_a)^\lambda$ fixing λ.

The Harish-Chandra map provides an isomorphism from the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ to the algebra $S(\mathfrak{h}_a)^{W_a}$ of Weyl group-invariant polynomials in $\mathfrak{h}$. In this way the homomorphisms from $\mathfrak{Z}(\mathfrak{g})$ to $\mathbb{C}$-equivalently, the maximal ideals in $\mathfrak{Z}(\mathfrak{g})$—are identified with Weyl group orbits on $\mathfrak{h}^*$. We write $\chi(\lambda)$ for the homomorphism (or *infinitesimal character*) corresponding to λ. We say that $\chi(\lambda)$ is *integral* or *regular* if λ is; as the notation indicates, these depend only on the map $\chi(\lambda)$; that is, on the orbit $W_a \cdot \lambda$.

Write $\mathrm{Ad}(\mathfrak{g})$ for the complex semisimple group of inner automorphisms of $\mathfrak{g}$. The weights of any finite-dimensional representation F of $\mathrm{Ad}(\mathfrak{g})$ lie in the lattice Q_a generated by R_a. Q_a is a lattice in $(\mathfrak{h}_a)^*$ of rank equal to the rank of the group $\mathrm{Ad}(\mathfrak{g})$.

Fix now a maximal compact subgroup K of G. We will state our results for the category of $(\mathfrak{g}, K)$-modules. *There is no difficulty in formulating them for any other category with a reasonable theory of coherent continuation* (including the category of highest weight modules, for example). We have chosen $(\mathfrak{g}, K)$-modules because the technical problems with coherent continuation in that case are slightly more serious. Finite-dimensional continuous representations of G correspond precisely to the finite-dimensional $(\mathfrak{g}, K)$-modules. In particular, any representation of $\mathrm{Ad}(\mathfrak{g})$ gives a representation of G (by composition with Ad) and hence a finite-dimensional $(\mathfrak{g}, K)$-module.

Write $\mathscr{F}(\mathfrak{g}, K)$ for the category of $(\mathfrak{g}, K)$-modules of finite length, and $\mathscr{V}(\mathfrak{g}, K)$ for its Grothendieck group. Recall that $\mathscr{F}(\mathfrak{g}, K)$ is closed under tensor product with a finite-dimensional $(\mathfrak{g}, K)$-module F. Because it is exact, the functor $\otimes F$ passes to $\mathscr{V}(\mathfrak{g}, K)$. Any $(\mathfrak{g}, K)$-module X of finite length decomposes as a finite direct sum over infinitesimal characters χ:

$$X = \bigoplus X_\chi,$$

with X_χ having generalized infinitesimal character χ. Write $\mathscr{F}(\mathfrak{g}, K; \chi)$ for the category of $(\mathfrak{g}, K)$-modules of finite length and generalized infinitesimal character χ, and $\mathscr{V}(\mathfrak{g}, K; \chi)$ for its Grothendieck group. Clearly

$$\mathscr{V}(\mathfrak{g}, K) = \bigoplus \mathscr{V}(\mathfrak{g}, K; \chi).$$

Definition 2.1 (Schmid [6]). Fix λ in $(\mathfrak{h}_a)^*$. A *coherent family* of $(\mathfrak{g}, K)$-modules based on $\lambda + Q_a$ is a function

$$\Theta: \lambda + Q_a \to \mathscr{V}(\mathfrak{g}, K)$$

with the following properties:

(a) for all μ in $\lambda + Q_a$, $\Theta(\mu) \in \mathscr{V}(\mathfrak{g}, K; \chi(\mu))$; and

(b) for any finite-dimensional representation F of $\mathrm{Ad}(\mathfrak{g})$, and any μ in $\lambda + Q_a$,

$$\Theta(\mu) \otimes F = \sum m(\gamma, F)\, \Theta(\mu + \gamma).$$

The sum extends over the weights γ of F, and the coefficient is the multiplicity of γ as a weight of F. The coherent family is called *irreducible* if $\Theta(\mu)$ is irreducible for one (or, equivalently, for all) regular dominant weight(s) μ in $\lambda + Q_a$.

Here is the basic theorem about coherent families.

Theorem 2.2 (Speh and Vogan [7]). *Suppose X is an irreducible $(\mathfrak{g}, K)$-module with infinitesimal character $\chi(\lambda)$, and λ is dominant. Then there is a unique irreducible coherent family Θ based on $\lambda + Q_a$ such that $\Theta(\lambda) = X$.*

Definition 2.3 (cf. [11, Chap. 7]). Suppose λ is a regular dominant weight. Then the integral Weyl group $W_a(\lambda)$ acts on the lattice $\mathscr{V}(\mathfrak{g}, K; \chi(\lambda))$ of virtual $(\mathfrak{g}, K)$-modules having infinitesimal character λ, as follows. Suppose $V \in \mathscr{V}(\mathfrak{g}, K; \chi(\lambda))$. Let Θ be the unique coherent family such that $\Theta(\lambda) = V$. Then

$$w \cdot V = \Theta(w^{-1} \cdot \lambda).$$

This action is called the *coherent continuation representation* of $W_a(\lambda)$.

In order to formulate our result, we need two basic facts about the coherent continuation representation. The first, due to Jantzen in the case of highest weight modules, is this.

Theorem 2.4 (Speh and Vogan [7]; see [11, Chap. 7]). *Suppose λ is a regular dominant weight, X is an irreducible $(\mathfrak{g}, K)$-module of infinitesimal character $\chi(\lambda)$, and $\phi \in \Pi(\lambda)$ is an integral simple root. (We do not assume that ϕ is simple in $(R_a)^+$.) Then either*

(a) $s_\phi \cdot X = X + U_\phi(X)$, *or*

(b) $s_\phi \cdot X = -X$.

In case (a), *$U_\phi(X)$ is a sum (with positive coefficients) of irreducible $(\mathfrak{g}, K)$-modules Z satisfying $s_\phi \cdot Z = -Z$.*

As is explained in [9], determination of the characters of irreducible $(\mathfrak{g}, K)$-modules is equivalent (at least if G is linear) to determination of all the various $U_\phi(X)$. The Kazhdan–Lusztig conjecture (now a theorem, although a detailed proof has appeared only in the case of integral infinitesimal character) tells how to do this in principle.

Definition 2.5 (Borho and Jantzen and Duflo). In the setting of Theorem 2.4, the *τ-invariant* of X is

$$\tau(X) = \{\phi \in \Pi(\lambda) \mid s_\phi \cdot X = -X\}.$$

We can now formulate the next result we need.

Theorem 2.6. *In the setting of Theorem 2.4, suppose ψ is another integral simple root. Assume that ϕ and ψ span a subsystem of type A_2, and that*

$(*)(\phi, \psi) \quad \phi \notin \tau(X),\ \psi \in \tau(X).$

Then there is a unique irreducible $(\mathfrak{g}, K)$-module Y such that

(a) $s_\phi \cdot X = X + Y + U_{\phi\psi}(X)$,

with $U_{\phi\psi}(X)$ a sum of irreducible $(\mathfrak{g}, K)$-modules having both ϕ and ψ in their τ-invariants. In addition, Y has the following two properties:

(b) $\psi \notin \tau(Y)$, $\phi \in \tau(Y)$;

(c) $s_\psi \cdot Y = Y + X + U_{\psi\phi}(Y)$,

with $U_{\psi\phi}(Y)$ a sum of irreducible $(\mathfrak{g}, K)$*-modules having both ψ and ϕ in their* τ*-invariants.*

This result is an easy formal consequence of Theorem 2.4 and the relation $s_\phi s_\psi s_\phi = s_\psi s_\phi s_\psi$. The argument may be found in [8].

If X is an irreducible $(\mathfrak{g}, K)$-module satisfying condition $(*)(\phi, \psi)$ of Theorem 2.6, and Y is defined as in the theorem, we write

$$Y = T_{\phi\psi}(X). \tag{2.7}$$

Part (c) of Theorem 2.6 can be rewritten in this notation as $T_{\psi\phi}(T_{\phi\psi}(X)) = X$.

At last we can begin to formulate the main result.

Setting 2.8. With notation as at the beginning of this section, fix a dominant regular weight λ in $(\mathfrak{h}_a)^*$. Assume that there are four simple roots α, β, γ, and δ in $\Pi(\lambda)$ spanning a subsystem of type D_4, with α the middle root. If X is an irreducible $(\mathfrak{g}, K)$-module of infinitesimal character λ, we defined the *little τ-invariant* of X, $\tau_0(X)$, to be the intersection of $\tau(X)$ with $\{\alpha, \beta, \gamma, \delta\}$.

In order to emphasize the adjacency relations among the simple roots, we may sometimes write the little τ-invariant as a schematic Dynkin diagram, with the roots in the τ-invariant labeled by their names and the roots not in the τ-invariant indicated with an asterisk. Thus the little τ-invariant $\{\alpha, \beta, \delta\}$ would be represented by $\beta\,\alpha\,{}^{*}_{\delta}$.

We need to consider various hypotheses on $(\mathfrak{g}, K)$-modules analogous to $(*)(\phi, \psi)$ in Theorem 2.6. These hypotheses come in four versions (or eight, depending on how one accounts for the symmetries of the Dynkin diagram of D_4).

2.9. *Hypothesis* $\mathscr{A}_\beta$. (We work in Setting 2.8.) An irreducible $(\mathfrak{g}, K)$-module A_β is said to be of *type* $\mathscr{A}_\beta$ if it has the following two properties.

(1) $\tau_0(A_\beta) = \beta\,\alpha\,{}^{*}_{*}$.

As a consequence of (1), A_β belongs to the domains of both $T_{\gamma\alpha}$ and $T_{\delta\alpha}$ (cf. (2.7)). The second property required of A_β is

$(2)_\gamma$ $\tau_0(T_{\gamma\alpha}(A_\beta)) = \beta * {}^{\gamma}_{*}$.

This is equivalent (as we will explain in a moment) to

$(2)_\delta$ $\tau_0(T_{\delta\alpha}(A_\beta)) = \beta * {}^{*}_{\delta}$

or to

$(2)'$ $T_{\delta\alpha}(A_\beta) \neq T_{\gamma\alpha}(A_\beta)$.

We leave to the reader the formulation of Hypotheses $\mathscr{A}_\gamma$ and $\mathscr{A}_\delta$, by permutation of β, γ, and δ.

The equivalence of $(2)_\delta$, $(2)_\gamma$, and $(2)'$ depends on the following lemma.

LEMMA 2.10. (a) *In the setting of Theorem 2.4, suppose ψ is another integral simple root. Assume that ϕ and ψ span a subsystem of type $A_1 \times A_1$, and that*

$(*)(\phi, \psi)$ $\phi \notin \tau(X)$, $\psi \in \tau(X)$.

If Y is any irreducible $(\mathfrak{g}, K)$-module occurring in $U_\phi(X)$, then $\psi \in \tau(Y)$.

(b) *Suppose ϕ, ψ, and χ are integral simple roots spanning a system of type A_3, with ϕ perpendicular to ψ. Let X be an irreducible $(\mathfrak{g}, K)$-module with $\phi, \psi \in \tau(X)$ but $\chi \notin \tau(X)$. Then $\psi \notin \tau(T_{\chi\phi}(X))$ if and only if $T_{\chi\phi}(X) = T_{\chi\psi}(X)$.*

(c) *In the setting of (b), let Y be an irreducible $(\mathfrak{g}, K)$-module with $\phi, \psi \notin \tau(Y)$ but $\chi \in \tau(Y)$. Then $\psi \in \tau(T_{\phi\chi}(Y))$ if and only if $T_{\phi\chi}(Y) = T_{\psi\chi}(Y)$.*

Proof. Part (a) follows from Theorem 2.4 and the fact that s_ϕ and s_ψ commute. Part (b) follows from Theorem 2.6(a) and (b). For (c), the "if" part is simply the statement of Theorem 2.6(b). The converse is proved by applying part (b) of this lemma to $T_{\phi\chi}(Y)$, and then using Theorem 2.6(c). Q.E.D.

We can now prove the equivalence of the three versions of Hypothesis $\mathscr{A}_\beta(2)$. Assuming Hypothesis $\mathscr{A}_\beta(1)$, Lemma 2.10(a) says that β must belong to $\tau_0(T_{\gamma\alpha}(\mathscr{A}_\beta))$ and to $\tau_0(T_{\delta\alpha}(A_\beta))$. The equivalence of $(2)_\gamma$, $(2)_\delta$, and $(2)'$ is therefore a consequence of Lemma 2.10(c).

Here is the second version of the main hypothesis.

2.11. *Hypothesis* $\mathscr{B}_\beta$. (We work in Setting 2.8.) An irreducible $(\mathfrak{g}, K)$-module B_β is said to be of *type* $\mathscr{B}_\beta$ if it has the following two properties.

(1) $\tau_0(B_\beta) = *\,*\,{}^{\gamma}_{\delta}$.

As a consequence of (1), B_β belongs to the domains of both $T_{\alpha\gamma}$ and $T_{\alpha\delta}$ (cf. (2.7)). The second property required of B_β is

$(2)_\gamma$ $\tau_0(T_{\alpha\gamma}(B_\beta)) = *\,\alpha\,{}^{*}_{\delta}$.

This is equivalent to

$(2)_\delta \quad \tau_0(T_{\alpha\delta}(B_\beta)) = * \alpha {}^{\gamma}_{*}$

or to

$(2)' \quad T_{\alpha\gamma}(B_\beta) \neq T_{\alpha\delta}(B_\beta)$.

The equivalence can be proved along the same lines as for Hypothesis $\mathscr{A}_\beta$. To prove $\beta \notin \tau_0(T_{\alpha\gamma}(B_\beta))$, we assume the contrary. Lemma 2.10(a) and Theorem 2.6(c) then imply that $\beta \in \tau_0(B_\beta)$, contradicting Hypothesis $\mathscr{B}_\beta(1)$. For the rest we use Lemma 2.10(b).

2.12. *Hypothesis* $\mathscr{C}$. (We work in Setting 2.8.) An irreducible $(\mathfrak{g}, K)$-module C is said to be of *type* $\mathscr{C}$ if it has the following two properties.

(1) $\tau_0(C) = \beta * {}^{\gamma}_{\delta}$.

As a consequence of (1), C belongs to the domains of $T_{\alpha\beta}$, $T_{\alpha\gamma}$, and $T_{\alpha\delta}$ (cf. (2.7)). The second property required of C is

$(2)_\beta \quad \tau_0(T_{\alpha\beta}(C)) = * \alpha {}^{*}_{*}$.

This is equivalent to the analogous $(2)_\gamma$ and to $(2)_\delta$, as well as to

$(2)' \quad T_{\alpha\beta}(C) = T_{\alpha\gamma}(C) = T_{\alpha\delta}(C)$.

The equivalence of the various versions of (2) follows from Lemma 2.10(b).

2.13. *Hypothesis* $\mathscr{D}$. (We work in Setting 2.8.) An irreducible $(\mathfrak{g}, K)$-module D is said to be of *type* $\mathscr{D}$ if it has the following two properties.

(1) $\tau_0(D) = * \alpha {}^{*}_{*}$.

As a consequence of (1), D belongs to the domains of $T_{\beta\alpha}$, $T_{\gamma\alpha}$, and $T_{\delta\alpha}$ (cf. (2.7)). The second property required of D is

$(2)_\beta \quad \tau_0(T_{\beta\alpha}(D)) = \beta * {}^{\gamma}_{\delta}$.

This is equivalent to each of its obvious analogues $(2)_\gamma$ and $(2)_\delta$, or to

$(2)' \quad T_{\beta\alpha}(D) = T_{\gamma\alpha}(D) = T_{\delta\alpha}(D)$.

In addition to these four "interesting" classes of $(\mathfrak{g}, K)$-modules, we need to describe certain "boring" classes.

DEFINITION 2.14. In Setting 2.8, an irreducible $(\mathfrak{g}, K)$-module satisfying one of the Hypotheses $\mathscr{A}$–$\mathscr{D}$ is said to be *type* 8. We say that an irreducible $(\mathfrak{g}, K)$-module is *type sub*-8 if it satisfies one of the following four hypotheses. (In the first two cases, (x, y, z) denotes a permutation of (β, γ, δ).)

(a) $\mathscr{E}_x : \tau_0(E_x) = \{\alpha, x\}$, but E_x does not satisfy Hypothesis $\mathscr{A}_x$.

(b) $\mathscr{F}_x : \tau_0(F_x) = \{\alpha, y, z\}$.

(c) $\mathscr{G} : \tau_0(G) = \{\beta, \gamma, \delta\}$, but G does not satisfy Hypothesis $\mathscr{C}$.

(d) $\mathscr{H} : \tau_0(H) = \{\alpha, \beta, \gamma, \delta\}$.

The $(\mathfrak{g}, K)$-modules of type sub-8 may be thought of as "smaller" in some sense than those of type 8. If $R(\lambda)$ is itself of type D_4, then this "smallness" is made precise by the Gelfand–Kirillov dimension. Otherwise it is not precise.

We have found it convenient to put much of the main theorem into two illustrations (Figs. 1 and 2). The theorem explains most of their interpretation, but some preliminary words may be helpful. Each diagram is a graph, whose vertices are irreducible $(\mathfrak{g}, K)$-modules. The little τ-invariant of each $(\mathfrak{g}, K)$-module appears next to it. In Fig. 1 the pairs of vertices

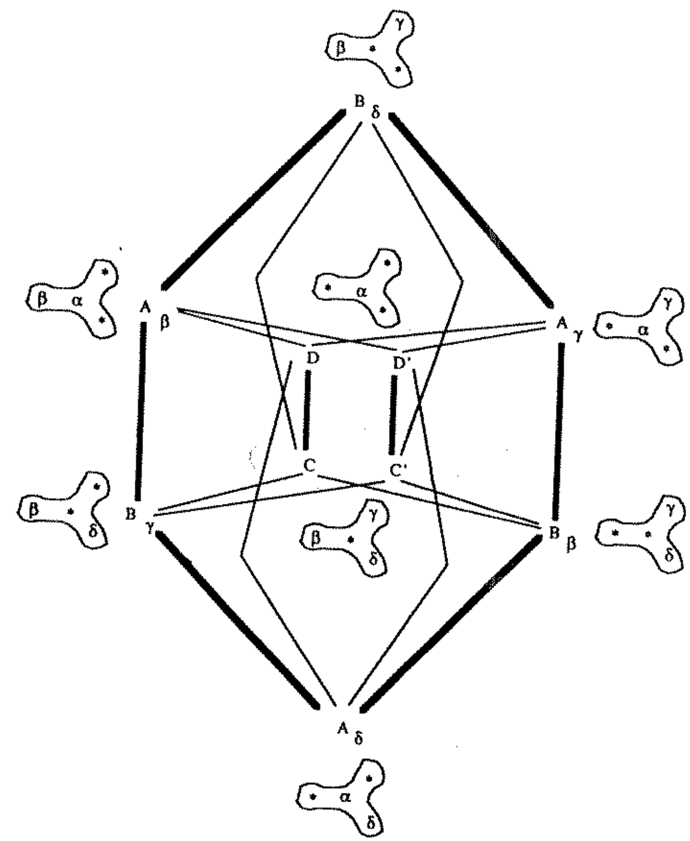

FIG. 1. Type 8-2.

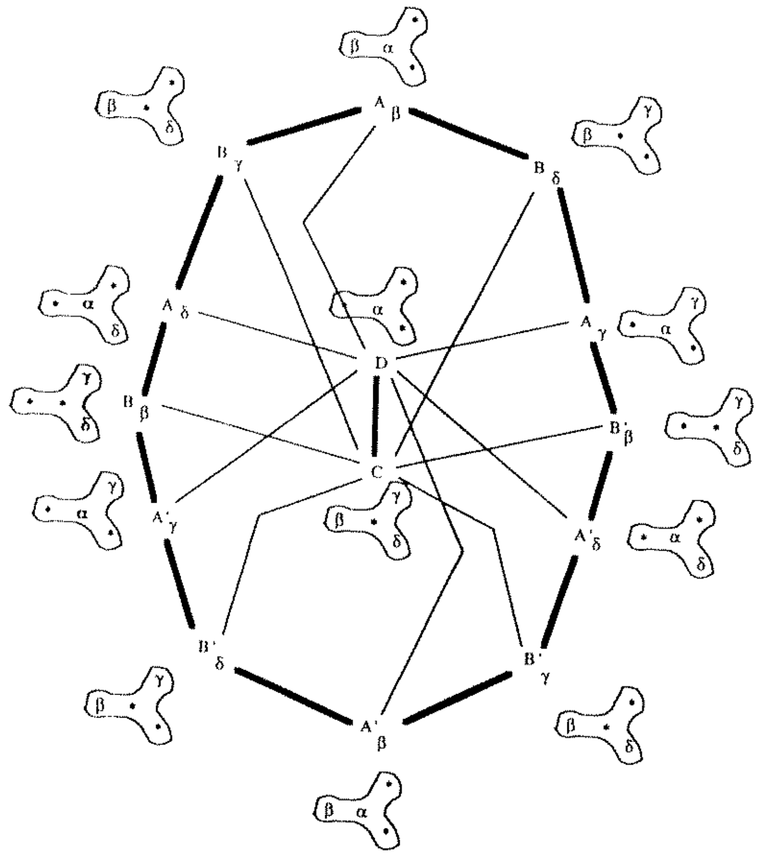

FIG. 2. Type 8-6.

(C, C') and (D, D') share the same little τ-invariants; these have been shown only once. The significance of the heavy edges is explained after the theorem.

THEOREM 2.15. *In Setting 2.8, suppose X is an irreducible $(\mathfrak{g}, K)$-module of type 8 (Definition 2.14)); that is, that X satisfies one of the Hypotheses $\mathscr{A}$–$\mathscr{D}$ stated at 2.9 and 2.11–2.13. Then X belongs to a unique set $\mathscr{C}_0$ of precisely 10 or 14 irreducible $(\mathfrak{g}, K)$-modules, illustrated by Fig. 1 or 2, respectively. The coherent continuation action on X has the following property. Suppose ϕ is one of the four roots $\alpha, \beta, \gamma, \delta$, and ϕ is not in $\tau_0(X)$. Then*

$$s_\phi(X) = X + U_8 + U_{\phi, \text{sub-8}}(X).$$

Here U_8 is a sum of certain irreducible $(\mathfrak{g}, K)$-modules Y in $\mathscr{C}_0$ such that $\phi \in \tau_0(Y)$. Such a Y occurs in U_8 exactly when $X - Y$ is an edge of the graph; in that case it appears exactly once. Every irreducible constituent Z of $U_{\phi, \text{sub-8}}(X)$ is of type sub-8 (Definition 2.14), and $\tau_0(Z) \supset \tau_0(X) \cup \{\phi\}$.

This result should be compared to Theorem 2.6. An analogous formulation of the earlier result would have declared X to be of "type 2" if its τ-invariant contained exactly one of ϕ and ψ, and of "type sub-2" if it contained both. The graph analogous to those in Figs. 1 and 2 would be

$$(*, \psi)\, A_\psi \;-\; A_\phi\, (\phi, *).$$

There are many such subgraphs in Figs. 1 and 2; they are indicated by heavy edges. That is, two elements of $\mathscr{C}_0$ are related by some $T_{\phi\psi}$ (with ϕ and ψ in the D_4 subsystem) exactly when they are joined by a heavy line in the graph.

We record here three immediate corollaries of this theorem, without explaining the various technical concepts to which they refer.

COROLLARY 2.16. *In the setting of Theorem 2.15, suppose X and Y belong to $\mathscr{C}_0$. Then there is a finite-dimensional representation F of $\mathrm{Ad}(\mathfrak{g})$ such that Y is a subquotient of $X \otimes F$ (and conversely). In particular, X and Y have the same Gelfand–Kirillov dimension, associated variety, and wavefront set (in the sense of Howe).*

COROLLARY 2.17. *In the setting of Theorem 2.15, suppose G is a linear group, so that Kazhdan–Lusztig polynomials and related Hecke algebra structure can be defined. Then the graph in Figs. 1 and 2 is the full subgraph of the Kazhdan–Lusztig graph generated by $\mathscr{C}_0$. The Kazhdan–Lusztig function $\mu(X, Y)$ (the coefficient of the highest term in an appropriate Kazhdan–Lusztig polynomial) is 1 if $X - Y$ is an edge and zero otherwise.*

COROLLARY 2.18. *In the setting of Theorem 2.15, suppose X is in $\mathscr{C}_0$. Let $\mathscr{S}$ be one of the eight types of 2.9 and 2.11–2.13. Then the set*

$$\{\mathrm{Ann}(Y) \mid Y \in \mathscr{C}_0,\ Y \text{ of type } \mathscr{S}\}$$

(a set of primitive ideals in $U(\mathfrak{g})$) depends only on $\mathrm{Ann}(X)$.

3. PROOF OF THE MAIN THEOREM

We will divide the argument into four steps. We work always in Setting 2.8. Write W_0 for the Weyl group of the D_4 subsystem. An irreducible $(\mathfrak{g}, K)$-module satisfying the conditions in Hypothesis $\mathscr{A}_\beta$ will be said to be of type $\mathscr{A}_\beta$, etc. Here is an outline of the argument. Fix a type 8 irreducible $(\mathfrak{g}, K)$-module X (Definition 2.14).

Step 1. *The subspace of $\mathscr{V}(\mathfrak{g}, K)$ spanned by the $(\mathfrak{g}, K)$-modules of type 8 or sub-8 (Definition 2.14) is stable under the coherent continuation action of W_0. So is the span of the $(\mathfrak{g}, K)$-modules of type sub-8.*

The proof of this fact uses only ideas from Section 2.

We can therefore study the coherent continuation representation "modulo type sub-8." We define $\mathscr{C}_0$ to be the smallest set of type 8 $(\mathfrak{g}, K)$-modules such that $\mathscr{C}_0$ contains X and the span of $\mathscr{C}_0$ in $\mathscr{V}(\mathfrak{g}, K)$ is W_0-invariant modulo type sub-8. This means that if Y belongs to $\mathscr{C}_0$ and w is in W_0, then every irreducible constituent of the virtual representation $w \cdot Y$ either belongs to $\mathscr{C}_0$ or is of type sub-8.

STEP 2. *Fix an element B_β in $\mathscr{C}_0$ of type $\mathscr{B}_\beta$, and write $\{C^i\}$ for the set of elements $\mathscr{C}_0$ of type $\mathscr{C}$. Define*

$$A_\beta = T_{\alpha\delta} T_{\beta\alpha} T_{\alpha\gamma}(B_\beta)$$
$$(A_\beta)' = T_{\alpha\gamma} T_{\beta\alpha} T_{\alpha\delta}(B_\beta)$$
$$D^i = T_{\alpha\beta}(C^i)$$
$$m_i = \text{multiplicity of } C^i \text{ in } U_\beta(B_\beta)$$
$$n_i = \text{multiplicity of } A_\beta \text{ in } U_\beta(D^i).$$

Then $\sum m_i n_i$ is equal to two if A_β is equal to $(A_\beta)'$, and one otherwise.

This follows from some of the defining relations in W_0, using the ideas of Section 2. In the same way we will prove some related (simpler) numerical results about the action of W_0, and draw somewhat stronger conclusions about the structure of $\mathscr{C}_0$.

STEP 3. *$\mathscr{C}_0$ contains elements of each of the eight types $\mathscr{A}$–$\mathscr{D}$. Specifically,*

(1) *if B_β is of type $\mathscr{B}_\beta$, then $U_\beta(B_\beta)$ has at least one constituent of type $\mathscr{C}$; and*

(2) *if D is of type $\mathscr{D}$, then $U_\beta(D)$ has at least one constituent of type $\mathscr{A}_\beta$.*

The main assertion is an easy consequence of the two numbered ones. Part (1) follows from Step 2. Part (2) lies somewhat deeper. Using primitive ideal theory (as in the proof of Theorem 3.10(c) in [8]) we will reduce it to the case of highest weight modules. Although it is almost certainly possible to avoid this by working much harder, we will then use the Kazhdan–Lusztig conjecture for Verma modules to reduce it to (1).

At this stage we could prove a version of Theorem 2.15, with a large set of possible graphs. The extra possibilities all lack the symmetry of Figs. 1 and 2 with respect to interchanging types $\mathscr{A}_\gamma$ and $\mathscr{B}_\gamma$, and $\mathscr{C}$ and $\mathscr{D}$. They are ruled out by the next step.

STEP 4. *In the notation of Step 2, $m_i = n_i$.*

The proof we give is short, but unfortunately it uses a very deep fact from the proof of the Kazhdan–Lusztig conjectures (for $(\mathfrak{g}, K)$-modules this time). We show that each number is related to the dimension of $\operatorname{Ext}^2(A_\beta, C^i)$.

Until the last step we used only techniques that apply to any category of $\mathfrak{g}$-modules in which Theorem 2.15 can be formulated. It seems unlikely that other graphs can occur for such a category; but we cannot rule them out. There is an analogous question for a subsystem of type B_2: can the analogue of $\mathscr{C}_0$ have exactly two elements? (For $(\mathfrak{g}, K)$-modules there must be three elements, by Theorem 3.10(c) in [8].)

We turn now to the details of the proof.

Proof of Step 1. We will start with an arbitrary irreducible $(\mathfrak{g}, K)$-module of type 8 and a simple reflection s in W_0 not in $\tau_0(X)$, and describe $s \cdot X$ as completely as we can using Theorem 2.6. (We omit repetitions that are obviously symmetric; for example, we discuss $\mathscr{A}_\beta$ but not $\mathscr{A}_\gamma$.)

Suppose first that A_β is of type $\mathscr{A}_\beta$. Then we claim that

$$s_\gamma \cdot A_\beta = A_\beta + T_{\gamma\alpha}(A_\gamma) + U_{\gamma\alpha\beta}(A_\beta). \tag{3.1}$$

Here $T_{\gamma\alpha}(A_\gamma)$ is of type $\mathscr{B}_\delta$, and the irreducible constituents of $U_{\gamma\alpha\beta}(A_\beta)$ are of types $\mathscr{F}_\delta$ and $\mathscr{H}$ (Definition 2.14). This is a consequence of Theorem 2.6, part $(2)_\gamma$ of Hypothesis $\mathscr{A}_\beta$ (using Lemma 2.10), and part $(2)_\gamma$ of Hypothesis $\mathscr{B}_\delta$.

It will be more convenient when more precision is not needed to write formulas like (3.1) in the form

$$s_\gamma \cdot A_\beta = A_\beta + B_\delta + \sum a_i (F_\delta)^i + \sum b_j H^j, \tag{3.1$'$}$$

letting the notation suggest the precise statement.

Next, we claim that

$$s_\alpha \cdot B_\beta = B_\beta + A_\gamma + A_\delta + \sum a_i (F_\beta)^i + \sum b_j H^j. \tag{3.2}$$

The proof is exactly like that of (3.1), and we omit it. Even easier is

$$s_\beta \cdot B_\beta = B_\beta + \sum m_i C^i + \sum a_j G^j + \sum b_k H^k; \tag{3.3}$$

this is immediate from Theorem 2.4 and Lemma 2.10. Theorem 2.6 and Hypotheses $\mathscr{C}$ and $\mathscr{S}$ give

$$s_\gamma \cdot C = C + D + \sum a_i H^i. \tag{3.4}$$

Last but not least, Theorem 2.6 gives

$$s_\beta \cdot D = D + C + \sum p_i (A_\beta)^i + \sum a_j (E_\beta)^j + \sum b_k (F_\gamma)^k + \sum c_l (F_\delta)^l + \sum d_m H^m. \tag{3.5}$$

Together these formulas imply the first claim in Step 1. The second is similar, and we omit it.

Before continuing with the proof of Step 2, we examine what Step 1 says about the structure of the set of type 8 $(\mathfrak{g}, K)$-modules. In what follows x, y, and z denote distinct elements of $\{\beta, \gamma, \delta\}$. First, those of type $\mathscr{C}$ are in one-to-one correspondence with those of type $\mathscr{D}$, by any of the three maps $T_{\alpha x}$. The three maps implement the same bijection, and it is inverted by any $T_{x\alpha}$. The situation with types $\mathscr{A}$ and $\mathscr{B}$ is more interesting. Each irreducible $(\mathfrak{g}, K)$-module of type $\mathscr{A}_x$ has exactly one neighbor of each type $\mathscr{B}_y$ and $\mathscr{B}_z$, obtained from an appropriate T. Similarly each of type $\mathscr{B}_x$ has neighbors of types $\mathscr{A}_y$ and $\mathscr{A}_z$. Continuing in the same way, we get a doubly infinite family of irreducible $(\mathfrak{g}, K)$-modules. Starting with A_β of type $\mathscr{A}_\beta$, for example, we get

$$\cdots B_\beta \text{ --- } A_\delta \text{ --- } B_\gamma \text{ --- } A_\beta \text{ --- } B_\delta \text{ --- } A_\gamma \text{ --- } (B_\beta)' \cdots \tag{3.6}$$

Because there are only finitely many irreducible $(\mathfrak{g}, K)$-modules of infinitesimal character λ, each such string must eventually repeat. Because of its construction, it is therefore periodic. Clearly the period is a multiple of six; we will see after Step 2 that it is six or twelve.

To summarize, the irreducible $(\mathfrak{g}, K)$-modules of types $\mathscr{C}$ and $\mathscr{D}$ appear in pairs. Those of types $\mathscr{A}$ and $\mathscr{B}$ appear in "loops" of $6k$ elements.

Proof of Step 2. Because the roots α and β span an A_2, the corresponding reflections satisfy the braid relation

$$s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta.$$

We will apply both sides of this identity to B_β and use the formulas (3.1)–(3.5). By Step 1, we may as well ignore terms of type sub-8. Then

$$\begin{aligned} s_\alpha B_\beta &= B_\beta + A_\gamma + A_\delta \\ s_\beta s_\alpha B_\beta &= \left(B_\beta + \sum m_i C^i\right) + (A_\gamma + B_\delta) + (A_\delta + B_\gamma) \\ s_\alpha s_\beta s_\alpha B_\beta &= (B_\beta + A_\gamma + A_\delta) + \sum m_i (C^i + D^i) - A_\gamma \\ &\quad + (B_\delta + A_\gamma + (A_\beta)') - A_\delta + (B_\gamma + A_\delta + A_\beta) \end{aligned} \tag{3.7}$$

modulo terms of type sub-8. The coefficients m_i and the representations A_β and $(A_\beta)'$ were defined in the original statement of Step 2. Similarly,

$$\begin{aligned} s_\beta B_\beta &= B_\beta + \sum m_i C^i \\ s_\alpha s_\beta B_\beta &= (B_\beta + A_\gamma + A_\delta) + \sum m_i (C^i + D^i) \\ s_\beta s_\alpha s_\beta B_\beta &= \left(B_\beta + \sum m_i C^i\right) + (A_\gamma + B_\delta) + (A_\delta + B_\gamma) - \left(\sum m_i C^i\right) \\ &\quad + \sum m_i (D^i + C^i + n_i A_\beta + Q_i). \end{aligned} \tag{3.8}$$

Here Q_i is a sum of terms of type $\mathscr{A}_\beta$, but not equal to A_β. Comparing coefficients of A_β gives the claim.

The coefficients m_i and n_i appear to depend on which representations in the cell $\mathscr{C}_0$ are used to define them. We need to know that this is not the case. Define $B_\gamma = T_{\beta\alpha} T_{\alpha\gamma}(B_\beta)$ and $A_\delta = T_{\alpha\beta} T_{\delta\alpha}(A_\beta)$; this is consistent with the notation in (3.6)–(3.8). Let p_i be the multiplicity of C^i in $U_\gamma(B_\gamma)$, and q_i the multiplicity of A_δ in $U_\delta(D^i)$. We claim that

$$m_i = p_i \quad \text{and} \quad n_i = q_i. \tag{3.9}$$

The first claim follows from applying the relation $s_\beta s_\gamma = s_\gamma s_\beta$ to A_δ. The second follows from the relation $s_\beta s_\delta = s_\delta s_\beta$ applied to D^i. We omit the (easy) calculation.

Let us see what Step 2 adds to our understanding of the structure of cells. Fix a loop $\mathscr{L}$ of modules of types $\mathscr{A}$ and $\mathscr{B}$ as in (3.6). For each pair (C^i, D^i) in $\mathscr{C}_0$, let

(3.10)(a) $\qquad m_i = \text{multiplicity of } C^i \text{ in } U_x(B_x)$

(3.10)(b) $\qquad n_i = \text{multiplicity of } A_x \text{ in } U_x(D^i)$.

Here A_x and B_x are elements of $\mathscr{L}$ of types $\mathscr{A}_x$ and $\mathscr{B}_x$, and x is β, γ, or δ. These integers are well-defined by (3.9). Comparing (3.7) and (3.8) shows that

$$\Big(\sum_i m_i n_i\Big)\Big(\sum_{A_x \in \mathscr{L}} A_x\Big) = (A_x)_1 + (A_x)_2;$$

here $(A_x)_1$ and $(A_x)_2$ are neighboring elements of type $\mathscr{A}_x$ in the loop $\mathscr{L}$. This means that the loop falls into one of two cases: either

(3.10)(c) $\mathscr{L}$ has 12 elements, and $\sum m_i n_i = 1$;

or

(3.10)(c)′ $\mathscr{L}$ has 6 elements, and $\sum m_i n_i = 2$.

Proof of Step 3. By Step 1, it is enough to prove (1) and (2). Part (1) follows from Step 2. Now consider (2). In light of Hypothesis $\mathscr{A}_\beta$ and Step 1, it is equivalent to

(*) $(1+s_\gamma)(1+s_\beta)D$ has a constituent Z with α and δ not in $\tau(Z)$. As is shown in [8], such an assertion (as well as Hypothesis $\mathscr{D}$) amounts to a property of the primitive ideal $\mathrm{Ann}(D)$ in $U(\mathfrak{g})$. By Duflo's theorem [2], any primitive ideal is the annihhilator of an irreducible highest weight module. So it suffices to prove (*) in case D is an irreducible highest weight module.

We are going to take advantage of a symmetry property of the Kazhdan–Lusztig polynomials. It may be formulated as follows [12, (1.3) and Corollary 14.9]: there is an involution $X \to X'$ on the set of irreducible highest weight modules of infinitesimal character $\chi(\lambda)$, with the following two properties:

(3.11)(a) $\tau(X') = \Pi(\lambda) - \tau(X)$;

and

(3.11)(b) if $\phi \notin \tau(X)$ but $\phi \in \tau(Y)$, then the multiplicity of Y in $s_\phi \cdot X$ is equal to the multiplicity of X' in $s_\phi \cdot Y'$.

One checks easily that the involution interchanges types $\mathscr{A}_x$ and $\mathscr{B}_x$, and types $\mathscr{C}$ and $\mathscr{D}$. Consequently (2) of Step 3 for highest weight modules is equivalent to (1), which we have already proved. This completes Step 3.

Before continuing with Step 4, let us see what we have already established. Fix a loop $\mathscr{L}$, and use the notation of (3.10). Write $(R^i)_x$ for the sum of the elements of $U_x(D^i)$ of type A_x but not occurring in $\mathscr{L}$. In light of (3.10), the equality of the final expressions in (3.7) and (3.8) says that

$$\sum m_i (R^i)_x = 0.$$

This says that if m_i is non-zero, then the only type-8 constituents of $U_x(D^i)$ belong to $\mathscr{L}$.

Proof of Step 4. We will try to compute various Ext groups in the category of $(\mathfrak{g}, K)$-modules using Theorem 3.9 of [8]. For the convenience of the reader we reproduce a part of the statement. Here for the first time we need to use the fact that $U_\phi(X)$ (which was introduced in Theorem 2.4 as a virtual character) has a natural realization as a $(\mathfrak{g}, K)$-module. We refer to [8] for the construction.

THEOREM 3.12. *In the setting of Theorem* 2.4, *suppose* Y *is an irreducible* $(\mathfrak{g}, K)$-*module of infinitesimal character* $\chi(\lambda)$, *and* $\phi \in \tau(Y)$. *Then*

$$\mathrm{Hom}(Y, U_\phi(X)) \cong \mathrm{Ext}^1(Y, X)$$
$$\mathrm{Hom}(U_\phi(X), Y) \cong \mathrm{Ext}^1(X, Y).$$

More generally, suppose that $\mathrm{Ext}^i(Y, X) = 0$. *Then*

$$\mathrm{Ext}^i(Y, U_\phi(X)) \cong \mathrm{Ext}^{i+1}(Y, X) \oplus \mathrm{Ext}^{i-1}(Y, X).$$

Similarly, if $\mathrm{Ext}^i(X, Y) = 0$, *then*

$$\mathrm{Ext}^i(U_\phi(X), Y) \cong \mathrm{Ext}^{i+1}(X, Y) \oplus \mathrm{Ext}^{i-1}(X, Y).$$

This result is quite easy to prove. What makes it powerful is the fact

(3.13) each $(\mathfrak{g}, K)$-module $U_\phi(X)$ is completely reducible.

Unfortunately no complete proof of this fact has been published except when λ is integral (in which case see [12] and the references therein). In general it seems to be due to Beilinson and Bernstein; it is a consequence of the "decomposition theorem" for direct images of perverse sheaves. (The reader who really wishes to pursue this point should note that the methods of [7] reduce one to the "rational case," when some integer multiple of λ lies in the root lattice.)

We use now the notation of the proof of Step 2. In light of (3.13), what we are trying to show is

(3.14) $\dim \mathrm{Hom}(A_\beta, U_\beta(D^i)) = \dim \mathrm{Hom}(U_\beta(B_\beta), C^i)$.

By (3.9), this is equivalent to

(3.14)′ $$\dim \operatorname{Hom}(A_\beta, U_\beta(D')) = \dim \operatorname{Hom}(U_\gamma(B_\gamma), C').$$

We propose to show that both sides of (3.14)′ are equal to

(3.15) $$\dim \operatorname{Ext}^2(A_\beta, C^i) - \dim \operatorname{Hom}(U_\delta(A_\beta), U_\alpha(C^i)).$$

We begin with the Ext^2 group. By Theorem 3.12,

$$\operatorname{Ext}^1(A_\beta, C^i) \cong \operatorname{Hom}(U_\delta(A_\beta), C^i).$$

The right side is zero by the analogue of (3.1)′. Now Theorem 3.12 and Theorem 2.6 give

(3.16) $$\begin{aligned}\operatorname{Ext}^2(A_\beta, C^i) &\cong \operatorname{Ext}^1(U_\delta(A_\beta), C^i)\\ &= \operatorname{Ext}^1(B_\gamma, C^i) \oplus \operatorname{Ext}^1(U_{\delta\alpha}(A_\beta), C^i);\end{aligned}$$

the second equality uses (3.13) again. Each summand here can be computed by Theorem 3.12. The first is $\operatorname{Hom}(U_\gamma(B_\gamma), C^i)$. The second is $\operatorname{Hom}(U_{\delta\alpha}(A_\beta), U_\alpha(C^i))$. Since B_γ cannot occur in $U_\alpha(C^i)$ (by (3.4)), this is equal to $\operatorname{Hom}(U_\delta(A_\beta), U_\alpha(C^i))$. We have therefore shown that

(3.17) $$\operatorname{Ext}^2(A_\beta, C^i) \cong \operatorname{Hom}(U_\gamma(B_\gamma), C^i) \oplus \operatorname{Hom}(U_\delta(A_\beta), U_\alpha(C^i)).$$

This says that (3.15) is equal to the right side of (3.14)′. To get the left side, we compute the Ext^2 beginning instead with $U_\alpha(C^i)$. We omit the details.

We turn now to the proof of Theorem 2.15. Fix a pair (C, D). By (2) of Step 3, we can find a loop $\mathscr{L}$ so that the corresponding multiplicity n (cf. (3.10)(b)) is not zero. By Step 4, m is not zero either. The remark after the proof of Step 3 now shows that the only type-8 constituents of $U_x(D)$ belong to $\mathscr{L}$. This proves

Lemma 3.18. *Each pair (C, D) is attached to a unique loop $\mathscr{L}$. This loop is characterized by either of the following conditions:*

(i) *C occurs in $U_x(B_x)$ ($B_x \in \mathscr{L}$ of type $\mathscr{B}_x$); or*

(ii) *A_x occurs in $U_x(D^i)$ ($A_x \in \mathscr{L}$ of type $\mathscr{A}_x$).*

We see now that a cell $\mathscr{C}_0$ consists of exactly one loop $\mathscr{L}$ together with all the pairs (C^i, D^i) attached to it. The multiplicities m_i and n_i of (3.10) are all non-zero. By Step 4, (3.10)(c) and (3.10)(c)′ may be rewritten as

(3.19) $\mathscr{L}$ has 12 elements, and $\sum (m_i)^2 = 1$;

or

(3.19)′ $\mathscr{L}$ has 6 elements, and $\sum (m_i)^2 = 2$.

When the loop has 12 elements, it follows that there is just one C^i, and that $m_i = 1$. The conclusions of Theorem 2.15 follow easily, with the graph given by Fig. 2. When the loop has 6 elements, there must be exactly two C^i, and $m_i = 1$. Again we get the conclusions of Theorem 2.15, this time with the graph in Fig. 1.

This completes the proof of Theorem 2.15.

4. Explicit Calculations

In this section we consider the problem of determining explicitly the sets $\mathscr{C}_0$ described by Theorem 2.15. More precisely, there is a concrete parametrization (the Langlands classification) of the set of irreducible $(\mathfrak{g}, K)$-modules. Suppose we are in Setting 2.8, and that X is an irreducible $(\mathfrak{g}, K)$-module, specified in terms of its "Langlands parameters." How can we compute the Langlands parameters of the other $(\mathfrak{g}, K)$-modules in $\mathscr{C}_0$? When G is not linear, there are serious problems with this calculation (as with everything else related to the Kazhdan–Lustztig conjectures); we will not attempt to resolve them.

Assume therefore that G is linear. Write $\mathscr{P}(\lambda)$ for the set of Langlands parameters for irreducible $(\mathfrak{g}, K)$-modules of infinitesimal character $\chi(\lambda)$. (One good way to think of $\mathscr{P}(\lambda)$ is as a set of conjugacy classes of regular characters of Cartan subgroups (see Sect. 3 of [13], or [8, 7], or [11]).) If ρ is in $\mathscr{P}(\lambda)$, write $M(\rho)$ for the corresponding standard $(\mathfrak{g}, K)$-module and $L(\rho)$ for its irreducible Langlands subquotient. (These are "generalized principal series representations"; see the references mentioned above.) There is on $\mathscr{P}(\lambda)$ an integer-valued function l^I [13, Sect. 12]. For each simple root ϕ in $\Pi(\lambda)$ there is a relation $\xrightarrow{\phi}$ on $\mathscr{P}(\lambda)$ [13, Definition 12.7] whose properties are summarized in the next proposition.

Proposition 4.1 (See [13], or Sects. 8.5 and 8.6 of [11]). *Suppose G is linear. Use the notation of the preceding paragraph, and let ξ and ρ be elements of $\mathscr{P}(\lambda)$. Then the relation*

(*) $$\xi \xrightarrow{\phi} \rho$$

is equivalent to

(i) $l^I(\rho) = l^I(\xi) + 1$;

(ii) $\phi \in \tau(L(\rho))$, $\phi \notin \tau(L(\xi))$; *and*

(iii) *$L(\rho)$ is a constituent of $U_\phi(L(\xi))$.*

The third condition may be replaced by

(iii)′ $L(\xi)$ *is a composition factor of* $M(\rho)$.

If ρ *is fixed, then there are exactly* 0, 1, *or* 2 *elements* ξ *of* $\mathscr{P}(\lambda)$ *such that* (*) *holds. They may be computed explicitly. If* ξ *is fixed, there are exactly* 0, 1, *or* 2 *elements* ρ *such that* (*) *holds. They may be computed explicitly.*

Perhaps the most important fact for us is a partial converse to the first part of this proposition.

PROPOSITION 4.2 ([8]; see [11, Theorem 8.5.18]). *In the setting of Proposition* 4.1, *suppose* $L(\rho)$ *is a constituent of* $U_\phi(L(\xi))$. *Then either* $\xi \xrightarrow{\phi} \rho$, *or* $l^I(\rho) \leqslant l^I(\xi) - 1$. *In the latter case* $L(\rho)$ *is a composition factor of* $M(\xi)$.

This result is used in [8] to calculate $T_{\phi\psi}$ (cf. (2.7)). The main point is that Theorem 2.6 and Proposition 4.2 together imply

COROLLARY 4.3 [8]. *In the settings of Theorem* 2.6 *and Proposition* 4.2, *suppose that* $\tau(L(\xi))$ *contains* ψ *but not* ϕ, *and* $\tau(L(\rho))$ *contains* ϕ *but not* ψ. *Then*

(*) $L(\rho) = T_{\phi\psi}(L(\xi))$.

is equivalent to

(**) $\xi \xrightarrow{\phi} \rho$ *or* $\rho \xrightarrow{\psi} \xi$.

We turn now to the calculation of the sets $\mathscr{C}_0$ of Theorem 2.15. What we want is something analogous to Corollary 4.3. Because the result is probably of interest only to experts, we will assume some familiarity with the combinatorial aspects of the Kazhdan–Lusztig conjecture for $(\mathfrak{g}, K)$-modules. It will be convenient to write the length function l^I and the relation $\xrightarrow{\phi}$ on the level of $(\mathfrak{g}, K)$-modules as well as parameters. If we know (the Langlands parameters of) a single element of type $\mathscr{A}$ or $\mathscr{B}$ in $\mathscr{C}_0$, Corollary 4.3 allows us to calculate all the others of types $\mathscr{A}$ and $\mathscr{B}$. Similarly, an element of type $\mathscr{C}$ or $\mathscr{D}$ determines the corresponding element of type $\mathscr{D}$ or $\mathscr{C}$. Suppose for definiteness that the elements of types $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{C}_0$ are known; we consider the problem of finding those of type $\mathscr{C}$. Fix one such element C, and write D for the corresponding type $\mathscr{D}$ element.

PROPOSITION 4.4. *In Setting* 2.8, *suppose that* G *is linear. Fix irreducible* $(\mathfrak{g}, K)$-*modules* A_β *and* C *of types* $\mathscr{A}_\beta$ *and* $\mathscr{C}$, *respectively, and write* $B_\gamma = T_{\delta\alpha}(A_\beta)$, $D = T_{\alpha\delta}(C)$. *If* A_β *and* C *belong to the same* $\mathscr{C}_0$ (*Theorem* 2.15) *then at least one of the following possibilities occurs*:

(i) $B_\gamma \xrightarrow{\gamma} C$;

(ii) $D \xrightarrow{\beta} A_\beta$; *or*

(iii) *there is an irreducible* $(\mathfrak{g}, K)$-*module* X *such that* $X \xrightarrow{\alpha} A_\beta$ *and* $X \xrightarrow{\delta} C$.

Conversely, suppose at least one of (i)–(iii) *holds. Then* A_β *and* C *belong to the same* $\mathscr{C}_0$.

Proof. Suppose first that A_β and C belong to the same $\mathscr{C}_0$. We know from Theorem 2.15 that C occurs in $U_\gamma(B_\gamma)$. If $B_\gamma \xrightarrow{\gamma} C$, we are done. By Proposition 4.2, we may therefore assume

(4.5)(a) $\quad l^I(C) \leqslant l^I(B_\gamma) - 1$.

Similarly, we may as well assume

(4.5)(b) $\quad l^I(D) \geqslant l^I(A_\beta) + 1$.

By Corollary 4.3, the elements C and D (respectively A_β and B_γ) must differ in length by exactly one. In light of (4.5) and Corollary 4.3, this implies

(4.6)(a) $\quad A_\beta \xrightarrow{\delta} B_\gamma$

(4.6)(b) $\quad l^I(C) = l^I(B_\gamma) - 1$

(4.6)(c) $\quad C$ is a composition factor of $M(B_\gamma)$.

In (c), $M(B_\gamma)$ denotes the standard representation with Langlands subquotient B_β.

Now (4.6)(a) allows us to study the composition series of $M(B_\gamma)$ in terms of that of $M(A_\beta)$ (cf. [11, proof of Proposition 8.6.19]). In particular, (4.6) implies that there is an irreducible $(\mathfrak{g}, K)$-module X such that

(4.7)(a) $\quad X \xrightarrow{\delta} C$

(4.7)(b) $\quad l^I(X) = l^I(A_\beta) - 1$

(4.7)(c) $\quad X$ is a composition factor of $M(A_\beta)$.

Consider the τ-invariant of X. By (4.6)(a), it does not contain δ. If it contains α, then (4.7)(a) would say that $X = T_{\delta\alpha}(C)$, and so $X = D$; but this contradicts (4.7)(b) and (4.5)(b). So $\tau(X)$ does not contain α. Since α is in $\tau(A_\beta)$, (4.7)(c) now implies that

(4.8) $\quad X \xrightarrow{\alpha} A_\beta$.

This proves the first half of Proposition 4.4.

For the converse, notice first that we can reverse the arguments above to conclude that C satisfies (4.6). To finish, we only need to know that C occurs in $U_\gamma(B_\gamma)$. This is a consequence of

LEMMA 4.9. *Suppose G is linear, and ξ and ρ are elements of $\mathscr{P}(\lambda)$. Assume that*

(i) $l^I(\xi) = l^I(\rho) + 1$

(ii) $\phi \notin \tau(L(\rho))$

(iii) $\phi \in \tau(L(\xi))$.

Then $L(\xi)$ occurs in $M(\rho)$ and $U_\phi(L(\rho))$ with the same multiplicity.

The lemma is an easy consequence of the Lazhdan–Lusztig conjectures: the common multiplicity is the Kazhdan–Lustztig polynomial $P_{\xi\rho}$ (which must be a constant by (i)). Now use for example Lemma 14.7 of [13].

5. An Example

Let $G = SO_e(4, 4)$, the identity component of $SO(4, 4)$. In this section we describe the representations of type 8 of G having the same infinitesimal character as the trivial representation. (At the conclusion we discuss the representations of type 8 in this infinitesimal character for two other groups with $SO_e(4, 4)$ as identity component.) It follows from primitive ideal theory that the wavefront set of any such representation must contain as its largest part one or more nilpotent coadjoint orbits with Jordan blocks of sizes 3, 3, 1, and 1. It turns our that there are exactly five such orbits. (This number is very sensitive to the disconnectedness of the group; for the group of real points of the adjoint group, which has four components, there are just two orbits.) One might (in light of Corollary 2.16) hope to have at least one set $\mathscr{C}_0$ as in Theorem 2.15 for each of these five orbits. We will exhibit five sets in the block of the trivial representation [11, Definition 9.2.1], and one more in another block. (There are at least two others on the linear double cover of G.) The techniques for making these calculations have been described in Section 4, and we omit any additional details.

To begin, we need a little notation. It is convenient at first to consider simultaneously the identity components of various groups $SO_e(2p, 2q)$, with $p + q = 4$. Each such group has a standard compact torus $T = SO(2)^4$. The group of characters of T is naturally identified with $\mathbb{Z}^4$, and the purely imaginary linear functionals on T with $\mathbb{R}^4 \supset \mathbb{Z}^4$. The 24 roots of T are

$$(\pm 1, \pm 1, 0, 0), (\pm 1, 0, \pm 1, 0), (\pm 1, 0, 0, \pm 1),$$

$$(0, \pm 1, \pm 1, 0), (0, \pm 1, 0, \pm 1), (0, 0, \pm 1, \pm 1).$$

We will specify simultaneously a character of T and a real form G by a *signed compact weight*, an element of $\mathbb{Z}^4$ with a plus or minus after each coordinate. (This extra sign is entirely independent of the positivity or negativity of the integer to which it is attached.) Each sign indicates the signature of the quadratic form ((+, +) or (−, −)) on the two corresponding coordinates in $\mathbb{R}^8$. Thus for example $(-3^+, 2^-, 1^+, 0^-)$ indicates a character of the compact torus of $SO_e(2, 6)$. Notice that the group is determined up to isomorphism by the number of plus or minus signs alone; it will be convenient to identify both $(3^+, 2^+, 1^-, 0^-)$ and $(3^+, 1^-, 2^+, 0^-)$ with the same character of the compact torus in $SO_e(4, 4)$, and similarly for other interchanges of adjacent integers labelled with opposite signs. The root $\pm e_i \pm e_j$ is non-compact for the real form specified by a signed compact weight if and only if the signs after the ith and jth coordinates are opposite. A discrete series representation may be specified as a signed regular compact weight ρ, its Harish-Chandra parameter; write $L(\rho)$ for the corresponding Harish-Chandra module. This has the infinitesimal character of the trivial representation if and only if the coordinates of ρ are (up to permutation and sign change) (3, 2, 1, 0). (In this case, our four simple roots α, β, γ, and δ become $e_2 - e_3$, $e_1 - e_2$, $e_3 + e_4$, and $e_3 - e_4$, respectively.) For $SO_e(4, 4)$ we may as well assume that the signs are (+, +, −, −). There are 192 such ρ. Two give the same discrete series if and only if they are conjugate by the compact Weyl group, which has order 16; so there are 12 distinct discrete series representations for $SO_e(4, 4)$ with the same infinitesimal character as the trivial representation.

Fix a signed compact weight ρ, and two adjacent coordinates to which opposite signs are attached; say for definiteness the second and the third. Because the root $\eta = (0, 1, -1, 0)$ is non-compact, we can apply to any discrete series parameter $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)$ a Cayley transform through η [11, Definition 8.3.6]. The result is a Langlands parameter that we write as

$$c^\eta(\rho) = (\rho_1, \underline{\rho_2, \rho_3}, \rho_4).$$

It is attached to a Cartan subgroup T^η with a one-dimensional split part. It turns out that $c^\eta(\rho)$ is independent of which of the coordinates ρ_2 and ρ_3 was labelled plus and which minus, so we can omit these labels from the underlined coordinates. For example,

$$c^\eta(3^+, 2^+, 1^-, 0^-) = (3^+, \underline{2, 1}, 0^-) = c^\eta(3^+, 2^-, 1^+, 0^-).$$

Our first four sets $\mathscr{C}_0$ contain a discrete series representation C for which the simple roots β, γ, and δ are all compact. There are exactly four such representations (of the infinitesimal character we want); consider for definiteness

$$C = L(3^+, 2^+, 1^-, 0^-).$$

(The other possibilities are $L(3^+, -2^+, 1^-, 0^-)$ and $L(1^+, 0^+, 3^-, \pm 2^-)$.) The set $\mathscr{C}_0$ is of type $8-2$; the representations in it are displayed in Fig. 5.1.

(As in Sect. 4, we write $L(\rho)$, for ρ any of the Langlands parameters introduced, to indicate the corresponding Langlands quotient.) The integral length function l' is zero on C, 1 on D, 2 on the A's, 3 on the B's, 4 on C', and 5 on D'. The connecting lines are of the type described in Section 4. Those which connect a representation of type A to a representation of type B, or a representation of type C to a representation of type D, are $T_{\phi\psi}$'s.

In addition to the discrete series representation C, several of these representations are interesting unitary representations. To see this, we need to recall a little about cohomological induction. Suppose $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{q}$ containing $\mathfrak{t}$ corresponds to a connected reductive subgroup L of G. Attached to $\mathfrak{q}$ there is a unitary irreducible Harish-Chandra module $X_\mathfrak{q}$, having the same infinitesimal character as the trivial representation of G; it is obtained by

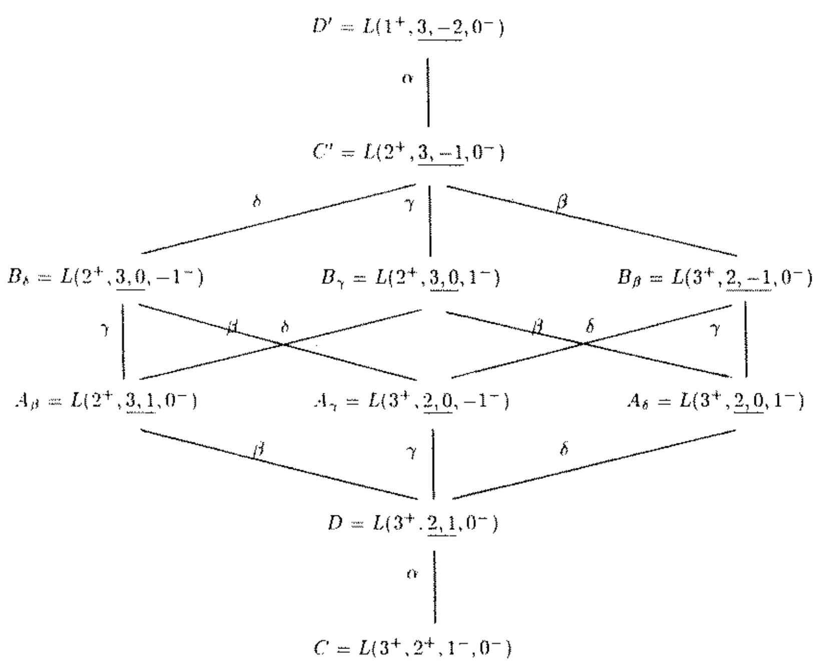

FIGURE 5.1

cohomological induction of the trivial representation of L. The integral length of $X_\mathfrak{q}$ is half the dimension of the symmetric space of L. Now our discrete series representation C defines a Borel subalgebra $\mathfrak{b}$ (characterized by the fact that $C = X_\mathfrak{b}$). For each simple root $\eta \in \{\beta, \gamma, \delta\}$ there is a parabolic subalgebra $\mathfrak{q}(\eta)$ with Levi factor corresponding to the two adjacent simple roots α and η. The corresponding group $L(\eta)$ is isomorphic to $U(2, 1) \times SO(2)$. It turns out that

$$A_\eta = X_{\mathfrak{q}(\eta)}.$$

Finally, D is equal to $X_{\mathfrak{q}(\alpha)}$, with $\mathfrak{q}(\alpha)$ the parabolic corresponding to the non-compact) simple root α; the corresponding Levi subgroup is $U(1, 1) \times SO(2)^2$. (These facts are of interest in part because the associated varieties of the representations $X_\mathfrak{q}$ admit a fairly simple description.)

To describe the representations in our remaining examples, we need to extend our previous notation somewhat. First, we can perform a Cayley transform through two pairs of coordinates (with opposite signs attached) as well as one, obtaining Langlands parameters such as $(\underline{2}, \underline{0}, \underline{3}, \underline{1})$ (attached to a Cartan subgroup with two-dimensional split part, or to the cuspidal parabolic subgroup with Levi factor $GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$). A more subtle possibility is to perform first a Cayley transform through $e_2 - e_3$, then a second through $e_2 + e_3$. What makes this interesting is that the resulting Cartan subgroup may not be connected; so to specify a character of it, we need a way to specify the action of some elements of order two. It is most convenient to use a split Cartan subgroup as a point of departure. To each pair of coordinates in $\mathbb{R}^8$ on which the quadratic form has different signs, there is associated a subgroup $SO(1, 1) \simeq \mathbb{R}^\times$ of $SO(4, 4)$. We can write a character of $\mathbb{R}^\times$ as a signed complex number z^ε; this maps a non-zero real number t to $|t|^z$ (if $\varepsilon = +$) or $\operatorname{sgn}(t)|t|^z$ (if $\varepsilon = -$). The product of four of these is a split Cartan subgroup of $SO(4, 4)$, so a character of it may be parametrized by a set of four signed complex numbers, a *signed split weight*. We write these with double underlining to distinguish them from the signed compact weights, as for example

$$(\underline{\underline{3^+}}, \underline{\underline{1^-}}, \underline{\underline{2^-}}, \underline{\underline{0^+}}).$$

The corresponding split Cartan subgroup H for $SO_e(4, 4)$ has index two in that for $SO(4, 4)$. In terms of characters, this means that we may change every sign in a signed split weight without affecting its restriction to H.

Given two coordinates for a signed split weight, we may try to perform a Cayley transform to pass to a character on a more compact Cartan subgroup. Say for definiteness the coordinates are the second and third; we will use a Cayley transform through the root $\eta = e_2 + e_3$. The Cayley transform of $(\underline{\underline{\rho_1}}, \underline{\underline{\rho_2}}, \underline{\underline{\rho_3}}, \underline{\underline{\rho_4}})$ exists if and only if $\rho_2 + \rho_3$ is an integer m, and $(-1)^{m+1}$ is the product of the signs attached to ρ_2 and ρ_3 [11, Definition 8.3.16]. (Notice that the condition is unchanged if all the signs are changed.) When the Cayley transform exists, we write it with a single underline beneath the affected coordinates:

$$c_\eta(\rho) = (\underline{\underline{\rho_1}}, \underline{\rho_2}, \underline{\rho_3}, \underline{\underline{\rho_4}}).$$

It turns out that the Cauley transform is independent of the signs attached to ρ_2 and ρ_3 (subject to the condition for it to exist), so we may drop them. We write for example

$$c_\eta(\underline{\underline{3^+}}, \underline{\underline{1^-}}, \underline{\underline{2^-}}, \underline{\underline{0^+}}) = (\underline{\underline{3^+}}, \underline{1, 2}, \underline{\underline{0^+}}).$$

This is a Langlands parameter attached to a Cartan subgroup with one-dimensional compact part. We can even perform a second Cayley transform through the other two coordinates, obtaining in this example a Langlands parameter

$$(\underline{3, 0}, \underline{1, 2})$$

on a Cartan subgroup with two-dimensional compact part. This notation is consistent with that based on a compact Cartan subgroup.

There are two other conjugacy classes of Cartan subgroups of G having two-dimensional compact part. The characters of one of these (coming from a split Cartan in one $SO(2, 2)$ and a compact Cartan in another) may be written as (for example)

$$(\underline{\underline{3^+}}, \underline{\underline{2^-}}, 1^+, 0^-);$$

we leave the precise definitions to the reader. (As usual it is permitted to change both signs attached to the doubly underlined coordinates.) The remaining conjugacy class arises from H by (for example) successive Cayley transforms in the real roots $e_1 + e_2$ and $e_3 - e_4$. Its Langlands parameters might be written by underlining these pairs of coordinates, but the notation must distinguish them from the Cayley transforms through $e_1 + e_2$ and $e_3 + e_4$. Since we do not need to use them, we will not specify such a notation.

Our fifth set $\mathscr{C}_0$ is again of type $8-2$. It contains the Langlands quotient, D, of a principal series representation (of the same infinitesimal character as the trivial representation) for which the Cayley transforms through the simple roots β, γ, and δ all fail to exist, but that through α does. (It follows that, up to problems about disconnectedness, this example arises from the first four by application of the duality of [13].) According to the criterion above, this means that the coordinate pairs (3, 2) and (1, 0) must be attached to opposite signs, and (2, 1) to the same sign. Up to the Weyl group action there are just two such signed split weights, and these differ by changing all signs. Accordingly there is just one representation

$$D = L(\underline{\underline{3^+}}, \underline{\underline{2^-}}, \underline{\underline{1^-}}, \underline{\underline{0^+}}).$$

(If we had used instead the group of real points of the adjoint group, we would have found four such representations, all with the same restriction to the identity component.) The other representations in $\mathscr{C}_0$ are

$$C = L(\underline{\underline{3^+}}, \underline{2, -1}, \underline{\underline{0^+}}),$$

$$B_\beta = L(\underline{\underline{2_-}}, \underline{3, -1}, \underline{\underline{0^+}}), \quad B_\gamma = L(\underline{\underline{3^+}}, \underline{2, 0}, \underline{\underline{1^-}}), \quad B_\delta = L(\underline{\underline{3^+}}, \underline{2, 0}, \underline{\underline{-1_-}})$$

$$A_\beta = L(\underline{\underline{3^+}}, \underline{2, 1}, \underline{\underline{0^+}}), \quad A_\gamma = L(\underline{\underline{2^+}}, \underline{3, 0}, \underline{\underline{-1^+}}), \quad A_\delta = L(\underline{\underline{2^+}}, \underline{3, 0}, \underline{\underline{1^+}})$$

$$D' = L(\underline{\underline{2^+}}, \underline{3, 1}, \underline{\underline{0_-}}), \quad C' = L(\underline{\underline{1^+}}, \underline{3, 0}, \underline{\underline{0^+}}).$$

The integral length function takes the value 8 on D, 7 on C, 6 on the B's, 5 on the A's, 4 on D', and 3 on C'. The only unitary representation among them is C': it is in fact equal to $X_{\mathfrak{q}}$, with $\mathfrak{q}$ the maximal parabolic subalgebra having Levi subgroup $L = U(1, 1) \times SO_e(2, 2)$. (Up to conjugacy by K, there is exactly one such parabolic subalgebra; it is the one determined by the highest non-compact root.)

Our final set (again of type $8-2$) is perhaps best characterized by its elements of type D. These are

$$D = L(\underline{\underline{3^+}}, \underline{\underline{2^-}}, 1^+, 0^-), \qquad D' = L(\underline{\underline{3^+}}, \underline{\underline{2^-}}, 1^-, 0^+).$$

(The analogous Langlands parameters for the other two conjugacy classes of Cartan subgroup with two-dimensional split part live only on Spin(4, 4).) From this we calculate immediately

$$C = L(\underline{\underline{3^+}}, \underline{\underline{1^+}}, 2^+, 0^-), \qquad C' = L(\underline{\underline{3^+}}, \underline{\underline{1^+}}, 2^-, 0^+).$$

The other six elements of $\mathscr{C}_0$ are

$$A_\beta = L(\underline{\underline{3^+}}, \underline{\underline{0^-}}, \underline{2, 1}), \quad A_\gamma = L(\underline{\underline{3^+}}, \underline{\underline{-2^-}}, \underline{1, 0}), \quad A_\delta = L(\underline{\underline{3^+}}, \underline{\underline{2^-}}, \underline{1, 0})$$

$$B_\beta = L(\underline{\underline{3^+}}, \underline{\underline{2^-}}, \underline{\underline{1^+}}, \underline{\underline{0^+}}), \quad B_\gamma = L(\underline{\underline{3^+}}, \underline{\underline{1^+}}, \underline{2, 0}), \quad B_\delta = L(\underline{\underline{3^+}}, \underline{\underline{-1^+}}, \underline{2, 0}).$$

The elements C, C', and A_β have integral length 5; D, D', B_γ, and B_δ have integral length 6; A_γ and A_δ have integral length 7; and B_β (a Langlands quotient of a principal series) has integral length 8. They are pictured in Fig. 5.2.

None of these representations seems to be particularly interesting in its own right (for example, none is unitary). One reason for studying them can be found in Arthur's conjectures about unipotent representations. In the

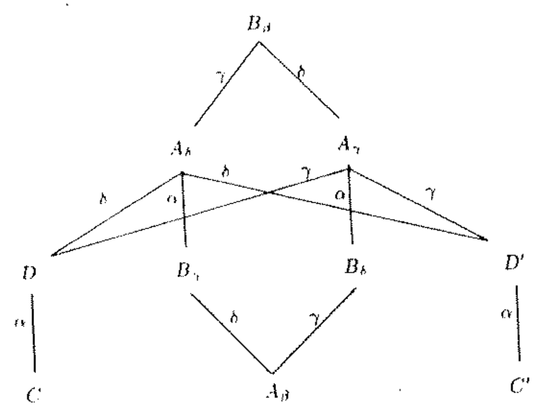

FIGURE 5.2

case of real forms of $SO(8)$ these conjectures predict (among other things) that if X is any irreducible Harish-Chandra module of type D, then the translation Z of X to the infinitesimal character $(1, 1, 0, 0)$ (which is singular with respect to β, γ, and δ) should be unitary. Here we get two such representations, which we write as

$$Z = L(\underline{\underline{1}}^+, \underline{\underline{1}}^+, 0^+, 0^-), \qquad Z' = L(\underline{\underline{1}}^+, \underline{\underline{1}}^+, 0^-, 0^+).$$

These are the two Langlands quotients of the principal series representation $M(\underline{\underline{1}}^+, \underline{\underline{1}}^+, \underline{\underline{0}}^+, \underline{\underline{0}}^-)$. It is easy to check that this principal series contains the unitary degenerate series induced from the character $1 \otimes (\mathrm{sgn}(\det))$ of the parabolic subgroup with Levi factor $GL(3, \mathbb{R}) \times GL(1, \mathbb{R})$, and to conclude that Z and Z' are in fact unitary. (It turns out that Z and Z' are the only constituents of the degenerate series.)

We have displayed six sets $\mathscr{C}_0$ for $SO_e(4, 4)$ (call them $\mathscr{C}_0^1, ..., \mathscr{C}_0^6$, in the order presented) with infinitesimal character that of the trivial representation, and in fact there are no others. The first five are in the block of the trivial representation, the last is not. For $SO(4, 4)$ there are again six sets $\mathscr{C}_0$ with infinitesimal character that of the trivial representation; call them $\mathscr{C}_0^{1'}, ..., \mathscr{C}_0^{6'}$. The first four are in the block of the trivial representation, the last two are not. Again, they are all of type $8-2$. The $\mathscr{C}_0^{1'}$ corresponds to the sets $\mathscr{C}_0^1$ and $\mathscr{C}_0^2$ in that each representation in $\mathscr{C}_0^{1'}$ decomposes on restriction to $SO_e(4, 4)$ into a sum of two irreducible representations, one from $\mathscr{C}_0^1$ and one from $\mathscr{C}_0^2$ (and type A_β restricts to the sum of the two type A_β's, etc.). In the same way the set $\mathscr{C}_0^{2'}$ corresponds to the sets $\mathscr{C}_0^3$ and $\mathscr{C}_0^4$. The next two sets, $\mathscr{C}_0^{3'}$ and $\mathscr{C}_0^{4'}$, correspond to $\mathscr{C}_0^5$ in that each representation of $\mathscr{C}_0^{3'}$ (respectively $\mathscr{C}_0^{4'}$), on restriction to $SO_e(4, 4)$, is an irreducible representation in $\mathscr{C}_0^5$, and again, the types correspond. In the same way the sets $\mathscr{C}_0^{5'}$ and $\mathscr{C}_0^{6'}$ (which lie in two different blocks) correspond to the set $\mathscr{C}_0^6$.

We now consider the group G'' which is the preimage in $SO(8, \mathbb{C})$ of the group of real points of the adjoint group (this is a disconnected group with four components). For this group we have five sets $\mathscr{C}_0$ (with the same infinitesimal character as the trivial representation), call them $\mathscr{C}_0^{1''}, ..., \mathscr{C}_0^{5''}$, in the block of the trivial representation, and two sets $\mathscr{C}_0^{6''}$ and $\mathscr{C}_0^{7''}$ which are not. The set $\mathscr{C}_0^{1''}$ corresponds on restriction to $SO(4, 4)$ to the sets $\mathscr{C}_0^{1'}$ and $\mathscr{C}_0^{2'}$, the sets $\mathscr{C}_0^{2''}$ and $\mathscr{C}_0^{3''}$ correspond to $\mathscr{C}_0^{3'}$, and the sets $\mathscr{C}_0^{4''}$ and $\mathscr{C}_0^{5''}$ correspond to $\mathscr{C}_0^{4'}$, as in previous cases. These five sets $\mathscr{C}_0$ are also of type $8-2$. We see that for G'' we have in the block of the trivial representation one set $\mathscr{C}_0$ with a discrete series representation in it, and four sets $\mathscr{C}_0$ containing a Langlands quotient of a principal series. We expect this, since the block of the trivial representation for G'' is dual (in the sense of [13]) to the block of the trivial representation for $SO_e(4, 4)$. (The block of the trivial representation for $SO(4, 4)$ is self dual.) The sets $\mathscr{C}_0^{6''}$ and $\mathscr{C}_0^{7''}$ are sets of type $8-6$. The set $\mathscr{C}_0^{6''}$ corresponds to the set $\mathscr{C}_0^{5'}$ in a more complicated fashion: a representation of type C or D in $\mathscr{C}_0^{6''}$ decomposes on restriction to $SO(4, 4)$ into the sum of the two irreducible representations of that type in $\mathscr{C}_0^{5'}$, whereas each of the two representations in $\mathscr{C}_0^{6''}$ of a given type A_x or B_x restricts to the one representation of that type in $\mathscr{C}_0^{5'}$. The set $\mathscr{C}_0^{7''}$ is related to the set $\mathscr{C}_0^{6'}$ in the same way. Since we have not introduced Langlands parameters for the group G'', we will illustrate the sets $\mathscr{C}_0^{6''}$ and $\mathscr{C}_0^{7''}$ by describing a final set $\mathscr{C}_0$ (we will call this set $\mathscr{C}_0^*$) which is contained in the block of the trivial representation of the group $SO_e(6, 2)$. This block is dual to the block containing $\mathscr{C}_0^{6''}$, and also to the block containing $\mathscr{C}_0^{7''}$. The representations in $\mathscr{C}_0^*$ are:

$$A_\beta = (3^+, 2^+, 1^+, 0^-), \qquad A_\gamma = (3^+, -1^+, \underline{-2, 0}),$$
$$A_\delta = (3^+, 1^+, \underline{-2, 0}), \qquad A'_\beta = (3^+, 2^+, -1^+, 0^-),$$
$$A'_\gamma = (3^+, -1^+, \underline{2, 0}), \qquad A'_\delta = (3^+, 1^+, \underline{2, 0}),$$
$$B_\beta = (3^+, 0^+, \underline{-2, 1}), \qquad B_\gamma = (3^+, 2^+, \underline{1, 0}),$$
$$B_\delta = (3^+, -2^+, \underline{-1, 0}), \qquad B'_\beta = (3^+, 0^+, \underline{2, -1}),$$
$$B'_\gamma = (3^+, 2^+, \underline{-1, 0}), \qquad B'_\delta = (3^+, -2^+, \underline{1, 0}),$$
$$C = (3^+, 2^+, \underline{\underline{1}}^+, \underline{\underline{0}}^+), \qquad D = (3^+, 1^+, \underline{\underline{2}}^+, \underline{\underline{0}}^+).$$

They are pictured in Fig. 5.3.

Duality reverses a picture like Fig. 5.3, that is, the picture is first flipped from top to bottom, and then the representations of type C become

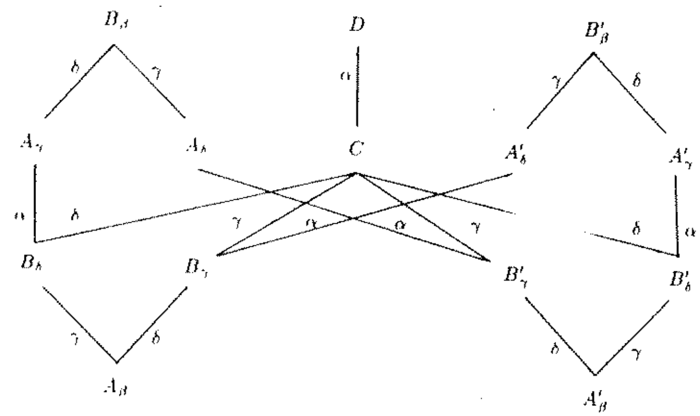

FIGURE 5.3

representations of type D, and representations of type A_x become representations of type B_x, and vice versa. We would thus obtain a diagram of the sets $\mathscr{C}_0^{\delta''}$ and $\mathscr{C}_0^{\gamma''}$.

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On Bass Orders

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INTRODUCTION

Drozd *et al.* [3], extending the earlier results for commutative rings due to Bass [1], introduced the notion of Bass orders and investigated their properties. One of the most prominent results accomplished by them is the classification of Bass orders (see [3, Sects. 5 and 6] or [5, Chap. IX, Sect. 6]). Recently, Hijikata informed the author that he doubted some of the proofs in [3, 5]. Indeed, after a careful reading of [3, 5], we find that the classification of Bass orders in $(D)_2$ is incomplete, where $(D)_2$ is the ring of all 2×2 matrices over a separable division algebra D.

In [3], they claim the following.

LEMMA A [3, Proposition 14.1; 5, Chap. IX, Lemma 6.22]. *Let Λ be a local Bass order in $(D)_2$. Then a conjugate of Λ contains $\mathbb{E}\Delta$, where Δ is the unique maximal order in D and $\mathbb{E}$ is the 2×2 identity matrix.*

Using this lemma, they assert that Bass orders are classified according to their basis over $\mathbb{E}\Delta$ and these bases are given in a list (see [5, Chap. IX, Sect. 6, p. 286]). We have, however, discovered local Bass orders in $(D)_2$ such that no conjugate of them contains $\mathbb{E}\Delta$.

In Section 1, we give some preliminary lemmas. In Sections 2 and 3, we construct chains of Bass orders in $(D)_2$ (Theorems 2.4 and 3.4) and under a special D, we provide local Bass orders which are counterexamples to Lemma A (Examples 2.5 and 3.6). Finally, in Section 4, looking over the orders in the list of [5, Chap. IX, Sect. 6, p. 286], we conclude that many orders constructed in Sections 2 and 3 are new Bass orders.

Throughout the paper, we preserve the following notation. Let R be a complete discrete valuation ring, π its prime element, and K its quotient field. Let Λ be an (R)-order in a separable K-algebra A; that is, Λ is a subring of A which is an R-algebra and a finitely generated R-module such

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