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Unipotent representations of complex semisimple groups

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Introduction

In [A], Arthur introduced some families of representations of a semisimple algebraic group over **R** or **C**. In this paper, we restrict attention to **C**. We generalize Arthur's definition slightly (or perhaps simply make it more precise). All of the resulting representations, except for a finite set, are then unitarily induced from representations of the same kind on proper parabolic subgroups. We call the finite set remaining *special unipotent representations*; a precise definition will be given later (Definition 1.17). Our main result (Theorem III of this introduction) is a character formula for any special unipotent representation. Of course such a formula can be deduced from the Kazhdan-Lusztig conjecture (cf. [V3]). The advantages of Theorem III are that it is in closed form, and that it lends itself to verification of some conjectures of Arthur in [A].

So let G be a connected complex semisimple Lie group, and $\mathfrak{g}$ its Lie algebra. Choose

$$\text{(1.1)}\qquad \begin{aligned} &B \subseteq G \text{ a Borel subgroup; } \mathfrak{b} = \mathrm{Lie}(B),\\ &K \subseteq G \text{ a maximal compact subgroup; } \mathfrak{k}_0 = \mathrm{Lie}(K). \end{aligned}$$

Define

$$\text{(1.2)}\qquad \begin{aligned} T &= K \cap B, \text{ a maximal torus in } K;\ \mathfrak{t}_0 = \mathrm{Lie}(T),\\ H &= \text{centralizer of } T \text{ in } G, \text{ a Cartan subgroup; } \mathfrak{h} = \mathrm{Lie}(H),\\ W &= W(\mathfrak{g}, \mathfrak{h}) \text{ the Weyl group,}\\ \mathfrak{a}_0 &= \sqrt{-1}\,\mathfrak{t}_0, \qquad A = \exp(\mathfrak{a}_0),\\ \Delta(\mathfrak{g}, \mathfrak{h}) &= \text{roots of } \mathfrak{h} \text{ in } \mathfrak{g} \subseteq \mathfrak{h}^*,\\ \Delta^+ &= \Delta(\mathfrak{b}, \mathfrak{h}),\\ \langle\, ,\rangle &= \text{restriction of Killing form to } \mathfrak{h}, \mathfrak{h}^*, \text{ etc.} \end{aligned}$$

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Choose a Chevalley automorphism

$$\sigma\colon \mathfrak{k}_0 \to \mathfrak{k}_0, \sigma|_{\mathfrak{t}_0} = -\,\mathrm{Id}, \tag{1.3}$$

and extend it to $\mathfrak{g}$ by complex linearity. As in [V2], Section 7.1, we can identify

$$\mathfrak{g}_{\mathbf{C}} \cong \mathfrak{g} \times \mathfrak{g} \tag{1.4}$$

in such a way that the following properties hold. Write

$$\mathfrak{k} = (\mathfrak{k}_0)_{\mathbf{C}}, \quad \mathfrak{t} = (\mathfrak{t}_0)_{\mathbf{C}}, \quad \mathfrak{a} = (\mathfrak{a}_0)_{\mathbf{C}}. \tag{1.5}$$

Then

(1.6)(a) $$\mathfrak{k} = \{(x, \sigma x) | x \in \mathfrak{g}\},$$

(b) $$\mathfrak{h}_{\mathbf{C}} \cong \mathfrak{h} \times \mathfrak{h},$$

(c) $$\mathfrak{t} \cong \{(x, -x) | x \in \mathfrak{h}\} \cong \mathfrak{h},$$

(d) $$\mathfrak{a} \cong \{(x, x) | x \in \mathfrak{h}\} \cong \mathfrak{h}.$$

Definition 1.7. Suppose $\lambda, \mu \in \mathfrak{h}^*$, and $\lambda - \mu$ is a weight of a finite dimensional holomorphic representation of G. Using (1.6)(b), regard (λ, μ) as a real-linear functional on $\mathfrak{h}$, and write

(a) $$\mathbf{C}_{(\lambda,\mu)} = \text{character of } H \text{ with differential } (\lambda, \mu)$$

(which exists). In the identifications (1.6)(c) and (d),

(b) $$\mathbf{C}_{(\lambda,\mu)}|_T = \mathbf{C}_{\lambda-\mu},$$

(c) $$\mathbf{C}_{(\lambda,\mu)}|_A = \mathbf{C}_{\lambda+\mu}.$$

Extend $\mathbf{C}_{(\lambda,\mu)}$ to a character of B. Put

(d) $$X(\lambda, \mu) = K\text{-finite part of } \mathrm{Ind}_B^G(\mathbf{C}_{(\lambda,\mu)}),$$

the principal series representation with parameter (λ, μ). Set

$$\overline{X}(\lambda, \mu) = \text{unique irreducible subquotient containing the } K \text{ representation of extremal weight } \lambda - \mu,$$

the *Langlands subquotient*.

PROPOSITION 1.8 (Zhelobenko). *In the setting of Definition 1.7, fix* (λ, μ) *and* (λ', μ') *such that* $\lambda - \mu$ *and* $\lambda' - \mu'$ *are weights of finite dimensional holomorphic representations of G. Then the following are equivalent*:

(a) $X(\lambda, \mu)$ *and* $X(\lambda', \mu')$ *have the same composition factors and multiplicities*;

(b) $\overline{X}(\lambda, \mu) \cong \overline{X}(\lambda', \mu')$;

(c) $(\lambda', \mu') = w(\lambda, \mu)$, *for some* $w \in W$.

Any irreducible $(\mathfrak{g}, K)$ *module is equivalent to some* $\overline{X}(\lambda, \mu)$.

In order to motivate the definition of unipotent representations, we include a brief discussion of some questions connected with the unitarity of the $\overline{X}(\lambda, \mu)$.

Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a parabolic subalgebra such that $\mathfrak{h} \subseteq \mathfrak{m}$ and let $P = MN$ be the corresponding parabolic subgroup. Then it is well-known that, if X_M is a unitary representation,

$$X = \operatorname{Ind}_P^G(X_M \otimes 1) \qquad \text{(unitary induction)}$$

is also unitary.

It would be desirable to have a solution to the following problem.

Problem A. Find all representations that are fundamental in the sense that:

1. They cannot be obtained by unitary induction from any proper parabolic subgroup P.

2. Any other unitary representation is obtained by induction from some fundamental representation (possibly by considering complementary series).

Of course this is beyond our reach. A somewhat weaker formulation would be the following. It is known by [B-V1], [H], how to attach to any representation X a set in the nilpotent cone in $\mathfrak{g}^*$, denoted by $WF(X)$ and called the *wavefront set*. In the case when X is irreducible, $WF(X)$ is the closure of one nilpotent orbit.

There is a simple relation between induction and the wavefront set. If $X = \operatorname{Ind}(X_M \otimes 1)$, then

$$WF(X) = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} WF(X_M) \tag{1.9}$$

where induction of nilpotent orbits is as in [L-Sp] (see (4.13)). This suggests the following problem.

Problem B. Find the unitary irreducible Harish-Chandra modules X such that $WF(X)$ is not induced from any nilpotent orbit of a proper parabolic subgroup.

The Dirac inequality suggests that if $\overline{X}(\lambda, \mu)$ is to be unitary, then $\operatorname{Re}(\lambda + \mu)$ ought to be small. Therefore Problem B suggests:

Problem C. Fix a non-induced nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}^*$. Find all irreducible Harish-Chandra modules $X = \overline{X}(\lambda, \mu)$ such that

(a) $WF(X) = \overline{\mathcal{O}}$,

(b) λ and μ are integral,

(c) $\|\lambda\| + \|\mu\|$ is minimal subject to (a) and (b).

(We will discuss (b) at the end of the introduction.)

Using the theory of primitive ideals, we determined the spherical representations in Problem C for $\mathrm{Sp}(2n, \mathbf{C})$ some time ago. The answer was found to coincide with the set of representations considered in [A]. Arthur's formulation suggests how to solve Problem C in general. In addition, his considerations lend some philosophical support to the conjecture that such representations are unitary, that is, that Problem C is really a part of Problem B.

From this point of view, the purpose of this paper is to solve Problem C, and to compute the distribution characters of the representations it describes.

Definition 1.10. A nilpotent orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ is called *special* if there is an irreducible Harish-Chandra module $X = \overline{X}(\lambda, \mu)$, such that

(a) $WF(X) = \overline{\mathcal{O}}$

(b) λ and μ are integral.

In [B-V2] and [B-V3] it is shown that this definition agrees with Lusztig's original one in [L1]. Obviously we only need to consider special orbits $\mathcal{O}$ in Problem C; for otherwise there will be no Harish-Chandra modules with the desired properties. Put

(1.11) $$\mathcal{S} = \text{set of special nilpotent orbits in } \mathfrak{g}^*.$$

The next step is to describe the non-induced elements of $\mathcal{S}$.

Definition 1.12. Suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$. Let $h_\alpha \in \mathfrak{h}$ be the corresponding Chevalley generator, defined by

(a) $$\lambda(h_\alpha) = 2\langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle = \langle \check{\alpha}, \lambda \rangle.$$

We call h_α (or sometimes $\check{\alpha}$) the *coroot corresponding to* α. Define

(b) $$ {}^L\mathfrak{h} = \mathfrak{h}^*, \quad {}^L\Delta = \{ h_\alpha | \alpha \in \Delta \} \subseteq \mathfrak{h} \cong ({}^L\mathfrak{h})^*,$$

the *coroot system*. It is a root system. We fix once and for all a semisimple Lie algebra

(c) $${}^L\mathfrak{g} \supseteq {}^L\mathfrak{h}$$

such that

(d) $$\Delta({}^L\mathfrak{g}, {}^L\mathfrak{h}) = {}^L\Delta.$$

The ${}^L\mathfrak{g}$ is called the *dual Lie algebra* to $\mathfrak{g}$. Notice that its Weyl group is canonically isomorphic to that of $\mathfrak{g}$:

(e) $$W({}^L\mathfrak{g}, {}^L\mathfrak{h}) \cong W(\mathfrak{g}, \mathfrak{h}).$$

Put

(1.13) $${}^L\mathcal{S} = \text{set of special nilpotent orbits in } ({}^L\mathfrak{g})^*.$$

Using the theory of primitive ideals, we can define an order-reversing bijection

$$(1.14)\qquad \eta\colon \mathcal{S} \to {}^L\mathcal{S},\ \eta(\mathcal{O}) = {}^L\mathcal{O}$$

(Corollary 3.25(a)); this map was first defined in another way in [Sp].

PROPOSITION I (Proposition 5.1 below). *Suppose $\mathcal{O} \subseteq \mathfrak{g}^*$ is a special nilpotent orbit, not induced from any orbit in a proper parabolic subalgebra. Then ${}^L\mathcal{O}$ (cf. (1.14)) is even (Definition 2.11).*

We actually solve the analogue of Problem C assuming only that $\mathcal{O}$ is special, and ${}^L\mathcal{O}$ is even. So fix such an $\mathcal{O}$. Let

$$(1.15)(a)\qquad {}^Lh \in {}^L\mathfrak{h}$$

be the semisimple element corresponding to ${}^L\mathcal{O}$ (cf. Definition 2.7). By Definition 1.12(b), ${}^Lh$ corresponds to an element of $\mathfrak{h}^*$, which we call

$$(1.15)(b)\qquad 2\lambda_{\mathcal{O}} \in \mathfrak{h}^*.$$

By Definition 2.11 and Definition 1.12(b),

$$(1.16)\qquad \lambda_{\mathcal{O}}(h_\alpha) \in \mathbf{Z},\quad \text{for all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h});$$

that is, $\lambda_{\mathcal{O}}$ is an integral weight.

PROPOSITION II (Corollary 5.18 below). *Suppose $\mathcal{O} \subseteq \mathfrak{g}^*$ is a special nilpotent orbit such that ${}^L\mathcal{O}$ is even (cf. (1.14)). Define $\lambda_{\mathcal{O}} \in \mathfrak{h}^*$ by (1.15). Let $\overline{X} = X(\lambda, \mu)$ be an irreducible Harish-Chandra module such that λ and μ are integral, and*

$$WF(X) \subseteq \overline{\mathcal{O}}.$$

Then

$$\|\lambda\| \geq \|\lambda_{\mathcal{O}}\|,$$

$$\|\mu\| \geq \|\lambda_{\mathcal{O}}\|.$$

Suppose that (say) $\|\lambda\| = \|\lambda_{\mathcal{O}}\|$. Then $\lambda = w\lambda_{\mathcal{O}}$ for some $w \in W$, and $WF(X) = \overline{\mathcal{O}}$.

Definition 1.17. Suppose $\mathcal{O}$ is a special nilpotent orbit with ${}^L\mathcal{O}$ even; define $\lambda_{\mathcal{O}}$ by (1.15). An (*integral*) *special unipotent representation* attached to $\mathcal{O}$ is an irreducible Harish-Chandra module $X = \overline{X}(\lambda, \mu)$ such that

a) λ and μ are both conjugate to $\lambda_{\mathcal{O}}$ under W; and
b) $WF(X) = \overline{\mathcal{O}}$.

(General special unipotent representations are discussed at the end of this section.) By Propositions I and II, the representations of Problem C are (integral) special unipotent representations. Because we consider no others in this paper, we generally drop the adjective "integral." We turn now to their parametrization and character formulas.

Let

(1.18)(a) $A(\mathcal{O})$ = group of components of the centralizer of an element of $\mathcal{O}$

(cf. Definition 2.3), and

(1.18)(b) $\bar{A}(\mathcal{O})$ = Lusztig's quotient of $A(\mathcal{O})$ (cf. (4.4)(c)).

Define

(1.19)
$$[\bar{A}(\mathcal{O})] = \text{set of conjugacy classes in } \bar{A}(\mathcal{O}),$$
$$[x] = \text{conjugacy class of } x \in \bar{A}(\mathcal{O}).$$

According to [L3] (see Theorem 4.7(c)) there is a one-to-one correspondence

(1.20)
$$[\bar{A}(\mathcal{O})] \to \text{subset of } \hat{W}, \qquad [x] \to \sigma_x.$$

Set

(1.21)
$$W_{\lambda_{\mathcal{O}}} = \{ w \in W \mid w\lambda_{\mathcal{O}} = \lambda_{\mathcal{O}} \}.$$

For $[x] = [\bar{A}(\mathcal{O})]$, define

(1.22)
$$R_x = R_{\sigma_x} = \frac{1}{|W_{\lambda_{\mathcal{O}}}|} \sum_{w \in W} \operatorname{tr}(\sigma_x(w)) X(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}});$$

this is to be regarded as an element of the complexified Grothendieck group of Harish-Chandra modules.

THEOREM III. *Fix a special unipotent orbit $\mathcal{O}$ with ${}^L\mathcal{O}$ even; define $\lambda_{\mathcal{O}}$ by* (1.15). *There is a bijection*

$$\pi \mapsto X_\pi$$

between the set of irreducible representations of the finite group $\bar{A}(\mathcal{O})$ (cf. (1.18)), *and the set of special unipotent representations of G attached to $\mathcal{O}$ (Definition* 1.17). *We have character formulas*

(a)
$$X_\pi = \frac{1}{|\bar{A}(\mathcal{O})|} \sum_{[x] \in [\bar{A}(\mathcal{O})]} \operatorname{tr} \pi(x) |[x]| R_x,$$

(b)
$$R_x = \sum_{\pi \in (\bar{A}(\mathcal{O}))^{\wedge}} \operatorname{tr} \pi(x) X_\pi.$$

If π is trivial, X_π is one of Arthur's representations from [A].

In [L3], Lusztig proves formulas like (b) for unipotent representations of finite Chevalley groups. (This is the source of our terminology and notation.) However, in his case there were more X_π's than R_x's; so his formulas cannot be inverted to get an analogue of (a).

Here is an outline of the paper. Section 2 collects some basic facts about nilpotent orbits. Section 3 recalls the theory of primitive ideals as developed in [B-V2] and [B-V3]. These results (which rely in turn on the Kazhdan-Lusztig conjecture) are our basic tool. Section 4 describes ideas of Lusztig relating that theory to the Springer correspondence. Section 5 begins the proof of Theorem III; the main result is Proposition 5.31. Section 6 outlines the proof. The first major reduction is carried out in Section 7, and the next in Section 8. We are left with a small number of very interesting cases, which are dealt with in Sections 9 through 11.

Except for the last three sections, we have tried to minimize the appearance of case-by-case considerations. This is misleading; even the definitions entering Theorem III ((1.20, for example) are made on a case-by-case basis. At various critical points in the argument, one must verify simple facts in all examples (if only because the definitions are not "uniform"). Here is a list of our basic references for such arguments, (which will generally be left to the reader): for the explicit calculation of the Springer correspondence: [Sho], [A-L], and the references there. A different formulation of Shoji's results is given in [L4]; for the parametrization of cells in $\hat{W}$: [L2] and Chapter 4 of [L3]; for the calculation of the order relation on nilpotent orbits: [Spa]; for induction of nilpotent orbits in the exceptional algebras: the tables of Elashvili reproduced in [Spa]; for facts about induction of representations of exceptional Weyl groups: [Al].

We have confined our attention to integral infinitesimal character. Arthur does not, and gets a larger class of representations than the ones we have called integral special unipotent; we call these *special unipotent*. Theorem III immediately gives character formulas for these as well, when applied to the subroot system on which the parameter is integral. We have not bothered to make this explicit, however, because we believe that it is a little misleading. There are unitary representations which seem obviously to be attached to certain (non-special) nilpotent orbits, but which are not given by Arthur's construction. (The metaplectic representation is an example.) In a treatment of non-integral infinitesimal character, these objects (for which we are reserving the unmodified term "unipotent representation") ought to play a central part. We hope to pursue this in a future paper. For now, we can write down a large number of representations for which an analogue of Theorem III holds; what remains is to decide which of them ought to be called unipotent.

2. Nilpotent elements

In this section we collect some of the basic results on nilpotent elements in semisimple Lie algebras. The references have been chosen for convenience, and are often not to original sources.

Definition 2.1. Suppose $e \in \mathfrak{g}$ is a nilpotent element. Write

(a) $$\mathcal{O}_e = \mathrm{Ad}(G) \cdot e \subseteq \mathfrak{g},$$

the orbit of e in $\mathfrak{g}$;

(b) $$G^e = \{g \in G | \mathrm{Ad}(g)e = e\},$$

the centralizer of e in G;

(c) $$\mathfrak{g}^e = \mathrm{Lie}(G^e),$$

the centralizer of e in $\mathfrak{g}$;

(d) $$\mathfrak{u}^e = (\mathrm{ad}(e)\mathfrak{g}) \cap \mathfrak{g}^e,$$

(e) $$U^e = \exp(\mathfrak{u}^e).$$

THEOREM 2.2. *Suppose* $e \in \mathfrak{g}$ *is a nilpotent element.*
(a) *There is a homomorphism*

$$\phi \colon \mathfrak{sl}(2) \to \mathfrak{g}$$

such that

$$\phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e.$$

(b) *Suppose* ϕ' *is another homomorphism from* $\mathfrak{sl}(2)$ *into* $\mathfrak{g}$ *with the property in* (a). *Then there is a unique element* $u \in U^e$ *such that*

$$\mathrm{Ad}(u) \circ \phi = \phi'.$$

Part (a) is the Jacobson-Morozov theorem, and (b) is due to Kostant. A proof of the entire result may be found in [Ko]. For a variety of reasons, it will usually be more convenient for us to work with the homomorphism ϕ than with e itself. Consequently, it is helpful to introduce some redundant notation.

Definition 2.3. Suppose $\phi \colon \mathfrak{sl}(2) \to \mathfrak{g}$ is a Lie algebra homomorphism. Set

(a) $$e = e_\phi = \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

(b) $$h = h_\phi = \phi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(c) $$G^{\phi} = \{g \in G \mid \mathrm{Ad}(g)z = z, \text{ for all } z \in \phi(\mathfrak{sl}(2))\} \subseteq G^{e},$$

(d) $$A(\mathcal{O}_e) = G^e/G_0^e \cdot Z(G)$$ (the component group of G^e in $\mathrm{Ad}(G)$; recall that $\mathcal{O}_e$ is the orbit of e in $\mathfrak{g}$).

Because $A(\mathcal{O}_e)$ need not be abelian, it depends (like a fundamental group) on the choice of a base point e. Ignoring this will cause no difficulty, however.

PROPOSITION 2.4. *Suppose ϕ: $\mathfrak{sl}(2) \to \mathfrak{g}$ is a Lie algebra homomorphism. Define $e = \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and use the notation of Definitions* 2.1 *and* 2.3.

a) $$G^e = G^{\phi} \cdot U^e,$$

a semidirect product. The group G^{ϕ} is reductive, and U^e is unipotent; it is the unipotent radical of G^e.

b) *The component groups $G^{\phi}/G_0^{\phi}Z(G)$ and $A(\mathcal{O}_e)$ are canonically isomorphic under the map induced by the inclusion $G^{\phi} \hookrightarrow G^e$.*

Proof. Because U^e is defined in terms of e, it is normal in the centralizer G^e of e. By the representation theory of $\mathfrak{sl}(2)$, applied to the representation $\mathrm{ad} \circ \phi$ of $\mathfrak{sl}(2)$ on $\mathfrak{g}$, $\mathfrak{u}^e$ is contained in the sum of the eigenspaces of h (Definition 2.3(b)) with positive eigenvalues. It is therefore nilpotent, so U^e is unipotent. As the centralizer of a reductive subgroup of a reductive group, G^{ϕ} is reductive. (This well-known fact follows, for example, from Weyl's unitary trick.) Since G^{ϕ} can have no unipotent normal subgroup but $\{1\}$, we deduce

$$G^{\phi} \cap U^e = \{1\}. \tag{2.5}$$

To prove (a), fix $g \in G^e$; and set

$$\phi' = \mathrm{Ad}(g)^{-1} \circ \phi.$$

Then

$$\begin{aligned}\phi'\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= \mathrm{Ad}(g)^{-1}\phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \mathrm{Ad}(g)^{-1}(e) \\ &= e,\end{aligned}$$

since $g \in G^e$. By Theorem 2.2(b), there is a $u \in U^e$ such that

$$\mathrm{Ad}(u) \circ \phi = \phi' = \mathrm{Ad}(g)^{-1} \circ \phi$$

so that

$$\mathrm{Ad}(gu) \circ \phi = \phi.$$

Therefore gu belongs to G^ϕ. In conjunction with (2.5), this proves (a). Because U^e is connected by definition, (b) follows. Q.E.D.

The splitting in Proposition 2.4(a) was discovered by Kostant for that nilpotent orbit in E_8 for which $A(\mathcal{O})$ is isomorphic to the symmetric group S_5. (In that case, the group G^ϕ is discrete, and so isomorphic to S_5. Kostant also proved in this case that $\phi(\mathrm{SL}(2))$ is the full centralizer of G^ϕ in G, which is very unusual.) Part (a) suggests the question of whether any (disconnected) algebraic group in characteristic zero has a Levi decomposition. This ought to be well-known, but we have not found a reference.

We turn now to Dynkin's list of all nilpotent conjugacy classes, which we will need at several critical points.

THEOREM 2.6 (Malcev—see [Ko]). *Let ϕ and ϕ' be two homomorphisms from $\mathfrak{sl}(2)$ into $\mathfrak{g}$, and suppose that h_ϕ is conjugate to $h_{\phi'}$ under G (Definition 2.3(b)). Then there is an element $g \in G$ such that*

1) $\mathrm{Ad}(g)(h_\phi) = h_{\phi'}$,
2) $\mathrm{Ad}(g) \circ \phi = \phi'$.

Definition 2.7 (Dynkin). Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. Choose a homomorphism

$$\phi\colon \mathfrak{sl}(2) \to \mathfrak{g}, \quad \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{O}$$

(cf. Theorem 2.2). Replace ϕ by a conjugate under G, in such a way that

$$h_\phi = \phi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

belongs to the closure of the positive Weyl chamber in the Cartan subalgebra $\mathfrak{h}$. (Since h_ϕ is automatically semisimple, this is possible.) *The Dynkin diagram of $\mathcal{O}$* is a labelling of the Dynkin diagram of $\mathfrak{g}$: the vertex corresponding to a simple root α is labelled with $\alpha(h_\phi)$, the eigenvalue of h_ϕ on the α weight space.

By the representation theory of $\mathfrak{sl}(2)$, the labels in the Dynkin diagram of $\mathcal{O}$ are non-negative integers. An extremely easy argument shows that 0, 1, and 2 are the only possibilities.

COROLLARY 2.8. *Suppose $\mathcal{O}$ and $\mathcal{O}'$ are nilpotent orbits having the same Dynkin diagram. Then $\mathcal{O} = \mathcal{O}'$.*

In [Dy], Dynkin has determined all possible Dynkin diagrams of nilpotent orbits. To our knowledge, there are no general results providing *a priori* restrictions on the form of the answer.

Example 2.9. Let $\mathcal{O} = \{0\} = \{e\}$, the zero nilpotent orbit in $\mathfrak{g}$. Then the map ϕ of Theorem 2.2 is just the zero map; so h_ϕ is also zero. The Dynkin diagram of $\mathcal{O}$ has all vertices labelled zero.

Example 2.10. Kostant established in [Ko] the fundamental properties of the unique nilpotent orbit $\mathcal{O}$ of largest dimension, the *principal nilpotent orbit*. (An element of it is called a *principal* or *regular* nilpotent element.) The Dynkin diagram of $\mathcal{O}$ has every vertex labelled 2.

Both of these nilpotent orbits are of the special kind in the next definition.

Definition 2.11. A nilpotent orbit $\mathcal{O}$ is called *even* if any one of the following equivalent conditions is satisfied. Choose ϕ as in Theorem 2.2.

a) The exponentiated map ϕ: SL(2) $\to$ Ad(G) factors through PSL(2).
b) All eigenvalues of ad(h_ϕ) (Definition 2.3) are even.
c) In the Dynkin diagram of $\mathcal{O}$, each vertex is labelled 0 or 2.

We conclude this section with a summary of the Springer correspondence (see [Spr]). Define

$$\mathcal{B} = \text{set of Borel subalgebras of } \mathfrak{g}, \tag{2.12}$$

the *flag variety of* $\mathfrak{g}$. There are isomorphisms

$$G/B \cong \mathcal{B}, \qquad gB \to \mathrm{Ad}(g)(\mathfrak{b}), \tag{2.13}$$

$$K/T \cong \mathcal{B}, \qquad kT \to \mathrm{Ad}(k)(\mathfrak{b}), \tag{2.14}$$

(Here $\mathfrak{b}$ is our fixed Borel subalgebra.)

Definition 2.15. Suppose $e \in \mathfrak{g}$ is a nilpotent element. The *Dynkin variety of* e is

$$\mathcal{B}_e = \text{set of Borel subgroups containing } e \subseteq \mathcal{B}.$$

Obviously G^e acts on $\mathcal{B}_e$ (Definition 2.1). A standard argument shows that the induced action on cohomology is trivial on G_0^e; so

$$A(\mathcal{O}_e) \text{ acts on } H^*(\mathcal{B}_e, \mathbf{C}) \tag{2.16}$$

(cf. Definition 2.3(e)). For an irreducible representation (π, V_π) of $A(\mathcal{O}_e)$, define

$$H^*(\mathcal{B}_e, \mathbf{C})_\pi = \mathrm{Hom}_{A(\mathcal{O}_e)}(V_\pi, H^*(\mathcal{B}_e, \mathbf{C})). \tag{2.17}$$

THEOREM 2.18 ([Spr]). *For any nilpotent element e, there is a natural action of W on $H^*(\mathcal{B}_e, \mathbf{C})$.*

(a) *The actions of W and $A(\mathcal{O}_e)$ commute; so W acts on each $H^*(\mathcal{B}_e, \mathbf{C})_\pi$ $(\pi \in A(\mathcal{O}_e)\hat{\ })$.*

(b) *The natural maps*

$$H^*(\mathscr{B}, \mathbf{C}) \to H^*(\mathscr{B}_e, \mathbf{C}),$$

induced by $\mathscr{B}_e \to \mathscr{B}$, *are* W-*equivariant.*

(c) *For* $\pi \in A(\mathcal{O}_e)^\wedge$, *the representation* $\sigma(e,\pi)$ *of* W *on* $H^{\dim_{\mathbf{R}}\mathscr{B}_e}(\mathscr{B}_e, \mathbf{C})_\pi$ *is irreducible or zero.*

(d) *If* π *is trivial*, $\sigma(e,\pi) \neq 0$.

(e) *Suppose* $\sigma \in \hat{W}$. *Then there are: a nilpotent element* $e \in \mathfrak{g}$, *unique up to* $\mathrm{Ad}(G)$; *and a unique* $\pi \in A(\mathcal{O}_e)^\wedge$, *such that*

$$\sigma = \sigma(e,\pi).$$

The correspondence

$$(G \cdot e, \pi) \to \sigma(e,\pi) \tag{2.19}$$

is called the *Springer correspondence.* We write

$$\sigma(\mathcal{O}) = \sigma(e,1) \qquad (e \in \mathcal{O}). \tag{2.20}$$

In this way, each nilpotent orbit $\mathcal{O}$ is attached to a finite set of Weyl group representations, having a distinguished element $\sigma(\mathcal{O})$. There is some question of normalization here: we require

$$\begin{aligned} \sigma(\{0\}) &= \text{sign representation of } W, \\ \sigma(\text{principal nilpotent}) &= \text{trivial representation.} \end{aligned} \tag{2.21}$$

Recall from (1.10) the set of special nilpotent orbits. A Weyl group representation σ is called *special* if $\sigma = \sigma(\mathcal{O})$ for some special nilpotent orbit $\mathcal{O}$. Because the zero orbit and the principal nilpotent orbit are both special, (2.21) shows that the sign and trivial representations of W are special.

From the invariant bilinear form, orbits of G on $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified. Those on $\mathfrak{g}^*$ corresponding to nilpotent orbits on $\mathfrak{g}$ will also be called nilpotent orbits, and freely identified with their counterparts on $\mathfrak{g}$. In particular, we may write $\sigma(\mathcal{O})$ for the Weyl group representation attached to $\mathcal{O} \subseteq \mathfrak{g}^*$. In fact the Springer correspondence is perhaps most naturally made directly in terms of nilpotents on $\mathfrak{g}^*$ only, using the moment map

$$\pi\colon T^*(\mathscr{B}) \to \mathfrak{g}^*$$

and defining

$$\mathscr{B}_e = \pi^{-1}(e)$$

for $e \in \mathfrak{g}^*$ a nilpotent element. The reader may notice that we use the Jacobson-Morozov theory (which is suited to nilpotents in the Lie algebra) mostly on ${}^L\mathfrak{g}$, and the Springer theory (suited to nilpotents in the dual) on $\mathfrak{g}$. This is probably significant, but in what way we do not know.

3. Primitive ideals and characters of complex groups

We summarize in this section some of the main results we need from [B-V2] and [B-V3] (see also [L1], [J2]).

Fix a negative integral weight $\mu \in \mathfrak{h}^*$, and define, for $w \in W$,

$$M(w\mu) = U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_{w\mu-\rho}, \tag{3.1}$$

$$L(w\mu) = \text{unique irreducible quotient of } M(w\mu),$$

$$\mathrm{Prim}_\mu U(\mathfrak{g}) = \text{set of primitive ideals of infinitesimal character } \mu,$$

$$I(w\mu) = \mathrm{Ann}\, L(w\mu) \in \mathrm{Prim}_\mu U(\mathfrak{g}).$$

By [D], the map from W to $\mathrm{Prim}_\mu U(\mathfrak{g})$ given by

$$w \mapsto I(w\mu) \tag{3.2}$$

is surjective. Notice that it is constant on left cosets of

$$W_\mu = \{w \in W | w\mu = \mu\}. \tag{3.3}$$

For any $w \in W$, the τ-invariant is

$$\tau(w) = \{\text{simple roots } \alpha \in \Delta^+ \colon w\alpha \notin \Delta^+\}. \tag{3.4}$$

Each left W_μ coset contains a unique minimal element w_1 for the Bruhat order, which may be characterized by the fact that

$$\alpha \in \Delta^+, \qquad \langle \alpha, \mu \rangle = 0 \Rightarrow w_1(\alpha) \in \Delta^+.$$

For such a minimal w_1, the Borho-Jantzen-Duflo τ-invariant of $I(w_1\lambda)$ is

$$\tau(I(w_1\lambda)) = \tau(w_1); \tag{3.5}$$

it depends only on the ideal, as the notation indicates. Notice that for a simple root α,

$$\langle \alpha, \mu \rangle = 0 \Rightarrow \alpha \notin \tau(I), \quad \text{for any } I \in \mathrm{Prim}_\mu U(\mathfrak{g}). \tag{3.6}$$

Suppose now that X is an irreducible Harish-Chandra module for the complex group G. The annihilator of X is of the form

$$\mathrm{Ann}\, X = I_1 \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes I_2,$$

and we define the left and right annihilators and τ-invariants by

$$L\,\mathrm{Ann}\, X = I_1, \qquad R\,\mathrm{Ann}\, X = I_2, \tag{3.7}$$

$$\tau_L(X) = \tau(I_1), \qquad \tau_R(X) = \tau(I_2).$$

Assume that X has infinitesimal character (λ, μ), with λ and $-\mu$ dominant integral weights. Then

$$X \cong \overline{X}(\lambda, w\mu), \tag{3.8}$$

for some $w \in W$. Only the double coset $W_\lambda w W_\mu$ is determined by X; and if w is chosen to be minimal in that double coset, then

$$\begin{aligned} \tau_L(X) &= \tau(w^{-1}) = \{\alpha \text{ simple: } w^{-1}\alpha \notin \Delta^+\}, \\ \tau_R(X) &= \tau(w) = \{\alpha \text{ simple: } w\alpha \notin \Delta^+\}. \end{aligned} \tag{3.9}$$

Still assuming (3.8) and that w is minimal in its double coset, Joseph has computed (with w_0 the longest element of W) ([J1] Theorem 5.2)

$$\begin{aligned} L \operatorname{Ann} X &= \check{I}(-w^{-1}\lambda), \\ R \operatorname{Ann} X &= \check{I}(-w_0 w\mu); \end{aligned} \tag{3.10}$$

here the caret denotes the principal antiautomorphism of $U(\mathfrak{g})$, which is -1 on $\mathfrak{g}$.

Assume now that λ and $-\mu$ are dominant, integral, and regular. We define an order relation $\underset{L}{\leq}$ on W by

$$\begin{aligned} w_1 \underset{L}{\leq} w_2 &\Leftrightarrow R \operatorname{Ann}(\overline{X}(\lambda, w_1\mu)) \subseteq R \operatorname{Ann}(\overline{X}(\lambda, w_2\mu)) \\ &\Leftrightarrow I(w_0 w_1(-\mu)) \subseteq I(w_0 w_2(-\mu)) \\ &\Leftrightarrow I(w_1(\mu)) \subseteq I(w_2(\mu)). \end{aligned} \tag{3.11}$$

(The last equivalence is not quite formal.) Similarly, define

$$\begin{aligned} w_1 \underset{R}{\leq} w_2 &\Leftrightarrow L \operatorname{Ann}(\overline{X}(\lambda, w_1\mu)) \subseteq L \operatorname{Ann}(\overline{X}(\lambda, w_2\mu) \\ &\Leftrightarrow I(w_1^{-1}(-\lambda)) \subseteq I(w_2^{-1}(-\lambda)) \\ &\Leftrightarrow w_1^{-1} \underset{L}{\leq} w_2^{-1}. \end{aligned} \tag{3.12}$$

Because the Kazhdan-Lusztig conjectures are true, [V1] shows that these relations coincide with the ones defined by Kazhdan and Lusztig in [K-L]. The smallest relation containing $\underset{L}{\leq}$ and $\underset{R}{\leq}$ is denoted by $\underset{LR}{\leq}$. Define

$$\begin{aligned} w_1 \underset{L}{\approx} w_2 &\Leftrightarrow w_1 \underset{L}{\leq} w_2 \quad \text{and} \quad w_2 \underset{L}{\leq} w_1 \\ &\Leftrightarrow I(w_1\mu) = I(w_2\mu). \end{aligned} \tag{3.13)(a}$$

Similarly we define $\underset{R}{\approx}$ and $\underset{LR}{\approx}$. Set

$$\overline{\mathscr{C}}^L(w) = \left\{w' \in W \mid w' \underset{L}{\geq} w\right\} = \{w' \in W \mid I(w'\mu) \supseteq I(w\mu)\}, \tag{3.13)(b}$$

and similarly $\overline{\mathscr{C}}^R(w), \overline{C}^{LR}(w)$. They are called the *left*, *right*, *and double cones* *over* w. The equivalence classes of w under $\underset{L}{\approx}$, etc., are denoted by $\mathscr{C}^L(w)$, etc. They are called *left*, *right*, *and double cells.*

That the relations just defined are independent of the choices of λ and μ is the first of many uses we will make of the Borho-Jantzen-Zuckerman *translation principle*. All the results of this kind that we need may be found in [B-J] and [V2]. We need too many different results to list them all, but here are some typical ones. Suppose λ and $-\mu$ are dominant integral; $w \in W$ is minimal in the double coset $W_\lambda w W_\mu$; and λ_r and $-\mu_r$ are dominant, integral, and regular. Then there is a finite dimensional representation F of G such that

$$\overline{X}(\lambda, w\mu) \text{ is a direct summand of } \overline{X}(\lambda_r, w\mu_r) \otimes F,$$

and

$$\overline{X}(\lambda_r, w\mu_r) \text{ is a submodule of } \overline{X}(\lambda, w\mu) \otimes F^*.$$

An immediate consequence is that $WF(\overline{X}(\lambda, w\mu)) = WF(\overline{X}(\lambda_r, w\mu_r))$. Under the same assumptions, write

$$\overline{X}(\lambda_r, w\mu_r) = \sum_{Y \in W} a_y X(\lambda_r, y\mu_r)$$

as in (3.14) below. Then

$$\overline{X}(\lambda, w\mu) = \sum_{Y \in W} a_y X(\lambda, y\mu).$$

The connection of such ideas with primitive ideal theory, which we also exploit, is the subject of [B-J].

The complexified Grothendieck group $\mathscr{G}(\lambda, \mu)$ of formal characters of G having infinitesimal character (λ, μ) has as a basis the various principal series $X(\lambda, w\mu)$. We may therefore identify it with $\mathbf{C}[W]$:

$$\sum_{w \in W} c_w \leftrightarrow \sum c_w X(\lambda, w\mu). \tag{3.14}$$

The Weyl group of the complexified Lie algebra of G is $W \times W$; so $W \times W$ acts by coherent continuation on this Grothendieck group. In the identification with $\mathbf{C}[W]$, the action is identified with the regular representation:

$$\begin{aligned}(w_1, w_2) \cdot \left(\sum_w c_w w \right) &= \sum_w c_w \left(w_1 w w_2^{-1} \right) \\ &= \sum_w c_{w_1^{-1} w w_2} w.\end{aligned} \tag{3.15}$$

The main fact about cones is that they are invariant under this action:

$$(w_1, 1) \cdot \overline{X}(\lambda, w\mu) = \sum_{w' \in \overline{C}^L(w)} a_{w'} \overline{X}(\lambda, w'\mu). \tag{3.16}$$

Set

$$\begin{aligned}(3.17)\qquad \bar{V}^L(w) &= \operatorname{span}\{\bar{X}(\lambda, w'\mu) | w' \in \bar{\mathscr{C}}^L(w)\} \subseteq \mathscr{G}(\lambda, \mu),\\ K^L(w) &= \operatorname{span}\{\bar{X}(\lambda, w'\mu) | w' \in \bar{C}^L(w),\ w' \notin \mathscr{C}^L(w)\},\\ V^L(w) &= \bar{V}^L(w)/K^L(w).\end{aligned}$$

These all carry representations of W arising from the action of the first factor in the coherent continuation representation. We call $V^L(w)$ a left cell representation. Similarly we define $V^R(w)$ and $V^{LR}(w)$. Thus

$$(3.18)\qquad \mathbf{C}[W] \cong \bigoplus_{\text{left cells}} V^L(w)$$

as a left representation of W, and

$$(3.19)\qquad \mathbf{C}[W] \cong \bigoplus_{\text{double cells}} V^{LR}(w)$$

as a representation of $W \times W$. (These isomorphisms are not canonical; the right side in each case is the associated graded module for a filtration of the left.)

THEOREM 3.20 ([B-V2], [B-V3]). *Fix $w \in W$. Then $V^L(w)$ contains a unique special representation $\sigma(w) \in \hat{W}$, with multiplicity one. Write $\mathcal{O}(w)$ or $\mathcal{O}(\sigma)$ for the nilpotent orbit in $\mathfrak{g}^*$ associated to σ by the Springer correspondence (cf. Theorem 2.18). Then for any dominant integral weights λ and $-\mu$, such that w is minimal in $W_\lambda w W_\mu$,*

$$(\mathrm{a})\qquad WF(\bar{X}(\lambda, w\mu)) = \overline{\mathcal{O}(w)}.$$

Every special representation of W arises as $\sigma(w)$, for some $w \in W$.

Fix $\sigma \in \hat{W}$. By (3.19), σ occurs in a unique double cell representation, say $V^{LR}(w)$. Define $\mathcal{O}(w)$ as in Theorem 3.20, and set

$$(3.21)(\mathrm{a})\qquad a(\sigma) = |\Delta^+| - 1/2 \dim \mathcal{O}(w).$$

For σ_1 and σ_2 in $\hat{W}$, we write

$$\begin{aligned}(3.21)(\mathrm{b})\qquad &\sigma_1 \underset{LR}{\leq} \sigma_2 \Leftrightarrow \sigma_i \text{ occurs in } V^{LR}(w_i), \text{ and } w_1 \underset{LR}{\leq} w_2.\\ &\sigma_1 \underset{LR}{\approx} \sigma_2 \Leftrightarrow \sigma_1 \text{ and } \sigma_2 \text{ occur in a common } V^{LR}(w).\end{aligned}$$

Define

$$\begin{aligned}(3.22)(\mathrm{a})\qquad \bar{\mathscr{C}}^{LR}(\sigma) &= \left\{\sigma' | \sigma' \underset{LR}{\geq} \sigma\right\},\\ \mathscr{C}^{LR}(\sigma) &= \left\{\sigma' | \sigma' \underset{LR}{\approx} \sigma\right\}.\end{aligned}$$

If $\mathcal{O}$ is a special nilpotent orbit, we may also write

(3.22)(b) $$\mathscr{C}^{LR}(\mathcal{O}) = \mathscr{C}^{LR}(\sigma)$$

whenever $\mathcal{O} = \mathcal{O}(\sigma)$.

PROPOSITION 3.23. *The Springer correspondence*

$$\sigma \to \mathcal{O}(\sigma)$$

of (2.20) *defines an order isomorphism from the set of special representations of* W (*ordered by* (3.21)(b)) *onto the set* $\mathscr{S}$ *of special nilpotent orbits in* $\mathfrak{g}$.

Proof. That the map is bijective follows from Theorem 3.20. That it respects the order relations is an elementary consequence of the definition of WF. That every inclusion $\mathcal{O} \subseteq \mathcal{O}'$ for special nilpotent orbits arises from a relation $\sigma \underset{LR}{\leq} \sigma'$ may be checked case by case, using [Spa] to describe all the inclusions, and the techniques of [B-V2] and [B-V3] to find relations $\sigma \underset{LR}{\leq} \sigma'$. Because we will not really use this fact, we omit the details. Q.E.D.

PROPOSITION 3.24. *Fix regular integral negative weights* $\lambda \in \mathfrak{h}^*$, ${}^L\lambda \in ({}^L\mathfrak{h})^*$. *Then the map*

(a) $$I(w\lambda) \to I\big(ww_0({}^L\lambda)\big)$$

is a well-defined order-reversing bijection

(b) $$\mathrm{Prim}_\lambda U(\mathfrak{g}) \to \mathrm{Prim}_{{}^L\lambda} U({}^L\mathfrak{g})$$

(*notation* (3.1), (3.2)). *In particular, the map* $w \to ww_0$ *is order-reversing for* $\underset{L}{\leq}$. *On the level of cells, there is a natural isomorphism*

(c) $$V^L(w) \cong V^L(ww_0)^* \otimes \mathrm{sgn};$$

and similarly for right and double cells. (*Here* $*$ *denotes the contragredient representation.*)

Because W is the same for $\mathfrak{g}$ and ${}^L\mathfrak{g}$, this is a reformulation of 2.23, 2.24 and 2.25 of [B-V3].

COROLLARY 3.25. *There is an order-reversing bijection*

(a) $$\eta\colon \mathcal{O} \to {}^L\mathcal{O}$$

from special nilpotent orbits in $\mathfrak{g}$ *to special nilpotent orbits in* ${}^L\mathfrak{g}$, *such that*

(b) $$\eta(\mathcal{O}_{\mathfrak{g}}(w)) = \mathcal{O}_{{}^L\mathfrak{g}}(ww_0)$$

(*cf. Theorem* 3.20). *The corresponding special representations are related by*

(c) $$\sigma({}^L\mathcal{O}) = \textit{unique special representation in } \mathscr{C}^{LR}(\sigma(\mathcal{O}) \otimes \mathrm{sgn})$$

(*cf.* (3.22)).

For the classical groups, and most cases in the exceptional groups, $\sigma(\mathcal{O}) \otimes \mathrm{sgn}$ is itself special. Further discussion of this duality may be found in Appendix A.

If X is any non-zero finitely generated $\mathfrak{g}$ module, then a choice of generating subspace gives rise to a filtration of X compatible with the standard filtration of $U(\mathfrak{g})$, and hence to a graded $S(\mathfrak{g})$ module $\operatorname{gr} X$. The *Gelfand-Kirillov dimension of* X is defined to be

$$\text{(3.26)} \qquad \operatorname{Dim} X = \text{Krull dimension of } \operatorname{gr} X,$$

and the *multiplicity* $c(X)$ (called Bernstein degree in [V1]) is the multiplicity of $\operatorname{gr} X$: that is, the leading coefficient of the Hilbert-Samuel polynomial, times $(\operatorname{Dim} X)!$. Therefore $c(X)$ is a positive integer. For λ and $-\mu$ dominant, regular, and integral, and $w \in W$, define

$$\text{(3.27)} \qquad c(w)(\lambda, \mu) = c(\overline{X}(\lambda, w\mu)).$$

By [J2], $c(w)$ extends to a harmonic polynomial on $\mathfrak{h}^* \times \mathfrak{h}^*$, homogeneous of degree $a(\sigma(w))$ (Theorem 3.20 and (3.21)) in each variable separately, and transforming according to $\sigma(w) \times \sigma(w)$ under $W \times W$. It may also be computed as follows [Ki]. Write

$$\text{(3.28)(a)} \qquad \overline{X}(\lambda, w\mu) = \sum_{w'} a_{w'} X(\lambda, w'\mu)$$

in the Grothendieck group $\mathcal{G}(\lambda, \mu)$. The numerator of the global character of $\overline{X}(\lambda, w\mu)$, lifted to a holomorphic function of the complexified Lie algebra $\mathfrak{h} \times \mathfrak{h}$, is

$$\text{(3.28)(b)} \qquad \phi(a, b) = \sum_{x, w'} a_{w'} e^{(x\lambda, xw'\mu)(a, b)} \qquad (a, b \in \mathfrak{h}).$$

Define

$$\text{(3.28)(c)} \qquad d(w)(\lambda, \mu) = \lim_{t \to 0^+} t^{-2a(\sigma(w))} \phi(ta_0, tb_0),$$

with a_0 and b_0 fixed dominant regular elements of $\mathfrak{h}$.

PROPOSITION 3.29 ([Ki]). *With notation* (3.27) *and* (3.28), *there is a non-zero constant* A, *depending only on* a_0, b_0 *and the double cell* $\mathcal{C}^{LR}(w)$, *such that*

$$c(w) = A \cdot d(w).$$

We need two more general facts about multiplicities.

PROPOSITION 3.30 ([J2]). *With notation* (3.27), *suppose* λ_0 *and* μ_0 *are dominant and integral, but possibly singular; and that* w *is minimal in* $W_{\lambda_0} w W_{\mu_0}$. *Then*

$$c(\overline{X}(\lambda_0, w\mu_0)) = c(w)(\lambda_0, \mu_0).$$

PROPOSITION 3.31. *Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$, and $X_{\mathfrak{m}}$ an irreducible Harish-Chandra module for $\mathfrak{m}$. Then there is a constant B, depending only on $\mathfrak{p}$ and $WF(X_{\mathfrak{m}})$, such that*

$$c\left(\mathrm{Ind}_P^G X_{\mathfrak{m}}\right) = B \cdot c(X_{\mathfrak{m}}).$$

This is more or less known; it may be deduced from the two preceding propositions.

4. The group $\bar{A}(\mathcal{O})$

We now have two rough correspondences between Weyl group representations and nilpotent orbits, given by Theorem 2.18 and Theorem 3.20. The goal of this section is to describe Lusztig's extension of the latter correspondence to something as detailed as Theorem 2.18, and to relate the two correspondences.

Definition 4.1. Fix $\sigma \in \hat{W}$. Put

a) $$\mathcal{O}(\sigma) = \text{unique nilpotent orbit } G \cdot e \text{ such that } \sigma = \sigma(e, \pi)$$

(Theorem 2.18). Next let σ' be the unique special representation of W such that $\sigma' \underset{LR}{\approx} \sigma$ (Theorem 3.20 and (3.21)(b)). Set

b) $$\mathcal{O}_{sp}(\sigma) = \mathcal{O}(\sigma').$$

Another way to say (b) is this: fix $w \in W$ so that σ occurs in $V^L(w)$ (cf. (3.17)). Then, in the notation of Theorem 3.20

b)′ $$\overline{\mathcal{O}_{sp}(\sigma)} = WF(\bar{X}(\lambda, w\mu)).$$

The first result is included as motivation for (4.4).

PROPOSITION 4.2 (Lusztig). *Suppose $\sigma_1, \sigma_2 \in \hat{W}$, and $\sigma_1 \underset{LR}{\leq} \sigma_2$. Then*

$$\overline{\mathcal{O}_{sp}(\sigma_1)} \supseteq \mathcal{O}(\sigma_2).$$

If σ_1 and σ_2 are special, this is a consequence of Proposition 3.23. If $\sigma_1 \underset{LR}{\approx} \sigma_2$, it may be checked case-by-case from the calculation of $\underset{LR}{\approx}$ in [B-V2] and [B-V3], and the known explicit form of the Springer correspondence. The general case is a consequence of these two particular ones. Now fix a special nilpotent orbit

(4.3)(a) $$\mathcal{O} \in \mathcal{S} \qquad \text{(cf.(1.11))},$$

and an element

(4.3)(b) $$e \in \mathcal{O}.$$

Define

$$\sigma_1 = \sigma(\mathcal{O}) \tag{4.3)(c}$$

(cf. (2.20)), the special Weyl group representation attached to $\mathcal{O}$. Now define

(4.4)(a)

$$\begin{aligned} Z(\mathcal{O}) &= \left\{\sigma \in \hat{W} | \mathcal{O}(\sigma) = \mathcal{O}_{sp}(\sigma) = \mathcal{O} \text{ (Definition 4.1)}\right\} \\ &= \left\{\sigma \in \hat{W} | \sigma \underset{LR}{\approx} \sigma_1, \text{ and } \sigma = \sigma(e,\pi) \text{ for some } \pi \in A(\mathcal{O})^{\wedge} \text{ (Theorem 2.18)}\right\}; \end{aligned}$$

$$K(\mathcal{O}) = \bigcap_{\substack{\pi \in A(\mathcal{O})^{\wedge} \\ \sigma(e,\pi) \in Z(\mathcal{O})}} (\ker \pi) \subseteq A(\mathcal{O}), \tag{4.4)(b}$$

$$\bar{A}(\mathcal{O}) = A(\mathcal{O})/K(\mathcal{O}). \tag{4.4)(c}$$

The group $\bar{A}(\mathcal{O})$ is *Lusztig's canonical quotient of* $A(\mathcal{O})$. By definition, some of the characters of $\bar{A}(\mathcal{O})$ parametrize some of the Weyl group representations in $\mathscr{C}^{LR}(\mathcal{O})$, the $\underset{LR}{\approx}$ equivalence class of the special representations σ_1.

We want to describe $\mathscr{C}^{LR}(\mathcal{O})$ in terms of $\bar{A}(\mathcal{O})$. There are a few cases where this description is a little less pleasant.

Definition 4.5 [L1]. A special nilpotent orbit $\mathcal{O}$ is called *exceptional* if $\mathfrak{g}$ contains a simple factor of type E_7 or E_8, and the Dynkin diagram of $\mathcal{O}$ on this factor is one of the following:

$$\begin{matrix} 010101 \\ 0 \end{matrix}$$

$$\begin{matrix} 2010101 \\ 0 \end{matrix}$$

$$\begin{matrix} 1010001 \\ 0 \end{matrix}$$

This set is closed under duality; the first orbit is self-dual, and the last two are dual to each other. (Note that even orbits and their duals are never exceptional.)

We turn now to Lusztig's description of $\mathscr{C}^{LR}(\mathcal{O})$.

Definition 4.6. In the setting (4.3), define

$$\begin{aligned} M(\mathcal{O}) &= \{(x,\xi_x) | x \in \bar{A}(\mathcal{O}), \text{ and } \xi_x \text{ is an irreducible representation of} \\ &\qquad \text{the centralizer of } x \text{ in } \bar{A}(\mathcal{O})\}; \\ [M(\mathcal{O})] &= \text{set of } \bar{A}(\mathcal{O}) \text{ orbits on } M(\mathcal{O}) \\ &\supseteq [\bar{A}(\mathcal{O})], \bar{A}(\mathcal{O})^{\wedge}. \end{aligned}$$

If $\bar{A}(\mathcal{O})$ is a product of copies of $\mathbf{Z}/2\mathbf{Z}$ (as it is for classical G), then

$$M(\mathcal{O}) = \bar{A}(\mathcal{O}) \times \bar{A}(\mathcal{O})^{\wedge},$$

a symplectic vector space over $\mathbf{Z}/2\mathbf{Z}$.

THEOREM 4.7 (Lusztig [L3]). *Suppose $\mathcal{O}$ is a special nilpotent orbit. With notation as in Definition 4.6, there is an injective correspondence*

$$\mathscr{C}^{LR}(\mathcal{O}) \to [M(\mathcal{O})], \qquad \sigma \to [m(\sigma)],$$

(notation (3.22)(b)) with the following properties. Recall that σ_1 is the special representation attached to $\mathcal{O}$.

(a) $m(\sigma_1) = [(1, 1)]$.

(b) *If $\pi \in \bar{A}(\mathcal{O})^{\wedge}$, and $\sigma(e, \pi) \neq 0$ (Theorem 2.18), then*

$$m(\sigma(e, \pi)) = [(1, \pi)] \qquad (cf.\ (4.4)).$$

(c) *Suppose $\mathcal{O}$ is not exceptional (Definition 4.5). Then every element of $[M(\mathcal{O})]$ of the form $[(x, 1)]$ (that is, an element of $[\bar{A}(\mathcal{O})] \subseteq [M(\mathcal{O})]$) is in the image of the correspondence; we write*

$$\sigma_x \in \hat{W} \qquad (x \in \bar{A}(\mathcal{O}))$$

for its preimage.

(d) *Suppose $\mathcal{O}$ is not exceptional. Then there is an isomorphism*

$$i\colon \bar{A}(\mathcal{O}) \to \bar{A}({}^L\mathcal{O})$$

such that the diagram

$$\begin{tikzcd} \mathcal{O}^{LR}(\mathcal{O}) \arrow[r, "m"] \arrow[d, "\otimes\,\mathrm{sgn}"] & {[M(\mathcal{O})]} \arrow[d, "i"] \\ \mathscr{C}^{LR}({}^L\mathcal{O}) \arrow[r, "m"] & {[M({}^L\mathcal{O})]} \end{tikzcd}$$

commutes.

We turn now to the Lusztig-Spaltenstein notion of induced nilpotent orbits. If $\mathcal{O}$ is any nilpotent orbit, define

$$d(\mathcal{O}) = |\Delta^+| - \tfrac{1}{2}\dim \mathcal{O}. \tag{4.8}$$

For $\sigma \in \hat{W}$, Definition 4.1 and (3.21)(a) give

$$a(\sigma) = d(\mathcal{O}_{sp}(\sigma)). \tag{4.9}$$

We have

$$\mathcal{O} \subseteq \bar{\mathcal{O}}' \Rightarrow d(\mathcal{O}) \geq d(\mathcal{O}'). \tag{4.10)(a}$$

Proposition 3.23 therefore implies

$$(4.10)(b) \qquad \sigma_1 \underset{LR}{\leq} \sigma_2 \Rightarrow a(\sigma_1) \leq a(\sigma_2).$$

Definition 4.11. Suppose $W_0 \subseteq W$ is the Weyl group of a root system contained in Δ. We define *truncated induction* J, from an irreducible representation σ_0 of W_0 to a representation of W, as follows. Define

$$A = \left\{ \sigma \in \hat{W} | \sigma \text{ occurs in } \operatorname{Ind}_{W_0}^W(\sigma_0'), \text{ for some } \sigma_0' \underset{LR}{\approx} \sigma_0 \right\},$$

$$a = \min_{\sigma \in A} a(\sigma).$$

Then

$$J_W^W(\sigma_0) = \bigoplus_{\substack{\sigma \in W \\ a(\sigma) = a}} \left[\sigma_0 \colon \sigma|_{W_0}\right] \sigma \subseteq \operatorname{Ind}_{W_0}^W(\sigma_0).$$

The most important case of this definition is when W_0 is the Weyl group of a Levi factor of a parabolic subalgebra of $\mathfrak{g}$. Let us consider that case, and use the notation of Proposition 4.14 below. Write σ_1 for the unique special representation of W in $V^L(w)$. By Proposition 4.14(a), each constituent σ of $\operatorname{Ind}_{W_0}^W(\sigma_0)$ satisfies

$$\sigma \underset{LR}{\geq} \sigma_1,$$

and so by (4.10)(b)

$$a(\sigma) > a(\sigma_j).$$

Hence $a = a(\sigma_1)$. Combining Proposition 4.14(c), Definition 4.12(b), and (4.9), we deduce

$$a = a(\sigma_1) = a(\sigma).$$

When W_0 is not a Levi subgroup, we know no simple formula for the constant a of Definition 4.11. We will use the definition in only finitely many other cases (Lemma 10.17 below), where the properties we need are simply verified. Nevertheless, we record here a few general properties in what seems to be the most general interesting case (which includes the examples in Lemma 10.17). Suppose W_0 is the integral Weyl group for some regular $\lambda \in \mathfrak{h}^*$:

$$W_0 = \{ w \in W | w\lambda - \lambda \text{ is a sum of roots} \}.$$

Fix $\sigma_0 \in \hat{W}_0$, and let $\sigma_0' \in \hat{W}_0$ be the special representation with $\sigma_0' \underset{LR}{\approx} \sigma_0$. By [J2], σ_0' may be realized uniquely on a subspace V_0' of

$$S^{a(\sigma_0')}(\mathfrak{h}).$$

Let $V' \subseteq S^{a(\sigma_0')}(\mathfrak{h})$ be the W-invariant subspace generated by V_0' and σ', the representation of W there; it is irreducible. By [L1], it is the Springer representation for a nilpotent orbit $\mathcal{O}'$, with

$$d(\mathcal{O}') = a(\sigma_0').$$

Let $\mathcal{O} = \mathcal{O}_{sp}(\sigma')$ be the smallest special nilpotent orbit containing $\mathcal{O}'$ in its closure. Then the constant of Definition 4.11 is

$$a = d(\mathcal{O}) = a(\sigma').$$

This follows from the next more precise statement. Suppose $\sigma_0'' \underset{LR}{\geq} \sigma_0$, and σ'' occurs in $\mathrm{Ind}_{W_0}^{W}(\sigma_0'')$. Then the nilpotent orbit $\mathcal{O}''$ attached to σ'' by the Springer correspondence satisfies

$$\mathcal{O}'' \subseteq \bar{\mathcal{O}}';$$

and consequently,

$$\sigma'' \geq \sigma', \qquad a(\sigma'') \geq a(\sigma').$$

This in turn follows from the theory of primitive ideals with non-integral infinitesimal character. Since we will not use these results, and the proofs would lead us even further astray, we omit details.

Definition 4.12 ([L-Spa]). Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be a parabolic subalgebra. The nilpotent orbit $\mathcal{O}$ for $\mathfrak{g}$ is said to be *induced* from $\mathcal{O}_{\mathfrak{m}} \subseteq \mathfrak{m}$ if either of the following equivalent conditions is satisfied:

a) $\overline{\mathcal{O} \cap (\mathcal{O}_{\mathfrak{m}} + \mathfrak{n})} = \bar{\mathcal{O}}_{\mathfrak{m}} + \mathfrak{n}$; or

b) $\mathcal{O}$ meets $\mathcal{O}_{\mathfrak{m}} + \mathfrak{n}$, and $d(\mathcal{O}) = d(\mathcal{O}_{\mathfrak{m}})$ (cf. (4.8)).

Given $\mathcal{O}_{\mathfrak{m}}$, Lusztig and Spaltenstein show that there is a unique $\mathcal{O}$ induced from it; we write

(4.13)(a) $$\mathcal{O} = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{m}}).$$

They also show that $\mathcal{O}$ depends only on $\mathfrak{m}$ and $\mathcal{O}_{\mathfrak{m}}$ and not on $\mathfrak{p}$; so we may also write

(4.13)(b) $$\mathcal{O} = \mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{m}}).$$

PROPOSITION 4.14 ([B-V3]). *Suppose $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$; write $W(\mathfrak{m})$ for the Weyl group of $\mathfrak{h}$ in $\mathfrak{m}$. Fix*

$$w \in W(\mathfrak{m}), \qquad \sigma_{\mathfrak{m}} \in W(\mathfrak{m})\hat{};$$

and assume that

$$\sigma_{\mathfrak{m}} \textit{ occurs in } V_{\mathfrak{m}}^{L}(w)$$

(*notation* (3.17)). *Define*

$$\mathcal{O}_{\mathfrak{m}} = \mathcal{O}_{sp}(\sigma_{\mathfrak{m}}) = \mathcal{O}_{\mathfrak{m}}(w) \qquad (\textit{Definition } 4.1).$$

Then

a) $\overline{V}^L(w) = \mathrm{Ind}_{W(\mathfrak{m})}^W(\overline{V}_{\mathfrak{m}}^L(w))$
b) $V^L(w) = J_{W(\mathfrak{m})}^W(V_{\mathfrak{m}}^L(w))$
c) $\mathcal{O}(w) = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{m}}(w))$ (*notation* (4.13))
d) $J_{W(m)}^W(\mathscr{C}^{LR}(\mathcal{O}_{\mathfrak{m}})) \subseteq \mathscr{C}^{LR}(\mathcal{O})$ (*notation* (3.22)(b)).

Definition 4.15 (Lusztig). Suppose $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a parabolic subalgebra of $\mathfrak{g}$, and $\mathcal{O}_{\mathfrak{m}}$ is a special nilpotent orbit for $\mathfrak{m}$. Set

$$\mathcal{O} = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{m}}).$$

We say that $\mathcal{O}$ is *smoothly induced* from $\mathcal{O}_{\mathfrak{m}}$ if $J_{W(\mathfrak{m})}^W$ is a bijection from $\mathscr{C}^{LR}(\mathcal{O}_{\mathfrak{m}})$ onto $\mathscr{C}^{LR}(\mathcal{O})$ (see Proposition 4.14(d)).

PROPOSITION 4.16 (Lusztig [L3]). *In the setting of Proposition* 4.14, $\mathcal{O}$ *is smoothly induced from* $\mathcal{O}_{\mathfrak{m}}$ *if and only if* $\overline{A}(\mathcal{O}) \cong \overline{A}(\mathcal{O}_{\mathfrak{m}})$. *In that case, the isomorphism may be chosen in such a way that the corresponding bijection* $[M(\mathcal{O}_{\mathfrak{m}})] \to [M(\mathcal{O})]$ (*Definition* 4.6) *induces the bijection*

$$J_{W(\mathfrak{m})}^W\colon \mathscr{C}^{LR}(\mathcal{O}_{\mathfrak{m}}) \to \mathscr{C}^{LR}(\mathcal{O})$$

via Theorem 4.7.

5. Some distinguished nilpotent orbits

PROPOSITION 5.1. *Let* $\mathcal{O}$ *be a special nilpotent orbit in* $\mathfrak{g}$, *and* ${}^L\mathcal{O}$ *the dual special orbit in* ${}^L\mathfrak{g}$ (*Corollary* 3.25). *Consider the properties*:

a) $\mathcal{O}$ *is not induced from any proper Levi subalgebra of* $\mathfrak{g}$ (*Definition* 4.12).
b) ${}^L\mathcal{O}$ *does not meet any proper Levi subalgebra of* ${}^L\mathfrak{g}$.
c) ${}^L\mathcal{O}$ *is even.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Because the proof is a simple case by case verification, we omit it. For the balance of this section, we fix a special nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, and assume that

(5.2) $\qquad {}^L\mathcal{O}$ is even.

By Theorem 2.2, we can find a homomorphism

(5.3)(a) $\qquad \phi\colon \mathfrak{sl}(2,\mathbf{C}) \to {}^L\mathfrak{g}$

such that

(5.3)(b) $\qquad \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = {}^Le \in {}^L\mathcal{O}.$

We may also assume that

$$\text{(5.3)(c)} \qquad {}^L h = \phi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in {}^L\mathfrak{h},$$

the fixed Cartan subalgebra of ${}^L\mathfrak{g}$, and ${}^L h$ is dominant. It is then uniquely determined by ${}^L\mathcal{O}$ (Theorem 2.6). As in (1.15), we define

$$\text{(5.4)} \qquad \lambda_{\mathcal{O}} = \text{element of } \mathfrak{h}^* \text{ corresponding to } \tfrac{1}{2}({}^L h);$$

this is a dominant integral weight for $\mathfrak{g}$.

Our next goal is Proposition II of the introduction. We will prove a result about conjugacy classes in ${}^L\mathfrak{g}$, and translate it into a result about infinitesimal characters in $\mathfrak{g}$ using Proposition 3.24. To simplify the notation, however, the results about conjugacy classes will be formulated in terms of $\mathfrak{g}$.

PROPOSITION 5.5 [L-Spa]. *Suppose the Dynkin diagram of the nilpotent orbit $\mathcal{O}$ has node α labelled 2. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the maximal parabolic subalgebra corresponding to this node. Then there is a nilpotent orbit $\mathcal{O}_{\mathfrak{m}}$ in $\mathfrak{m}$ such that*

$$\mathcal{O} = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{m}}.$$

COROLLARY 5.6. *Suppose $\mathcal{O}$ is an even nilpotent orbit. Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ be the parabolic subalgebra such that the simple roots in $\mathfrak{l}$ are the simple roots labelled 0 in the Dynkin diagram of $\mathcal{O}$. Then*

$$\mathcal{O} = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\{0\});$$

that is, $\mathcal{O} \cap \mathfrak{u}$ is dense in $\mathfrak{u}$.

In this situation, therefore, the semisimple element h attached to $\mathcal{O}$, and $\mathcal{O}$ itself, are both closely related to $\mathfrak{p}$. Any other nilpotent in $\mathfrak{u}$ lies in $\bar{\mathcal{O}}$, and so should have a smaller h. The next result is motivated by these facts.

LEMMA 5.7. *Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ be any parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{p} \supseteq \mathfrak{b}$ and $\mathfrak{l} \supseteq \mathfrak{h}$. Let S be the set of simple roots of $\mathfrak{h}$ in $\mathfrak{l}$. Define $\xi \in \mathfrak{h}$ by*

$$\alpha(\xi) = 0, \qquad \alpha \in S,$$

$$\alpha(\xi) = 1, \quad \alpha \text{ simple but not in } S.$$

Let $e \in \mathfrak{u}$ be arbitrary, and fix ϕ: $\mathfrak{sl}(2) \to \mathfrak{g}$ and h as in Definition 2.3. Then

$$\|\xi\| \geq \|\tfrac{1}{2}h\|,$$

with equality if and only if ξ is conjugate to $\frac{1}{2}h$. In that case e is even, and $\mathfrak{p}$ is the parabolic attached to it by Corollary 5.6.

Proof. Choose a conjugate ϕ' of ϕ so that h' is a dominant element of $\mathfrak{h}$. Define $\lambda_1 \in \mathfrak{h}^*$ by

$$\lambda_1(x) = (h', x) \qquad x \in \mathfrak{h}.$$

Since h' is integral on roots instead of coroots, λ_1 need not be an integral weight; but some multiple

$$\lambda = k\lambda_1$$

will be integral. Write (F_λ, π_λ) for the finite-dimensional representation of $\mathfrak{g}$ of highest weight λ. Define N to be the unique integer such that

$$\pi_\lambda(e)^N \neq 0, \qquad \pi_\lambda(e)^{N+1} = 0.$$

Of course this could be calculated with e' replacing e.

By the representation theory of $\mathfrak{sl}(2)$,

$$\begin{aligned} N &= \max\{\nu(h') | \nu \text{ a weight of } F_\lambda\} \\ &= \lambda(h') \\ &= k\langle h', h'\rangle = \frac{1}{k}\langle \lambda, \lambda\rangle. \end{aligned} \tag{5.8}$$

If X_β is any root vector in $\mathfrak{u}$, then β involves a simple root not in S; so

$$\langle \nu, \xi\rangle \leq \langle \nu + \beta, \xi\rangle - 1, \quad \text{for all } \nu \in \mathfrak{h}^*.$$

It follows that if we filter F_λ by the inner products of its weights with ξ, then $\mathfrak{u}$ acts by raising degrees by at least 1. The length of this filtration is

$$M = \langle \lambda - w_0\lambda, \xi\rangle.$$

It follows that if $X \in \mathfrak{u}$ then

$$\pi_\lambda(X)^{M+1} = 0.$$

Since e belongs to $\mathfrak{u}$, $M \geq N$; or (by (5.8))

$$\begin{aligned} \frac{1}{k}\langle \lambda, \lambda\rangle &\leq \langle \lambda - w_0\lambda, \xi\rangle \\ &= \langle \lambda, \xi\rangle - \langle w_0\lambda, \xi\rangle \\ &\leq 2\langle \lambda, \lambda\rangle^{1/2}\langle \xi, \xi\rangle^{1/2}. \end{aligned} \tag{5.9}$$

Consequently,

$$\frac{1}{2k}\langle \lambda, \lambda\rangle^{1/2} \leq \langle \xi, \xi\rangle^{1/2},$$

or

$$\tfrac{1}{2}\langle h, h\rangle^{1/2} \leq \langle \xi, \xi\rangle^{1/2},$$

as we wished to show. If equality holds, then the condition for equality in the last step of (5.9) makes ξ a positive multiple of λ, and therefore a positive multiple of h'. The equality of lengths then gives $\xi = \frac{1}{2}h'$. The last assertions are clear.

PROPOSITION 5.10. *Let X be an irreducible module for $\mathfrak{g}$ of dominant integral infinitesimal character $\lambda \in \mathfrak{h}^*$. Let $\mathcal{O}'$ be the nilpotent orbit attached to the primitive ideal* Ann X. *Fix any special nilpotent orbit $\mathcal{O}$ such that ${}^L\mathcal{O}$ is even, and $\mathcal{O}' \subseteq \mathcal{O}$. Then*

$$\|\lambda\| \geq \|\lambda_{\mathcal{O}}\|$$

(*notation* (5.4)). *Equality holds if and only if $\mathcal{O} = \mathcal{O}'$, and $\lambda = \lambda_{\mathcal{O}}$.*

Proof. We begin by clarifying slightly what $\mathcal{O}'$ is supposed to be. As in (3.2), choose $w \in W$ so that

$$\mathrm{Ann}(X) = I(w_0 w \lambda)$$

and so that w is minimal in the left coset wW_λ. Choose a regular dominant integral weight λ_r. Then Ann X will be related to $I(w_0 w \lambda_r)$ by a translation functor (cf. [B-J]); and $\mathcal{O}'$ means the nilpotent orbit $\mathcal{O}(w)$ defined in Theorem 3.20. By Corollary 3.25,

(5.11)(a) $$\overline{{}^L\mathcal{O}'} \supseteq {}^L\mathcal{O}$$

(5.11)(b) $$\overline{{}^L\mathcal{O}'} = WF(\overline{X}(w^{-1}\mu, \mu)).$$

Here μ is any regular dominant integral weight for ${}^L\mathfrak{g}$. (The reader may wonder why there is no w_0 explicit here, as in Corollary 3.25. The point is that $w_0\mu$ is a negative regular integral weight; so (5.11)(b) may be written as

$${}^L\overline{\mathcal{O}}' = WF(\overline{X}(\mu, (ww_0)(w_0\mu))),$$

which follows directly from Corollary 3.25(b).) Since w is minimal in the left coset wW_λ, (3.6) guarantees that for any positive α orthogonal to λ,

(5.12)(a) $$\langle \check{\alpha}, w^{-1}\mu \rangle > 0.$$

Furthermore,

(5.12)(b) $$\langle \check{\alpha}, \mu \rangle > 0$$

for such roots since μ is dominant and regular. Now let

$${}^L\mathfrak{p} = {}^L\mathfrak{m} + {}^L\mathfrak{n}$$

be the parabolic subalgebra of ${}^L\mathfrak{g}$ corresponding to the simple roots orthogonal to λ. By (5.12), the Langlands quotient

(5.13) $$\overline{X}_{{}^L\mathfrak{m}}(w^{-1}\mu, \mu) = F$$

is a finite dimensional representation of ${}^L\mathfrak{m}$. By induction by stages

$$\overline{X}(w^{-1}\mu, \mu) \subseteq \operatorname{Ind}_{{}^L\mathfrak{p}}^{{}^L\mathfrak{g}}(F). \tag{5.14}$$

Since $WF(F) = \{0\}$, (1.9) implies

$$WF(\overline{X}(w^{-1}\mu, \mu)) \subseteq \text{closure of } \operatorname{Ind}_{{}^L\mathfrak{p}}^{{}^L\mathfrak{g}}(\{0\}). \tag{5.15}$$

By (5.11), therefore, ${}^L\mathcal{O}$ meets ${}^L\mathfrak{n}$. Lemma 5.7 now guarantees that

$$\|\lambda_{\mathcal{O}}\| \leq \|\xi\| \tag{5.16}$$

with equality if and only if $\xi = \lambda_{\mathcal{O}}$. Here $\xi \in {}^L\mathfrak{h} \cong \mathfrak{h}^*$ is defined by

$$\langle \check{\alpha}, \xi \rangle = 1 \text{ if } \alpha \text{ is simple and } \langle \check{\alpha}, \lambda \rangle > 0,$$
$$\langle \check{\alpha}, \xi \rangle = 0 \text{ if } \alpha \text{ is simple and } \langle \check{\alpha}, \lambda \rangle = 0.$$

Since λ is integral, obviously

$$\|\xi\| \leq \|\lambda\|, \tag{5.17}$$

with equality if and only if $\xi = \lambda$. The proposition follows from (5.16) and (5.17).

Q.E.D.

COROLLARY 5.18 (*Proposition* II *of the introduction*). *Fix a special nilpotent orbit $\mathcal{O}$ such that ${}^L\mathcal{O}$ is even, and define $\lambda_{\mathcal{O}}$ by* (5.4). *Let $\overline{X}(\lambda, \mu)$ be an irreducible Harish-Chandra module for $\mathfrak{g}$ such that*

$$WF(\overline{X}) \subseteq \mathcal{O},$$

and λ and μ are integral. Then

$$\|(\lambda, \mu)\| \geq \|(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})\|$$

with equality if and only if λ and μ are both conjugate to $\lambda_{\mathcal{O}}$. In that case $WF(\overline{X}) = \overline{\mathcal{O}}$.

This follows from Proposition 5.10 applied to the irreducible $\mathfrak{g} \times \mathfrak{g}$ module $\overline{X}(\lambda, \mu)$.

COROLLARY 5.19. *Let X be a Harish-Chandra module for $\mathfrak{g}$ of infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ (cf.* 5.4). *Assume that*

$$\dim WF(X) < \dim \mathcal{O}.$$

Then $X = \{0\}$.

COROLLARY 5.20. *Fix a special nilpotent orbit $\mathcal{O}$ such that ${}^L\mathcal{O}$ is even, and define $\lambda_{\mathcal{O}}$ by* (5.4). *Let $w_{\mathcal{O}}$ be the longest element of the Weyl group fixing $\lambda_{\mathcal{O}}$; that is, if $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ is the parabolic defined by $\lambda_{\mathcal{O}}$, then $w_{\mathcal{O}}$ is the longest element of $W(\mathfrak{l})$. Let λ be a dominant regular integral weight for $\mathfrak{g}$, and μ such a weight for ${}^L\mathfrak{g}$. Fix an irreducible Harish-Chandra module X of infinitesimal* *character* $\lambda_{\mathcal{O}}$ *and wavefront set* $\bar{\mathcal{O}}$; *and choose* $w \in W$ *so that* X *is obtained from* $X(\lambda, w_0 w \lambda)$ *by a translation functor. Then*

a) $L \operatorname{Ann} X = R \operatorname{Ann} X = I(\lambda_{\mathcal{O}}) = I(w_{\mathcal{O}}\lambda_{\mathcal{O}})$

b) $L \operatorname{Ann} \bar{X}(\lambda, w_0 w \lambda) = R \operatorname{Ann} \bar{X}(\lambda, w_0 w \lambda) = I(w_{\mathcal{O}}\lambda)$

c) $L \operatorname{Ann} \bar{X}_{L_{\mathfrak{g}}}(\mu, w\mu) = R \operatorname{Ann} \bar{X}_{L_{\mathfrak{g}}}(\mu, w\mu) = \check{I}(-w_{\mathcal{O}}\mu)$.

Proof. For (a), $I(\lambda_{\mathcal{O}})$ is the unique maximal element of $\operatorname{Prim}_{\lambda_{\mathcal{O}}}(U(\mathfrak{g}))$; so

$$I(\lambda_{\mathcal{O}}) \supseteq L \operatorname{Ann} X. \tag{5.21}$$

On the other hand, Proposition 5.10 guarantees that X has minimal Gelfand-Kirillov dimension among Harish-Chandra modules with infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$. Therefore

$$\operatorname{Dim} U(\mathfrak{g})/L \operatorname{Ann} X = \operatorname{Dim} U(\mathfrak{g})/I(\lambda_{\mathcal{O}}). \tag{5.22}$$

One cannot have a proper containment of prime ideals in $U(\mathfrak{g})$ of the same Gelfand-Kirillov dimension ([BK], Korollar 3.5); so (5.21) and (5.22) give the first equalities of (a). The last is trivial. Part (b) follows from the translation principle of [B-J], and (c) from Proposition 3.24(a). Q.E.D.

We can now improve a little on Definition 1.17.

Definition 5.23. Let $\mathcal{O}$ be a special nilpotent orbit such that ${}^L\mathcal{O}$ is even, and X an irreducible Harish-Chandra module with infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ (notation (5.4)). We say that X is a *special unipotent representation* attached to $\mathcal{O}$ if any of the following equivalent conditions hold:

a) $WF(X) \subseteq \bar{\mathcal{O}}$.

b) $\operatorname{Ann} X$ is the maximal primitive ideal with infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$.

c) $\operatorname{Dim} X = \dim_{\mathbf{C}}(\mathcal{O})$.

There are just two easy examples for this definition. First, let $\mathcal{O} = \{0\}$, so that ${}^L\mathcal{O}$ is the principal nilpotent orbit in ${}^L\mathfrak{g}$ (Example 2.10). Since the Dynkin diagram of ${}^L\mathcal{O}$ has each node labelled 2, $\lambda_{\mathcal{O}}$ must take the value one on each simple coroot; so $\lambda_{\mathcal{O}}$ is ρ, half the sum of the positive roots. Any of the three conditions in Definition 5.23 now shows that the only special unipotent representation attached to $\mathcal{O}$ is the trivial representation.

Next, let $\mathcal{O}$ be the principal nilpotent orbit in $\mathfrak{g}$ (Example 2.10). Then ${}^L\mathcal{O} = \{0\}$, so $\lambda_{\mathcal{O}} = 0$. The only special unipotent representation attached to $\mathcal{O}$ is therefore the irreducible tempered spherical principal series representation $X(0, 0)$.

We now begin the determination of the set of special unipotent representations.

COROLLARY 5.24. *In the notation of Corollary* 5.20, *the cardinality of the set of special unipotent representations attached to* $\mathcal{O}$ *is the same as that of*

$$\{w \in W \mid \text{the left and right annihilators of } \overline{X}(\lambda, w_0 w\lambda) \text{ are both } I(w_{\mathcal{O}}\lambda)\}$$
$$= \mathscr{C}^L(w_0 w_{\mathcal{O}}) \cap \mathscr{C}^R(w_0 w_{\mathcal{O}})$$

(*cf.* (3.13)); *and this in turn has the same cardinality as*

$$\mathscr{C}^L(w_{\mathcal{O}}) \cap \mathscr{C}^R(w_{\mathcal{O}}).$$

The first assertion is Corollary 5.20(b); the second is a definition (cf. (3.12)–(3.13)); and the last is Proposition 3.24. To compute these numbers, we use a result of Lusztig.

PROPOSITION 5.25 ([L3]).

$$\operatorname{card}(\mathscr{C}^L(w_1) \cap \mathscr{C}^R(w_2)) = \dim \operatorname{Hom}_W(V^L(w_1), V^R(w_2)).$$

Because of this proposition, we need to understand $V^L(w_{\mathcal{O}})$. With notation as in Corollary 5.20, $-w_{\mathcal{O}}\mu$ is dominant, integral, and regular for ${}^L\mathfrak{l}$; so $L_{L_\mathfrak{l}}(-w_{\mathcal{O}}\mu)$ is finite dimensional, and $\check{I}_{L_\mathfrak{l}}(-w_{\mathcal{O}}\mu)$ has finite codimension in $U({}^L\mathfrak{l})$. Hence $w_{\mathcal{O}}$ is the unique maximal element of $W({}^L\mathfrak{l}) = W(\mathfrak{l})$ for $\underset{LR}{\leq}$. The left, right, and double cell and cone representations of $W(\mathfrak{l})$ corresponding to $w_{\mathcal{O}}$ all coincide:

$$V^L_{L_\mathfrak{l}}(w_{\mathcal{O}}) = \overline{V}^{LR}_{L_\mathfrak{l}}(w_{\mathcal{O}}) = \text{sign representation of } W(\mathfrak{l}).$$

Now Proposition 4.14 shows that

$$V^L(w_{\mathcal{O}}) = \bigoplus_{\substack{\sigma \in \hat{W} \\ a(\sigma) = |\Delta^+(\mathfrak{l})|}} [\text{sgn}: \sigma|_{W(\mathfrak{l})}]\sigma; \tag{5.26}$$

and by Proposition 3.24(c),

$$\begin{aligned} V^L(w_0 w_{\mathcal{O}}) &= V^L(w_{\mathcal{O}}) \otimes \text{sgn} \\ &= \bigoplus_{\substack{\sigma \in W \\ a(\sigma \otimes \text{sgn}) = |\Delta^+(\mathfrak{l})|}} [\text{trivial}: \sigma|_{W(\mathfrak{l})}]\sigma. \end{aligned} \tag{5.27}$$

(We use the fact that representations of W are self-dual.) The Weyl group representation $V^L(w_{\mathcal{O}})$ may be computed explicitly from (5.26). Lusztig has done this in [L2], and in [L3] has formulated the answer as follows.

PROPOSITION 5.28. *Suppose* ${}^L\mathcal{O}$ *is an even nilpotent orbit, and define* $w_{\mathcal{O}}$ *as in Corollary* 5.20. *Then, in the notation of Theorem* 4.7(c),

$$V^L(w_{\mathcal{O}}) = \bigoplus_{[x] \in [\bar{A}({}^L\mathcal{O})]} \sigma_x,$$

$$V^L(w_0 w_{\mathcal{O}}) = \bigoplus_{[x] \in [\bar{A}(\mathcal{O})]} \sigma_x.$$

Combined with Corollary 5.24 and Proposition 5.25, this gives immediately:

COROLLARY 5.29. *Suppose $\mathcal{O}$ is a special orbit, and ${}^L\mathcal{O}$ is even. Then the number of special unipotent representations attached to $\mathcal{O}$ (Definition 5.23) is the number of conjugacy classes in $\bar{A}(\mathcal{O})$.*

COROLLARY 5.30. *Suppose $\mathcal{O}$ is a special nilpotent orbit and ${}^L\mathcal{O}$ is even. Fix $\sigma \in \hat{W}$, and suppose*

$$\sigma \underset{LR}{\geq} \sigma(\mathcal{O})$$

(cf. (3.21)); that is, that σ occurs in $\bar{V}^{LR}(w_0 w_{\mathcal{O}})$ (cf. Corollary 5.20).

a) *The representation σ occurs in $V^L(w_0 w_{\mathcal{O}})$ if and only if $\sigma = \sigma_x$, for some $x \in \bar{A}(\mathcal{O})$.*

b) *If σ occurs in $V^L(w_0 w_{\mathcal{O}})$, then it has a unique $W_{\lambda_\mathcal{O}}$ fixed vector. (Here $W_{\lambda_\mathcal{O}} = W(\mathfrak{l})$ is the stabilizer of $\lambda_\mathcal{O}$ in W; see Corollary 5.20.)*

c) *If σ does not occur in $V^L(w_0 w_{\mathcal{O}})$, then it has no $W_{\lambda_\mathcal{O}}$ fixed vector.*

Part (a) is Proposition 5.28. Part (b) follows from (5.27) and the fact (contained in Proposition 5.28) that $V^L(w_0 w_{\mathcal{O}})$ decomposes with multiplicity one. For (c), suppose σ has such a fixed vector. Then (5.27) implies that

$$(*) \qquad a(\sigma \otimes \mathrm{sgn}) \neq |\Delta^+(\mathfrak{l})|.$$

On the other hand, $\sigma \otimes \mathrm{sgn}$ must contain the sign representation of $W_{\lambda_\mathcal{O}}$; so Proposition 4.14 shows that $\sigma \otimes \mathrm{sgn}$ occurs in $\bar{V}^L(w_\mathcal{O})$. Consequently

$$\sigma \otimes \mathrm{sgn} \underset{LR}{\geq} \sigma({}^L\mathcal{O}).$$

By Proposition 3.24,

$$\sigma \underset{LR}{\leq} \sigma(\mathcal{O}).$$

Together with the first hypothesis of the corollary, this gives

$$\sigma \underset{LR}{\approx} \sigma_\mathcal{O},$$

and so

$$\sigma \otimes \mathrm{sgn} \underset{LR}{\approx} \sigma({}^L\mathcal{O}),$$

$$a(\sigma \otimes \mathrm{sgn}) = d({}^L\mathcal{O}) = |\Delta^+(\mathfrak{l})|$$

(cf. (4.9)). This contradicts $(*)$, and proves (c).

We can now prove a preliminary version of Theorem III of the introduction.

PROPOSITION 5.31. *Fix a special nilpotent orbit $\mathcal{O}$ such that ${}^L\mathcal{O}$ is even, and define $\lambda_\mathcal{O}$ by (5.4). Write $\mathcal{G}_\mathbf{Z}$ for the Grothendieck group of all Harish-Chandra* *modules of finite length, and*

$$\mathscr{G}_{\mathbf{C}} = \mathscr{G}_{\mathbf{Z}} \underset{\mathbf{Z}}{\otimes} \mathbf{C}.$$

Define

(a) $\quad U(\mathscr{O})$ = *span of all special unipotent representations attached to* $\mathscr{O}$ $\subseteq \mathscr{G}_{\mathbf{C}}$.

Define $\bar{A}(\mathscr{O})$ *by* (4.4)(c), *and the various* $R_x \in \mathscr{G}_{\mathbf{C}}$ *by* (1.22). *Then*

(b) $$\{R_x | [x] \in [\bar{A}(\mathscr{O})]\}$$

is a basis of $U(\mathscr{O})$. *Consequently, any special unipotent representation* X *attached to* $\mathscr{O}$ *may be written*

(c) $$X = \sum_{x \in \bar{A}(\mathscr{O})} c(x) R_x$$

in $\mathscr{G}_{\mathbf{Q}}$. *Here* $c(x)$ *is a class function on* $\bar{A}(\mathscr{O})$, *and depends on* X.

Proof. Recall the subspaces $\mathscr{G}(\lambda, \mu) \subseteq \mathscr{G}_{\mathbf{C}}$ defined before (3.14). We will use the standard translation functor

(5.32) $$T\colon \mathscr{G}(\rho, \rho) \to G(\lambda_{\mathscr{O}}, \lambda_{\mathscr{O}})$$

(see for example [V2], Definition 4.5.7). The space $\mathscr{G}(\rho, \rho)$ has as a basis the set of standard representations

$$\{X(\rho, w\rho) | w \in W\}$$

(cf. (3.14)). We consider the projection onto $W(\lambda_{\mathscr{O}}) \times W(\lambda_{\mathscr{O}})$-fixed vectors (cf. (3.15)), which is

(5.33) $$M\colon \mathscr{G}(\rho, \rho) \to \mathscr{G}(\rho, \rho),$$

$$M(X(\rho, w\rho)) = \frac{1}{|W_{\lambda_{\mathscr{O}}}|^2} \sum_{x, y \in W_{\lambda_{\mathscr{O}}}} X(\rho, xwy^{-1}\rho).$$

Now T satisfies

(5.34) $$T(X(\rho, w\rho)) = X(\lambda_{\mathscr{O}}, w\lambda_{\mathscr{O}})$$

(see for example [V2], Proposition 8.2.12); so we conclude that

(5.35) $$T = TM.$$

Consider now the double cone

(5.36) $$\bar{V}^{LR}(w_0 w_{\mathscr{O}}) \subseteq \mathscr{G}(\rho, \rho)$$

(cf. (3.17)). If $\sigma \in \hat{W}$ and $w' \in W$, set

(5.37) $$\tilde{R}_\sigma(w') = \sum_{x \in W} (\operatorname{tr} \sigma(x)) X(\rho, xw'\rho).$$

Using (3.16), we find:

Observation 5.38. If σ occurs in $\overline{V}^{LR}(w_0 w_{\mathcal{O}})$ and $w' \in W$, then $\tilde{R}_\sigma(w') \in \overline{V}^{LR}(w_0 w_{\mathcal{O}})$, and such elements span $\overline{V}^{LR}(w_0 w_{\mathcal{O}})$.

On the other hand, Definition 5.23(a) and the translation principle show that

$$T\big(\overline{V}^{LR}(w_0 w_{\mathcal{O}})\big) = U(\mathcal{O}). \tag{5.39}$$

So we need to apply T to the $\tilde{R}_\sigma(w')$. We have, for σ occurring in $\overline{V}^{LR}(w_0 w_{\mathcal{O}})$,

$$T(\tilde{R}_\sigma(w')) = TM(\tilde{R}_\sigma(w')) \qquad \text{(by (5.35))}.$$

By Corollary 5.30(c), this is zero unless σ is equal to some σ_x, with $x \in \overline{A}(\mathcal{O})$. So assume that $\sigma = \sigma_x$. By Corollary 5.30(b), (5.34), and the definition of R_x,

$$T\tilde{R}_\sigma(w') = a_{w'} R_x.$$

By observation 5.38 and (5.39), the various R_x span $U(\mathcal{O})$. By Corollary 5.29, they are linearly independent. Q.E.D.

6. Outline of the proof of Theorem III

Theorem III of the introduction will be proved in the course of Sections 6 through 11. Throughout, *we will assume that the result is known for groups of strictly smaller dimension*. We use two basic reduction techniques, given in Sections 7 and 8. The cases to which neither applies seem to us to be of special interest; they are treated in Sections 9 through 11. This section contains several lemmas which are needed at various points in the argument.

LEMMA 6.1. *Let $\mathcal{O}$ be a special nilpotent orbit with ${}^L\mathcal{O}$ even. Then there is a constant $m = m(\mathcal{O})$, with the following property. Let X be any special unipotent representation attached to $\mathcal{O}$, and write*

$$X = \sum_{x \in \overline{A}(\mathcal{O})} c(x) R_x$$

in accordance with Proposition 5.31(*c*). *Then the multiplicity of X* (*defined after* (3.26)) *is*

$$m \cdot c(1).$$

Proof. By Propositions 3.29 and 3.30, it is enough to prove this with multiplicity replaced by the limit in (3.28). So fix a_0 and b_0 as in (3.28), and write ϕ_x for the numerator of the character of R_x. Recall $d(\mathcal{O})$ from (4.8). What we have to show is that

$$\lim_{t \to 0} t^{-2d(\mathcal{O})} \phi_x(ta_0, tb_0) = 0$$

for $x \neq 1$. But by [J2], for example, the Weyl group representation σ_x does not occur in $S^{d(\mathcal{O})}(\mathfrak{h})$ unless it is special; that is, unless $x = 1$. Since ϕ_x transforms according to $\sigma_x \otimes \sigma_x$, the result follows. Q.E.D.

LEMMA 6.2. *Suppose $\lambda \in \mathfrak{h}^*$ is an integral weight. Define W_λ to be the stabilizer of λ in W. Suppose $\sigma \in \hat{W}$ and $z \in W$. Define*

$$R_\sigma(z) = \frac{1}{|W_\lambda|} \sum_{w \in W} \operatorname{tr} \sigma(w) X(z\lambda, w\lambda),$$

an element of the Grothendieck group of Harish-Chandra modules.

a) *If σ does not contain the trivial representation of W_λ, then $R_\sigma(z) = 0$.*

b) *If σ contains the trivial representation of W_λ exactly once, then $R_\sigma(z) = c(z)R_\sigma(1)$.*

This is an elementary formal result about representations of finite groups, using only the fact that $X(w\lambda, w\mu) = X(\lambda, \mu)$ in the Grothendieck group. Details are left to the reader (cf. Proposition 5.31).

LEMMA 6.3. *In the setting of (1.22), let E be any finite dimensional Harish-Chandra module. Consider the functor*

$$TX = P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(X \otimes E)$$

on Harish-Chandra modules of finite length; here P denotes projection on the indicated infinitesimal character. Then

$$TR_x = m(T)R_x.$$

Proof. We may assume that, as a $\mathfrak{g} \times \mathfrak{g}$ module,

$$E = F \otimes \mathbf{C},$$

with F a holomorphic representation. (Then symmetry treats $\mathbf{C} \otimes F$, and the two cases combine to give the general result.) Then

$$\begin{aligned} TX(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) &= \sum_{\substack{\mu \text{ a weight} \\ \text{of } F \\ \lambda_{\mathcal{O}} + \mu \in W \cdot \lambda_{\mathcal{O}}}} X(\lambda_{\mathcal{O}} + \mu, w\lambda_{\mathcal{O}}) \\ &= \sum_{z \in W/W_{\lambda_{\mathcal{O}}}} (\text{multiplicity of } z\lambda_{\mathcal{O}} - \lambda_{\mathcal{O}} \text{ in } F) X(z\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}). \end{aligned}$$

Consequently,

$$TR_x = \sum_{z \in W/W_{\lambda_{\mathcal{O}}}} (\text{multiplicity of } z\lambda_{\mathcal{O}} - \lambda_{\mathcal{O}}) \Big(\sum_{w \in W} \operatorname{tr} \sigma_x(w) X(z\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) \Big).$$

Lemma 6.2 shows that each inner sum on the right is a multiple of R_x. Q.E.D.

LEMMA 6.4. *In the setting of* (1.22), *the multiplicity of* $\overline{X}(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ *in the virtual representation* R_x *is one.*

Proof. Because $X(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ is the only standard representation in which $\overline{X}(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ can occur, this is equivalent to expressing R_x in terms of standard representations, and asking for the coefficient of $X(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$. It is

$$\frac{1}{|W_{\lambda_{\mathcal{O}}}|} \sum_{w \in W_{\lambda_{\mathcal{O}}}} (\operatorname{tr} \sigma_x)(w) = \left[\text{trivial: } \sigma_x|_{W_{\lambda_{\mathcal{O}}}}\right] = 1,$$

by Corollary 5.30. Q.E.D.

LEMMA 6.5. *Let* σ *be any representation of* W *such that* $a(\sigma) > d(\mathcal{O})$. *Then, for any* $z \in W$,

$$\sum_{w \in W} \operatorname{tr} \sigma(w) X(z\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) = 0.$$

Proof. By Theorem 3.20 and (4.8), the sum is a combination of irreducible characters with wavefront sets of dimension at most $2(|\Delta^+| - a(\sigma))$. This number is assumed to be less than

$$2(|\Delta^+| - d(\mathcal{O})) = \dim \mathcal{O}.$$

The lemma now follows from Corollary 5.19. Q.E.D.

PROPOSITION 6.6. *In the setting of Definition* 4.11 *and Proposition* 5.31, *assume that* $\sigma_0 \in \hat{W}_0$, *and*

$$\sigma_x = J_{W_0}^W \sigma_0,$$

for some $x \in \overline{A}(\mathcal{O})$. *Then*

$$\sum_{w \in W_0} \operatorname{tr} \sigma_0(w) X(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) = cR_x,$$

for some real number c.

Proof. Let $\sigma_x, \sigma_1, \sigma_2, \dots, \sigma_r$ be the irreducible constituents of $\operatorname{Ind}_{W_0}^W(\sigma_0)$. By hypothesis and Definition 4.11,

$$a(\sigma_i) > d(\mathcal{O}), \qquad i = 1, \dots, r. \tag{6.7}$$

A formal calculation in the group ring of W gives an expression

$$\sum_{w \in W_0} \operatorname{tr} \sigma_0(w) w = \sum_{z \in W} \sum_{w \in W} \left[a_{x,z} \operatorname{tr} \sigma_x(w) + \sum_{i=1}^{r} a_{i,z} \operatorname{tr} \sigma_i(w) \right] z^{-1} w.$$

Consequently, in the notation of Lemma 6.2,

$$\sum_{w \in W_0} \operatorname{tr} \sigma_0(w) X(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) = \sum_z \left(a_{x,z} R_{\sigma_x}(z) + \sum_{i=1}^{r} a_{i,z} R_{\sigma_i}(z) \right).$$

By Lemma 6.5 and (6.7), the last terms on the right are zero. By Lemma 6.2, the first terms are all multiples of R_x. Q.E.D.

7. Reduction when ${}^L\mathcal{O}$ is smoothly induced

In this section we give the first main induction step (Proposition 7.20) in the proof of Theorem III of the introduction. It applies under three hypotheses. The first is:

Hypothesis 7A. There is a parabolic subalgebra ${}^L\mathfrak{p} = {}^L\mathfrak{m} + {}^L\mathfrak{n}$ in ${}^L\mathfrak{g}$ such that ${}^L\mathcal{O}$ is smoothly induced from a special nilpotent orbit ${}^L\mathcal{O}_{\mathfrak{m}} \subseteq {}^L\mathfrak{m}$ (Definition 4.15).

It may be that this implies the other two hypotheses, but we have not checked this and do not need it. The second is:

Hypothesis 7B. The orbit ${}^L\mathcal{O}_{\mathfrak{m}}$ is itself even.

Now (1.15) provides elements $\lambda_{\mathcal{O}_{\mathfrak{m}}}$ and $\lambda_{\mathcal{O}}$. Set

$$(7.1) \qquad {}^L\mathfrak{l}^1 = \text{centralizer of } \lambda_{\mathcal{O}_{\mathfrak{m}}} \text{ in } {}^L\mathfrak{m},$$

$${}^L\mathfrak{l} = \text{centralizer of } \lambda_{\mathcal{O}} \text{ in } {}^L\mathfrak{g}.$$

By Hypothesis 7A and Corollary 5.6,

$$(7.2) \qquad {}^L\mathcal{O} = \operatorname{Ind}_{L_{\mathfrak{l}^1}}^{L_{\mathfrak{g}}}(\{0\}) = \operatorname{Ind}_{L_{\mathfrak{l}}}^{L_{\mathfrak{g}}}(\{0\}).$$

This formula makes ${}^L\mathfrak{l}$ and ${}^L\mathfrak{l}^1$ very closely related—for example, they have the same dimension—but they need not coincide. Almost all of the work in this case arises when they are different. Our last hypothesis says that they are not too different.

Definition 7.3. Two Levi factors $\mathfrak{r}$ and $\mathfrak{r}'$ (generated by simple root vectors for the same positive root system) are said to be *adjacent* if they differ by transposing two type A factors. More precisely, we require that $\mathfrak{r}$ and $\mathfrak{r}'$ lie in a larger Levi factor $\mathfrak{s}$, and that

$$\mathfrak{s} = \mathfrak{sl}(p+q) \times \mathfrak{s}^0,$$

$$\mathfrak{r} = \mathfrak{sl}(p) \times \mathfrak{sl}(q) \times \mathfrak{s}^0,$$

$$r' = \mathfrak{sl}(q) \times \mathfrak{sl}(p) \times \mathfrak{s}^0.$$

We say that $\mathfrak{r}$ and $\mathfrak{r}'$ are *linked* if there is a sequence of Levi factors, beginning with $\mathfrak{r}$ and ending with $\mathfrak{r}'$, such that any two terms are adjacent.

Hypothesis 7C. The Levi factors $\mathfrak{l}$ and $\mathfrak{l}^1$ in $\mathfrak{g}$ (corresponding to ${}^L\mathfrak{l}$ and ${}^L\mathfrak{l}^1$; cf. (7.1)) are linked.

Obviously linked Levi factors are conjugate, and the converse is probably true as well.

Choose a chain $\mathfrak{l}^1, \dots, \mathfrak{l}^r = \mathfrak{l}$ of Levi factors as in the definition of linked, and define

(7.4)(a) $\lambda^p =$ sum of fundamental weights for $\mathfrak{g}$ corresponding to simple roots not in $\mathfrak{l}^p$;

(7.4)(b) $$w^p = \text{longest element of } W(\mathfrak{l}^p).$$

It follows that

(7.5) $$\lambda^r = \lambda_{\mathcal{O}}, \qquad w^r = w_{\mathcal{O}}, \qquad w^1 = w_{\mathcal{O}_m}.$$

(cf. Corollary 5.20).

By Corollary 5.30, we are interested in $V^L(w_0 w^r)$. We will study it by studying $V^L(w_0 w^1)$, then relating the various $V^L(w_0 w^p)$.

LEMMA 7.6. *For any p, in the notation of Theorem* 4.7(c),

$$V^L(w_0 w^p) = \bigoplus_{[x] \in A(\mathcal{O})} \sigma_x.$$

Proof. By the argument for (5.27),

$$V^L[w_0 w^p] = \bigoplus_{\substack{\sigma \in \hat{W} \\ a(\sigma \otimes \mathrm{sgn}) = |\Delta^+(\mathfrak{l}^p)|}} [\text{trivial: } \sigma|_{W(\mathfrak{l}^p)}]\,\sigma.$$

Since all the $\mathfrak{l}^p$ are conjugate, this shows that the left side is independent of p. To prove the lemma, we take $p = r$. By (7.5), $w^r = w_{\mathcal{O}}$; so the result follows from Proposition 5.28. Q.E.D.

Now define for each p, and $[x] \in [\bar{A}(\mathcal{O})]$,

(7.7)(a) $$R_x^p = \frac{1}{|W_{\lambda^p}|} \sum_{w \in W} \operatorname{tr}(\sigma_x(w)) X(\lambda^p, w\lambda^p),$$

in analogy with (1.22). By (7.5),

(7.7)(b) $$R_x^r = R_x.$$

Using Lemma 7.6 and the proof of Proposition 5.31, we get

PROPOSITION 7.8. *For fixed p, the various R_x^p constitute a basis of the subspace of $\mathcal{G}(\lambda^p, \lambda^p)$ (notation before* (3.14)) *spanned by representations X*

with the following properties:

a) X *has infinitesimal character* (λ^p, λ^p); *and*

b) $WF(X) = \bar{\mathcal{O}}$.

Our next goal is to study the representations of Proposition 7.8 when $p = 1$, relating them to their analogues for $\mathfrak{m}$. Let k be the number of conjugacy classes in $\bar{A}(\mathcal{O}) = \bar{A}(\mathcal{O}_{\mathfrak{m}})$. By inductive hypothesis, there are exactly k representations of $\mathfrak{m}$ having infinitesimal character $(\lambda_{\mathcal{O}_{\mathfrak{m}}}, \lambda_{\mathcal{O}_{\mathfrak{m}}})$ and wavefront set $\bar{\mathcal{O}}_{\mathfrak{m}}$ (Theorem III of the introduction). Write them as

$$(7.9)(\mathrm{a}) \qquad \left\{ \bar{X}_{\mathfrak{m}}\left(\lambda_{\mathcal{O}_{\mathfrak{m}}}, w_j \lambda_{\mathcal{O}_{\mathfrak{m}}}\right) | j = 1, 2, \ldots, k \right\},$$

with $w_j \in W(\mathfrak{m})$. We may assume that

$$(7.9)(\mathrm{b}) \qquad w_j \text{ is maximal in its } W(\mathfrak{m}) \cap W_{\lambda_{\mathcal{O}_{\mathfrak{m}}}}\text{-double coset.}$$

By (7.1) and (7.4), this is the same as

$$(7.9)(\mathrm{c}) \qquad w_j \text{ is maximal in its } W_{\lambda^1} \text{ double coset.}$$

Let μ be a dominant integral regular weight for ${}^L\mathfrak{g}$. By Proposition 3.24,

$$(7.10)(\mathrm{a}) \qquad \left\{ \bar{X}_{L_{\mathfrak{m}}}(\mu, -w_j\mu) \right\}$$

is the set of irreducible representations of ${}^L\mathfrak{m}$ having wavefront set ${}^L\bar{\mathcal{O}}_{\mathfrak{m}}$, infinitesimal character (μ, μ), and every root in $\mathfrak{l}^1$ in the τ invariant. By [S-V],

$$(7.10)(\mathrm{b}) \qquad \bar{X}_{L_{\mathfrak{g}}}(\mu, -w_j\mu) = \mathrm{Ind}_{L_{\mathfrak{p}}}^{L_{\mathfrak{g}}} \bar{X}_{L_{\mathfrak{m}}}(\mu, -w_j\mu).$$

Consequently, these representations of ${}^L\mathfrak{g}$ have infinitesimal character (μ, μ), every root in $\mathfrak{l}^1$ in the τ invariant, and wavefront set

$$\mathrm{Ind}_{L_{\mathfrak{p}}}^{L_{\mathfrak{g}}}\left({}^L\bar{\mathcal{O}}_{\mathfrak{m}}\right) = {}^L\bar{\mathcal{O}}$$

(cf. [B-V1]). By Proposition 3.24 for $\mathfrak{m}$, we deduce:

LEMMA 7.11. *In the notation* (7.9), *the representations of Proposition* 7.8 *for* $p = 1$ *are*

$$\left\{ \bar{X}_{\mathfrak{g}}\left(\lambda^1, w_j\lambda^1\right) | j = 1, \ldots, k \right\}.$$

(We have shown that these representations have the desired properties. That there are no others follows from Proposition 7.8 and the fact that k is the number of conjugacy classes in $\bar{A}(\mathcal{O})$.)

For $x \in \bar{A}(\mathcal{O}_{\mathfrak{m}}) \cong \bar{A}(\mathcal{O})$, write

$$(7.12) \qquad \sigma_x^{\mathfrak{m}} \in W(\mathfrak{m})\hat{}$$

for the corresponding Weyl group representation (Theorem 4.7(c)). By Hypothesis 7A,

$$\sigma_x \otimes \mathrm{sgn} \subseteq \mathrm{Ind}_{W(\mathfrak{m})}^{W}\left(\sigma_x^{\mathfrak{m}} \otimes \mathrm{sgn}(\mathfrak{m})\right).$$

Consequently

(7.13) $$\sigma_x \subseteq \operatorname{Ind}_{W(\mathfrak{m})}^{W} \sigma_x^{\mathfrak{m}}.$$

LEMMA 7.14. *With notation as above and in* (7.9), *fix* j *between* 1 *and* k. *Write*

$$\overline{X}_{\mathfrak{m}}(\lambda_{\mathcal{O}_{\mathfrak{m}}}, w_j\lambda_{\mathcal{O}_{\mathfrak{m}}}) = \sum_{[x]\in[\overline{A}(\mathcal{O}_{\mathfrak{m}})]} a_x^{\mathfrak{m}} R_x^{\mathfrak{m}},$$

$$\overline{X}(\lambda^1, w_j\lambda^1) = \sum_{[x]} a_x R_x^1$$

(*as is possible by inductive hypothesis Lemma* 7.11 *and Proposition* 7.8). *Then* $a_x^{\mathfrak{m}} = a_x$.

Proof. Define

(7.15)(a) $$A = \text{set of } W_{\lambda_{\mathcal{O}_{\mathfrak{m}}}} \cap W(\mathfrak{m})\text{-double cosets in } W(\mathfrak{m}),$$

$$B = \text{set of } W_{\lambda_1}\text{-double cosets in } W.$$

By (7.1) and (7.4), $A \subseteq B$; put

(7.15)(b) $$C = A - B.$$

Write

(7.16)(a) $$\overline{X}_{\mathfrak{m}}(\lambda_{\mathcal{O}_{\mathfrak{m}}}, w_j\lambda_{\mathcal{O}_{\mathfrak{m}}}) = \sum_{z\in A} a_z X_{\mathfrak{m}}(\lambda_{\mathcal{O}_{\mathfrak{m}}}, z\lambda_{\mathcal{O}_{\mathfrak{m}}}).$$

A straightforward argument using either intertwining operators or highest weight modules (see [B-V3]) shows that then

(7.16)(b) $$\overline{X}_{\mathfrak{g}}(\lambda^1, w_j\lambda^1) = \sum_{z\in A} a_z X_{\mathfrak{g}}(\lambda^1, z\lambda^1) + \sum_{w\in C} b_w X_{\mathfrak{g}}(\lambda^1, w\lambda^1).$$

Similarly, write

(7.17)(a) $$R_x^{\mathfrak{m}} = \sum_{z\in A} c_z X_{\mathfrak{m}}(\lambda_{\mathcal{O}_{\mathfrak{m}}}, z\lambda_{\mathcal{O}_{\mathfrak{m}}})$$

$$R_x^1 = \sum_{z\in A} c_z' X_{\mathfrak{g}}(\lambda^1, z\lambda^1) + \sum_{w\in C} d_w X_{\mathfrak{g}}(\lambda^1, w\lambda^1).$$

We can compute c_z, c_z', and d_w directly from the definitions. For c_z', for example, let v be the (unique up to scalars) unit vector in the space of σ_x which is fixed by W_{λ^1} (cf. proof of Lemma 7.6). Then

$$c_z' = |W_{\lambda^1} z W_{\lambda^1}| / |W_{\lambda^1}| \langle zv, v\rangle.$$

A similar formula holds for c_z; so by (7.13),

(7.17)(b) $$c_z = c_z'.$$

Combining (7.16) and (7.17), we can write

$$\overline{X}_{\mathfrak{g}}(\lambda^1, w_j\lambda^1) - \sum_{[x]} a_x^{\mathfrak{m}} R_x^1 = \sum_{w \in C} X_{\mathfrak{g}}(\lambda^1, w\lambda^1).$$

When expressed in terms of irreducible representations, the left side involves only the various $\overline{X}_{\mathfrak{g}}(\lambda^1, w_k\lambda^1)$, and the right involves none of them (Proposition 7.8 and Lemma 7.11). So both sides are zero. Q.E.D.

The next step is to relate the representations of Proposition 7.8 for different values of p. This requires a variation on standard translation principles.

LEMMA 7.18. *With notation* (7.4), *fix p with $1 \leq p < r$. Denote by F the finite dimensional holomorphic representation of $\mathfrak{g}$ of extremal weight $\lambda^{p+1} - \lambda^p$. Let E be the $\mathfrak{g} \times \mathfrak{g}$ module $F \otimes F$, regarded as a Harish-Chandra module for G. Consider the functors*

$$T(Y) = P_{(\lambda^{p+1}, \lambda^{p+1})}(Y \otimes E),$$
$$S(Z) = P_{(\lambda^p, \lambda^p)}(Z \otimes E^*);$$

here P denotes projection on the indicated generalized infinitesimal character. Then

$$TR_x^p = R_x^{p+1},$$
$$SR_x^{p+1} = R_x^p$$

(*notation* (7.7)).

We will prove this in a moment.

COROLLARY 7.19. *Let X be one of the irreducible representations of Proposition 7.8 for p. Then TX has the same properties for $p + 1$.*

Because T and S are exact, this is a formal consequence of Lemma 7.18.

PROPOSITION 7.20. *Suppose $\mathcal{O}$ is a special nilpotent orbit, ${}^L\mathcal{O}$ is even, and Hypotheses* 7A–7C *are satisfied. Assume that Theorem* III *of the introduction holds for $\mathfrak{m}$ and $\mathcal{O}_{\mathfrak{m}}$. To each character π of $\overline{A}(\mathcal{O}) = \overline{A}(\mathcal{O}_{\mathfrak{m}})$, attach a representation X_π as follows. Begin with the special unipotent representation $X_\pi^{\mathfrak{m}}$ for $\mathfrak{m}$. Let X_π^1 be the representation of $\mathfrak{g}$ corresponding to it via* (7.9) *and Lemma* 7.11. *Assume X_π^p is defined for some $p \leq r - 1$; define*

$$X_\pi^{p+1} = TX_\pi^p$$

using the functor of Lemma 7.18. *Finally, set*

$$X_\pi = X_\pi^r.$$

With this definition, Theorem III *holds for $\mathfrak{g}$ and $\mathcal{O}$.*

This is immediate from Proposition 7.8, Lemma 7.14 and Lemma 7.18.

We turn now to the proof of Lemma 7.18. For convenience, write

(7.21) $$W_j = \text{stabilizer of } \lambda^j \text{ in } W.$$

LEMMA 7.22. *Suppose μ is a weight of the representation F of Lemma 7.18. Then $\lambda^p + \mu$ belongs to $W \cdot \lambda^{p+1}$ if and only if*

$$\mu = \sigma\lambda^{p+1} - \lambda^p = \sigma(\lambda^{p+1} - \lambda^p)$$

for some $\sigma \in W_p$. Two such weights μ are equal if and only if the corresponding σ's have the same image in $W_p/W_p \cap W_{p+1}$.

Proof. Write

(7.23) $$\lambda^p + \mu = w\lambda^{p+1}.$$

Then

$$\mu = w\lambda^{p+1} - \lambda^p,$$

so that

$$\begin{aligned}\langle \mu, \mu \rangle &= \langle \lambda^p, \lambda^p \rangle + \langle \lambda^{p+1}, \lambda^{p+1} \rangle - 2\langle \lambda^p, w\lambda^{p+1} \rangle \\ &= \langle \lambda^p - \lambda^{p+1}, \lambda^p - \lambda^{p+1} \rangle + 2\langle \lambda^p, \lambda^{p+1} - w\lambda^{p+1} \rangle.\end{aligned}$$

Since λ^{p+1} is dominant, $\lambda^{p+1} - w\lambda^{p+1}$ is a sum of positive roots; so the second term is non-negative. Since μ is a weight of F, it cannot be longer than the extremal weight $\lambda^p - \lambda^{p+1}$. The conclusion is that

(7.24) $$\lambda^{p+1} - w\lambda^{p+1} \text{ is a sum of roots orthogonal to } \lambda^p.$$

Now suppose w is chosen to have minimal length in the coset wW_{p+1}. Write a reduced expression

$$w = s_{\alpha_1} \cdots s_{\alpha_t},$$

with each α_i a positive root. Then

$$s_{\alpha_i}(s_{\alpha_{i+1}} \cdots s_{\alpha_t})\lambda^{p+1} = (s_{\alpha_{i+1}} \cdots s_{\alpha_t})\lambda^{p+1} - m_i\alpha_i.$$

Here m_i is non-negative because the expression is reduced, and m_i is nonzero by the minimality of w in wW_{p+1}. By (7.24), each of the α_i is orthogonal to λ^p; that is, s_{α_i} belongs to W_p. So w belongs to W_p. The lemma is now clear from (7.23).

Q.E.D.

Proof of Lemma 7.18. By symmetry, we need consider only T. On the level of characters,

$$T(X(\lambda^p, w\lambda^p)) = \sum_{\substack{\mu, \nu \text{ weights of } F \\ \lambda^p + \mu, \lambda^p + \nu \in W \cdot \lambda^{p+1}}} X(\lambda^p + \mu, w(\lambda^p + \nu)).$$

Using Lemma 7.22, we can rewrite this as

$$T(X(\lambda^p, w\lambda^p)) = |W_p \cap W_{p+1}|^{-2} \sum_{y, z \in W_p} X(\lambda^{p+1}, ywz\lambda^{p+1}).$$

By (7.7)(a),

$$\text{(7.25)(a)} \quad TR_x^p = |W_p|^{-1}|W_p \cap W_{p+1}|^{-2} \sum_{\substack{y, z \in W_p \\ w \in W}} \operatorname{tr}(\sigma_x(w))X(\lambda^{p+1}, ywz\lambda^{p+1})$$

$$= |W_p|^{-1}|W_p \cap W_{p+1}|^{-2} \sum_{\substack{y, z \in W_p \\ w \in W}} \operatorname{tr}(\sigma_x(ywz))X(\lambda^{p+1}, w\lambda^{p+1}).$$

On the other hand, the proof of Lemma 6.2 shows that

$$\text{(7.25)(b)} \qquad TR_x^p = mR_x^p,$$

for some constant m. Because $X(\lambda^{p+1}, \lambda^{p+1})$ occurs exactly once in R_x^p, m is equal to its multiplicity on the right in (7.25)(a):

$$\text{(7.25)(c)} \qquad m = |W_p|^{-1}|W_p \cap W_{p+1}|^{-2} \sum_{\substack{y, z \in W_p \\ w \in W_{p+1}}} \operatorname{tr}(\sigma_x(ywz)).$$

We may replace ywz by zyw inside the trace. Then zy runs over W_p with multiplicity $|W_p|$; so

$$m = |W_p \cap W_{p+1}|^{-2} \sum_{\substack{y \in W_p \\ W \in W_{p+1}}} \operatorname{tr} \sigma_x(yw)$$

$$= |W_p \cap W_{p+1}|^{-2} \operatorname{tr}\left[\left(\sum_{y \in W_p} \sigma_x(y)\right)\left(\sum_{w \in W_{p+1}} \sigma_x(w)\right)\right].$$

Each of the inner sums is a multiple of the orthogonal projection Q_s on the unique W_s-invariant line L_s in σ_x. So

$$\text{(7.26)} \qquad m = \left(|W_p||W_{p+1}|/|W_p \cap W_{p+1}|^2\right)\operatorname{tr}(Q_pQ_{p+1})$$

$$= \left(|W_p||W_{p+1}|/|W_p \cap W_{p+1}|^2\right)\cos^2\theta;$$

here θ is the angle between L_p and L_{p+1}.

Now we use the hypothesis that the Levi factors $\mathfrak{l}^p$ and $\mathfrak{l}^{p+1}$ are adjacent (Definition 7.3). In that notation, all the contributions to m from $W(\mathfrak{s}^0)$ cancel, and we are left with the following calculation. Write $n = p + q$, and let σ be the representation of the symmetric group S_n attached to this partition of n. Choose fixed vectors v and w for $S_p \times S_q$ and $S_q \times S_p$, respectively. Then (if say $p \geq q$)

$$\text{(7.27)} \qquad m = \left([p!q!]^2/[(q!)^2(p-q)!^2]\right)\left(|\langle v, w\rangle|^2/\langle v, v\rangle\langle w, w\rangle\right).$$

Since σ has a very simple realization, one might hope to calculate directly that $m = 1$. In fact it seems to be easier to view the preceding calculation as reducing Lemma 7.18 to the case

$$\begin{aligned} \mathfrak{g} &= \mathfrak{gl}(p+q) \\ \lambda^p &= (1,\ldots,1,0\cdots 0) \qquad (p\ 1\text{'s}) \\ \lambda^{p+1} &= (1,\ldots,1,0\cdots 0) \qquad (q\ 1\text{'s}). \end{aligned}$$

Then F has highest weight

$$(1,\ldots,1,0\cdots 0) \qquad (p-q\ 1\text{'s}),$$

and R^p_σ is the spherical irreducible representation X with infinitesimal character (λ^p, λ^p). This X is induced from a one dimensional representation of

$$[\mathfrak{gl}(2)]^p \times [\mathfrak{gl}(1)]^{p-q},$$

which allows TX to be calculated easily. Details are left to the reader. Q.E.D.

8. Reduction when $\mathcal{O}$ is smoothly induced

In this section we give the second induction step (Proposition 8.10) in the proof of Theorem III. It requires two hypotheses:

Hypothesis 8A. There is a parabolic subalgebra ${}^L\mathfrak{p} = {}^L\mathfrak{m} + {}^L\mathfrak{n}$ in ${}^L\mathfrak{g}$, such that ${}^L\mathcal{O}$ meets ${}^L\mathfrak{m}$. Write ${}^L\mathcal{O}_\mathfrak{m}$ for an orbit of ${}^LM$ on ${}^L\mathcal{O} \cap {}^L\mathfrak{m}$.

Hypothesis 8B. $\bar{A}(\mathcal{O}) \cong \bar{A}(\mathcal{O}_\mathfrak{m})$. Henceforth we assume this isomorphism chosen as in Proposition 4.16. By Corollary A4,

$$\mathcal{O} = \operatorname{Ind}_\mathfrak{p}^\mathfrak{g}(\mathcal{O}_\mathfrak{m}). \tag{8.1}$$

By Hypothesis 8A,

$$\lambda_{\mathcal{O}_\mathfrak{m}} = \lambda_{\mathcal{O}}. \tag{8.2}$$

LEMMA 8.3. *Suppose $X_\mathfrak{m}$ is a special unipotent representation of $\mathfrak{m}$ attached to $\mathcal{O}_\mathfrak{m}$. Then every irreducible constituent of*

$$X = \operatorname{Ind}_\mathfrak{p}^\mathfrak{g} X_\mathfrak{m}$$

is a special unipotent representation of $\mathfrak{g}$ attached to $\mathcal{O}$.

Proof. Condition (a) of Definition 5.23 follows from (8.1), and the infinitesimal character condition from (8.2). Q.E.D.

LEMMA 8.4. *If* $x \in \bar{A}(\mathcal{O}) \cong \bar{A}(\mathcal{O}_{\mathfrak{m}})$ (*Hypothesis* 8B), *then*
a) $\mathrm{Ind}_{W(\mathfrak{m})}^{W} \sigma_x^{\mathfrak{m}} = \sigma_x + \sum_{\sigma' \underset{LR}{>} \sigma_x} m_{\sigma'}\sigma'$;
b) $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} R_x^{\mathfrak{m}} = R_x$.

Proof. Part (a) is a consequence of Proposition 4.16. For (b), clearly

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} R_x^{\mathfrak{m}} = \frac{1}{|W_{\lambda_{\mathcal{O}}} \cap W(\mathfrak{m})|} \sum_{w \in W(\mathfrak{m})} \mathrm{tr}\, \sigma_x^{\mathfrak{m}}(w) X(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}). \tag{8.5}$$

Consider the element of $\mathbf{C}[W]$ defined by

$$f(w) = \begin{cases} \mathrm{tr}\, \sigma_x(w), & w \in W(\mathfrak{m}) \\ 0, & w \notin W(\mathfrak{m}). \end{cases}$$

In the decomposition

$$\mathbf{C}[W] \cong \sum_{\sigma \in \hat{W}} V_\sigma \otimes V_\sigma,$$

part (a) allows us to write

$$f(w) = v_x + \sum_{\sigma' \underset{LR}{>} \sigma_x} v_{\sigma'}.$$

Here v_x belongs to $V_{\sigma_x} \otimes V_{\sigma_x}$ and $v_{\sigma'}$ to $V_{\sigma'} \otimes V_{\sigma'}$. In the notation of Lemma 6.2, (8.5) now gives (with $\lambda = \lambda_{\mathcal{O}}$)

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} R_x^{\mathfrak{m}} = \sum_{z \in W} \Big(c(z) R_{\sigma_x}(z) + \sum_{\sigma' > \sigma_x} c_{\sigma'}(z) R_{\sigma'}(z) \Big).$$

By Lemma 6.2, this amounts to

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} R_x = a_x R_x.$$

Lemma 6.4 implies that $a_x = 1$. Q.E.D.

List the special unipotent representations attached to $\mathcal{O}_{\mathfrak{m}}$ as

$$\big\{ X_{\mathfrak{m}}^{j} = \bar{X}_{\mathfrak{m}}(\lambda_{\mathcal{O}_{\mathfrak{m}}}, w_j \lambda_{\mathcal{O}_{\mathfrak{m}}}) | j = 1, \ldots, r \big\}, \tag{8.6}$$

in such a way that "$\mathfrak{a}$-parameters" increase with j:

$$|w_j \lambda_{\mathcal{O}_{\mathfrak{m}}} + \lambda_{\mathcal{O}_{\mathfrak{m}}}| > |w_{j'} \lambda_{\mathcal{O}_{\mathfrak{m}}} + \lambda_{\mathcal{O}_{\mathfrak{m}}}| \Rightarrow j > j'. \tag{8.7}$$

Define

$$X_{\mathfrak{g}}^{j} = \bar{X}_{\mathfrak{g}}(\lambda_{\mathcal{O}}, w_j \lambda_{\mathcal{O}}), \tag{8.8}$$

$$I_{\mathfrak{g}}^{j} = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} X_{\mathfrak{m}}^{j}. \tag{8.9}$$

Case by case, one verifies that the $X_{\mathfrak{g}}^j$ are all distinct (as j varies). Therefore, they exhaust the special unipotent representations attached to $\mathcal{O}$.

PROPOSITION 8.10. *Suppose $\mathcal{O}$ is a special nilpotent orbit, ${}^L\mathcal{O}$ is even, and Hypotheses* 8A *and* 8B *are satisfied. Assume that Theorem* III *of the introduction holds for* $\mathfrak{m}$ *and* $\mathcal{O}_{\mathfrak{m}}$. *For each character π of $\bar{A}(\mathcal{O}) \cong \bar{A}(\mathcal{O}_{\mathfrak{m}})$, define*

$$X_\pi^{\mathfrak{g}} = X_{\mathfrak{g}}^j$$

(*notation* (8.8)), *in such a way that*

$$X_\pi^{\mathfrak{m}} = X_{\mathfrak{m}}^j$$

(*notation* (8.6). *Then*

a) $X_\pi = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} X_\pi^{\mathfrak{m}}$

b) *Theorem* III *holds for* $\mathfrak{g}$ *and* $\mathcal{O}$.

Proof. Part (b) is a consequence of (a) and Lemma 8.4(b). To prove (a), recall that every constituent of $I_{\mathfrak{g}}^j$ (cf. (8.9)) is one of the $X_{\mathfrak{g}}^k$, by Lemma 8.3. By (8.7)

$$I_{\mathfrak{g}}^j = X_{\mathfrak{g}}^j + \sum_{k<j} b_{jk} X_{\mathfrak{g}}^k, \tag{8.11}$$

for some integers b_{jk}. What we must show is that all the b_{jk} are equal to zero. Write

$$\pi_j \in \bar{A}(\mathcal{O}_{\mathfrak{m}})^{\wedge} \tag{8.12}$$

for the representation corresponding to $X_{\mathfrak{m}}^j$. By Lemma 6.1 and Theorem III for $\mathfrak{m}$, the multiplicity of $X_{\mathfrak{m}}^j$ is

$$c\left(X_{\mathfrak{m}}^j\right) = c_{\mathcal{O}_{\mathfrak{m}}} \cdot \dim \pi_j. \tag{8.13}$$

By Proposition 3.31,

$$c\left(I_{\mathfrak{g}}^j\right) = c_{\mathcal{O}} \cdot \dim \pi_j. \tag{8.14}$$

Because the formulas (8.11) are obviously invertible, it follows that

$$c\left(X_{\mathfrak{g}}^j\right) = c_{\mathcal{O}} \cdot a_j, \tag{8.15}$$

with a_j a positive integer less than or equal to $\dim \pi_j$. Now (8.11), (8.14), and (8.15) give

$$\dim \pi_j = a_j + \sum_{k<j} b_{jk} a_k. \tag{8.16}$$

The proposition therefore comes down to proving that

(8.17) $$a_j = \dim \pi_j.$$

To prove this, we may assume $\mathfrak{g}$ is simple. If $\dim \pi_j = 1$, we are done; and if $\mathfrak{g} = \mathfrak{m}$, the proposition is trivial. So assume $\mathfrak{g} \neq \mathfrak{m}$, and $\dim \pi_j > 1$. By inspection of cases, this forces $\mathfrak{g}$ to be of type E_7 or E_8, and $\bar{A}(\mathcal{O}) \cong S_3$ (the symmetric group on three letters). This group has three representations π_1, π_2, and π_3 (the ordering forced by (8.7)); they have dimensions 1, 2 and 1, respectively. Our assumption forces $j = 2$. Suppose the proposition is false. By (8.11) and (8.16), $a_2 = 1$, and

(8.18) $$I_{\mathfrak{g}}^2 = X_{\mathfrak{g}}^2 + X_{\mathfrak{g}}^1$$

on the level of characters. Because the induction is unitary (by (8.2)) and $X_{\mathfrak{m}}^2$ admits a non-degenerate invariant Hermitian form, the induced representation $I_{\mathfrak{g}}^2$ does as well. Consequently (8.18) must actually be a direct sum on the level of representations; so

(8.19) $$\operatorname{Hom}_{\mathfrak{g}\times\mathfrak{g},K}\left(I_{\mathfrak{g}}^2, X_{\mathfrak{g}}^1\right) \neq 0.$$

Now let F_2 be the lowest K-type of $X_{\mathfrak{g}}^2$, regarded as a holomorphic representation of $\mathfrak{g}$. By inspection in each of the two cases, one sees that the highest weight of F_2 is shorter than the corresponding weight for $X_{\mathfrak{g}}^1$. So

(8.20) $$\operatorname{Hom}_K\left(F_2, X_{\mathfrak{g}}^1\right) = 0.$$

Write E_2 for the finite dimensional Harish-Chandra module $F_2 \otimes \mathbf{C}$ (for $\mathfrak{g} \times \mathfrak{g}$). Since $\dim \pi_3 = 1$,

(8.21) $$X_{\mathfrak{g}}^3 = I_{\mathfrak{g}}^3;$$

this representation is the unique special unipotent representation attached to $\mathcal{O}$ having a K-fixed vector (by (8.7)). By (8.20), therefore,

(8.22) $$\operatorname{Hom}_{\mathfrak{g}\times\mathfrak{g},K}\left(X_{\mathfrak{g}}^3 \otimes E_2, X_{\mathfrak{g}}^1\right) = 0.$$

Because $X_{\mathfrak{g}}^3$ is an induced representation (cf. (8.21)), $X_{\mathfrak{g}}^3 \otimes E_2$ has a computable filtration (coming from a $\mathfrak{p}$-invariant filtration of F_2) whose subquotients are induced representations. After projecting on the infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ to eliminate most of these, a standard calculation exhibits $I_{\mathfrak{g}}^2$ as a quotient of $X_{\mathfrak{g}}^3 \otimes E_2$. Therefore,

(8.23) $$\operatorname{Hom}_{\mathfrak{g}\times\mathfrak{g},K}\left(X_{\mathfrak{g}}^3 \otimes E_2, X_{\mathfrak{g}}^1\right) \supseteq \operatorname{Hom}_{\mathfrak{g}\times\mathfrak{g},K}\left(I_{\mathfrak{g}}^2, X_{\mathfrak{g}}^1\right).$$

Now (8.19), (8.22), and (8.23) together give a contradiction. Q.E.D.

9. Cases when neither $\mathcal{O}$ nor ${}^L\mathcal{O}$ is smoothly induced: Parametrization of unipotent representations

Throughout sections 9 through 11, we assume

Hypothesis 9A. $\mathcal{O}$ is a special nilpotent orbit; ${}^L\mathcal{O}$ is even; the hypotheses of Sections 7 and 8 both fail for any proper parabolic subalgebra; and $\mathfrak{g}$ is simple.

Such orbits are very rare indeed. (They play a distinguished role also in Lusztig's work on characters of finite Chevalley groups.) Here is a list of all of them. There are none in type A. In types B and C, they exist only if n is twice a triangular number: $n = m(m+1)$, with $m \geq 1$. For such n there is a unique $\mathcal{O}$ in B_n and in C_n; it has symbol (cf. [L3] or [L1])

(9.1)(a)
$$\begin{pmatrix} 0 & & 2 & & & & 2m \\ & 1 & & 3 \cdots 2m-1 & & & \end{pmatrix}.$$

In $\mathfrak{so}(2n+1)$, this is the nilpotent with Jordan blocks

(9.1)(b) $\quad (1+1)+(3+3)\cdots+(2m-1+2m-1)+2m+1.$

In $\mathfrak{sp}(n)$, the Jordan blocks are

(9.1)(c)
$$(2+2)+(4+4)\cdots+(2m+2m).$$

In both cases,

(9.1)(d)
$$A(\mathcal{O}) = \bar{A}(\mathcal{O}) \cong (\mathbf{Z}/2\mathbf{Z})^m.$$

These nilpotents are dual to each other in the duality of Corollary 3.25.

In type D, Hypothesis 9A is satisfied only for n a perfect square: $n = (m+1)^2$, with $m \geq 1$. The symbol (cf. [L3] or [L1]) is

(9.2)(a)
$$\begin{pmatrix} 1 & 3 & & 2m+1 \\ 0 & 2 & \cdots & 2m \end{pmatrix};$$

in $\mathfrak{so}(2n)$, $\mathcal{O}$ has Jordan blocks

(9.2)(b) $\quad (1+1)+(3+3)\cdots+[(2m+1)+(2m+1)].$

It is self-dual, and

(9.2)(c)
$$A(\mathcal{O}) = \bar{A}(\mathcal{O}) \cong (\mathbf{Z}/2\mathbf{Z})^m.$$

In G_2 the self-dual subregular nilpotent satisfies 9A. It has

(9.3)
$$A(\mathcal{O}) = \bar{A}(\mathcal{O}) \cong S_3.$$

In F_4 the unique self-dual nilpotent, with Dynkin diagram (0 2 0 0), satisfies 9A. It has

(9.4)
$$A(\mathcal{O}) = \bar{A}(\mathcal{O}) \cong S_4.$$

In E_6 the unique self-dual nilpotent, with Dynkin diagram

$$(9.5)(a) \qquad \begin{matrix} 0 & 0 & 2 & 0 & 0 \\ & & 0 & & \end{matrix}$$

satisfies 9A. It has

$$(9.5)(b) \qquad A(\mathcal{O}) = \bar{A}(\mathcal{O}) \cong S_3.$$

Hypothesis 9A is never satisfied in E_7. (The unique self-dual nilpotent, which is exceptional in the sense of Definition 4.5, is not even.)

In E_8 there are two self-dual nilpotents. One has $\bar{A}(\mathcal{O}) = \{e\}$, and may be treated by the reduction techniques of Section 7 or Section 8. The other has diagram

$$(9.6)(a) \qquad \begin{matrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ & & & & 0 & & \end{matrix}$$

and satisfies Hypothesis 9A. It has

$$(9.6)(b) \qquad A(\mathcal{O}) = \bar{A}(\mathcal{O}) \cong S_5.$$

In this section, we will construct the special unipotent representations attached to $\mathcal{O}$, and establish the parametrization stated in Theorem III. The character formulas will be proved in Sections 10 (for the exceptional groups) and 11 (for the classical groups). The main results of this section are Proposition 9.27 and Definition 9.28. We first parametrize unipotent representations by a set $P(\mathcal{O})$ of weights (Lemma 9.10 and Proposition 9.11), then identify $P(\mathcal{O})$ with $\bar{A}(\mathcal{O})\hat{}$.

LEMMA 9.7. *Suppose that neither $\mathcal{O}$ nor ${}^L\mathcal{O}$ is smoothly induced. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the parabolic subalgebra of $\mathfrak{g}$ defined by $\lambda_{({}^L\mathcal{O})}$ (that is, by the torus in an $\mathfrak{sl}(2) \subseteq \mathfrak{g}$ meeting $\mathcal{O}$). Define*

$$P(\mathcal{O}) = \{ w\lambda_{\mathcal{O}} | w \in W, \text{ and } w\lambda_{\mathcal{O}}|_{\mathfrak{m}} \text{ is dominant and regular} \}.$$

a) $\mathcal{O} = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ *(zero)*

b) $|P(\mathcal{O})|$ = *number of conjugacy classes in $\bar{A}(\mathcal{O})$.*

c) *There is a unique element $\lambda^1_{\mathcal{O}} \in P(\mathcal{O})$ such that for any $\mu \in P(\mathcal{O})$, $\mu - \lambda^1_{\mathcal{O}}$ is a sum of negative roots. For every simple root α of $\mathfrak{h}$ in $\mathfrak{m}$, we have*

$$\langle \check{\alpha}, \lambda^1_{\mathcal{O}} \rangle = 1.$$

d) *Suppose μ and μ' belong to $P(\mathcal{O})$; and $\mu' = w\mu$ for some w in the stabilizer of $\lambda^1_{\mathcal{O}}$ in W. Then $\mu' = \mu$.*

(Here P stands for "parameters"; this set will parametrize the representations.) This is a completely straightforward case-by-case verification. Let $\rho_{\mathfrak{m}}$ be half the sum of the positive roots in $\mathfrak{m}$. For $\mu \in P(\mathcal{O})$, let $F^{\mathfrak{m}}_{\mu}$ denote the finite dimensional holomorphic representation of $\mathfrak{m}$ of highest weight $\mu - \rho_{\mathfrak{m}}$; and set

(9.8) $$E_\mu = F_\mu^{\mathfrak{m}} \otimes F_{\lambda_{\mathcal{O}}}^{\mathfrak{m}_1},$$

a Harish-Chandra module for $\mathfrak{m}$ of infinitesimal character $(\mu, \lambda_{\mathcal{O}}^1)$. Put

(9.9)(a) $$I_\mu = \mathrm{Ind}_P^G(E_\mu) \qquad (\mu \in P(\mathcal{O})),$$

(9.9)(b) X_μ = unique irreducible subquotient of I_μ containing the K-type $\mu - \lambda_{\mathcal{O}}^1$.

LEMMA 9.10. *With notation as above,*

a) *All I_μ and X_μ have infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$, and wavefront set $\bar{\mathcal{O}}$.*

b) *The various X_μ are distinct, and exhaust the special unipotent representations attached to $\mathcal{O}$.*

c) *The irreducible composition factors of I_μ are X_μ and various other $X_{\mu'}$ with*

$$|\mu' - \lambda_{\mathcal{O}}^1| > |\mu - \lambda_{\mathcal{O}}^1|.$$

d) *Every X_μ may be written as an integral combination of the $I_{\mu'}$ (on the level of characters).*

e) *There is a constant $c_{\mathcal{O}}$ depending only on $\mathcal{O}$, such that the multiplicity of I_μ is*

$$c(I_\mu) = c_{\mathcal{O}} \cdot \dim F_\mu$$

f) *The multiplicity of each X_μ is an integral multiple of $c_{\mathcal{O}}$.*

Proof. Part (a) is a consequence of Lemma 9.7 (a) and the definitions. Part (b) follows from Lemma 9.7 (d) and (b). Parts (c), (d), and (e) follow exactly as in the proof of Proposition 8.10; and (f) is immediate from (d) and (e). Q.E.D.

PROPOSITION 9.11. *With notation* (9.9), *I_μ is irreducible, so that $I_\mu = X_\mu$.*

Proof. We may assume $\mathfrak{g}$ is simple. If it is classical, then all the $F_\mu^{\mathfrak{m}}$ are of dimension 1; so Lemma 9.10 implies that X_μ and I_μ both have multiplicity $c_{\mathcal{O}}$. By Lemma 9.10(c), I_μ is irreducible.

So we may assume $\mathfrak{g}$ is of exceptional type. We can list the elements of $P(\mathcal{O})$ as

(9.12)(a) $$P(\mathcal{O}) = \{\lambda_{\mathcal{O}}^1, \lambda_{\mathcal{O}}^2, \ldots, \lambda_{\mathcal{O}}^r\}$$

in such a way that

(9.12)(b) $$|\lambda_{\mathcal{O}}^i - \lambda_{\mathcal{O}}^1| < |\lambda_{\mathcal{O}}^{i+1} - \lambda_{\mathcal{O}}^1|$$

for $i = 1, \ldots, r-1$. (This depends on the explicit form of $P(\mathcal{O})$, computed in the verification of Lemma 9.7.) In addition, we have

(9.12)(c) $\qquad \lambda_{\mathcal{O}}^{i+1} - \lambda_{\mathcal{O}}^{i}$ is a sum of negative roots.

We now define some modules in category $\mathcal{O}$ for $\mathfrak{g}$: for $\mu \in P(\mathcal{O})$, set

(9.13) $$M_\mu = U(\mathfrak{g}) \bigotimes_{\mathfrak{p}} \left(F_\mu^{\mathfrak{m}} \otimes \mathbf{C}_{\rho(\mathfrak{n})}\right).$$

$$L_\mu = \text{unique irreducible quotient of } M_\mu.$$

For simplicity, we write

(9.14) $$M_i = M_{\lambda_{\mathcal{O}}^i}$$

and similarly for L.

Lemma 9.15. *Suppose $\mathfrak{g}$ is an exceptional simple Lie algebra. With notation as above,*

a) *M_μ and L_μ have infinitesimal character $\lambda_{\mathcal{O}}$; they have Gelfand-Kirillov dimension* $\dim(\mathfrak{n}) = \frac{1}{2}\dim_{\mathbf{C}}\mathcal{O}$; *and they are locally finite for $\mathfrak{m}$ (for $\mu \in P(\mathcal{O})$).*

b) *The L_μ exhaust the irreducibles in category $\mathcal{O}$ having the properties in* (a).

c) *The multiplicity of M_μ is the dimension of $F_\mu^{\mathfrak{m}}$.*

d) *On the level of characters*

$$M_i = L_i + \sum_{j>i} a_{ij} L_j$$

for some integers a_{ij}.

This is just the category $\mathcal{O}$ version of Lemma 9.10; it relies on the definition of $P(\mathcal{O})$ and (for (d)) on (9.12)(c). We leave the trivial details to the reader.

Lemma 9.16. *Suppose $i < j$, and L_i has a non-split extension with L_j. Then L_j is a composition factor of M_i.*

Proof. Because of the existence of a composition-series reversing duality in category $\mathcal{O}$, we may assume the extension E looks like

$$0 \to L_j \to E \to L_i \to 0.$$

Because of (9.12)(c), $\lambda_{\mathcal{O}}^i - \rho$ is a highest weight of E; so the universality property of algebraic induction gives a non-zero map

$$M_i \xrightarrow{\phi} E.$$

If ϕ is not surjective, its image must be L_j. Since M_i has a *unique* irreducible quotient, this is impossible; so ϕ is onto, and L_j occurs in M_i. Q.E.D.

LEMMA 9.17. *M_i is irreducible.*

Proof. We proceed by induction on i. For $i = 1$, M_1 has multiplicity 1 (by Lemma 9.15(c) and Lemma 9.7(c)); so it can have only L_1 as a composition factor. Now suppose $i > 1$, and the result is known for all M_k with $k < i$. Lemma 9.16 then tells us that:

(9.18) If $k < i$, L_k has no extensions with L_j in category $\mathcal{O}$, for any j.

Let

(9.19)(a) $V_i =$ finite dimensional holomorphic representation of $\mathfrak{g}$ of extremal weight $\lambda^i_{\mathcal{O}} - \lambda^1_{\mathcal{O}}$,

(9.19)(b) $V_i^{\mathfrak{m}} =$ representation of $\mathfrak{m}$ of highest weight $\lambda^i_{\mathcal{O}} - \lambda^1_{\mathcal{O}}$.

Because of (9.12)(b),

$$V_i|_{\mathfrak{m}} = V_i^{\mathfrak{m}} \oplus \sum_{j<i} a_{ij} V_j^{\mathfrak{m}} \oplus W_i, \tag{9.19)(c}$$

and none of the $V_k^{\mathfrak{m}}$ occurs in W_i. Consider the translation functor

$$\tilde{T}_i(X) = P_{\lambda_{\mathcal{O}}}(X \otimes V_i)$$

on representations of $\mathfrak{g}$. Because of (9.19)(c), $\tilde{T}_i(M_1)$ has a filtration whose subquotients are M_i (once) and, for $j < i$, a_{ij} occurrences of $M_j = L_j$. By (9.18),

$$\tilde{T}_i(M_1) = M_i \oplus \sum_{j<i} a_{ij} L_j$$

on the level of representations. Since $M_1 = L_1$ admits a non-degenerate contravariant form, $\tilde{T}_i(M_1)$ does; so M_i does as well. Since L_i is the unique irreducible quotient of M_i, this implies the lemma. Q.E.D.

These two lemmas together give

COROLLARY 9.20. *Let M be an object in category $\mathcal{O}$, such that*
a) M has generalized infinitesimal character $\lambda_{\mathcal{O}}$.
b) M is $\mathfrak{m}$-locally finite.
Then M is a direct sum of various L_μ, with $\mu \in P(\mathcal{O})$.

This corollary and the preceding lemma hold even if $\mathfrak{g}$ is not exceptional. In the general case, (9.12)(b) and (c) fail; but we used these only to prove the irreducibility of the M_i (which is obvious for classical $\mathfrak{g}$ by the multiplicity one argument).

We return now to the proof of Proposition 9.11, still assuming that $\mathfrak{g}$ is exceptional. By a *left translation functor* in the category of Harish-Chandra modules for $\mathfrak{g}$—that is, $(\mathfrak{g} \times \mathfrak{g}, K)$ modules—we understand any functor of the form

$$T(X) = P_{(\xi, \lambda_{\mathcal{O}})}(X \otimes (F \otimes \mathbf{C})).$$

Here F is a holomorphic representation of $\mathfrak{g}$, and $F \otimes \mathbf{C}$ is regarded as a $(\mathfrak{g} \times \mathfrak{g}, K)$ module trivial on the second factor.

LEMMA 9.21. *If I_μ is as in (6.42)(a), and T is any composite of left translation functors, then*

$$P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(T(I_\mu)) = \bigoplus_{\mu' \in P(\mathcal{O})} a_{\mu'} I_{\mu'},$$

a direct sum on the level of representations.

Proof. I_μ is the K-finite dual of a highest weight module of the form $M_i \otimes M_1$. It is therefore clear that $P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(T(I_\mu))$ is the K-finite dual of an object in category $\mathcal{O}$ of the form

$$\left[P_{\lambda_{\mathcal{O}}}(\tilde{T}(M_i))\right] \otimes M_1.$$

The first factor satisfies the hypotheses of Corollary 9.20 and is therefore a direct sum of various M_j. The lemma follows. Q.E.D.

Now let

$$T_i(X) = P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(X \otimes (V_i \otimes 1))$$

(cf. (9.19)). By (9.19)(c) and Lemma 9.21,

$$T_i(I_{\lambda^1_{\mathcal{O}}}) = I_{\lambda^i_{\mathcal{O}}} \oplus S. \tag{9.22}$$

Since $I_{\lambda^1_{\mathcal{O}}}$ is irreducible, it follows first of all that $I_{\lambda^i_{\mathcal{O}}}$ is self-dual. If it is not irreducible, we can therefore find a $j > i$ so that

$$\operatorname{Hom}(I_{\lambda^i_{\mathcal{O}}}, X_{\lambda^j_{\mathcal{O}}}) \neq 0$$

(cf. (9.9)(b)). By (9.22), it follows that

$$\operatorname{Hom}(I_{\lambda^1_{\mathcal{O}}} \otimes (V_i \otimes 1), X_{\lambda^j_{\mathcal{O}}}) \neq 0$$

and therefore that

$$\operatorname{Hom}\left(I_{\lambda^1_{\mathcal{O}}}, X_{\lambda^j_{\mathcal{O}}} \otimes \left(V_i^* \otimes 1\right)\right) \neq 0.$$

Since $I_{\lambda^1_{\mathcal{O}}}$ is irreducible, this implies that the trivial representation of K (the lowest K-type of $I_{\lambda^1_{\mathcal{O}}}$) must occur in $X_{\lambda^j_{\mathcal{O}}} \otimes (V_i^* \otimes 1)$; and thus that V_i occurs in

$X_{\lambda^i_{\mathcal{O}}}|_K$. Since V_i has extremal weight $\lambda^i_{\mathcal{O}} - \lambda^1_{\mathcal{O}}$, and the lowest K-type of $X_{\lambda^i_{\mathcal{O}}}$ is V_j, (9.12)(b) now implies that $j \leq i$. This contradiction proves Proposition 9.11.

Although the definitions in (9.8) and (9.9) are not symmetric in the two factors of $\mathfrak{g} \times \mathfrak{g}$, Proposition 9.11 shows that the induced representations do not depend on this choice. We may therefore strengthen Lemma 9.21:

COROLLARY 9.23. *Suppose $\mathcal{O}$ satisfies Hypothesis* 9A, *and X is a special unipotent representation attached to $\mathcal{O}$. Let T be any composition of translation functors. Then*

$$P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(TX)$$

is a direct sum of special unipotent representations attached to $\mathcal{O}$.

Probably this is true whenever ${}^L\mathcal{O}$ is even.

Next we reparametrize the special unipotent representations attached to $\mathcal{O}$, in accordance with Theorem III of the introduction. Recall (Lemma 9.7) that $\mathfrak{m}$ is the centralizer in $\mathfrak{g}$ of an element in $\mathfrak{h}$ of a special form: There is a homomorphism

(9.24)(a) $$\psi\colon \mathfrak{sl}(2) \to \mathfrak{g}, \qquad \psi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e \in \mathcal{O}$$

such that if we define

(9.24)(b) $$h = \psi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}$$

then

(9.24)(c) $$\mathfrak{m} = \text{centralizer of } h \text{ in } \mathfrak{g}.$$

In accordance with Definition 2.3(c), set

(9.25) $$\begin{aligned} G^{\psi} &= \text{centralizer of the image of } \psi \text{ in } G \\ &\subseteq M = \text{centralizer of } h. \end{aligned}$$

Finally, in the notation established before (9.8), we define

(9.26) $$D_{\mu} = F_{\mu}^{\mathfrak{m}} \otimes \left(F_{\lambda^1_{\mathcal{O}}}^{\mathfrak{m}}\right)^* \qquad (\mu \in P(\mathcal{O})),$$

a finite dimensional holomorphic representation of M.

PROPOSITION 9.27. *With notation as above, fix $\mu \in P(\mathcal{O})$.*

a) $\bar{A}(\mathcal{O}) \cong G^{\psi}/G_0^{\psi}Z(G)$;

b) *D_{μ} is trivial on the subgroup $G_0^{\psi}Z(G)$ of M.*

c) *The representation π_{μ} of $\bar{A}(\mathcal{O})$ on D_{μ} (defined because of* a), b), *and* (9.25)) *is irreducible.*

d) *The correspondence $\mu \to \pi_{\mu}$ is a bijection from $P(\mathcal{O})$ onto $\bar{A}(\mathcal{O})^{\wedge}$.*

This is proved by a case-by-case verification. (Part a), in light of Proposition 2.4(c), says simply that $A(\mathcal{O}) \cong \bar{A}(\mathcal{O})$. This has already been noted, in (9.1)–(9.6). What makes the calculation fairly easy is that M is always of classical type. We omit the details.

Definition 9.28. Suppose $\mathcal{O}$ satisfies Hypothesis 9A. We define a bijection from $\bar{A}(\mathcal{O})\hat{}$ to the set of special unipotent representations of G attached to $\mathcal{O}$, as follows. Fix $\pi \in A(\mathcal{O})\hat{}$. Let $\mu \in P(\mathcal{O})$ (Lemma 9.7) be such that $\pi \cong \pi_\mu$ (Proposition 9.27). Then we define

$$X_\pi = X_\mu = \operatorname{Ind}_P^G(E_\mu)$$

(cf. (9.9) and Proposition 9.11).

10. Proof of the character formulas for exceptional groups

We now have the special unipotent representations parametrized as required by Theorem III. Since they are given explicitly as induced representations, we have explicit character formulas. It remains only to verify that these agree with the ones stated in Theorem III. This verification is quite hard to do by brute force, although a computer would probably make it possible. We will therefore introduce some additional ideas, which reduce the actual calculation required substantially. They apply in general, but it is easier to establish some of them just for the exceptional groups. The classical groups will be treated in Section 11 by a different trick.

PROPOSITION 10.1. *Under Hypothesis* 9A, *suppose* $\mathfrak{g}$ *is exceptional; and let* π *be a virtual representation of* $\bar{A}(\mathcal{O})$. *Write* $U(\mathcal{O})$ *for the linear span of the special unipotent characters attached to* $\mathcal{O}$ (*Proposition* 5.31(*a*)). *There is a linear transformation* T_π *on* $U(\mathcal{O})$ *with the following properties*:

a) $$T_\pi X_\rho = \sum_{\gamma \in \bar{A}(\mathcal{O})\hat{}} [\gamma\colon \pi \otimes \rho] X_\gamma \qquad (\rho \in A(\mathcal{O})\hat{}),$$

b) $$T_\pi R_x = \operatorname{tr}\pi(x) R_x \qquad (x \in \bar{A}(\mathcal{O})).$$

Assuming this for a moment, let us complete the proof of Theorem III when G is exceptional. Fix $x \in \bar{A}(\mathcal{O})$, and define

$$P_x = \sum_{\rho \in \bar{A}(\mathcal{O})\hat{}} \operatorname{tr}\rho(x) X_\rho. \tag{10.2}$$

By Proposition 10.1(b),

$$\begin{aligned} T_\pi(P_x) &= \sum_{\rho,\gamma} \operatorname{tr}\rho(x)[\gamma\colon \pi \otimes \rho] X_\gamma \\ &= \frac{1}{|\bar{A}(\mathcal{O})|} \sum_{\rho,\gamma} \sum_{z \in \bar{A}(\mathcal{O})} (\operatorname{tr}\rho(x))(\operatorname{tr}\gamma(z))(\operatorname{tr}\pi(z))(\operatorname{tr}\rho(z)) X_\gamma \end{aligned}$$

by the orthogonality relations for characters of $\bar{A}(\mathcal{O})$. Now

$$\sum_\rho \operatorname{tr}\rho(z)\operatorname{tr}\rho(x) = \begin{cases} 0, & x \text{ not conjugate to } z \\ |\bar{A}(\mathcal{O})|/|[x]|, & x \text{ conjugate to } z. \end{cases}$$

So

$$\begin{aligned} T_\pi(P_x) &= \sum_\gamma \operatorname{tr}\gamma(x)\operatorname{tr}\eta(x) X_\gamma \\ &= \operatorname{tr}\pi(x) P_x. \end{aligned}$$

By Proposition 10.1(b), the R_x are the unique simultaneous eigenvectors for all T_π. Hence

$$P_x = c_x R_x.$$

By (10.2), P_x contains X_1 with multiplicity one; and by Lemma 6.4, R_x does as well. So $P_x = R_x$, proving Theorem III(b). Theorem III(a) follows, by the orthogonality relations for characters. This proves Theorem III in the exceptional case.

We turn now to the proof of Proposition 10.1.

LEMMA 10.3. *In the setting of Proposition* 10.1, *fix virtual representations π_1 and π_2 of $\bar{A}(\mathcal{O})$. Suppose that T_{π_1} and T_{π_2} satisfy* (a) *and* (b) *of Proposition* 10.1. *Then the linear transformations $T_{\pi_1+\pi_2}$ and $T_{\pi_1\otimes\pi_2}$ defined by*

$$\begin{aligned} T_{\pi_1+\pi_2} &= T_{\pi_1} + T_{\pi_2}, \\ T_{\pi_1\otimes\pi_2} &= T_{\pi_1}T_{\pi_2} \end{aligned}$$

satisfy Proposition 10.1(a) *and* (b) *as well.*

This is trivial. Consequently, it is enough to produce the T_π for a set of π generating the ring of virtual representations of $\bar{A}(\mathcal{O})$. Recall from (9.3)–(9.6) that $\bar{A}(\mathcal{O})$ is isomorphic to S_3, S_4, or S_5. Define

$$\begin{aligned} &(10.4) && \pi_r = \text{reflection representation of } S_r, \\ & && \varepsilon = \text{sign representation of } S_r. \end{aligned}$$

(Here π_r has dimension $r-1$.) Then π_r (and 1) generate the ring of virtual characters of S_3; and π_r and ε generate this ring for S_4 and S_5. This establishes

LEMMA 10.5. *It is enough to prove Proposition* 10.1 *for the particular virtual representations* $\pi_r + b \cdot 1$ *and* (*if* G *is not of type* E_6) ε (*cf.* (10.4)). (*Here* b *may be any fixed integer.*)

We begin with ε. Put

(10.6)(a)

$\mathcal{G}(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ = complexified Grothendieck group of Harish-Chandra modules with infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}}) \supseteq U(\mathcal{O})$

(cf. Proposition 10.1). This space has as a basis the set

(10.6)(b) $$\{X(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) | w \in W_{\lambda_{\mathcal{O}}} \backslash W / W_{\lambda_{\mathcal{O}}}\}.$$

Definition 10.7. Suppose the longest element w_0 of W is -1. Define a linear transformation T_ε on $\mathcal{G}(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ by

$$\begin{aligned} T_\varepsilon X(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) &= a \cdot X(\lambda_{\mathcal{O}}, -w\lambda_{\mathcal{O}}) \\ &= a \cdot X(\lambda_{\mathcal{O}}, ww_0\lambda_{\mathcal{O}}) \\ &= a \cdot X(-\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}). \end{aligned}$$

Here

$$a = (-1)^{|\Delta^+(\mathfrak{m})|}.$$

LEMMA 10.8. *In the setting of Proposition* 10.1, *suppose* $w_0 = -1$. *Let* $w_0(\mathfrak{m})$ *denote the longest element of* $W(\mathfrak{m})$, *and define* a *as in Definition* 10.7.

a) $$T_\varepsilon X_\mu = X_{-w_0(\mathfrak{m})\mu} \qquad (\mu \in P(\mathcal{O})),$$

b) $$T_\varepsilon R_x = a \cdot \sigma_x(w_0) R_x \qquad (x \in \bar{A}(\mathcal{O})).$$

Proof. Part (a) follows from Proposition 9.11 and the formula for induced characters. Part (b) is immediate from the definitions. Q.E.D.

LEMMA 10.9. *In the setting of Lemma* 10.8,

a) $$\pi_{-w_0(\mathfrak{m})\mu} = \pi_\mu \otimes \varepsilon \qquad (\mu \in P(\mathcal{O});\ \text{cf. Proposition 9.27(d)}),$$

b) $$\sigma_x(w_0) = a \cdot \varepsilon(x) \qquad (x \in \bar{A}(\mathcal{O})).$$

Both of these facts are simply verified case by case. In conjunction with Lemma 10.8, this proves Proposition 10.1 when $\pi = \varepsilon$.

We turn now to the reflection representation of $\bar{A}(\mathcal{O})$. Define

(10.10)(a) $$b = \begin{cases} 1 \text{ if there is only one root length in } \mathfrak{g} \\ 0 \text{ if there are two root lengths} \end{cases}$$

(10.10)(b) $$\tilde{\pi}_r = \pi_r + b \cdot 1,$$

a virtual representation of $\bar{A}(\mathcal{O})$ (cf. (9.32)). Set

(10.11)(a) $$F = \text{irreducible holomorphic representation of } \mathfrak{g} \text{ whose highest weight is a short root,}$$

$$E = (\mathfrak{g} \times \mathfrak{g}, K) \text{ module } F \otimes \mathbf{C}.$$

Define a translation functor on $\mathcal{G}(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ by

(10.11)(b) $$T_{\tilde{\pi}_r} X = P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(X \otimes E).$$

Finally, fix $\gamma \in \bar{A}(\mathcal{O})^\wedge$, and let $\mu \in P(\mathcal{O})$ be the corresponding element (Proposition 9.27(d)). Set

(10.12) $$D_\gamma = D_\mu$$

(notation (9.26)), a finite dimensional holomorphic representation of $\mathfrak{m}$.

Lemma 10.13. *With notation as above,*

(a) $$F|_{\mathfrak{m}} = D_{\pi_r} \oplus b \cdot D_1 \oplus Y,$$

and none of the D_ρ occur in Y. More generally,

(b) $$F|_{\mathfrak{m}} \otimes D_\gamma = \left(\sum_{\rho \in \bar{A}(\mathcal{O})^\wedge} [\rho : \tilde{\pi}_r \otimes \gamma] D_\rho \right) \oplus Y_\gamma,$$

and none of the D_ρ occur in Y_γ.

This is verified case by case. One has to compute only the restriction of the small representation F to the large Levi factor $\mathfrak{m}$; so it is quite easy even in E_8.

Corollary 10.14. *With notation* (10.10) *and* (10.11),

$$T_{\tilde{\pi}_r} X_\gamma = \sum_{\rho \in \bar{A}(\mathcal{O})^\wedge} [\rho : \tilde{\pi}_r \otimes \gamma] X_\rho \qquad (\gamma \in \bar{A}(\mathcal{O})^\wedge).$$

Because of Proposition 9.11, X_γ is induced from $\mathfrak{m}$; so one applies the standard computation of translation functors on induced representations, and Lemma 10.13(b).

It remains to show that $T_{\tilde{\pi}_r}$ satisfies (b) of Proposition 10.1. This is quite difficult. We will obtain R_x from an R_1 for a smaller group S, and show that the necessary calculation may be made on S. This reduces to the case $x = 1$, but Hypothesis 9A is lost in the process; so we need to treat quite a few cases separately.

We begin by describing the smaller group S.

Definition 10.15. Suppose Hypothesis 9A holds, and $\mathfrak{g}$ is exceptional. Fix $x \in \bar{A}(\mathcal{O})$. We attach to x a new group S, as follows. We require S to be a complex semisimple group sharing the Cartan subgroup H with G, and having root system

(a) $$\Delta(\mathfrak{s}, \mathfrak{h}) \subseteq \Delta(\mathfrak{g}, \mathfrak{h}),$$

defined below. (Such a group exists for any subsystem of $\Delta(\mathfrak{g}, \mathfrak{h})$; since $\Delta(\mathfrak{s}, \mathfrak{h})$ need not be closed under addition in $\Delta(\mathfrak{g}, \mathfrak{h})$, S may not be a subgroup of G.) Consider the extended Dynkin diagram Γ obtained from the Dynkin diagram of G by adjoining a vertex for the lowest short root β. Write

(b) $$-\check{\beta} = \sum_{\alpha \text{ simple}} m_\alpha \check{\alpha};$$

label the vertex α in Γ by m_α, and β by 1. Attach to x a vertex γ_x of Γ, subject to

(c) $$\text{label of } \gamma_x = \text{order of x in } \bar{A}(\mathcal{O}).$$

(This does not always specify γ_x; we eliminate the ambiguity below.) Finally, set

(d) $$\Delta(\mathfrak{s}, \mathfrak{h}) = \text{subsystem of } \Delta(\mathfrak{g}, \mathfrak{h}) \text{ with simple roots } \tilde{\Gamma} - \{\gamma_x\}.$$

To eliminate the ambiguity in (c), we will simply tabulate the type of $\tilde{\Gamma} - \{\gamma_x\}$ in a few cases. The number we need to list is reduced by

(e) If x and y are not conjugate in $\bar{A}(\mathcal{O})$, then the corresponding root systems are non-isomorphic.

Here are the ambiguous cases. Elements of $\bar{A}(\mathcal{O})$ (which is always a symmetric group) are given by their cycle decomposition.

TABLE 10.16

G	x	$\Delta(\mathfrak{s}, \mathfrak{h})$
F_4	(12)(34)	C_4
E_8	(12)(34)(5)	D_8
E_8	(123)(4)(5)	A_8
E_8	(1234)(5)	$D_5 \times A_3$

It is possible to give a much simpler definition of S: let $\tilde{x}$ be the element of $\bar{A}({}^L\mathcal{O})$ corresponding to x (Theorem 4.7(d)). Choose a semisimple representative s of $\tilde{x}$ in ${}^LG^0$. Let ${}^LS^0$ be the identity component of the centralizer of s in ${}^LG^0$, and S its dual group. (In fact this depends on the choice of s, but the choice may be made to give the same answer as Definition 10.15.) Since all of our arguments require an explicit knowledge of S, and few of them use the fact that it arises this way, the simpler definition leads to substantially more work in the proof.

LEMMA 10.17. *In the setting of Definition* 10.15, *the inclusion* $\Delta({}^L\mathfrak{s}, {}^L\mathfrak{h}) \subseteq \Delta({}^L\mathfrak{g}, {}^L\mathfrak{h})$ *defines an inclusion* ${}^LS^0 \subseteq {}^LG^0$. *The orbit* ${}^L\mathcal{O}$ *in* ${}^L\mathfrak{g}^0$ *meets* ${}^L\mathfrak{s}^0$; *write*

(a) ${}^L\mathcal{O}_S = {}^L\mathcal{O} \cap {}^L\mathfrak{s}^0$,

(b) $\mathcal{O}_S$ = *dual orbit in* S,

(c) σ_S = *special representation of* $W(S)$ *attached to* $\mathcal{O}_S$.

Then

(d) $J^W_{W(S)}(\sigma_S) = \sigma_x$ (*Definition* 4.11).

Outline of Proof. The first assertion is fairly clear from Definition 10.15. The second (which would be trivial if we had used the alternate definition of S) can be verified from Dynkin's tables in [Dy]. (Notice that, since ${}^L\mathcal{O}$ is even, ${}^L\mathcal{O}_S$ is as well.) Part (d) may be verified from Alvis's tables in [Al]. Q.E.D.

This lemma must be proved case by case, because σ_x is defined (by Lusztig) case by case. However, it suggests the possibility of *defining* σ_x by Lemma 10.17(d).

COROLLARY 10.18. R_x *is equal to*

$$\frac{1}{|W_{\lambda_{\mathcal{O}}} \cap W(S)|} \sum_{w \in W(s)} \mathrm{tr}(\sigma_S(w)) X_G(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}).$$

This uses Proposition 6.6.

Definition. 10.19. *Use the notation of Lemma* 10.17. *Since S and G share the Cartan subgroup H, and* $W(S) \subseteq W$, *there is a virtual holomorphic representation* $\tilde{F}_S$ *of* S, *whose H-character is equal to that of F* (*cf.* (10.11)(*a*)). *Set*

$$F_S = \text{sum of all constituents of } \tilde{F}_S \text{ on which } Z(S) \text{ acts trivially},$$

$$E_S = \text{virtual } (\mathfrak{s} \times \mathfrak{s}, K_S) \text{ module } F_S \otimes \mathbf{C},$$

$$T_S X = P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(X \otimes E_S),$$

a translation functor on virtual $(\mathfrak{s} \times \mathfrak{s}, K_S)$ *modules.*

The representation F_S is a sum of one copy of the representation whose highest weight is a short root, for each simple factor of S having such a root, plus or minus some copies of the trivial representation.

LEMMA 10.20. *In the setting of Definition* 10.19, *suppose*

$$T_S R_{\sigma_S} = c R_{\sigma_S} \qquad (\textit{notation } (1.22)).$$

Then

a) $T_{\tilde{\pi}_r} R_x = cR_x$ (*cf.* (10.11)(b)),

b) T_S *multiplies the multiplicity of any special unipotent representation of* S *attached to* $\mathcal{O}_S$, *by the constant* c.

Part (b) of this lemma is just a special case of Lemma 6.1. We will return to the proof of (a) in a moment. Assuming it, let us complete the proof of Proposition 10.1(b) for $\pi = \tilde{\pi}_r$. On the one hand, by (10.10)(b),

$$\text{(10.21)} \quad \operatorname{tr} \tilde{\pi}_r(x) = (\text{number of fixed points of the permutation } x) + (b - 1).$$

This number is therefore easy to compute in each case. We must compute the constant c of Lemma 10.20, and check that it is equal to $\operatorname{tr} \tilde{\pi}_r(x)$. We compute c using Lemma 10.20(b). For every case except one in E_8, $\mathcal{O}_S$ is a Richardson orbit:

$$\mathcal{O}_S = \operatorname{Ind}_{\mathfrak{m}_S}^{\mathfrak{s}}(\{0\}).$$

In analogy with Lemma 9.7, define

$$P(\mathcal{O}_S) = \{ w\lambda_{\mathcal{O}} | w \in W(S), \text{ and } w\lambda_{\mathcal{O}}|_{\mathfrak{m}_S} \text{ is dominant and regular} \}.$$

Just as in Lemma 9.7(c), there is a distinguished element μ_0 such that

$$\langle \check{\alpha}, \mu_0 \rangle = 1, \text{ all simple } \alpha \in \Delta(\mathfrak{m}_S, \mathfrak{h}).$$

Define $F_\mu^{\mathfrak{m}_S}$, $E_\mu^{\mathfrak{m}_S}$, I_μ^S, and $D_\mu^{\mathfrak{m}_S}$ in analogy with (9.8), (9.9) and (9.26). Then I_μ^S is a combination of special unipotent representations attached to $\mathcal{O}_S$. The standard calculation of translation functors on induced representations gives

$$T_S I_{\mu_0} = \sum_{\mu \in P(\mathcal{O}_S)} \left[\dim \operatorname{Hom}_{\mathfrak{m}_S}\left(D_\mu^{\mathfrak{m}_S}, F_S\right) \right] \cdot I_\mu^S.$$

Consequently, the constant c of Lemma 9.48(b) is

$$\text{(10.22)} \qquad c = \sum_{\mu \in P(\mathcal{O}_S)} \left[\dim \operatorname{Hom}_{\mathfrak{m}_S}\left(D_\mu^{\mathfrak{m}_S}, F_S\right) \right] \dim\left(D_\mu^{\mathfrak{m}_S}\right).$$

Computing it involves only restricting the small representation F_S to the large Levi factor $\mathfrak{m}_S$, and so is very easy. In each case, the answer agrees with (10.21).

There remains the case when $\mathfrak{g}$ is of type E_8 and x has order 4. Then

$$\text{(10.23)} \qquad \begin{aligned} F &= \text{adjoint representation}, \\ \mathfrak{s} &= \mathfrak{so}(10) \times \mathfrak{so}(6), \\ \mathcal{O}_S &= (\text{minimal orbit}) \times (\{0\}), \\ \operatorname{tr} \tilde{\pi}_r(x) &= 1, \\ F_S &= \operatorname{ad}(\mathfrak{so}(10)) \oplus \operatorname{ad}(\mathfrak{so}(6)). \end{aligned}$$

There is a unique special unipotent representation X_S of $\mathfrak{s}$ attached to $\mathcal{O}_S$. It is trivial on $\mathfrak{so}(6)$, and

$$X_S = (U(\mathfrak{so}(10))/(\text{Joseph ideal}). \tag{10.24}$$

For any $\mathfrak{g}, U(\mathfrak{g})$ modulo the Joseph ideal contains the adjoint representation exactly once. Consequently,

$$T_S(X_S) = P_{(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})}(X_S \otimes (F_S \otimes \mathbf{C}))$$

can contain X_S at most once. But the action of $\mathfrak{so}(10)$ defines a non-zero map

$$X_S \otimes (F_S \otimes \mathbf{C}) \to X_S;$$

so

$$T_S X_S = X_S. \tag{10.25}$$

The constant c of Lemma 10.18 is therefore 1 in this case. Together with the next to last assertion of 10.23, this proves Proposition 10.1 for $\tilde{\pi}_r$. By Lemma 10.5, the general case follows.

It remains to prove (a) of Lemma 10.20.

LEMMA 10.26. *In the setting of Definition* 10.19, *suppose* $\phi \in W \cdot \lambda_{\mathcal{O}}$, *but* $\phi \notin W(S) \cdot \lambda_{\mathcal{O}}$. *Then*

$$\sum_{w \in W(S)} \operatorname{tr}(\sigma_S(w)) X_S(\phi, w\lambda_{\mathcal{O}}) = 0.$$

Proof. Obviously the sum is a combination of irreducible characters with wavefront set in $\bar{\mathcal{O}}_S$ (Theorem 3.20). Because ϕ belongs to $W \cdot \lambda_{\mathcal{O}}$, the length of the infinitesimal character $(\phi, \lambda_{\mathcal{O}})$ is equal to the length of $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$. However, the infinitesimal character is not equal to $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$, since $\phi \notin W(S) \cdot \lambda_{\mathcal{O}}$. The lemma therefore follows from Corollary 5.18. Q.E.D.

Proof of Lemma 10.20(a). We use the fact that

$$X_S(\lambda, \mu) = X_S(\lambda', \mu') \Rightarrow X_G(\lambda, \mu) = X_G(\lambda', \mu'), \tag{10.27}$$

both equalities understood on the level of characters. (This follows from Proposition 1.8.) Now (with a some normalizing constant)

$$\begin{aligned} T_S R_{\sigma_S} &= T_S\Big(a \sum_{w \in W(S)} \operatorname{tr}(\sigma_S(w)) X_S(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}})\Big) \\ &= a \sum_{w \in W(S)} \sum_{\substack{\nu \text{ weight of } F \\ \nu + \lambda_{\mathcal{O}} \in W(S)\cdot\lambda_{\mathcal{O}}}} \operatorname{tr}(\sigma_S(w)) X_S(\nu + \lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}) \end{aligned} \tag{10.28}$$

by Definition 10.19. By Lemma 10.26, this is

$$= a \sum_{w \in W(S)} \sum_{\substack{\nu \text{ weight of } F_S \\ \nu + \lambda_{\mathcal{O}} \in W\cdot\lambda_{\mathcal{O}}}} \operatorname{tr} \sigma_S(w)) X_S(\nu + \lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}).$$

If ν is a weight of F, and $\nu + \lambda_{\mathcal{O}}$ belongs to $W(S) \cdot \lambda_{\mathcal{O}}$, then ν is a sum of roots in $\Delta(\mathfrak{s}, \mathfrak{h})$; so ν is a weight of F_S. We may therefore write

$$T_S T_{\sigma_S} = a \sum_{w \in W(S)} \sum_{\substack{\nu \text{ weight of } F \\ \nu + \lambda_{\mathcal{O}} \in W \cdot \lambda_{\mathcal{O}}}} (\operatorname{tr} \sigma_S(w)) X_S(\nu + \lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}).$$

On the other hand, the hypothesis is that this is cR_{σ_S}:

$$ca \sum_{w \in W(S)} (\operatorname{tr} \sigma_S(w)) X_S(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}})$$

$$= a \sum_{w \in W(S)} \sum_{\substack{\nu \text{ weight of } F \\ \nu + \lambda_{\mathcal{O}} \in w \cdot \lambda_{\mathcal{O}}}} \operatorname{tr} \sigma_S(w) X_S(\nu + \lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}).$$

By (10.27), we may replace the subscript S by G everywhere. By a calculation like (10.28) for G, this amounts to

$$T_{\tilde{\pi}_r} R_x = cR_x;$$

here we use the formula in Corollary 10.18 for R_x. Q.E.D.

11. Proof of the character formulas for classical groups

In this section, we prove Theorem III for the classical cases of Hypothesis 9A. We emphasize again that Proposition 10.1 is still true; but we will use another approach which requires less calculation. So assume we are in one of the cases (9.1) or (9.2), so that $\bar{A}(\mathcal{O})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^m$. Define $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ as in Lemma 9.7. Put

(11.1)(a) β = highest short root,

(11.1)(b) $\mathfrak{l}$ = Levi factor generated by roots orthogonal to β

(11.1)(c) $= \mathfrak{l} + \mathfrak{u}$ corresponding maximal parabolic subalgebra,

(11.1)(d) $\Delta_0 = \Delta(\mathfrak{l}) \cup \{\pm\beta\}$,

(11.1)(e) $W_0 = W(\Delta_0) = W(\mathfrak{l}) \times \mathbf{Z}/2\mathbf{Z}$.

LEMMA 11.2. *The parabolic subalgebra* $\mathfrak{p}$ *defined by Lemma* 9.7 *is contained in the maximal parabolic* $\mathfrak{q}$ *of* (11.1)(c). *Set*

(a) $$\mathcal{O}_{\mathfrak{l}} = \operatorname{ind}_{\mathfrak{m}}^{\mathfrak{l}}(\{0\}).$$

Then ${}^L\mathcal{O}_{\mathfrak{l}}$ *is an even nilpotent in* ${}^L\mathfrak{l}$; *define*

(b) $$\lambda_{\mathfrak{l}} = \lambda_{\mathcal{O}_{\mathfrak{l}}}$$

as in (1.15). *Then* $\lambda_{\mathcal{O}}$ *is* (*up to conjugacy in* W)

(c) $$\lambda_{\mathcal{O}} = \tfrac{1}{2}\beta + \lambda_{\mathfrak{l}}.$$

(d) $$\bar{A}(\mathcal{O}_{\mathfrak{l}}) \cong (\mathbf{Z}/2\mathbf{Z})^{m-1}$$

(*with* m *as in* (9.1) *and* (9.2)).

This is verified case by case.

By inductive hypothesis, Theorem III is available for the unipotent representations of $\mathfrak{l}$ attached to $\mathcal{O}_{\mathfrak{l}}$. Case by case, this can be made explicit.

Write $\mathbf{C}_{1/2\beta}$ for the holomorphic representation of $\mathfrak{l}$ of weight $\frac{1}{2}\beta$ (cf. (11.1)(a)). Define two one-dimensional $(\mathfrak{l}, L \cap K)$ modules by

(11.3)
$$D_1 = \mathbf{C}_{1/2\beta} \otimes \mathbf{C}_{1/2\beta},$$
$$D_{-1} = \mathbf{C}_{1/2\beta} \otimes \mathbf{C}^*_{1/2\beta}.$$

(Notice that D_1 is trivial on $L \cap K$, and D_{-1} is trivial on $\mathfrak{l} \cap \mathfrak{p}$.)

Definition 11.4. For $\pi \in A(\mathcal{O}_{\mathfrak{l}})\hat{}$, let X^L_π be the corresponding unipotent representation of L attached to $\mathcal{O}_{\mathfrak{l}}$ (Theorem III, which is available for L by induction). For $\varepsilon = \pm 1$, set

$$X^L_{\pi,\varepsilon} = X^L_\pi \otimes D_\varepsilon$$

(cf. (11.3)). Finally, define

$$X_{\pi,\varepsilon} = \operatorname{Ind}_Q^G X^L_{\pi,\varepsilon}.$$

PROPOSITION 11.5. *In the setting of Definition* 11.4, $X_{\pi,\varepsilon}$ *is an irreducible unipotent representation of G attached to* $\mathcal{O}$. *This defines bijections*

$$\bar{A}(\mathcal{O}_{\mathfrak{l}})\hat{} \times \{\pm 1\} \leftrightarrow \{\text{unipotent representations attached to } \mathcal{O}\}$$
$$\leftrightarrow \bar{A}(\mathcal{O})\hat{},$$

the latter correspondence being that of Definition 9.28.

We will prove this in a moment. The idea is to use it to lift the known character formulas for X^L_π to formulas for $X_{\pi,\varepsilon}$. To do this, we must lift the other side of the formula as well.

Definition 11.6. For $x \in \bar{A}(\mathcal{O}_{\mathfrak{l}})$, let R^L_x be the virtual character of (1.22). For $\varepsilon = \pm 1$, define

$$R^L_{x,\varepsilon} = R^L_x \otimes D_\varepsilon$$

(cf. (11.3)). Regard W_0 as $W(\mathfrak{l}) \times \{\pm 1\}$ (cf. (11.1)). If τ is a character of $\{\pm 1\}$, define

$$R^L_{x,\tau} = \frac{1}{|W_{\lambda^1_{\mathcal{O}}} \cap W_0|} \sum_{w \in W_0} \operatorname{tr}\left(\sigma^L_x \otimes \tau\right)(w) X_L\left(\lambda^1_{\mathcal{O}}, w\lambda^1_{\mathcal{O}}\right)$$
$$= R^L_{x,1} + \tau(-1) R^L_{x,-1}.$$

Finally, define

$$R_{x,\tau} = \operatorname{Ind}_Q^G R^L_{x,\tau}$$
$$= \frac{1}{W_{\lambda^1_{\mathcal{O}}} \cap W_0} \sum_{w \in W_0} \operatorname{tr}\left(\sigma^L_x \otimes \tau\right)(w) X\left(\lambda^1_{\mathcal{O}}, w\lambda^1_{\mathcal{O}}\right).$$

PROPOSITION 11.7. *In the notation of Definitions* 11.4 *and* 11.6,

$$X_{\pi,\varepsilon} = \frac{1}{2^m} \sum_{(x,\tau)\in \bar{A}(\mathcal{O}_{\mathfrak{l}})\times\{\pm 1\}^{\wedge}} \pi(x)\tau(\varepsilon)R_{x,\tau}.$$

This will turn out to be the character formula in Theorem III.

Proof. By Definition 11.6, and Fourier inversion in the group $\{\pm 1\}$,

$$R^L_{x,\varepsilon} = \frac{1}{2} \sum_{\tau\in\{\pm 1\}^{\wedge}} \tau(\varepsilon)R^L_{x,\tau}. \tag{11.8}$$

By Theorem III for L,

$$X^L_{\pi,\varepsilon} = \frac{1}{2^{m-1}} \sum_{x\in A(\mathcal{O}_{\mathfrak{l}})} \pi(x)R^L_{x,\varepsilon}. \tag{11.9}$$

Inserting (11.8) in (11.9), and inducing, we get the proposition. Q.E.D.

LEMMA 11.10. *The map* $J^W_{W_0}$ *of Definition* 4.11 *is a bijection from*

$$\left\{\sigma^L_x \otimes \tau \mid x \in \bar{A}(\mathcal{O}_{\mathfrak{l}}),\ \tau \in \{\pm 1\}^{\wedge}\right\}$$

to

$$\left\{\sigma_y \mid y \in \bar{A}(\mathcal{O})\right\}.$$

The induced bijection

(a) $$\bar{A}(\mathcal{O}_{\mathfrak{l}}) \times \{\pm 1\}^{\wedge} \leftrightarrow \bar{A}(\mathcal{O})$$

is an isomorphism of groups. The dual bijection

(b) $$\bar{A}(\mathcal{O}_{\mathfrak{l}})^{\wedge} \times \{\pm 1\} \leftrightarrow \bar{A}(\mathcal{O})^{\wedge}$$

is the same as that of Proposition 11.5.

This is proved case by case, by an explicit calculation. By Proposition 6.6, the first assertion implies that if (x,τ) corresponds to y, then

$$R_{x,\tau} = R_y \tag{11.11}$$

(see the last formula of Definition 11.6). By the last assertion, the formula of Proposition 11.7 is Theorem III(a). The second formula of Theorem III follows by Fourier inversion in $(\mathbf{Z}/2\mathbf{Z})^m$.

It remains only to prove Proposition 11.5. Define

$$P(\mathcal{O}_{\mathfrak{l}}) = \{w\lambda_{\mathfrak{l}} \mid w \in W(\mathfrak{l}), \text{ and } w\lambda_{\mathfrak{l}}|_{\mathfrak{m}} \text{ is dominant and regular}\}. \tag{11.12}$$

For $\varepsilon = \pm 1$, set

$$P(\mathcal{O}_{\mathfrak{l}})_{\varepsilon} = \{\mu + \tfrac{1}{2}\varepsilon\beta \mid \mu \in P(\mathcal{O}_{\mathfrak{l}})\}. \tag{11.13}$$

By Lemma 11.2, and the definition in Lemma 9.7,

$$P(\mathcal{O}) \supseteq P(\mathcal{O}_{\mathfrak{l}})_1 \cup P(\mathcal{O}_{\mathfrak{l}})_{-1}. \tag{11.14}$$

Case by case, it is trivial to verify that equality actually holds in (11.14). Consequently, $P(\mathcal{O}_{\mathfrak{l}})$ has 2^{m-1} elements. Recall the element $\lambda^1_{\mathcal{O}}$ of Lemma 9.7. Clearly it belongs to $P(\mathcal{O}_{\mathfrak{l}})_1$, and we define $\lambda^1_{\mathfrak{l}}$ by

$$\lambda^1_{\mathcal{O}} = \lambda^1_{\mathfrak{l}} + \tfrac{1}{2}\beta. \tag{11.15}$$

Now define, for $\mu \in P(\mathcal{O}_{\mathfrak{l}})$,

$$\begin{aligned} F^L_\mu &= \text{holomorphic representation of } \mathfrak{m} \text{ of highest weight } \mu - \rho_{\mathfrak{m}}, \\ E^L_\mu &= (\mathfrak{l}, L \cap K) \text{ module } F^L_\mu \otimes F^L_{\lambda^1_{\mathfrak{l}}}, \\ I^L_\mu &= \operatorname{Ind}^L_{P \cap L} E^L_\mu. \end{aligned} \tag{11.16}$$

Now suppose $\varepsilon = \pm 1$, $\mu \in P(\mathcal{O}_{\mathfrak{l}})$. Then it is immediate from the definitions that

$$I_{\mu + 1/2\varepsilon\beta} = \operatorname{Ind}^G_Q\left(I^L_\mu \otimes D_\varepsilon\right). \tag{11.17}$$

We know from Section 9 that the representations on the right are all distinct and irreducible, and exhaust the unipotent representations of G attached to $\mathcal{O}$. Consequently, the various I^L_μ must be distinct and irreducible. As they obviously have wavefront set $\mathcal{O}_{\mathfrak{l}}$ and infinitesimal character $\lambda_{\mathfrak{l}}$, and there are 2^{m-1} of them, the I^L_μ must exhaust the unipotent representations of L attached to $\mathcal{O}_{\mathfrak{l}}$. Proposition 11.5 follows. Q.E.D.

12. Complements

The results in this section are obtained by straightforward applications of the techniques used in the paper. We have omitted the proofs which involve tedious case-by-case checking and only mention two results which seemed noteworthy to us.

The first is a generalization of Corollary 10.18. In the setting of Theorem III, let s be a semisimple element of ${}^LG$ centralizing an element of ${}^L\mathcal{O}$, and $\bar{s}$ its image in $\bar{A}({}^L\mathcal{O}) \cong \bar{A}(\mathcal{O})$. In a way similar to Definition 10.15, let ${}^LS = \operatorname{Cent}(s, {}^LG)$ and let S be a connected reductive group whose dual is ${}^LS$. Then ${}^L\mathcal{O}$ meets ${}^L\mathfrak{s}$ (the Lie algebra of ${}^LS$) in an orbit ${}^L\mathcal{O}_s$. Clearly ${}^L\mathcal{O}_s$ is also even. Let $\mathcal{O}_s$ and $\sigma_1^{(s)}$ be the dual orbit and the special representation attached to it, respectively. Let $W(S)$ be the Weyl group corresponding to S.

Proposition 12.1.

$$J^W_{W(S)}\sigma_1^{(s)} = \sigma_s.$$

COROLLARY 12.2.

$$R_{\bar{s}} = \frac{1}{|W_{\lambda_{\mathcal{O}}} \cap W(S)|} \sum_{w \in W(S)} \operatorname{tr}(\sigma_s(w)) X_G(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}}).$$

Proposition 12.1 and Corollary 12.2 are generalizations of Lemma 10.17(d) and Corollary 10.18 and proved in exactly the same way.

This result can be restated so as to match the conjectures in [A]. Let H be a Cartan subgroup of S with Lie algebra $\mathfrak{h}$.

PROPOSITION 12.3. *There is a correspondence*

$$f \mapsto f^S$$

from $C_c^\infty(G)$ *to* $C_c^\infty(S)$ *such that*

$$X_G(\lambda, \mu)(f) = X_S(\lambda, \mu)(f^S)$$

for all $\lambda, \mu \in \mathfrak{h}^*$.

Proof. This can be obtained from the Paley-Wiener theorem for complex groups ([Ze], [De]).

A generalization of the map $f \to f^S$ for real groups and different classes of functions is obtained in the work of D. Shelstad.

COROLLARY 12.4.

$$\sum_{\pi \in \hat{\bar{A}}(\mathcal{O})} \operatorname{tr}(\pi(\bar{s})) X_\pi^G(f) = \sum_{\rho \in \hat{\bar{A}}(\mathcal{O}_S)} \operatorname{tr}(\rho(1)) X_\rho^S(f^S).$$

Proof. This is immediate from Theorem III, Corollary 12.2 and Proposition 12.3.

Corollary 12.4 is mentioned for its similarity with Conjecture 1.3.3(iii) in [A].

The second result is a generalization of the reduction techniques in Section 8. Let $\mathcal{O} = \operatorname{Ind}_P^G \mathcal{O}_{\mathrm{m}}$ be an induced nilpotent such that ${}^L\mathcal{O}$ is even. We would like to compare the representations $\{X_\pi\}_{\pi \in \hat{\bar{A}}(\mathcal{O})}$ with $\{\operatorname{Ind}_P^G X_\rho^{(\mathrm{m})}\}_{\rho \in \hat{\bar{A}}(\mathcal{O}_{\mathrm{m}})}$.

PROPOSITION 12.5. *There is a subgroup* $\bar{A}^{(\mathrm{m})}(\mathcal{O}) \subseteq \bar{A}(\mathcal{O})$ *with the following properties*:

a) $\bar{A}(\mathcal{O}_{\mathrm{m}})$ *is a quotient of* $\bar{A}^{(\mathrm{m})}(\mathcal{O})$. *Write*

$$1 \to \kappa \to \bar{A}^{(\mathrm{m})}(\mathcal{O}) \to \bar{A}(\mathcal{O}_{\mathrm{m}}) \to 1.$$

b) *For any* $\pi \in \hat{\bar{A}}^{(\mathrm{m})}(\mathcal{O})$ *we can find an irreducible representation* $X_{(\pi)}^{(\mathrm{m})}$ *of* M *such that*

$$\operatorname{Ind}_P^G(X_\pi^{(\mathrm{m})}) = \sum_{\eta \in \hat{\bar{A}}(\mathcal{O})} [\eta|_{\bar{A}(\mathrm{m})(\mathcal{O})} \colon \pi] X_n.$$

The infinitesimal character of $X_\pi^{(m)}$ is of the form $(w\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$ and the left and right annihilators are maximal for their respective infinitesimal characters.

In other words, the composition series of representations induced from unipotent representations matches induction from $\bar{A}^{(m)}(\mathcal{O})$ to $\bar{A}(\mathcal{O})$. The quotient space $\bar{A}(\mathcal{O})/A^{(m)}(\mathcal{O})$ plays the role of the Knapp-Stein R-group.

The proof of this proposition involves a minor generalization of Theorem III to include the representations $X_\pi^{(m)}$. Then statement (b) is simply matching the known character of $\mathrm{Ind}_P^G(X_\pi^{(m)})$ to the right-hand side of the equation.

Finally, we remark that the group $\bar{A}^{(m)}(\mathcal{O})$ is the same as the one considered by [L3] in 10.7.

Appendix: Duality for nilpotent orbits

In [Spa], Spaltenstein defined an order-reversing map d from the partially ordered set $\mathcal{N}$ of nilpotent orbits in $\mathfrak{g}$, onto the set $\mathcal{S}$ of special nilpotent orbits. When restricted to $\mathcal{S}$, d may be seen (by calculation in each case) to coincide with the order-reversing bijection of Corollary 3.25. We propose here to define an analogue of d, called η, from $\mathcal{N}$ to the set ${}^L\mathcal{N}$ of nilpotent orbits in ${}^L\mathfrak{g}$. While our definition is much less elementary than Spaltenstein's, we believe it is simpler. We have not tried to find "natural" proofs of the basic properties of η; since the map is easily computed by known techniques (and Corollary 5.20), these properties can be simply checked case by case. Our main motivation for including the material is Corollary A4.

Definition A1. Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. Choose a map ϕ: $\mathfrak{sl}(2) \to \mathfrak{g}$ so that

$$\phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{O},$$

$$\phi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h \in h.$$

Define $\mu_{\mathcal{O}} \in {}^L\mathfrak{h}^* \cong \mathfrak{h}$ to be the weight corresponding to $\frac{1}{2}h$. Set

$${}^LI_{\mathcal{O}} = \text{maximal primitive ideal in } U({}^L\mathfrak{g}) \text{ with infinitesimal character } \mu_{\mathcal{O}}.$$

Let ${}^LX$ be the Harish-Chandra module $U({}^L\mathfrak{g})/{}^LI_{\mathcal{O}}$. Define

$${}^L\mathcal{O} = \eta(\mathcal{O}) = \text{unique open orbit in } WF({}^LX).$$

We may call this map $\eta_{\mathfrak{g}}$ for precision.

PROPOSITION A2. *Suppose* $\mathcal{O}$ *is a nilpotent orbit in* $\mathfrak{g}$.

a) *If* $\mathcal{O}'$ *is any other nilpotent orbit and* $\mathcal{O}' \subseteq \bar{\mathcal{O}}$, *then*

$$\eta(\mathcal{O}) \subseteq \overline{\eta(\mathcal{O}')}.$$

b) $\eta_{\mathfrak{g}} \circ \eta_{L_{\mathfrak{g}}} \circ \eta_{\mathfrak{g}} = \eta_{\mathfrak{g}}$.
Consequently, $\eta_{\mathfrak{g}}$ *is an order reversing bijection of the image of the* $\eta_{L_{\mathfrak{g}}}$ *onto the image of* $\eta_{\mathfrak{g}}$ *with inverse* $\eta_{L_{\mathfrak{g}}}$.

c) *Suppose* $\mathcal{O}$ *meets a Levi factor* $\mathfrak{m} \subseteq \mathfrak{g}$; *say it contains the* $\mathfrak{m}$ *orbit* $\mathcal{O}_{\mathfrak{m}}$. *Then*

$$\eta_{\mathfrak{g}}(\mathcal{O}) = \operatorname{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(\eta_{\mathfrak{m}}(\mathcal{O}_{\mathfrak{m}})).$$

d) *The image of* η *is precisely the set* ${}^L\mathcal{S}$ *of special nilpotent orbits in* ${}^L\mathfrak{g}$. *On* $\mathcal{S}$, η *is the duality of Corollary* 3.26.

Sketch of Proof. In the notation of Definition A1, define a subsystem R of the root system of $\mathfrak{g}$ by

$$R = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) | \alpha(h) \in 2\mathbf{Z}\}.$$

Obviously R is the root system of a subalgebra $\mathfrak{e} \supseteq \mathfrak{h}$ of $\mathfrak{g}$; and $\check{R}$ is the set of integral roots for $\mu_{\mathcal{O}}$ in ${}^L\mathfrak{g}$. Clearly $\mathcal{O}$ meets $\mathfrak{e}$ in an orbit $\mathcal{O}_{\mathfrak{e}}$, which is even. Let $\sigma_{\mathfrak{e}}$ be the representation of $W(\mathfrak{e}, \mathfrak{h})$ inside $S(\mathfrak{h})$, which is generated by the product of the roots orthogonal to h. Let $\sigma_{\mathfrak{e}}'$ be the unique special representation in the $W(\mathfrak{e})$ double cell of $\sigma_{\mathfrak{e}} \otimes \operatorname{sgn}$, and $V_{\mathfrak{e}}'$ its realization on harmonic polynomials of minimal degree. Let (σ', V') be the W representation generated by $(\sigma_{\mathfrak{e}}', V_{\mathfrak{e}}')$ in $S(\mathfrak{h})$. By [BV2, 3], σ' is the Springer representation attached to $\eta(\mathcal{O})$. In this way one can compute the map η explicitly, and so verify immediately (a) and (b). To minimize the work required in verifying (c), we may assume (by induction on $\dim \mathfrak{g}$ and $\dim \mathfrak{g}/\mathfrak{m}$) that $\mathfrak{m}$ is in the unique minimal conjugacy class of Levi factors meeting $\mathcal{O}$. The representation ${}^LX$ of Definition Al is just the unique irreducible spherical representation of ${}^L\mathfrak{g}$ of infinitesimal character $\mu_{\mathcal{O}}$. Consequently (since $\mu_{\mathcal{O}} = \mu_{\mathcal{O}_{\mathfrak{m}}}$)

$${}^LX \text{ is a subquotient of } \operatorname{Ind}_{\mathfrak{m}+\mathfrak{u}}^{\mathfrak{g}}({}^LX_{\mathfrak{m}}),$$

with obvious notation. It follows that

$$\eta_{\mathfrak{g}}(\mathcal{O}) \subseteq \overline{\operatorname{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(\eta_{\mathfrak{m}}(\mathcal{O}_{\mathfrak{m}}))}.$$

So we only need to check that both sides have the same dimension. For (d), Lusztig shows in [L1] that the class of special nilpotents is closed under induction. To prove that $\eta(\mathcal{N}) \subseteq {}^L\mathcal{S}$, it suffices therefore to consider an orbit $\mathcal{O}$ not meeting any Levi factor. Case by case, one can check that such an orbit is even. Therefore, $\mu_{\mathcal{O}}$ is integral, and $\eta(\mathcal{O})$ is special by Theorem 3.20. The last assertion of (d) may be verified by inspection. Q.E.D.

COROLLARY A3. $(\eta_{L_{\mathfrak{g}}} \circ \eta_{\mathfrak{g}})(\mathcal{O})$ *is the unique smallest special nilpotent orbit containing* $\mathcal{O}$. *If it is denoted* $s(\mathcal{O})$, *then*

$$\eta(\mathcal{O}) = \textit{dual of } s(\mathcal{O}) \textit{ in the sense of Corollary } 3.25.$$

COROLLARY A4. *Suppose* $\mathcal{O}$ *is a special nilpotent orbit in* $\mathfrak{g}$, ${}^L\mathcal{O}$ *is even, and* ${}^L\mathcal{O}$ *meets a Levi factor* ${}^L\mathfrak{m}$ *in* ${}^L\mathcal{O}_{\mathfrak{m}}$. *Write* $\mathcal{O}_{\mathfrak{m}}$ *for the* $\mathfrak{m}$ *orbit dual to* ${}^L\mathcal{O}_{\mathfrak{m}}$.

a) $\mathcal{O} = \mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{m}})$.

b) *If* $X_{\mathfrak{m}}$ *is a special unipotent representation of* $\mathfrak{m}$ *attached to* $\mathcal{O}_{\mathfrak{m}}$, *then all composition factors of* $\mathrm{Ind}_{\mathfrak{m}+\mathfrak{u}}^{\mathfrak{g}}(X_{\mathfrak{m}})$ *are special unipotent representations of* $\mathfrak{g}$ *attached to* $\mathcal{O}$.

Proof. Part (a) follows from (c) and (d) of Proposition A2. Since $\lambda_{\mathcal{O}}$ is obviously equal to $\lambda_{\mathcal{O}_{\mathfrak{m}}}$ (cf. (5.4)), part (b) follows from (a) and Definition 5.23. Q.E.D.

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REFERENCES

[Al] D. Alvis, Induce/restrict matrices for the exceptional Weyl groups, preprint.

[A-L] D. ALVIS and G. LUSZTIG, On Springer's correspondence for simple groups of type E_n ($n = 6, 7, 8$), Math. Proc. Camb. Phil. Soc. **92** (1982), 73–78.

[A] J. ARTHUR, On some problems suggested by the trace formula, to appear in Proc. Special Year in Harmonic Analysis 1982–83, University of Maryland.

[B-V1] D. BARBASCH and D. VOGAN, The local structure of characters, J. Funct. Anal. **37** (1980), 27–55.

[B-V2] ______, Primitive ideals and orbital integrals in complex classical groups, Math. Ann. **259** (1982), 153–199.

[B-V3] ______, Primitive ideals and orbital integrals in complex exceptional groups, J. Alg. **80** (1983), 350–382.

[B-G-R] W. BORHO, P. GABRIEL, and R. RENTSCHLER, *Primideale in Einhüllenden auflösbarer Lie-Algebren*, Lecture Notes in Math. **357**, Springer, Berlin-Heidelberg-New York, 1973.

[B-J] W. BORHO and J. C. JANTZEN, Über primitive Ideale in der Einhüllenden einer halbeinfacher Lie-algebra, Invent. Math. **39** (1977), 1–53.

[B-K] W. BORHO and H. KRAFT, Über die Gelfand-Kirillov-Dimension, Math. Ann. **220** (1976), 1–24.

[D] M. DUFLO, Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple, Ann. of Math. **105** (1977), 107–120.

[De] P. DELORME, Théorème de type Paley-Wiener pour les groupes de Lie semi-simples réels avec une seule classe de conjugation des sous-groupes de Cartan, J. Funct. Anal. **47** (1982), 26–63.

[Dy] E. B. DYNKIN, Semisimple subalgebras of semisimple Lie algebras, A.M.S. Transl., Ser. 2, **6** (1957), 111–245 (= Mat. Sb. (N.S.) **30** (1952), 349–462).

[H] R. HOWE, Wave front sets of representations of Lie groups, in *Automorphic Forms, Representation Theory, and Arithmetic*, Tata Inst. Fund. Res. Studies in Math. **10**, Tata Inst. Fund. Res., Bombay, 1981.

[J1] A. Joseph, On the annihilators of the simple subquotients of the principal series, Ann. Sci. École Norm. Sup. (4), **10** (1977), 419–439.

[J2] ______, Goldie rank in the enveloping algebra of a semisimple Lie algebra I, II, J. Alg. **65** (1980), 269–316.

[K-L] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. **53** (1979), 165–184.

[Ki] D. King, The character polynomial of the annihilator of an irreducible Harish-Chandra module, Amer. J. Math. **103** (1981), 1195–1240.

[Ko] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. **81** (1959), 973–1032.

[L1] G. Lusztig, A class of irreducible representations of a Weyl group, Proc. Kon. Nederl. Akad., A, **82** (1979), 323–335.

[L2] ______, A class of irreducible representations of a Weyl group II, Proc. Kon. Nederl. Akad., A, **85** (1982), 219–226.

[L3] ______, *Characters of a Reductive Group over a Finite Field*, Princeton University Press, Princeton, New Jersey, 1984.

[L4] ______, Intersection cohomology complexes on a reductive group, II, preprint.

[L-Spa] G. Lusztig and N. Spaltenstein, Induced unipotent classes, J. London Math. Soc. **19** (1979), 41–52.

[Sh] D. Shelstad, *Orbital Integrals, Endoscopic Groups and L-Indistinguishability for Real Groups*, Publ. Math. de l'Université de Paris, 1983.

[Sho] T. Shoji, On the Springer representations of the Weyl groups of classical algebraic groups, Comm. in Alg. **7** (1979), 1713–1745 and 2027–2033.

[Spa] N. Spaltenstein, *Classes Unipotentes et Sous-Groupes de Borel*, Lecture Notes in Math. **946**, Springer, Berlin-Heidelberg-New York, 1982.

[Spr] T. A. Springer, A construction of representations of Weyl groups, Invent. Math. **44** (1978), 279–293.

[V1] D. Vogan, Ordering of the primitive spectrum of a semisimple Lie algebra, Math. Ann. **248** (1980), 195–203.

[V2] ______, *Representations of Real Reductive Lie Groups*, Birkhäuser, Boston-Basel-Stuttgart, 1981.

[V3] ______, Irreducible characters of semisimple Lie groups III. Proof of the Kazhdan-Lusztig conjectures in the integral case, Invent. Math. **71** (1983), 381–417.

[V4] ______, Irreducible characters of semisimple Lie groups IV: character-multiplicity duality, Duke Math. J. **49** (1982), 943–1073.

[Ze] D. P. Zelobenko, *Harmonic Analysis on Complex Semisimple Lie Groups*, Mir, Moscow, 1974.

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