that $K$ is const :vature (i.e. $\delta R = dS = 0$). On the other hand, (1 constant scalar curvature, which is locally conf s equivalent to $W=0$) must have harmonic cu of finding new compact manifolds with harmon al case of Yamabe's conjecture, cf. [1, 8, 4].) H ith harmonic curvature, described above, do n of constant curvature. In fact, each of these manifolds look ed product $\mathbb{R} \times_F N$ which is, in turn, locally conformally equivalent to th y product $\mathbb{R} \times N$. It is now easy to see that, provided $\dim N \geqq 3$, $\mathbb{R} \times N$ satisfies $W=0$ if and only if $N$ has constant sectional curvature.

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# Primitive Ideals and Orbital Integrals in Complex Classical Groups

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## Introduction

Let $G$ be a complex connected semisimple Lie group. We are interested in two apparently unrelated problems. The first is Fourier inversion of orbital integrals. To explain this, let $g$ be any element of $G$, and

$$\mathcal{O}_g = \{xgx^{-1} \mid x \in G\}$$

its conjugacy class. The set $\mathcal{O}_g$ is a manifold, locally closed in $G$; and it carries a unique (up to scalar) nice measure $dx$ which is invariant under the action of $G$. Suppose that $f$ is a compactly supported smooth function on $G$. Since $\mathcal{O}_g$ need not be closed, the restriction of $f$ to $\mathcal{O}_g$ need not be compactly supported. Nevertheless, Harish-Chandra and Rao have shown that it is integrable with respect to $dx$; so we define

$$D_g(f) = \int_{\mathcal{O}_g} f dx. \tag{1}$$

The distribution $D_g$ is called an *orbital integral*. Because $dx$ is invariant under $G$, $D_g$ is fixed by the conjugation action; we call $D_g$ *invariant* or *central* for this reason.

Orbital integrals appear in the Selberg trace formula. In order to use that formula to study automorphic forms, one would like to relate them to group representations. More explicitly, write $\hat{G}_u$ for the set of irreducible unitary representations of $G$. Each member $\pi$ of $\hat{G}_u$ has a character $\Theta_\pi$, which is a distribution on $G$. What we want is a distribution $\mu_g$ on $\hat{G}_u$ such that

$$D_g(f) = \int_{\hat{G}_u} \Theta_\pi(f) d\mu_g(\pi) \tag{2}$$

for every compactly supported smooth function $f$ on $G$. The problem of finding such a distribution is Fourier inversion. For example, if $g$ is the identity element $e$, then $D_e(f)$ *is* $f(e)$ and (2) reads

$$f(e) = \int_{\hat{G}_u} \Theta_\pi(f) d\mu_e \pi. \tag{3}$$

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This is the Plancherel formula for $G$; the measure $\mu_e$ has been computed by Harish-Chandra. In general, $\mu_g$ is not explicitly known. The problem of computing it can be reduced to the case when $g$ is a unipotent element. To describe what is known, we fix a Cartan subgroup $H$ of $G$. Harish-Chandra has defined a map

$$C_c^\infty(G) \to C_c^\infty(H), \quad f \mapsto F_f$$

(see [H-C1, H-C2]), which is a sort of normalized orbital integral: if $h$ is a regular element of $H$, then $F_*(h)$ is a multiple of the orbital integral $D_h$; but if $h$ is singular, then $F_f(h)$ is defined to make $F_f$ continuous on $H$, and is not directly related to $D_h(f)$. For example, if $G = SL(2, \mathbb{C})$, and

$$n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then

$$F_f(e) = D_n(f) = \int_{G/N} f(gng^{-1})dg$$
$$\neq f(e)$$

in general. Harish-Chandra has shown that all unipotent orbital integrals arise in a similar way.

**Theorem A** (Harish-Chandra, unpublished). *Suppose $n$ is a unipotent element of $G$. Then there is a linear differential operator $\partial(n)$ on $H$, such that*

$$D_n(f) = \int_{\mathcal{O}_n} f dx = [\partial(n)F_f](e).$$

This says that a unipotent orbital integral may be obtained as a limit of semisimple ones in a nice way. To relate this to Fourier inversion, we need

**Theorem B** (Harish-Chandra). *There are explicitly computable measures $d\xi_h(\pi)$ on $\hat{G}_u$, depending smoothly on $h \in H$, such that*

$$F_f(h) = \int_{\hat{G}_u} \Theta_\pi(f) d\xi_h(\pi).$$

So once $\partial(n)$ is known, the Fourier transform of $D_n$ may be obtained by differentiating (in $h$) under the integral sign. This is, in fact, precisely how Harish-Chandra proves the Plancherel formula for $G$ (corresponding to the case $n = e$). The problem we consider, therefore, is that of computing $\partial(n)$. We solve it when $G$ is a classical (complex) simple group, and $n$ is a special unipotent element in the sense of Lusztig (see [L]); this includes all of the Richardson orbits (for which the answer is due to Harish-Chandra), and many other cases. To describe the answer, write $\mathfrak{h}$ for the Lie algebra of $H$ (which we regard as real), and $\mathfrak{h}_c$ for its complexification; then $\mathfrak{h}_c$ is naturally isomorphic to a sum of two copies of $\mathfrak{h}$:

$$\mathfrak{h}_c = \mathfrak{h}_L \oplus \mathfrak{h}_R.$$

So if $W$ is the Weyl group of $H$ in $G$, then $W \times W$ acts on $\mathfrak{h}_c$.

**Theorem C** (Corollary 34 below). *Suppose $G$ is a complex classical simple group, and $n \in G$ is a special unipotent element. Let $\sigma$ be the irreducible representation of $W$ attached by Springer to the conjugacy class of $n$ (and the trivial representation of the centralizer of $n$ – see [S]). Let $p \in S(\mathfrak{h}_c)$ be a polynomial of lowest degree satisfying*

*a) $p$ is invariant by the diagonal $W \subsetneqq W \times W$.*

*b) $p$ transforms by $\sigma \otimes \sigma$ under $W \times W$.*

*Then for an appropriate normalization of measures,*

$$[\partial_p F_f](e) = \int_{\mathcal{O}_n} f dx.$$

*Here $\partial_p$ is the linear differential operator on $H$ corresponding to $p$.*



Actually we compute $p$ explicitly; the characterization in the theorem then follows from [L]. The hypotheses that $G$ is classical and $n$ is special should not be necessary. The techniques in this paper apply to all unipotents in classical groups. For the exceptional groups, the difficulties appear to be mostly computational, but some new ideas may be necessary. (Our techniques apply to real groups as well; but the computational problems are increased enormously.)

The second problem we consider is entirely algebraic. Write $\mathfrak{g}$ for the Lie algebra of $G$, and $U(\mathfrak{g})$ for its universal enveloping algebra. A two-sided ideal in $U(\mathfrak{g})$ is called *primitive* if it is the annihilator of an irreducible (possibly infinite dimensional) representation of $\mathfrak{g}$. We want to describe the set Prim $(U(\mathfrak{g}))$ of all primitive ideals in $U(\mathfrak{g})$. This we do when $\mathfrak{g}$ is classical. [Previously, the answer was known for $\mathfrak{sl}(n, \mathbb{C})$ and for the other simple algebras of rank at most five.] To describe the answer, we need Harish-Chandra's description of the maximal ideals in the center $\mathfrak{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$; it says that they are parametrized by $W$ orbits in $\mathfrak{h}^*$. If $I$ is a primitive ideal, then $I \cap \mathfrak{Z}(\mathfrak{g})$ is a maximal ideal in $\mathfrak{Z}(\mathfrak{g})$, and so corresponds to some weight $\lambda \in \mathfrak{h}^*$. We express this by saying that $I$ has *infinitesimal character* $\lambda$; write

$$\mathrm{Prim}_\lambda(U(\mathfrak{g}))$$

for the set of such $I$. For $\lambda \in \mathfrak{h}^*$, define

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \text{ is a sum of roots}\},$$

the *integral Weyl group* for $\lambda$. Duflo defines in [D2] a surjection

$$W_\lambda \to \mathrm{Prim}_\lambda(U(\mathfrak{g})).$$

Here we determine the fibers of this map whenever $W_\lambda$ is a product of classical Weyl groups (Theorems 18 and 30). (Because of the Borho-Jantzen translation principle, we may assume $\lambda$ is regular.) The result is too complicated to restate completely here, but part of it is

**Theorem D.** *Suppose $\lambda \in \mathfrak{h}^*$ is regular, and $I \in \mathrm{Prim}_\lambda(U(\mathfrak{g}))$. Assume that $W_\lambda$ is a product of classical Weyl groups. Then Joseph's Goldie rank representation*

$$\sigma(I) \in \hat{W}_\lambda$$

*is a special representation of $W_\lambda$ in the sense of Lusztig (see [L, J1]); and all special representations occur. In particular, if we define*

$$S(\lambda) = \{\sigma \in \hat{W}_\lambda \mid \sigma \text{ is special}\}.$$

*then*

$$|\mathrm{Prim}_\lambda U(\mathfrak{g})| = \sum_{\sigma \in S(\lambda)} \dim \sigma .$$

This confirms conjectures of Joseph and Kazhdan-Lusztig. The corresponding result is also known for $G_2$ and $F_4$; in type $E$ a little more theoretical progress may be required to make the result accessible to hand calculation.

Finally, we need a reason for including these rather different problems in a single paper. That is provided by the theory of the wavefront set of a representation, and the asymptotic support of a character [B-V, H]. Specifically, fix an irreducible admissible representation $\pi$ of $G$. The *wavefront set of* $\pi$, $WF(\pi)$, defined by Howe in [H], is an $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}^*$; its definition is a little complicated, but it should be regarded as a measure of how infinite dimensional $\pi$ is. Write $\Theta$ for the distribution character of $\pi$, and $\theta$ the lift of $\Theta$ to $\mathfrak{g}$ via exp. In [B-V], an "asymptotic expansion" of $\theta$ was defined, of the form

$$\theta \sim \sum_{i=-r}^{\infty} D_i ,$$

with $D_i$ a tempered distribution supported on $WF(\pi)$. (In fact, this is a convergent series of distributions.) For our purposes, the main point is that the Fourier transform of the first term $D_{-r}$ is actually a measure; up to coordinate changes, it is one of the unipotent orbital integrals considered earlier. Now King has shown in [K] how to attach to $\theta$ a linear differential operator $\partial(\theta)$ on $H$; and he proves

**Theorem E** (King [K]). *$\partial(\theta)$ transforms under $W_\lambda \times W_\lambda$ by the Joseph Goldie rank representation $\sigma(\mathrm{Ann}\,\pi)$ attached to the annihilator of $\pi$ in the enveloping algebra of $G$.*

On the other hand, it is obvious from the definitions (cf. Proposition 5 below) that $\partial(\theta)$ is substantially the differential operator attached to the unipotent orbital integral $D_{-r}$ by Theorem A. So the problem we consider is this: given $\pi$, determine $WF(\pi)$ and the Weyl group representation defined by $\partial(\theta)$. This we solve for $G$ classical, when $\pi$ has integral infinitesimal character. It turns out that only special unipotent orbits arise in this way (the others presumably corresponding to non-integral infinitesimal character), which explains the hypotheses of Theorem C. Known techniques reduce the problem to a study of certain equivalence relations in $W$. More precisely, the set of representations with a fixed regular integral infinitesimal character is parametrized by $W$. Ideas of Duflo, Joseph, and Jantzen give some conditions under which

$$WF(\pi(w)) \subsetneq WF(\pi(w')) \qquad (w, w' \in W).$$

The theory of parabolic induction allows us to compute $WF(\pi)$ precisely for some $\pi$. (In this way we bypass some thorny technical problems in the purely algebraic theory of primitive ideals: $WF$ obviously behaves well under induction, but it is still not known whether "variety of the associated graded ideal" does.) Accordingly, we can compare unknown wavefront sets with known ones, and finally compute them all. The study of the equivalence relations on $W$ is facilitated by realizing $W$ in terms of tableaus, via the Robinson-Schensted correspondence; this is where the hypothesis that $W$ is classical (that is, nearly a symmetric group) enters. The theory is not nearly as clean as in the case of $SL(n, \mathbb{C})$, because the Robinson-Schensted map is no longer identical with the Duflo map; but the results are almost as good as for $SL(n, \mathbb{C})$.



This is due to a flawless insight into the nature of the results, which was provided to us by Lusztig on an almost daily basis as the paper was written. We would like to thank him for his generous and invaluable assistance. Conversations with R. Stanley were also very helpful.

## Notation and Preliminary Results

We begin by recalling some results in [B-V]. Let $G$ be a connected real semisimple Lie group with finite center. Let $\pi$ be an irreducible admissible representation on a Hilbert space $\mathscr{H}$ and $\Theta_\pi$ its distribution character, namely, if $f \in C_c^\infty(G)$,

$$\Theta_\pi(f) = \mathrm{Tr}\left(\int_G f(g)\pi(g)\,dg\right).$$

Due to the work of Harish-Chandra [H-C1, H-C2] it is known that $\Theta_\pi$ is given by integration against a function which is locally $L^1$, analytic on the set of regular semisimple elements of $G$. We recall the following theorem:

**Theorem 1** (1.1 in [B-V]). *Let $\theta_\pi$ be the lift of $\Theta_\pi$ to a neighborhood of the identity on $\mathfrak{g} = Lie(G)$. If $f \in C_c^\infty(\mathfrak{g})$ and $t > 0$, define*

$$f_t(x) = t^{-\dim \mathfrak{g}} f(t^{-1}x).$$

*Then there are an integer $r$ and tempered distributions $\{D_i\}_{i=-r}^{\infty}$ on $\mathfrak{g}$, such that for $f \in C_c^\infty(\mathfrak{g})$,*

$$\theta_\pi(f_t) \sim \sum_{i=-r}^{\infty} t^i D_i(f).$$

*(This expansion is understood in the same sense as a Taylor series for a smooth function of $t$.)*

The support of $\hat{D}_i$ (the Fourier transform of $D$) is a union of nilpotent orbits in $\mathfrak{g}^*$. We define $AS(\Theta_\pi) \subseteq \mathfrak{g}^*$ to be the union of the supports of the $\hat{D}_i$, and $\sum_{i=-r}^{\infty} t^i D_i$ the asymptotic expansion of $\theta$.

The key facts used in the determination of the Fourier transforms of nilpotent orbital integrals are the relation of $AS(\Theta_\pi)$ to the primitive ideals and the behaviour of $AS(\Theta_\pi)$ under coherent continuation.

Let $X \subseteq \mathscr{H}$ be the space of $K$-finite vectors in $\mathscr{H}$ ($K$ is the maximal compact subgroup of $G$). Then $X$ is a module for the (complexified) enveloping algebra $U(\mathfrak{g}_c)$ of $\mathfrak{g}$. We denote this action by $\pi$ also. The ideal $\mathscr{I}_\pi \subseteq U(\mathfrak{g}_c)$ is defined to be the annihilator of $X$. Let $\mathrm{gr}(\mathscr{I}_\pi) \subseteq S(\mathfrak{g}_c)$ be the associated graded ideal in the symmetric algebra of $\mathfrak{g}_c$. The associated cone $\mathscr{V}(\mathscr{I}_\pi)$ is defined to be the zero variety of $\mathrm{gr}(\mathscr{I}_\pi)$ inside $\mathfrak{g}_c^*$. The Gelfand-Kirillov dimension $d(\mathscr{I}_\pi)$ may be defined to be the complex dimension of $\mathscr{V}(\mathscr{I}_\pi)$. It is known that $\mathscr{V}(\mathscr{I}_\pi)$ is a finite union of nilpotent orbits of the complex adjoint group $G_c$ in $\mathfrak{g}_c^*$ [B-K].

**Theorem 2** (4.1 in [B-V]).

$$AS(\Theta_\pi) \subseteq \mathscr{V}(\mathscr{I}_\pi) \cap \mathfrak{g}^*.$$

*Furthermore,*

$$d(\mathscr{I}_\pi) = \dim_{\mathbb{R}} AS(\Theta_\pi) = \dim_{\mathbb{C}} \mathscr{V}(\mathscr{I}_\pi) = \dim_{\mathbb{R}} \mathscr{V}(\mathscr{I}_\pi) \cap \mathfrak{g}^*.$$

*The first term in the asymptotic expansion of the lift $\theta_\pi$ of $\Theta_\pi$ to $\mathfrak{g}$ is a linear combination of Fourier transforms of invariant measures on nilpotent orbits of dimension $d(\mathscr{I}_\pi)$, and hence, has homogeneity degree $-\frac{1}{2}d(\mathscr{I}_\pi)$.*

We turn now to the behaviour of $AS(\Theta_\pi)$ under coherent continuation (see [B-V] for more details of the definitions and main results). Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and some translate $\Lambda \subseteq \mathfrak{h}_c^*$ of the lattice of weights of finite dimensional representations of $G$. Let $\{\Theta(\lambda) : \lambda \in \Lambda\}$ be a coherent family of virtual characters of $G$. This means $\Theta(\lambda)$ is a linear combination of irreducible characters of infinitesimal character $\lambda$; and if $F$ is a finite dimensional representation of $G$ with weights $\Delta(F) \subseteq \mathfrak{h}_c^*$ counted with multiplicity

$$\Theta(\lambda) \cdot \Theta(F) = \sum_{\mu \in \Delta(F)} \Theta(\lambda + \mu).$$

Fix a positive system $\Delta^+ \subseteq \Delta(\mathfrak{g}_c, \mathfrak{h}_c)$. We also assume that, whenever $\lambda$ is dominant $\left(\dfrac{2\langle\alpha, \lambda\rangle}{\langle\alpha, \alpha\rangle}\right.$ is not a negative integer for $\left.\alpha \in \Delta^+\right)$ then $\Theta(\lambda)$ is either 0 or the character of an irreducible representation of $G$, and that the latter is true when $\lambda$ is nonsingular. Any irreducible representation $\pi$ occurs as a $\pi(\lambda)$ in some coherent family, unique up to equivalence. In this case the nonzero $\pi(\lambda)$ have annihilators $\mathscr{I}_{\pi(\lambda)}$ with a common Gelfand-Kirillov dimension $2d$. Every constituent of each virtual character $\Theta(\lambda)$ has Gelfand-Kirillov dimension at most $2d$. We can write

$$\theta_\lambda \sim \sum_{i=-d}^{\infty} t^i D_i(\lambda).$$

**Proposition 3** (4.7 in [B-V]). *Let $\{\Theta(\lambda) | \lambda \in \Lambda\}$ be a coherent family of virtual characters as above. For fixed i, the distributions*

$$\{D_i(\lambda) | \lambda \in \Lambda\}$$

*span a finite dimensional space. Let $\{E_{ij}\}$ be a basis of this space. Then*

$$D_i(\lambda) = \sum_j P_{ij}(\lambda) E_{ij}$$

*with $P_{ij}$ a polynomial function on $\mathfrak{h}^*$, homogeneous of degree $i + \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h})$. If $i = -d$, then the $P_{ij}$ are harmonic.*

Let $\pi$ be a not necessarily irreducible representation of $G$ and $\pi_1$ a constituent of $\pi$. We will need to know the relation between $AS(\pi_1)$ and $AS(\pi)$. In order to do this we will relate $AS(\pi)$ to the wave front set of $\pi$. This will be denoted by $WF(\pi)$ and is defined in [H] as follows.

Let $\pi$ be a representation of $G$ on a Hilbert space $\mathscr{H}$. Let $J_1(\mathscr{H})$ be the trace class operators on $\mathscr{H}$. Given $T \in J_1$, set

$$\operatorname{tr}_\pi(T)(g) = \operatorname{tr}(\pi(g)T), \quad g \in G.$$



We regard $\operatorname{tr}_\pi(T)$ as a distribution by putting

$$\operatorname{tr}_\pi(T)(f) = \int_G f(g) \operatorname{tr}_\pi(T)(g)\, dg$$

and define $WF(\pi)$ to be the closure of the union of the wave front sets at 0 $WF_0(\operatorname{tr}_\pi(T))$ for all $T \in J_1$.

If $WF_0(\theta_\pi)$ is the wave front set of $\theta_\pi$, then it is proved in [H] that

$$WF_0(\theta_\pi) \subseteq WF(\pi) \subseteq \mathscr{V}(\mathscr{I}_\pi) \cap \mathfrak{g}^*$$

with equality when $\pi$ is unitary.

**Proposition 4.** *With notation as before,*

$$AS(\theta_\pi) \subseteq WF_0(\theta_\pi).$$

*Proof.* Multiplying $\theta$ with a function of compact support $\phi$ such that $\phi(0) = 1$ does not change $AS(\theta)$ or $WF_0(\theta)$. Thus we may assume that $\theta$ has compact support. By the definition of an asymptotic expansion, for each $f \in C_c^\infty(\mathfrak{g})$ such that $\operatorname{supp} f \subseteq \mathscr{K}$, and each $N$, there is a constant $C_{\mathscr{K},N}$ and a $\mathfrak{k} > 0$ such that

$$\left|\theta(f_t) - \sum_{i=-d}^{N} t^i D_i(f)\right| \leq C_{\mathscr{K},N} t^{N+1} \sup_{|\alpha| \leq \mathfrak{k}} |D^\alpha f|.$$

We will first show that in the above estimate, the $C_c^\infty$ seminorm can be replaced by a Schwartz seminorm. This will imply that the asymptotic expansion holds for Schwartz functions as well.

We consider a partition of unity on the real line $\{\phi_j\}$ as in Lemma 2.2 in [B-V]. This means $\phi_j(x) = \phi(2^{-j}x)$ where $\phi \in C_c^\infty(\mathbb{R})$, $\phi \geq 0$ and $\operatorname{supp}\phi \subseteq (\frac{1}{2}, 2)$. Then define

$$f^j(x) = f(x)\phi_j(|x|).$$

We can write

$$f = f\left(1 - \sum_{j=1}^{\infty} \phi_j\right) + \sum_{j=1}^{\infty} f^j.$$

The desired estimate holds for the first term, since it has fixed support regardless of $\operatorname{supp} f$. We estimate each remaining term separately. Fix $j$. If $1 \geq t > 2^{-j+1}$, then $\operatorname{supp} f_t^j \cap \operatorname{supp}\theta = \emptyset$ so

$$\theta(f_t^j) = 0.$$

(We can assume without loss of generality that the support of $\theta$ is contained in the ball of radius 1 around the origin.) Then we only have to estimate

$$\left|\sum_{i=-d}^{N} t^i D_i(f^j)\right|.$$

But all the $D_i$'s are distributions, so we can write

$$\left|\sum_{i=-d}^{N} t^i D_i(f^j)\right| \leq C \cdot 2^{dj} \sup_{\substack{|\alpha| \leq \mathfrak{k} \\ 2^{j-1} \leq |x| \leq 2^{j+1}}} |D^\alpha f^j| \leq C_1 \cdot 2^{-(N+2)j} \sup_{|\alpha| \leq \mathfrak{k}} \left||x|^{\mathfrak{k}_1} D^\alpha(f)\right|$$

by choosing $\mathfrak{k}_1$ large enough.

Since $t>2^{-(j-1)}$, we obtain

$$\left|\theta(f_t)-\sum_{i\leqq N} t^i D_i(f)\right| \leqq C_1 t^{N+1} 2^{-j} \operatorname*{supp}_{|\alpha|\leqq \mathfrak{k}} |x^{t_1} D^\alpha f|.$$

For $t<2^{-(j-1)}$ we can write $t=z\cdot t_1$ where $2^{-j-1}\leqq|z|\leqq 2^{-j+1}$ and $t_1<1$. We get

$$\left|\theta(f^j_{z\cdot t_1})-\sum_{i\leqq N} t_1^i D_i(f_z^j)\right| \leqq C_1\cdot t_1^{N+1} \operatorname*{supp}_{|\alpha|\leqq \mathfrak{k}} |D^\alpha f_z^j|$$

since supp$f_z$ is contained in the unit ball centered at 0. It follows that

$$|\theta(f_t^j)-\sum_{i\leqq N} t^i D_i(f^j)| \leqq C_1\cdot t^{N+1}\cdot |z|^{-N-1-\mathfrak{k}} \operatorname*{supp}_{|\alpha|\leqq \mathfrak{k}} |D^\alpha f^j|.$$

But we have seen earlier that

$$\operatorname*{supp}_{|\alpha|\leqq \mathfrak{k}} |D^\alpha f^j| \leqq 2^{-Mj} \operatorname*{supp}_{|\alpha|\leqq \mathfrak{k}} ||x|^\beta D^\alpha f|$$

for any $M\geqq 0$, if $\beta$ is large enough. Thus

$$|\theta(f_t^j)-\sum_{j\leqq N} t^i D_i(f^j)| \leqq C\cdot t^{N+1} 2^{-j} \operatorname*{supp}_{|\alpha|\leqq \mathfrak{k}} |x^\beta D^\alpha f|$$

for any $t\leqq 1$ with $\beta$ and $\mathfrak{k}$ large enough. By summing over $j$, and using the estimate on the first term,

$$|\theta(f_t)-\sum_{i\leqq N} t^i D_i(f)| \leqq t^{N+1} v(f),$$

where $v(f)$ is a Schwartz seminorm.

Then $\xi\notin AS(\theta)$ is equivalent to

$$D_i(\hat{\phi})=0$$

for all $i$ and all $\phi$ such that $\hat{\phi}\in C_c^\infty(\mathfrak{g}^*)$, $\xi\in\operatorname{supp}\hat{\phi}$ and $\operatorname{supp}\hat{\phi}\cap AS(\theta)=\phi$. This can be rewritten as

$$\lim_{t\to 0^+} t^{-N}\theta(\hat{\phi}_t)=0$$

for all $N>0$, or

$$\lim_{t\to 0^+} t^{-N}\hat{\theta}(\phi_{t^{-1}})=0.$$

On the other hand, $\xi\notin WF_0(\theta)$ implies that there is a neighborhood $\{u:|u-\xi|<\varepsilon\}$ such that

$$t^N\hat{\theta}(tu)\to 0$$

as $t\to\infty$, for all $N>0$.

Thus $\xi\notin WF_0(\theta)$ implies that

$$t^{-N}\hat{\theta}(\phi_{t^{-1}})=t^{-N-n}\int_{\mathbb{R}^n}\hat{\theta}(x)\phi(tx)\,dx=t^{-N}\int_{\mathbb{R}^n}\hat{\theta}(t^{-1}x)\phi(x)\,dx$$

$$=t^{-N}\int_{|x-\xi|\leqq\varepsilon}\hat{\theta}(t^{-1}x)\phi(x)\,dx$$

tends to zero when $t\to 0$ and the proof is complete.



From here on we assume that $\mathfrak{g}$ is a complex semisimple algebra with simply connected Lie group $G$. Let $\mathfrak{h}\subseteq\mathfrak{g}$ be a Cartan subalgebra and $H$ the corresponding subgroup. Let $\Delta(\mathfrak{g},\mathfrak{h})$ be the root system and $W$ the Weyl group of $\mathfrak{h}$. Choose a positive system of roots $\Delta^+(\mathfrak{g},\mathfrak{h})$. Let $\alpha\in\Delta$ and $\mathfrak{n}_\alpha$ be the corresponding root space. We choose elements $x_\alpha\in\mathfrak{n}_\alpha$ and $h_\alpha\in\mathfrak{h}$ such that

$$[h_\alpha,x_\alpha]=2x_\alpha,\qquad [h_\alpha,x_{-\alpha}]=-2x_{-\alpha},\qquad [x_\alpha,x_{-\alpha}]=h_\alpha$$

and that $[x_\alpha,x_\beta]=N_{\alpha,\beta}x_{\alpha+\beta}$ where $N_{\alpha,\beta}$ are real numbers such that $N_{-\alpha,-\beta}=-N_{\alpha,\beta}$. We denote by $X\mapsto\bar{X}$ the conjugation with respect to the real form generated by $\{x_\alpha,h_\alpha,x_{-\alpha}\}_{\alpha\in\Delta}$ and by $x\mapsto{}^t x$ the antiautomorphism of $\mathfrak{g}$ defined by ${}^t h=h$ if $h\in\mathfrak{h}$ and ${}^t x_\alpha=x_{-\alpha}$.

We denote by

$$\mathfrak{k}=\{x\in\mathfrak{g}:x=-{}^t\bar{x}\}.$$

We denote by $\mathfrak{n}$ the space spanned by the $\{x_\alpha\}_{\alpha\in\Delta^+}$ and by $\mathfrak{n}^-$ the space spanned by the $\{x_{-\alpha}\}_{\alpha\in\Delta^+}$. Let $\mathfrak{m}=\mathfrak{k}\cap\mathfrak{h}$ and $\mathfrak{a}$ the real algebra generated by the $h_\alpha$. If we regard all the algebras defined, as real Lie algebras, then we have the Iwasawa decomposition

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}.$$

We identify $\mathfrak{g}_c$ with $\mathfrak{g}\times\mathfrak{g}$ by extending the map $x\mapsto(x,\bar{x})$ from $\mathfrak{g}$ to $\mathfrak{g}_c$. Then $\mathfrak{h}_c$ is identified with $\mathfrak{h}\times\mathfrak{h}$ via $h\mapsto(h,\bar{h})$. Then $\mathfrak{h}_c^*$ is isomorphic to $\mathfrak{h}^*\times\mathfrak{h}^*$ by the map $(p,q)(h_1,h_2)=p(h_1)+q(\bar{h}_2)$.

On the other hand $\mathfrak{h}=\mathfrak{m}+\mathfrak{a}$ so $\mathfrak{h}_c=\mathfrak{m}_c+\mathfrak{a}_c$. Then we denote by $\mu\oplus\lambda\in\mathfrak{h}_c^*$, the element

$$(\mu\oplus\lambda)(h+h')=\mu(h)+\lambda(h'),\qquad h\in\mathfrak{m},\ h'\in\mathfrak{a}.$$

Let $W$ be the Weyl group of $\mathfrak{h}$. Then the Weyl group of $(\mathfrak{g}_c,\mathfrak{h}_c)$ can be identified with $W\times W$.

The Weyl group $W$ embeds in $W_c$ as the diagonal.

Let $\theta_\lambda$ be a coherent family of virtual characters. Let $d>0$ be the smallest integer such that $\lim_{t\to 0^+} t^{-d}\theta_\lambda(f_t)$ is finite and nonzero for some $f\in C_c^\infty(\mathfrak{g})$. Since $\mathfrak{g}$ has no real roots, it follows from the results in [H-C2] that

$$\theta_\lambda(x)=\frac{\sum_{w\in W_c} a_w e^{w\lambda(x)}}{\pi(x)},\qquad x\in\mathfrak{h}$$

where $\pi(x)=\prod_{\alpha\in\Delta^+(\mathfrak{g}_c,\mathfrak{h}_c)}\alpha(x)$.

Using Proposition 3 and the results in [J-1, J-2] it was proved in [K] that

$$\lim_{t\to 0^+} t^{-d}\frac{\sum a_w e^{tw\lambda(x)}}{\pi(x)}=c(\lambda)\frac{p(x)}{\pi(x)},\tag{1}$$

where $c(\lambda)$ is a harmonic polynomial, belonging to an irreducible representation of $W_c$ in the space of harmonic polynomials. This representation occurs with multiplicity one in the space of harmonic polynomials, homogeneous of degree $d$. Also $d$ is the lowest degree in which this representation occurs in the space of harmonic polynomials.

**Proposition 5.** *With notation as before, $p(x)$ is a harmonic polynomial belonging to the same representation as $c(\lambda)$. In addition, $p(x)$ is invariant by the action of $W$ so it is unique up to a scalar.*

*Proof*. By L'Hopital's rule it is clear that

$$\sum_{w\in W_c} a_w\langle w\lambda, x\rangle^d = c(\lambda)p(x)$$

so $p(x)$ in particular has the same degree as $c(\lambda)$.

We want to show that $p(x)$ belongs to the same representation of $W_c$ as $c(\lambda)$. Then due to the properties of this representation and the decomposition of $S(\mathfrak{h}_c)$ into invariant and harmonic polynomials it will follow that $p$ is also harmonic.

Let $\chi$ be the character of an irreducible representation of $W_c$. Then

$$\begin{aligned} c(\lambda)\sum_{s\in W_c}\overline{\chi(s)}p(s^{-1}x) &= \sum_{w,s\in W_c}\overline{\chi(s)}a_w\langle w\lambda, s^{-1}x\rangle^d \\ &= \sum_{w,s\in W_c}\overline{\chi(s)}a_w\langle sw\lambda, x\rangle^d = \sum_{w,u\in W_c}\overline{\chi(wuw^{-1})}a_w\langle wu\lambda, x\rangle^d \\ &= p(x)\sum_{u\in W_c}\overline{\chi(u)}c(u\lambda) = p(x)\sum_{u\in W_c}\overline{\chi(u^{-1})}c(u\lambda), \end{aligned}$$

where the last equality follows from the fact that all characters of $W_c$ are real [S2]. These relations imply the desired result.

Let $\pi: W\to \mathrm{GL}(V_\pi)$ be an irreducible representation of $W$. Since $W_c = W\times W$, any irreducible representation of $W_c$ is of the form $\pi\otimes\varrho$, $\pi,\varrho\in\hat{W}$. Let $\pi\otimes\varrho$ be the representation to which $c(\lambda)$ and $p(x)$ belong. Then, since $\theta_\lambda(x)$ is invariant by $G$, $p(x)$ must be invariant by $W$. But

$$(\pi\otimes\varrho)_W \simeq \mathrm{Hom}_W(V_\pi, V_\varrho^*)$$

and the latter has dimension 0 if $\pi\neq\varrho$ and 1 if $\pi=\varrho$. This completes the proof.

Let $\tilde{\lambda}\in\mathfrak{h}$ be regular. Let

$$F_f(\tilde{\lambda}) = \pi(\tilde{\lambda})\int_{G/H} f(\mathrm{Ad}\,g)\tilde{\lambda})dg\,.$$

Then it is known that the Fourier transform of $F_f$ is a tempered invariant eigendistribution which restricts to a $C^\infty$ function on $\mathfrak{h}$:

$$\hat{F}(\tilde{\lambda}, x) = \frac{\sum_{w\in W}\det(w)e^{w\tilde{\lambda}(x)}}{\pi(x)}\,.$$

Differentiating in $\tilde{\lambda}$, we get

$$(\partial(p_{\tilde{\lambda}})F)\hat{}\,(0,x) = \frac{p(x)}{\pi(x)}\,.$$

On the other hand, formula (1) can be interpreted as

$$\lim_{t\to 0^+} t^{-d}\theta_\lambda(f_t) = c(\lambda)\hat{T}_d(f)\,,$$



where $\hat{T}_d$ is the Fourier transform of a measure supported on the nilpotent cone. Then $\hat{T}_d|_{\mathfrak{h}} = \frac{p(x)}{\pi(x)}$. This can be rewritten as

$$F_f(0;\partial(p)) = \mathrm{const}\, T_d(f)\,.$$

In a set of unpublished notes, Harish-Chandra has proved that such a formula must hold for any invariant distribution supported on the nilpotent cone.

The rest of this paper is devoted to associating coherent families of representations to nilpotent orbits $\mathrm{Ad}\,G\cdot X$ such that $AS(\theta) = \mathrm{cl}(\mathrm{Ad}\,G\cdot X)$ and finding the Weyl group representation that determines the polynomial $p$.

We now summarize the basic technical concepts which will appear repeatedly below. Suppose $\pi$ is an admissible representation of finite length of the (complex) group $G$. After Theorem 1 and before Proposition 4 we have defined four $G$-invariant closed nilpotent sets

$$AS(\theta_\pi) \subseteq WF_0(\theta_\pi) \subseteq WF(\pi) \subseteq \mathscr{V}(\mathscr{I}_\pi)\cap\mathfrak{g}^* \tag{$*$}$$

in $\mathfrak{g}^*$; all of these are defined by regarding $G$ as a real group. We are most interested in the first, because of its connection (just described) with orbital integrals. Conjecturally, all four coincide. At any rate (by Theorem 2) they have the same dimension. We will make a serious study only of the first three. This we will do by concentrating on $WF(\pi)$. Whenever we can show that it is an irreducible variety, it will follow that the first three coincide. What makes $WF(\pi)$ easier to study is the fact (obvious from the definition) that

$$WF(\pi) = WF(\pi\otimes F) \supseteq WF(\varrho) \tag{$**$}$$

whenever $F$ is a finite dimensional representation of $G$, and $\varrho$ is a subquotient of $\pi\otimes F$. [Such a property is shared by $\mathscr{V}(\mathscr{I}_\pi)\cap\mathfrak{g}^*$; but for this object the analogue of Lemma 8 is not known.] By [V1], it follows that for $\pi$ irreducible, $WF(\pi)$ depends only on

$$L\,\mathrm{Ann}(\pi) = \{u\in U(\mathfrak{g}) : u\otimes 1\in\mathscr{I}_\pi\}\,;$$

here we are using the identification $\mathfrak{g}_c = \mathfrak{g}\times\mathfrak{g}$ which gives $U(\mathfrak{g}_c) = U(\mathfrak{g})\otimes U(\mathfrak{g})$. By [D2], any primitive ideal is of the form $L\,\mathrm{Ann}(\pi)$ for some irreducible admissible $\pi$; so if $I = L\,\mathrm{Ann}(\pi)$, we define

$$WF(I) = WF(\pi)\,,$$

the *wavefront set of* $I$. By $(*)$

$$WF(I) \subseteq \mathscr{V}(I)$$

(the associated cone for $I$). Because of $(**)$, the wavefront set will be the same for ideals giving the same double cell (defined before Lemma 12 below).

## Induced Representations and Nilpotents

In [L-S] the notion of an induced nilpotent is defined and it is shown that it is compatible with the Springer map which associates a Weyl group representation to each conjugacy class of nilpotent elements. This representation is irreducible if one singles out the trivial character of the component group of the centralizer of the nilpotent. We show that the same is true for the differential operators introduced earlier.

Let $\mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ be the Levi decomposition ($\mathfrak{m} \supseteq \mathfrak{h}$ is the Levi component and $\mathfrak{n}$ the nilradical). Let $\mathscr{C}$ be a conjugacy class of nilpotents in $\mathfrak{m}$. Suppose that we have found a differential operator $\tilde{\omega}_{\mathscr{C}}$ such that

$$F_f^{(\mathfrak{m})}(0;\omega_{\mathscr{C}}) = T_{\mathscr{C}}^{(\mathfrak{m})}(f)$$

($T_{\mathscr{C}}^{(\mathfrak{m})}$ is the canonical measure on the orbit $\mathscr{C}$).

**Lemma 6.** *Let $T_{\mathscr{C}}$ be the measure supported on the orbit of the induced nilpotent. Then*

$$F_f(0;\omega_{\mathscr{C}}) = T_{\mathscr{C}}(f)$$

*for an appropriate normalization of the invariant measures.*

*Proof*. The induced nilpotent is defined as follows. Consider the set $\mathscr{C} + \mathfrak{n}$. The $\operatorname{Ad} G$ invariant set generated by this set consists of finitely many nilpotent orbits under the action of $G$ and there is a unique dense orbit. This orbit is by [L-S] the induced orbit of $\mathscr{C}$ from $\mathfrak{m}$ to $\mathfrak{g}$. Let $f \in C_c^\infty(\mathfrak{g})$. Define $f^K \in C_c^\infty(\mathfrak{g})$ by

$$f^K(x) = \int_K f(\operatorname{Ad} k \cdot x)\,dk$$

and

$$f_P(c) = \pi_1(c) \int_{\mathfrak{n}} f(c+x)\,dx \qquad c \in \mathfrak{m},$$

where $\pi_1(c) = |\det(\operatorname{ad} c)|_{\mathfrak{n}}|$.

Then the measure $T_{\mathscr{C}}$ can be normalized so that

$$T_{\mathscr{C}}(f) = T_{\mathscr{C}}^{(\mathfrak{m})}((f^K)_P).$$

On the other hand,

$$F_f(x) = F_{(f^K)_P}^{(\mathfrak{m})}(x) \qquad x \in \mathfrak{m}$$

and the assertion follows.

This is exactly the way in which irreducible representations of the Weyl group of $\mathfrak{m}$ are lifted to representations of the Weyl group of $\mathfrak{g}$ in [L-S].

This allows us to find the differential operators for a large class of nilpotent orbits by proceeding by induction. To do this in the case of the classical groups we need to know which nilpotents are induced. It is enough to consider the case when $\mathfrak{p}$ is a maximal parabolic since induction is transitive [L-S].

*Type* $(A_{n-1})$. Every nilpotent is induced. If the Jordan block decomposition of the nilpotent is $(n_1, \ldots, n_p)$ then it is induced from the trivial nilpotent coming from the associate class of parabolics corresponding to the transpose partition. For $(B_n)$, $(C_n)$, and $(D_n)$ it is convenient to parametrize the nilpotents by their symbols as in [L], or by their Jordan block decomposition.



*Type* $(B_n)$. The representations of the Weyl group of type $B_n$ can be parametrized by pairs of partitions $(\alpha_1, \ldots, \alpha_{m+1})$, $(\beta_1, \ldots, \beta_m)$ (see [L]) such that $\sum \alpha_i + \sum \beta_j = n$, $\alpha_i, \beta_j \geqq 0$. A symbol

$$\Lambda = \begin{pmatrix} \lambda_1 \ldots \lambda_{m+1} \\ \mu_1 \ldots \mu_m \end{pmatrix}$$

is obtained by taking $\lambda_i = \alpha_i + i - 1$, $\mu_i = \beta_i + i - 1$. If we identify

$$\begin{pmatrix} \lambda_1 \ldots \lambda_{m+1} \\ \mu_1 \ldots \mu_m \end{pmatrix} \sim \begin{pmatrix} 0 & \lambda_1 + 1 \ldots \lambda_{m+1} + 1 \\ 0 & \mu_1 + 1 \ldots \lambda_m + 1 \end{pmatrix},$$

then the equivalence classes of $\Lambda$'s such that

$$\lambda_i < \lambda_{i+1}, \mu_i < \mu_{i+1} \quad \text{and} \quad \sum_{i=1}^{m+1} \lambda_i + \sum_{i=1}^{m} \mu_i = n + m^2$$

parametrize the irreducible representations of the Weyl group.

Let $\bar{\mathscr{S}}_W$ be the set of symbols such that, $\lambda_i \leqq \mu_i$ and $\mu_i \leqq \lambda_{i+1} + 1$. Let $\mathscr{S}_W$ be the set of symbols such that $\lambda_i \leqq \mu_i \leqq \lambda_{i+1}$. $\mathscr{S}_W$ is called the set of special symbols. $\bar{\mathscr{S}}_W$ parametrizes the nilpotent orbits in the following way. Take the set $\{2\lambda_i + 1, 2\mu_i\}$ and order it in increasing order, say $\{v_j\}_{j=1\ldots 2m+1}$. Then $(v_j - j + 1)$ forms a partition of $2n+1$ such that every even part occurs an even number of times. This is the Jordan block decomposition of a nilpotent orbit in $B_n$. This correspondence, which is a bijection between $\bar{\mathscr{S}}_W$ and conjugacy classes, was first introduced in [L].

*Type* $C_n$. The Weyl group is the same as for $B_n$, so the symbols parametrize the irreducible representations as before. We let $\bar{\mathscr{S}}_W$ be the symbols such that $\lambda_i \leqq \mu_i + 1$ and $\mu_i \leqq \lambda_{i+1}$ and $\mathscr{S}_W$ be the symbols such that $\lambda_i \leqq \mu_i \leqq \lambda_{i+1}$.

Then $\bar{\mathscr{S}}_W$ is in a bijection with the nilpotent conjugacy classes as before, but the set $(v_j)_{j=1,\ldots,2m+1}$ is given by $\{2\lambda_i, 2\mu_i + 1\}$.

*Type* $D_n$. The irreducible representations of the Weyl group of type $D_n$ can be parametrized by two partitions $\alpha = (\alpha_1 \ldots \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ such that $\sum \alpha_i + \sum \beta_i = n$. This is obtained from taking the representation of the Weyl group of type $C_n$ with partitions $[(0, \alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m)]$ and restricting to $D_n$. Since $[(0, \alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m)$ or $(0, \beta_1, \ldots, \beta_m), (\alpha_1, \ldots, \alpha_m)]$ have the same restriction we regard $[\alpha, \beta]$ or $[\beta, \alpha]$ as identical. If $\alpha = \beta$ then the restriction from $C_n$ splits into 2 irreducible representations I and II. We take $\lambda_i = \alpha_i + i - 1$, $\mu_i = \beta_i + i - 1$ and form a symbol $\Lambda = \begin{pmatrix} \lambda_1 \ldots \lambda_m \\ \mu_1 \ldots \mu_m \end{pmatrix}$. Then the irreducible representations of $W_n$ are parametrized by such $\Lambda$, where $0 \leqq \lambda_i < \lambda_{i+1}$, $0 \leqq \mu_i < \mu_{i+1}$ and $\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \mu_i = n + m(m-1)$, subject to the identifications $\begin{pmatrix} \lambda \\ \mu \end{pmatrix} \sim \begin{pmatrix} \mu \\ \lambda \end{pmatrix}$ and

$$\begin{pmatrix} \lambda_1 \ldots \lambda_m \\ \mu_1 \ldots \mu_m \end{pmatrix} \sim \begin{pmatrix} 0 & \lambda_1 + 1 \ldots \lambda_m + 1 \\ 0 & \mu_1 + 1 \ldots \mu_m + 1 \end{pmatrix}$$

When $\lambda = \mu$ we have two representations I and II. Let $\bar{\mathscr{S}}_W$ be the symbols satisfying $\lambda_i \leqq \mu_i$ and $\mu_i \leqq \lambda_{i+1} + 1$, and $\mathscr{S}_W$ be the symbols satisfying $\mu_i \leqq \lambda_i \leqq \mu_{i+1}$, or $\lambda_i \leqq \mu_i \leqq \lambda_{i+1}$. $\bar{\mathscr{S}}_W$ is in a bijection with the conjugacy classes of nilpotents by taking $\{2\lambda_i+1, 2\mu_i\}$ and applying the procedure from before. When $\lambda=\mu$ there are two distinct nilpotents with the same Jordan block decomposition, I and II.

We now determine the induced nilpotents.

**Theorem 7.** *A symbol $\Lambda\in\mathscr{S}_W$ of type B, C or D corresponds to an induced nilpotent if and only if it is of the form*

$$\begin{pmatrix}\lambda_1\dots\lambda_{j-1}\lambda_j+1 & \lambda_{j+1}+1\dots\\ \mu_1 \quad \mu_{j-1}\mu_j+1 & \dots\end{pmatrix} \quad or \quad \begin{pmatrix}\lambda_1\dots\ \lambda_{j-1}\lambda_j & \lambda_{j+1}+1\dots\\ \mu_1 \quad \mu_{j-1}\mu_j+1 & \dots\end{pmatrix}$$

*where* $\begin{pmatrix}\lambda_1\dots\lambda_j\dots\\ \mu_1 \quad \mu_j\end{pmatrix}$ *is again a symbol.*

For type $D_n$ when $\lambda=\mu$ both nilpotents are induced.

*Proof.* We give details for type $C_n$ only. The other cases are entirely similar. This result was obtained independently by [Sl, Ke].

It is enough to find all the induced nilpotents from associate classes of maximal parabolics. An associate class is parametrized by a pair $(p,q)$ such that $p+q=n$ and $p\geqq 0$, $q\geqq 0$. The Levi component has semisimple part $A_{p-1}\times C_q$. A conjugacy class of nilpotents is given by a pair of partitions $[(p_1,\dots,p_s),(q_1,\dots,q_r)]$ such that $p_1+\dots+p_s=p$, $q_1+\dots+q_r=2q$, $0<q_1\leqq\dots\leqq q_r$, $0<p_1\leqq\dots\leqq p_s$ and the odd parts of $(q)$ occur an even number of times. Since we can induce in stages, and any nilpotent in an algebra of type $A$ is induced, it is enough to consider the case when the nilpotent in $A_{p-1}$ is trivial. (A nilpotent induced from the trivial nilpotent in the Levi component of a parabolic is called a Richardson orbit.) Consider the nilpotent corresponding to $(q_1\dots q_r)$ in $A_{2q-1}$. It is the Richardson orbit of the associate class of parabolics with dual partition

$$(\underbrace{r\dots r}_{q_1}, \underbrace{r-1, \quad\dots\quad, r-1}_{q_2-q_1}, \dots, \underbrace{1\dots 1}_{q_r-q_{r-1}}).$$

Then the associate class of parabolics corresponding to

$$(\underbrace{1\dots 1}_{q_r-q_{r-1}}, \underbrace{2\dots 2}_{q_{r-1}-q_{r-2}}, \underbrace{p\dots p}_{q_{-r-p+1}-q_{r-p}+2}, \dots, \underbrace{r\dots r}_{q_1})$$

has Richardson orbit given by the dual partition

$$(q_1,q_2,\dots,q_{r-p},q_{r-p+1}+2,q_{r-p+2}+2,\dots,q_r+2).$$

This partition corresponds to a nilpotent orbit in $C_n$ unless $q_{r-p}=q_{r-p+1}$ is odd. Then the partition

$$(q_1,\dots,q_{r-p}+1,q_{r-p+1}+1,q_{r-p+2}+2,\dots,q_r+2)$$

corresponds to a nilpotent in $C_n$ and has the largest dimension among such orbits in the closure of $(q_1,\dots,q_{r-p}, q_{1-p+1}+2,\dots,q_r+2)$ in $A_{2n-1}$. A dimension calculation and the previous lemma show that,

$$\begin{aligned}&\mathrm{Ind}^{C_n}_{(p,q)}((1\dots 1),(q_1\dots q_r))\\ &=\begin{cases}(q_1\dots q_{r-p}, q_{r-p+1}+2,\dots,q_r+2) & \text{if it is in } C_n\\ (q_1\dots q_{r-p}+1,q_{r-p+1}+1,q_{r-p+2}+2,\dots,q_r+2) & \text{otherwise.}\end{cases}\end{aligned}$$



To calculate the symbol of this partition we attach a zero, if necessary, to get a partition with an odd number of terms $(v_j)_{j=1}^{2m+1}$ and take $\left\{\left[\frac{v_j+j-1}{2}\right]\right\}$ ($[x]$ greatest integer less than or equal to $x$). The $\lambda$'s are the $\left[\frac{v_j+j-1}{2}\right]$ for $v_j+j-1$ even, the $\mu$'s for $v_j+j-1$ odd. But

$$\left[\frac{q_j+2+j-1}{2}\right]=\left[\frac{q_j+j-1}{2}\right]+1 \text{ and } \left[\frac{q_j+j-1+1}{2}\right]=\left[\frac{q_j+j-1}{2}\right]+1,$$

when $q_j+j-1$ is odd, and the proof is complete.

**Lemma 8.** *Let $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ be a parabolic subalgebra and $P=MN$ the corresponding subgroup. Let $\pi$ be an admissible representation of $M$. Then*

$$WF(\varrho)\subseteq \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{m}}(WF(\pi)),$$

*where $\varrho=\mathrm{Ind}^G_P(\pi\otimes 1)$.*

*Proof.* We use the results on the wave front set in [H]. Let $U\subseteq\mathfrak{g}^*$ be an open set such that $\mathrm{Ad}K\cdot U\subseteq U$ and

$$U\cap(AS(\pi)+\mathfrak{n})=\phi.$$

Let $\phi\in C_c^\infty(G)$ and $\psi\in C_c^\infty(G)$ be as in Theorem 1.4 in [H]. We must show that for $h$ and $f$, $C^\infty$-vectors in the representation space of $\varrho$,

$$I(\phi,\psi,t)=\int_G\langle\varrho(g)h,f\rangle\phi(g)e^{it\psi(g)}dg$$

is rapidly decreasing when $t\to\infty$.

Since $h(xmn)=\pi(m)h(x)$ and $\langle\varrho(g)h,f\rangle=\int_K\langle h(g^{-1}k_1),f(k_1)\rangle dk_1$ we can rewrite $I(\phi,\psi,t)$ as

$$I(\phi,\psi,t)=\int_{KNM}\int_K\langle\pi(m)h(k),f(k_1)\rangle\phi(k_1m^{-1}n^{-1}k^{-1})e^{it\psi(k_1m^{-1}n^{-1}k^{-1})}dk_1dkdndm.$$

Then, letting $\alpha=(k_1,n,k)$ be a parameter varying in a compact set, and letting $\psi_\alpha(m)=\psi(k_1m^{-1}n^{-1}k^{-1})$, $\psi_\alpha(m)=\phi(k_1m^{-1}n^{-1}k^{-1})$, $h_\alpha=h(k)$, $f_\alpha=f(k_1)$ we get

$$I^{(m)}(\phi_\alpha,\psi_\alpha,t)=\int_M\langle\pi(m)h_\alpha,f_\alpha\rangle\phi_\alpha(m)e^{it\psi_\alpha(m)}dm.$$

Since $U$ is $\mathrm{Ad}K$ invariant and we may take $k$ and $k_1$ close to the identity, $d\psi_\alpha(e)\in U^{(m)}$ where $U^{(m)}$ is an open set in $\mathfrak{m}^*$ such that $U^{(m)}\cap WF(\pi)=\phi$. Then $I^{(m)}(\phi_\alpha,\psi_\alpha,t)$ is rapidly decreasing, uniformly in $\alpha$. Since $I(\phi,\psi,t)$ is an integral of $I^{(m)}(\phi_\alpha,\psi_\alpha,t)$ over the parameter $\alpha$ which is contained in a compact set, $I(\phi,\psi,t)$ is also rapidly decreasing.

We now parametrize Harish-Chandra modules and their annihilators. For $\lambda\in\mathfrak{h}^*$, let $\mathfrak{b}=\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$ be the minimal parabolic and $M(\lambda)=U(\mathfrak{g})\bigotimes_{U(\mathfrak{b})}\mathbb{C}_{\lambda-\varrho}$ denote the Verma module of highest weight $\lambda-\varrho$. We write $\mathscr{Z}(\mathfrak{g})$ for the center of $U(\mathfrak{g})$ and $\chi_\lambda:\mathscr{Z}(\mathfrak{g})\to\mathbb{C}$ for the infinitesimal character of $M(\lambda)$. *We also define*

$$\Delta_\lambda=\left\{\alpha\in\Delta:\frac{2(\alpha,\lambda)}{(\lambda,\lambda)}\in\mathbb{Z}\right\}$$

$\Delta_\lambda^+=\Delta_\lambda\cap\Delta^+$, $W_\lambda=W(\Delta_\lambda)\subseteq W$.

We call $\lambda$ dominant if $2(\alpha,\lambda)/(\alpha,\alpha)$ is not a negative integer for any $\alpha\in\Delta^+$ and regular or nonsingular if $(\alpha,\lambda)\neq 0$ for all $\alpha\in\Delta$. The set of simple roots of $\Delta_\lambda$ will be called $\Sigma_\lambda$.

J3 §2 gives a better description

Set $\mathscr{U}=U(\mathfrak{g}\times\mathfrak{g})$. Then define $L(\lambda,\mu)$ for $\lambda,\mu\in\mathfrak{h}^*$ to be the $\mathscr{U}$-module consisting of $\mathfrak{k}$-finite elements of the dual of $M(-\lambda)\otimes M(-\mu)$. If $\mu\in\mathfrak{h}^*$ is integral, let $F_\mu$ be the irreducible representation of $\mathfrak{g}$ (or $\mathfrak{k}$) with extreme weight $\mu$.

**Proposition 9** (cf. [D1]). *$L(\lambda,\mu)=0$ unless $\lambda-\mu$ is integral. In that case, $F_{\lambda-\mu}$ as a representation of $\mathfrak{k}$ occurs exactly once in $L(\lambda,\mu)$. Let $V(\lambda,\mu)$ denote the unique irreducible subquotient of $L(\lambda,\mu)$ containing $F_{\lambda-\mu}$.*

*a) $V(\lambda,\mu)\simeq V(\lambda',\mu')$ if and only if $\lambda'=w\lambda$, $\mu'=w\mu$ for some $w\in W$;*

*b) if $-\lambda$ or $-\mu$ is dominant, then $V(\lambda,\mu)$ is the unique irreducible subrepresentation of $L(\lambda,\mu)$.*

According to [D1] any irreducible Harish-Chandra module is isomorphic to some $V(\lambda,\mu)$. Let $\lambda,\mu\in\mathfrak{h}^*$ be such that $-\lambda,-\mu$ are dominant. Then we can parametrize Harish-Chandra modules by $V(w\lambda,\mu)\simeq V(\lambda,w^{-1}\mu)$. [Since we are only interested in $AS(V(w\lambda,\mu))$, we may assume $\lambda=\mu$.]

Let $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$ and $I(\lambda)=\operatorname{Ann}L(\lambda)$.

If $V$ is any $\mathscr{U}$-module, put

$$L\operatorname{Ann}(V)=\{u\in U(\mathfrak{g}):u\otimes l\in\operatorname{Ann}V\}$$

$$R\operatorname{Ann}(V)=\{u\in U(\mathfrak{g}):l\otimes u\in\operatorname{Ann}V\}.$$

Let $u\mapsto\check{u}$ be the antiautomorphism of $U(\mathfrak{g})$ defined by $\check{x}=-x$ for $x\in\mathfrak{g}$. Then we have

**Corollary 10.**

$$L\operatorname{Ann}V(w\lambda,\lambda)=\check{I}(-w\lambda)$$

$$R\operatorname{Ann}V(w\lambda,\lambda)=\check{I}(-w^{-1}\lambda).$$

*Proof.* See [V1, Corollary 3.6] or [J3, Theorem 4.12].

**Corollary 11.**

$$\operatorname{Ann}V(w\lambda,\lambda)=U(\mathfrak{g})\otimes\check{I}(-w^{-1}\lambda)+I(-w\lambda)\otimes U(\mathfrak{g}).$$

*Proof.* This follows from the definition of the right and left annihilator and Corollary 10.

To each annihilator $I(w\lambda)$ of a Verma module we associate a left, right, and double cell in the following way. We say $w_1\underset{L}{\sim}w_2$ if $I(w_1\lambda)=I(w_2\lambda)$ and $w_1\underset{R}{\sim}w_2$ if $I(w_1^{-1}\lambda)=I(w_2^{-1}\lambda)$. Then

$$C_L(I(w\lambda))=\{w'\in W:w'\underset{L}{\sim}w\}$$

$$C_R(I(w\lambda))=\{w'\in W:w'\underset{R}{\sim}w\}$$

$$C(I(w\lambda))=\{\text{the smallest set generated by }\underset{L}{\sim},\underset{R}{\sim}\text{ from }w\}.$$



We can associate Weyl group representations to the cells in the following way. We denote by $\Theta(\lambda,\mu)$ the character of $V(\lambda,\mu)$. Then let

$$\hat{\Phi}_L(I(w\lambda))=\operatorname{lin\,span}\{\Theta(w'\lambda,\lambda):L\operatorname{Ann}V(w'\lambda,\lambda)\supseteqq I(w\lambda)\}$$

$$\Phi_L(I(w\lambda))=\hat{\Phi}_L(I(w\lambda))/\operatorname{lin\,span}\{\Theta(w'\lambda,\lambda)\in\hat{\Phi}_L(I(w\lambda)):\operatorname{Dim}V(w\lambda,\lambda)<\operatorname{Dim}U(\mathfrak{g})/I\}.$$

The Weyl group $W_\lambda$ acts on $\Phi_L$ by coherent continuation of the characters $\Theta(w\lambda,\lambda)$ on the left factor. Similarly, $W_\lambda$ acts on $\Phi_R$ and $W_\lambda\times W_\lambda$ acts on $\Phi$. As was pointed out earlier, the wavefront set of $I(-w\lambda)$ depends only on the double cell containing $w$; so we may speak of the wavefront set of a double cell.

We will call both the $C$'s and $\Phi$'s cells but we will use $C$ or $\Phi$ to denote which one we are referring to. For a more ample explanation of why we make this definition, see [J4] or [B-V2].

**Lemma 12.** *The $\hat{\Phi}_L(I)$ are mutually disjoint. The Weyl group $W_\lambda$ acts on each cell $\Phi_L(I)$ by coherent continuation in the second factor. The representation of $W_\lambda$ on the direct sum of the $\Phi_L(I)$ is isomorphic to the regular representation of $W_\lambda$. The same is true for the right cells. A double cell is a representation of the complex Weyl group $W_c\simeq W_\lambda\times W_\lambda$ and the representation of $W_\lambda\times W_\lambda$ on the direct sum of all the double cells is isomorphic to the representation on $L^2(W_\lambda)$.*

*Proof.* The fact that these cells are representations for the corresponding groups follows from [J4]. Clearly, they are disjoint. To see that their sum is the regular representation, we argue as follows. First we note that

$$\oplus\Phi_L(I)\simeq\operatorname{lin\,span}\{\Theta(w\lambda,\lambda)\}.$$

Let $P_{(w\lambda,\lambda)}$ be the harmonic part of $\pi\cdot\Theta(w\lambda,\lambda)|_{\mathfrak{h}}$. Then the $\{P_{(w\lambda,\lambda)}\}$ are linearly independent since, by Theorem 2, if the harmonic part of a virtual character is 0 then the virtual character is 0. Since there are $|W_\lambda|$ independent characters, the statement follows. The fact that the action is the same as the left regular representation follows from the fact that a representation of a finite group on a filtered vector space is isomorphic to the associated graded representation. We identify bases of the cells with subsets of the Weyl group via $\Theta(w\lambda,\lambda)\leftrightarrow w$.

**Corollary 13.** *Identify the cells with representations on the space of harmonic polynomials. The lowest degree occurring in $\Phi_L(I)$ is $|\Delta^+|-d(I)$ and the representation occurring in that degree is irreducible and occurs with multiplicity 1 in that degree and 0 in lower degree.*

*Proof.* This follows from the results in [K1] relating the lowest harmonic term calculated in Proposition 5 to the Goldie rank of $U(\mathfrak{g})/I$.

**Proposition 14.** *Let $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{h}\subseteqq\mathfrak{m}$. Then the restriction of a left cell $\Phi_L(I)$ to $W_\mathfrak{m}^\lambda$ consists of a direct sum of left cells of $W_\mathfrak{m}^\lambda$. The same is true for right cells.*

*Proof.* By [D2], every primitive ideal is of the form $I(w)=\operatorname{Ann}L(w\lambda)$. Then $\hat{\Phi}_L(I)$ can be identified with a subset of $W_\lambda$. We write $\hat{\Phi}_{w_0}=\hat{\Phi}_L(I(w_0))$ for any $w_0\in W_\lambda$. It is enough to prove that $W_{\mathfrak{m}}^{\lambda}\cdot w_0 \cap \hat{\Phi}_L(I(w_0))$ consists of a direct sum of $W_{\mathfrak{m}}^{\lambda}$-cells when we take the appropriate quotient. If we denote by $\theta(w\lambda,\lambda)$ the lift of the character $\Theta(w\lambda,\lambda)$ to a neighborhood of $0\in\mathfrak{g}$ then (if we set $\lambda'=w_0\lambda$)

$$\frac{\pi}{\pi_{\mathfrak{m}}}\theta(w\lambda,\lambda)|_{\mathfrak{h}}=\theta^{(\mathfrak{m})}(f(w)\lambda',\lambda)|_{\mathfrak{h}}+\text{other terms},$$

where $f(w)\in W_{\mathfrak{m}}^{\lambda}$ is uniquely defined by the fact that $\theta^{(\mathfrak{m})}(f(w)\lambda',\lambda)$ contains the term $e^{(w\lambda,\lambda)}$ in the numerator; $w\mapsto f(w)$ is a bijection of $W_{\mathfrak{m}}^{\lambda}\cdot w_0$ to $W_{\mathfrak{m}}^{\lambda}$. Since the "other terms" in the expression of $\theta(w\lambda,\lambda)$ do not contain any $W_{\mathfrak{m}}^{\lambda}$-translates of $w_0$, the action of $W_{\mathfrak{m}}^{\lambda}$ on the $\theta(w\lambda,\lambda)$ coincides with the action of $W_{\mathfrak{m}}^{\lambda}$ on the $\theta^{(\mathfrak{m})}(f(w)\lambda',\lambda)$.

Using the characteristic variety defined in [J5] one can prove that, for $w_1,w_2\in\hat{\Phi}_{w_0}$

$$f(w_1)\in\mathscr{V}(I(f(w_2)))\Rightarrow w_1\in\mathscr{V}(I(w_2)).$$

This and the fact that $w_1\in\mathscr{V}(I(w_2))$ if and only if $I(w_1)\supsetneqq I(w_2)$ then imply that if $I^{(\mathfrak{m})}(f(w_1))\subsetneqq I^{(\mathfrak{m})}(f(w_2))$ then $I(w_1)\subsetneqq I(w_2)$ so that $\theta^{(\mathfrak{m})}(f(w_2)\lambda,\lambda)$ occurs in $\hat{\Phi}_{w_0}$. On the other hand, if $I(w')\supsetneqq I(w_0)$ has strictly smaller Gelfand-Kirillov dimension, then $I(f(w'))$ has smaller Gelfand-Kirillov dimension than $I(f(w_0))$ so the map $\theta(w\lambda,\lambda)\mapsto\theta^{(\mathfrak{m})}(f(w)\lambda',\lambda)$ factors to a map from $\hat{\Phi}_{w_0}/\mathrm{lin\,span}\{\theta(w\lambda,\lambda),\ w\in\hat{\Phi}_{w_0}$ of smaller dimension$\}$ to a sum of left cells from $\mathfrak{m}$.

The proof is complete.

Using these results, it is clear that the equivalences between the $I(w)$ are crucial for the determination of the characteristic polynomials $c(\lambda)$.

We assume $\lambda$ is such that $-\lambda$ is dominant.

*Definition.* Let $\Sigma_\lambda$ be the simple roots in $\Delta_\lambda^+$ and $w\in W_\lambda$. Then we define

$$S_\lambda(w)=\{\alpha\in\Delta_\lambda^+ : w\alpha\notin\Delta_\lambda^+\}$$

$$\tau_\lambda(w)=S_\lambda(w)\cap\Sigma_\lambda.$$

**Proposition 15.** *Let $\alpha\in\Sigma_\lambda$ and $w\in W_\lambda$. Suppose $\alpha\in\tau_\lambda(w^{-1})$ is such that $\tau_\lambda(w^{-1}s_\alpha)\nsubseteq\tau_\lambda(w^{-1})$, where $s_\alpha$ is the reflection about $\alpha$. Then*

$$I(s_\alpha w)=I(w).$$

*Proof.* See [J5, Theorem 5.1].

**Proposition 16.** *Suppose $\alpha_0,\alpha_1,\alpha_2,\alpha_3\in\Sigma_\lambda$ span a root system of type $D_4$, with $\alpha_0$ the central root. Set $A=\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$. Fix $w\in W_\lambda$ with*

$$\tau_\lambda(w^{-1})\cap A=\{\alpha_0\},$$

$$\tau_\lambda(w^{-1}s_{\alpha_0})\cap A=\{\alpha_1,\alpha_2,\alpha_3\}.$$

*Suppose we are given a subset $B'$ of $A-\{\alpha_0\}$ consisting of 2 or 3 elements with the following property. For every $\alpha,\alpha'\in B'$*

$$\alpha_0\notin\tau_\lambda(w^{-1}s_\alpha s_{\alpha'}).$$



*Let $s_{B'}=\prod_{\alpha\in B'}s_\alpha$. Then*

$$I(s_{B'}w)=I(w).$$

*Proof.* See [V2, Theorem 4.1].

The set $\tau_\lambda(w)$ contains information about the left annihilator of a Harish-Chandra module and $\tau_\lambda(w^{-1})$ contains the same information about right annihilators. We say that $w_1\underset{L}{\sim}w_2$ if we can get from $w_1$ to $w_2$ by multiplying $w_1$ with simple reflections on the left such that the conditions in Propositions 15 or 16 are satisfied at each step. Similarly we define $w_1\underset{R}{\sim}w_2$ for multiplication on the right and $w_1\approx w_2$ for using both equivalences.

### The Robinson-Schensted Algorithm for Classical Groups

In this section we take an explicit realization for each of the classical Weyl groups and associate to each element in the Weyl group a pair of tableaus.

We assume that $\lambda$ is a regular integral and identify $w$ with $w\varrho$.

*Type $A_{n-1}$.* $w\varrho$ can be identified with a permutation of $\{n,n-1,\dots,1\}$. We associate to each such permutation a pair of tableaus by using the Robinson-Schensted algorithm, one for the permutation, one for its inverse. A convenient reference is [B]. We describe the results we need. We also note that our conventions differ somewhat from the usual ones. For example the tableau is usually formed so that its rows and columns are increasing, whereas ours are decreasing.

A tableau is a set of numbers $(\xi_{ij})$ arranged in rows $(\xi_{i1},\xi_{i2},\dots,\xi_{in_i})$ such that $\xi_{il}\geqq\xi_{il+1}$ and $n_i\geqq n_{i+1}$. In addition, the numbers in the columns $(\xi_{1j},\xi_{2j},\dots)$ are also in decreasing order. A tableau is associated to a permutation $(i_1,i_2,\dots,i_n)$ in the following way. Suppose $i_1\dots i_r$ have already been arranged in a tableau. Then $i_{r+1}$ will replace the largest element less than $i_{r+1}$ in the first row. If it is less than everything in the first row it is just added to the first row. The replaced element is added to the next row by the same rules. If an element is replaced in the last row it is just added as the next row.

*Example.* (1423)

$$\begin{array}{cccc} 1 & 4 & 4\,2 & 4\,3 \\ & 1 & 1 & 2 \\ & & & 1 \end{array}.$$

Let $w$ be an element in $W_n$ and $w^{-1}$ its inverse. Then $w$ and $w^{-1}$ have the same size tableaus. $W_n$ is in $1-1$ correspondence with pairs of tableaus, $w\mapsto(T(w),T(w^{-1}))$. To get from a pair of equal size tableaus to $w$ we proceed as follows. First, look for the position of 1 in the right tableau. Suppose $i$ is in the corresponding position in the left tableau. Then $i$ replaces the smallest element larger than $i$ in the next higher row and so on until we reach the first row. The number we end up with is $i_n$ in the permutation $w$. We cross out 1 in the right tableau and continue with $2\dots n$.

*Example.*

a) 
| | | | |
|---|---|---|---|
| 4 2 | 4 3 | 1→2 |
| 3 1 | 2 1 | |

d)
| | | |
|---|---|---|
| 3 | 4 x | 4→3 |
| | x x | |

b)
| | | |
|---|---|---|
| 4 1 | 4 3 | 2→4 |
| 3 | 2 x | |

c)
| | | | |
|---|---|---|---|
| 3 1 | 4 3 | 3→1 | 3142 |
| | x x | | inverse 2413 |

We now turn to the relation between tableaus and equivalences of primitive ideals. For type $A_n$ these results can be found in [J5, Theorem 7.9].

1. The mapping $w \to (T(w), T(w^{-1}))$ is a $1-1$ correspondence between $W_n$ and pairs of tableaus of the same size (Robinson, Schensted, and Schützenberger – cf. [B, 2.3]).

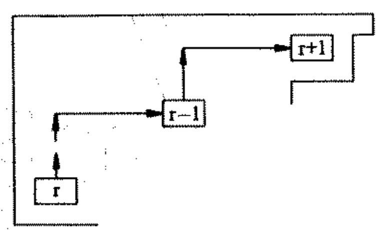
2. Let $w \in W_n$ and $\alpha \in \Sigma_\lambda$ be as in Proposition 15. Then the tableaus of $w$ and $s_\alpha w$ are related in the following way.

a) $T(w) = T(s_\alpha w)$.

b) $T(w^{-1}s_\alpha)$ can be obtained from $T(w^{-1})$ by a hook exchange.

We describe how this is done. If $w = (i_1 \dots i_r i_{r+1} \dots i_n)$ then $s_\alpha w = (i_1 \dots i_{r+1}, i_r \dots i_n)$, i.e., the $r^{\text{th}}$ and $(r+1)^{\text{st}}$ elements are interchanged. The condition on $\alpha$ and $w$ amounts to having $(i_r < i_{r-1} < i_{r+1}$ or $i_r < i_{r+2} < i_{r+1})$. For the right tableaus, if, say, $w^{-1} = (\dots r \dots r+1 \dots)$ then $w^{-1}s_\alpha = (\dots r+1 \dots r \dots)$ and the condition is that either $w^{-1} = (\dots r+1 \dots r-1 \dots r \dots)$ or $w^{-1} = (\dots r+1 \dots r+2 \dots r \dots)$.

This can be detected in the right tableau if the position of $r+1$ and $r$ can be joined by going vertically up and horizontally to the right, as below.




Then, $T(w^{-1}s_\alpha)$ is the tableau of $T(w^{-1})$ where $r$ and $r+1$ have been interchanged.

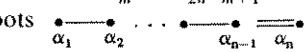
3. Two Harish-Chandra modules $V(w_1\lambda, \lambda)$ and $V(w_2\lambda, \lambda)$ have the same left annihilators if and only if $T(w_1) = T(w_2)$. They have the same right annihilators if and only if $T(w_1^{-1}) = T(w_2^{-1})$. If two elements $w_1$ and $w_2$ have the same left tableau then it is possible to pass from $w_1$ to $w_2$ by using equivalences as in Proposition 15 only [or using hook exchanges in $T(w_1^{-1})$].

Thus the left annihilators are parametrized by the left tableaus and the right annihilators by the right tableaus. The wavefront set of a representation is given by the nilpotent with Jordan cells given by the lengths of the rows of $T(w)$.

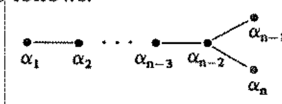
Since any nilpotent is induced in this case, Lemma 6 determines the left, right, and double cells.



*Type $B_n$, $C_n$.* Let $W_n = W_n(C) = W_n(B)$ be the Weyl group of type $C_n$ or $B_n$. By identifying $w$ with $w\varrho$ we may assume $w\varrho$ is a permutation of $\{n, n-1, \dots, 1, -1, \dots, -n+1, -n\}$ such that $i_m = -i_{2n-m+1}$.

Suppose we label the simple roots . To each $w \in W_n$ we associate a pair of tableaus $(T(w), T(w^{-1}))$ by using the Robinson-Schensted algorithm for $A_{2n-1}$ (we identify $n \leftrightarrow 2n$, $n-1 \leftrightarrow 2n-1 \dots -n+1 \leftrightarrow 2$, $-n \leftrightarrow 1$).

*Type $D_n$.* We denote the Weyl group of type $D_n$ by $W_n(D)$. We label the simple roots in the Dynkin diagram as follows.

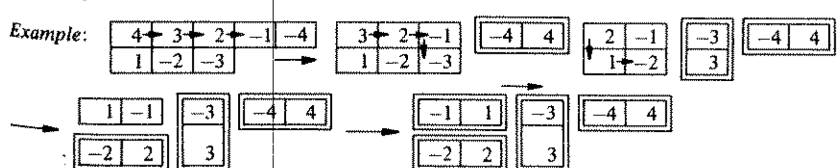



If we identify $w$ with $w\varrho$ then $w$ will be a permutation of $\{n-1, \dots, 1, 0, 0, -1, \dots, -(n-1)\}$ such that $i_m = -i_{n-m+1}$. We then embed $W_n(D)$ in $W_n$ by identifying $1 \leftrightarrow 2, \dots (n-1) \leftrightarrow n$ and $0 \leftrightarrow \pm 1$ such that, if $w = (i_1, \dots, i_n, -i_n, \dots, -i_1)$ then the number of positive numbers among $i_1 \dots i_n$ is even. Then we associate a pair $(T(w), T(w^{-1}))$ to each $w$ as for type $B_n, C_n$.

In order to be able to use induction it is necessary to use the Weyl group of $O(2n)$ instead. Under the identifications just described we can regard $W(O(2n))$ as $W_n$. Then the left annihilators of $(\dots 1 \dots -1 \dots)$ and $(\dots -1 \dots, +1 \dots)$ are the same. For the right annihilators, $(i_1 \dots i_n, -i_n, \dots -i_1)$ and $(i_1, \dots -i_n, i_n \dots -i_1)$ are equivalent. We include these two equivalences in the definitions of $\underset{L}{\sim}$, $\underset{R}{\sim}$ and $\approx$.

We now describe all the tableaus coming from $W_n$. The element $n$ appears in the first entry. We consider the following operation. We interchange $n$ with the larger element on its right or below it. We continue until this is no longer possible, i.e., $n$ has reached the corner of the tableau and there is no element on the right or below it. We say a tableau has property $C_n$ if the last element replaced was $-n$. We can cross out $\{n, -n\}$ from the tableau and get a $(2n-2)$-tableau. We say the tableau has property $(C)$ if all the successive smaller tableaus have the same property.

*Example:*



From the example it is clear that a tableau with property $(C)$ is formed by adding "dominoes" $\boxed{-i \mid i}$ or $\begin{array}{|c|}\hline -i \\ \hline i \\ \hline\end{array}$ for $i = 1, \dots, n$ and then reversing the procedure outlined before to move $i$ to the first entry of the tableau.

**Proposition 17.** *A pair of 2n-tableaus corresponds to an element in $W_n$ (i.e., it is of the form $(T(w), T(w^{-1}))$) if and only if the tableaus have property $(C)$. The partition $(n(1), \dots n(r))$ of $2n$ associated to the tableau is such that it can be obtained from a symbol $\binom{\lambda}{\mu}$ by the following process. Take the set $\{2\lambda_i, 2\mu_i + 1\}$ and order it in* *increasing sequence* $(v_1, \dots, v_{2m+1})$. *Then* $\{v_j - j + 1\}$ *is a partition of* $2n$ *differing from* $(n(1), \dots, (n(r)))$ *by possibly some* 0*'s*.

*Proof*. (The first statement is proved in [G].) We describe a way to get from $w$ to $(T(w), T(w^{-1}))$ that takes into account the symmetry of $W_n$. Suppose the tableau for $(i_2 \dots i_n, -i_n \dots -i_2)$ has been constructed. We then add $-i_1$ by the Robinson-Schensted algorithm. To add $i_1$ on the left, we do the operations described in the Robinson-Schensted algorithm but acting on columns instead of rows. For example, consider $(-1\ \ 3\ \ 2\ \ -2\ \ -3\ \ 1)$. We get the following steps

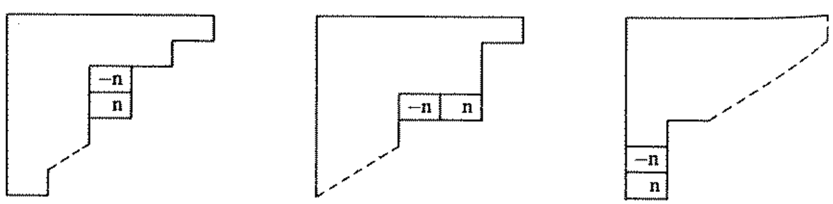
$$\begin{array}{|c|c|}\hline 2 & -2 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 2 & -2 & -3 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 3 & 2 & -2 & -3 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & -3 \\ \hline -2 \\ \cline{1-1}\end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & -3 \\ \hline -1 \\ \cline{1-1}\end{array} \qquad \text{$-2$ added to next column} \qquad \longrightarrow \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & -3 \\ \hline -1 & -2 \\ \cline{1-2}\end{array}$$

On the other hand, if we are given a pair of tableaus we can reverse the procedure to get $i_1$ and the tableau for $(i_2 \dots i_n | -i_n, -i_{n-1} \dots -i_2)$. First apply operation $(C_n)$ to the right tableau. To get $-i_1$ we take the element in the left tableau which is in the same place as $+n$ in the right tableau and put it in the place of the smallest element larger than it in the next higher row. After a finite number of steps we reach the first row. We are left with $-i_1$ and a $(2n-1)$ tableau. We now take the same element in the left tableau (which is in the same place as $-n$ in the right tableau) and repeat the operation described by using columns. This gives $i_1$ and the left tableau for $(i_2 \dots i_n, -i_n \dots -i_2)$. We cross out $n$ and $-n$ in the right tableau. We apply the same operations to the pair of $(2n-2)$ tableaus until we get all elements of the permutation for $w$.

Using the procedures described, the first statement of the proposition follows by induction on $n$.

Now suppose a tableau has property $(C_n)$. Then applying the operation $(C_n)$ to it we get a tableau with $(2n-2)$ elements and a domino attached to it, say,




Suppose by induction that the $(2n-2)$ tableau satisfies the second statement of the proposition. Then the partition associated to the $(2n)$-tableau is obtained by either

a) adding a 2 to the length of one row;
b) adding a 1 to the length of 2 consecutive rows
c) adding two rows of length 1.

The proposition follows from this observation.

*Definition*. To each $w \in W_n$ we associate a symbol in the following way:

*Type B, C*. We associate the symbol coming from Proposition 17.



*Type D or* $W(O(2n))$. Let $\begin{pmatrix} \lambda_1 \dots \lambda_{m+1} \\ \mu_1 \dots \mu_m \end{pmatrix}$ be the symbol attached to $w$ as in Proposition 17. By using the equivalences for symbols we may assume $\lambda_1 = 0$. Then $\begin{pmatrix} \lambda_2 - 1, \dots, \lambda_{m+1} - 1 \\ \mu_1, \quad \dots, \mu_m \end{pmatrix}$ is the symbol of type $D$ corresponding to $w$.

A Weyl group element has a special symbol if the corresponding symbol defined earlier corresponds to a special representation.

## The Main Theorem

This section is devoted to the proof of the following theorem.

**Theorem 18.** *Let* $w, w_1, w_2$ *be elements of* $W(B_n)$, $W(C_n)$ *or* $W(D_n)$.

1. $w_1, w_2 \in W_n$ *have the same left tableau only if* $w_1 \underset{L}{\sim} w_2$.
2. *If the symbol of* $w$ *is* $\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$ *then there is* $w' \in W_n$ *such that* $w \underset{L}{\sim} w'$ *and the symbol of* $w'$ *is special and a permutation of the symbol of* $w$ *(i.e.,* $\{\lambda, \mu\} = \{\lambda', \mu'\}$ *as sets).*
3. *If the symbol of* $w_1$ *is a permutation of the symbol of* $w_2$ *then* $w_1 \approx w_2$. *The wavefront set of* $I(w_1)$ *is the closure of the nilpotent whose symbol is a permutation of the symbol of* $w$ *and is special.*
4. *The regular representation of* $W_n$ *splits into double cells in the same manner as the equivalence* $\approx$. *More precisely, two representations,* $\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$ *and* $\begin{pmatrix} \lambda' \\ \mu' \end{pmatrix}$ *belong to the same cell if and only if* $\{\lambda, \mu\} = \{\lambda', \mu'\}$.

The proof is split up in a series of Lemmas 19–29. It is by induction on the rank of $\mathfrak{g}$. Since it is rather lengthy we give a brief outline. The main problem is that it is no longer true that one can go from a tableau to *any* other tableau of the same shape by hook exchanges only. We always assume the results true for $n-1$. Lemma 19 starts the induction. Lemma 20 relates hook exchanges in $B_n, C_n, D_n$ to the hook exchanges in $A_{2n-1}$. Lemma 21 gives a precise description of the shapes for which hook exchanges are enough to pass from one tableau to any other. Corollary 22 shows roughly how many tableaus one can obtain from a given one using the induction hypothesis and hook exchanges. Lemma 23, Corollary 24, Lemma 25, and Corollary 26 deduce from this a weak characterization of what the double cells are. Lemmas 27 and 28 characterize the left cells, Lemma 29 describes the splitting of regular representations of $W$ into double cells.

**Lemma 19.** *The statements of Theorem 18 hold for* $B_2$, $C_2$, *and* $D_4$.

*Proof*. *Type* $B_2$, $C_2$. There are 8 elements in the Weyl group and using the equivalences in Proposition 15, we get the following left tableaus and equivalences.

$$\text{I}\left\{ \begin{array}{|c|c|c|c|}\hline 2 & 1 & -1 & -2 \\ \hline\end{array} \right. \quad \text{1 elt.}$$

$$\text{II}\left\{ \begin{array}{l} \begin{array}{|c|c|}\hline 2 & -1 \\ \hline 1 & -2 \\ \hline\end{array} \quad \text{2 elts.} \\[2ex] \begin{array}{|c|c|c|}\hline 2 & 1 & -2 \\ \hline -1 \\ \cline{1-1}\end{array} \quad \text{1 elt.} \end{array} \right.$$

$$\text{III}\left\{ \begin{array}{l} \begin{array}{|c|c|}\hline 2 & 1 \\ \hline -1 & -2 \\ \hline\end{array} \quad \text{2 elts.} \\[2ex] \begin{array}{|c|c|}\hline 2 & -1 \\ \hline 1 \\ \cline{1-1} -2 \\ \cline{1-1}\end{array} \quad \text{1 elt.} \end{array} \right.$$

$$\text{IV}\left\{ \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline -1 \\ \hline -2 \\ \hline\end{array} \right. \quad \text{1 elt.}$$

It is clear what the wavefront sets of I and IV are. Also, II and III have the same wavefront set since II $\approx$ III. The element $(2\ -1\ 1\ -2)$ is the longest element in the Levi component of a parabolic so the nilpotent is the Richardson orbit for that associate class of parabolic. Thus, II and III have as a wavefront set the nilpotent with Jordan cells $(2,2)$. By Proposition 5, Lemma 6, and Corollary 13, the representation corresponding to $(2,2)$ occurs in the double cell of II and III. Since its dimension is 2, the left cells II and III are distinct. The remaining representations corresponding to $(2,1,1)$ and $(3,1)$ must belong to the cell defined by II and III. The assertions of the lemma follow.

*Type $D_4$.* There are 192 elements in the $W_n(D)$. Using the equivalences for $O(2n)$ on the left we get the following left tableaus:

I $\begin{array}{cccccccc} 4&3&2&1&-1&-2&-3&-4 \end{array}$ 1 elt.

II $\left\{ \begin{array}{l} \begin{array}{cccccc} 4&3&1&-1&-2&-4\\ 2&-3 \end{array} \quad \text{2 elts.} \\ \begin{array}{ccccc} 4&2&1&-2&-4\\ 3&-1&-3 \end{array} \quad \text{2 elts.} \end{array} \right.$

III $\left\{ \begin{array}{cccccc} 4&3&2&1&-3&-4\\ -1&-2 \end{array} \right.$ 1 elt. $\begin{array}{ccccc} 4&3&2&1&-4\\ -1&-2&-3 \end{array}$ 3 elts.

IV $\left\{ \begin{array}{cccccc} 4&3&2&-1&-3&-4\\ 1&-2 \end{array} \right.$ 3 elts. $\begin{array}{ccccc} 4&3&2&-1&-4\\ 1&-2&-3 \end{array}$ 1 elt.

V $\left\{ \begin{array}{cccccc} 4&2&1&-1&-2&-3\\ 3&-4 \end{array} \right.$ 3 elts. $\begin{array}{ccccc} 4&2&1&-2&-3\\ 3&-1&-4 \end{array}$ 1 elt.

VI $\left\{ \begin{array}{cccc} 4&1&-1&-2\\ 3&-3\\ 2&-4 \end{array} \right.$ 3 elts. $\begin{array}{ccc} 4&1&-2\\ 3&-1&-3\\ 2&-4 \end{array}$ 3 elts. $\begin{array}{ccc} 4&1&-2\\ 3&-1&-3\\ 2\\ -4 \end{array}$ 5 elts. $\begin{array}{cccc} 4&1&-1&-2\\ 3&-3\\ 2\\ -4 \end{array}$ 3 elts.

VII $\left\{ \begin{array}{cccc} 4&2&-1&-3\\ 3&-2\\ 1&-4 \end{array} \right.$ 3 elts. $\begin{array}{cccc} 4&2&-1&-3\\ 3&-2\\ 1\\ -4 \end{array}$ 3 elts. $\begin{array}{ccc} 4&2&-1\\ 3&-2&-3\\ 1&-4 \end{array}$ 3 elts. $\begin{array}{ccc} 4&2&-1\\ 3&-2&-3\\ 1\\ -4 \end{array}$ 5 elts.

VIII $\left\{ \begin{array}{cccc} 4&3&-1&-4\\ 2&-2\\ 1&-3 \end{array} \right.$ 3 elts. $\begin{array}{cccc} 4&3&-1&-4\\ 2&-2\\ 1\\ -3 \end{array}$ 3 elts. $\begin{array}{ccc} 4&3&-1\\ 2&-2&-4\\ 1\\ -3 \end{array}$ 5 elts. $\begin{array}{ccc} 4&3&-1\\ 2&-2&-4\\ 1&-3 \end{array}$ 3 elts.

IX $\left\{ \begin{array}{cccc} 4&3&1&-4\\ 2&-2\\ -1&-3 \end{array} \right.$ 5 elts. $\begin{array}{ccc} 4&3&1\\ 2&-2&-4\\ -1&-3 \end{array}$ 3 elts. $\begin{array}{cccc} 4&3&1&-4\\ 2&-2\\ -1\\ -3 \end{array}$ 3 elts. $\begin{array}{ccc} 4&3&1\\ 2&-2&-4\\ -1\\ -3 \end{array}$ 3 elts.



X $\left\{ \begin{array}{cccc} 4&2&1&-3\\ 3&-2\\ -1&-4 \end{array} \right.$ 5 elts. $\begin{array}{ccc} 4&2&1\\ 3&-2&-3\\ -1&-4 \end{array}$ 3 elts. $\begin{array}{ccc} 4&2&1\\ 3&-2&-3\\ -1\\ -4 \end{array}$ 3 elts. $\begin{array}{cccc} 4&2&1&-3\\ 3&-2\\ -1\\ -4 \end{array}$ 3 elts.

XI $\left\{ \begin{array}{cccc} 4&3&2&-4\\ 1&-1\\ -2&-3 \end{array} \right.$ 5 elts. $\begin{array}{cccc} 4&3&2&-4\\ 1&-3\\ -1\\ -2 \end{array}$ 3 elts. $\begin{array}{ccc} 4&3&2\\ 1&-1&-4\\ -2&-3 \end{array}$ 3 elts. $\begin{array}{ccc} 4&3&2\\ 1&-3&-4\\ -1\\ -2 \end{array}$ 3 elts.

XII $\begin{array}{cccc} 4&2&-1&-3\\ 3&1\\ -2&-4 \end{array}$ 5 elts. $\begin{array}{ccc} 4&2&-3\\ 3&1&-4\\ -1\\ -2 \end{array}$ 5 elts.

XIII $\begin{array}{cccc} 4&3&-1&-2\\ 2&1\\ -3&-4 \end{array}$ 5 elts. $\begin{array}{ccc} 4&3&-2\\ 2&1&-4\\ -1\\ -3 \end{array}$ 5 elts.

XIV $\begin{array}{cccc} 4&2&-1&-3\\ 3&1&-2&-4 \end{array}$ 3 elts.

XV $\begin{array}{cccc} 4&2&1&-3\\ 3&-1&-2&-4 \end{array}$ 3 elts.

XVI $\begin{array}{cccc} 4&3&-1&-2\\ 2&1&-3&-4 \end{array}$ 3 elts.

XVII $\begin{array}{cccc} 4&3&1&-2\\ 2&-1&-3&-4 \end{array}$ 3 elts.

XVIII $\begin{array}{cccc} 4&3&2&1\\ -1&-2&-3&-4 \end{array}$ 3 elts.

XIX $\begin{array}{cccc} 4&3&2&-1\\ 1&-2&-3&-4 \end{array}$ 3 elts.

XX $\begin{array}{cc} 4&-1\\ 3&-2\\ 2&-3\\ 1&-4 \end{array}$ 3 elts.

XXI $\begin{array}{cc} 4&2\\ 3&1\\ -1&-3\\ -2&-4 \end{array}$ 3 elts.

XXII $\begin{array}{cc} 4&1\\ 3&-2\\ 2&-3\\ -1&-4 \end{array}$ 3 elts.

XXIII $\begin{array}{cc} 4&2\\ 3&-1\\ 1&-3\\ -2&-4 \end{array}$ 3 elts.

XXIV $\begin{array}{cc} 4&3\\ 2&-1\\ 1&-2\\ -3&-4 \end{array}$ 3 elts.

XXV $\begin{array}{cc} 4&3\\ 2&1\\ -1&-2\\ -3&-4 \end{array}$ 3 elts.

XXVI $\begin{array}{ccccc} 4&3&1&-2&-4\\ 1\\ -1\\ -3 \end{array}$ 3 elts.

XXVII $\begin{array}{ccccc} 4&3&2&-3&-4\\ 1\\ -1\\ -2 \end{array}$ 3 elts.

XXVIII $\begin{array}{ccccc} 4&2&1&-2&-3\\ 3\\ -1\\ -4 \end{array}$ 3 elts.

XXIX $\begin{array}{llll} 4 & 3 & -1 & -4 \\ 2 \\ 1 \\ -2 \\ -3 \end{array}$ 3 elts.

XXX $\begin{array}{ll} 4 & 3 \\ 2 & -4 \\ 1 \\ -1 \\ -2 \\ -3 \end{array}$ 3 elts.

XXXI $\begin{array}{llll} 4 & 1 & -1 & -2 \\ 3 \\ 2 \\ -3 \\ -4 \end{array}$ 3 elts.

XXXII $\begin{array}{ll} 4 & 3 \\ 2 & -1 \\ 1 & -4 \\ -2 \\ -3 \end{array}$ 1 elt. $\begin{array}{ll} 4 & 3 \\ 2 & -4 \\ 1 \\ -1 \\ -2 \\ -3 \end{array}$ 3 elts.

XXXIII $\begin{array}{ll} 4 & 1 \\ 3 & -2 \\ 2 & -3 \\ -1 \\ -4 \end{array}$ 1 elt. $\begin{array}{ll} 4 & 1 \\ 3 & -2 \\ 2 \\ -1 \\ -3 \\ -4 \end{array}$ 3 elts.

XXXIV $\begin{array}{ll} 4 & -1 \\ 3 & -2 \\ 2 \\ 1 \\ -3 \\ -4 \end{array}$ 3 elts. $\begin{array}{ll} 4 & -1 \\ 3 & -2 \\ 2 & -3 \\ 1 \\ -4 \end{array}$ 1 elt.

XXXV $\begin{array}{ll} 4 & 2 \\ 3 & -1 \\ 1 & -3 \\ -2 \\ -4 \end{array}$ 1 elt. $\begin{array}{ll} 4 & 2 \\ 3 & -3 \\ 1 \\ -1 \\ -2 \\ -4 \end{array}$ 3 elts.

XXXVI $\begin{array}{l} 4 \\ 3 \\ 2 \\ 1 \\ -1 \\ -2 \\ -3 \\ -4 \end{array}$ 1 elt.

Using Propositions 15 and 16 we find the following decomposition into double cells:

{I}, {II, III, IV, V}, {VI–XIII}, {XIV, XVI, XVIII}, {XV, XVII, XIX},
{XX, XXIII, XXIV}, {XXI, XXII, XXV}, {XXVI, XXVII, XXVIII},
{XXIX–XXXI}
{XXXII–XXXV}, {XXXVI}.

In order to see that these are all equivalences we calculate the wavefront sets. Since the permutations

$(4\ 3\ -1\ -2\ 2\ 1\ -3\ -4)$, $(4\ 1\ 2\ 3\ -3\ -2\ -1\ -4)$, $(3\ 4\ -1\ -2\ 2\ 1\ -4\ -3)$,
$(1\ 2\ 3\ 4\ -4\ -3\ -2\ -1)$ $(4\ -3\ -2\ 1\ -1\ 2\ 3\ -4)$, $(4\ 3\ -2\ -1\ 1\ 2\ -3\ -4)$

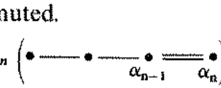
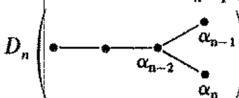
all correspond to Weyl group elements such that the corresponding primitive ideals are annihilators of induced representations this takes care of all double cells except for {XXXII–XXXV}, {XV, XVII, XIX}, {XXI, XXII, XXV}. Using the



outer isomorphism of $D_4$ we can also conclude that the wavefront sets of {XV, XVII, XIX} and {XIV, XVI, XVIII} are the nilpotents I and II with Jordan cells $(4,4)$ and the wavefront sets of {XXI, XXII, XXV} and {XX, XXIII, XXIV} are the nilpotents I and II for $(2,2,2,2)$. By using the Duflo ordering in the Weyl group defined in [D2] we can conclude that the wavefront set of {XXXII–XXXV} is contained in the closure of the orbit with Jordan cells $(1,1,1,1,1,3)$. Since this can be done using $w_1=(-4\ -3\ -1\ -2\ 2\ 1\ 3\ 4)$ in XXXIII and $w_2=(-4,-3,2,1,-1,-2,3,4)$ in XXXI, the $\tau_L$ invariants are not the same and the wavefront set of XXXIII is of strictly smaller dimension than the wavefront set of XXXI. Thus the wavefront set of {XXXII–XXXV} is the nilpotent with Jordan cells $(1,1,1,1,2,2)$.

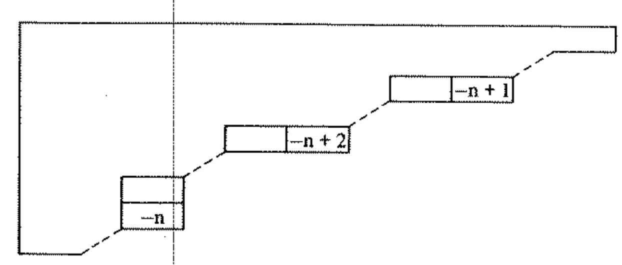
The decomposition of the regular representation into double cells can be obtained by using Proposition 14 to restrict to $W_m$ for m the Levi component of a parabolic subalgebra. We leave the details to the reader, since they are contained in Lemma 29 anyway.

In addition, we can check that, if we allow the equivalences from $O(2n)$, the tableaus change in such a way that the symbol is permuted.

**Lemma 20.** *Suppose $\alpha\in\Sigma$ such that $\alpha\neq\alpha_{n-1}$ for $B_n$, $C_n$* (  ) *and $\alpha\neq\alpha_{n-2},\alpha_{n-1}$ for type $D_n$* (  ). *If $w$ and $\alpha$ satisfy the conditions of Proposition 15 then $T(w)=T(s_\alpha w)$ and $T(w^{-1})$ and $T(w^{-1}s_\alpha)$ differ by hook exchanges.*

*Proof.* The lemma follows easily from the fact that, if $w=(\dots n_i n_{i+1} n_{i+2}\dots)$ is such that $n_i$ is between $n_{i+1}$ and $n_{i+2}$ then the same is true about $(\dots -n_2 -n_{i+1}, -n_i\dots)$.

We will need the following hook exchanges later. Suppose $w=(\dots i_{n-2}, i_{n-1}, i_n | -i_n, -i_{n-1}, -i_{n-2}\dots)$ is such that $i_{n-2}$ is between $i_{n-1}$ and $i_n$. Then $T(w^{-1})$ will look like




so that we can perform a hook exchange between $-n$ and $-n+1$. From the proof of Lemma 19 it follows that we can hook exchange $n$ and $n-1$. This means that the top left corner of the tableau is

$\begin{array}{|c|c|} \hline n & n-1 \\ \hline n-2 & \\ \hline \end{array}$ or $\begin{array}{|c|c|} \hline n & n-1 \\ \hline n-1 & \\ \hline \end{array}$

Then the hook exchanges will interchange the positions of $n-1$ and $n-2$ and of $-n$ and $-n+1$. The same hook exchange can be obtained by interchanging the dominoes $\boxed{\phantom{n}\,|\,-n}$ and $\boxed{\phantom{n}\,|\,-n+1}$ and using the operations $C_n$.

The other possibility is that $i_n$ is between $i_{n-1}$ and $i_{n-2}$ so that we can exchange $i_{n-1}$ and $i_{n-2}$. $T(w^{-1})$ will then look like

| | |
|---|---|
| n | n−1 |
| n−2 | |

$\boxed{\phantom{n}\,|\,n+2}$

$\boxed{\phantom{n}\,|\,-n}$

$\boxed{\phantom{n}\,|\,-n+1}$

and the same argument applies. It is also possible that the tableau will have shape

| | |
|---|---|
| | |
| | −n+1 |
| −n+2 | −n |

with dominoes

| | |
|---|---|
| −n+2 | −n+1 |
| n−2 | n−1 |
| −n | n |

In this case the hook exchange will interchange $-(n-1)$ and $-(n-2)$ and the new tableau will have dominoes

| | |
|---|---|
| −n+2 | n−2 |
| −n+1 | −n |
| n−1 | +n |

Let $w$ be a Weyl group element with symbol $\binom{\lambda}{\mu}$. We will say $\boxed{-n\,|\,n}$ is at the $\lambda_i$ or $\mu_j$ place if, after subtracting the domino $\boxed{-n\,|\,n}$ from the right tableau, the symbol of the remaining tableau is obtained by subtracting $-1$ from $\lambda_i$ or $\mu_j$. In the next lemmas we will need to move $\boxed{-n\,|\,n}$ to various places in the tableau. We first eliminate all the places that can be reached by hook exchanges. We give details for type $D_n$ only. For type $B_n$ and $C_n$ it will follow from this.

Suppose $\boxed{-n\,|\,n}$ is at $\lambda_i$ and we would like to move it to, say, $\lambda_j$. If $w=(i_n w'-i_n)$ then by the induction hypothesis in Theorem 18 we should be able to place $\boxed{-n+1\,|\,n-1}$ and $\boxed{-n+2\,|\,n-2}$ anywhere in the right tableau, provided the resulting pair of tableaus $(T(w'), T_1)$ gives a Weyl group element of the same parity as $w'$.

**Lemma 21.** *Suppose* $\binom{\lambda}{\mu}$ *is a symbol such that, we can never get the shape* $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$ $\left[\textit{with symbol}\binom{1}{1}\right]$ *by subtracting dominoes. Then all* $w\in W_n$ *with symbol* $\binom{\lambda}{\mu}$ *have the same parity. If* $T(w_1)=T(w_2)$ *then* $w_1 \underset{L}{\sim} w_2$.

*Proof.* We say that a symbol $\binom{\lambda}{\mu}$ is reducible to $\binom{1}{1}$ if we can get from $\binom{\lambda}{\mu}$ to $\binom{1}{1}$ by subtracting 1's from either $\lambda$ or $\mu$ such that the result is always a symbol. This is equivalent to subtracting dominoes in the tableau and is possible if and only if



$\lambda_m \geqq m$, $\mu_m \geqq m$. Thus the symbols that the lemma is concerned with, are

$$\text{I.}\begin{pmatrix}\lambda_1 & & \dots & \lambda_m\\ 0 & 1 & \dots & m-1\end{pmatrix}\qquad \text{II.}\begin{pmatrix}0 & 1 & \dots & m-1\\ \mu_1 & & \dots & \mu_m\end{pmatrix}.$$

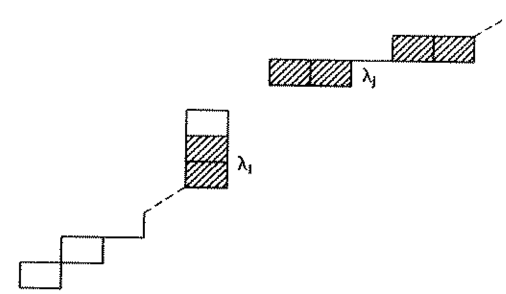
The shapes of the tableaus have symbols

$$\text{I.}\begin{pmatrix}0 & \lambda_1+1 & \dots & \lambda_m+1\\ & 0 & \dots & m-1\end{pmatrix}\sim\begin{pmatrix}\lambda_1 & & \dots & \lambda_m\\ 0 & 1 & \dots & m-2\end{pmatrix}$$

$$\text{II.}\begin{pmatrix}0 & 1 & \dots & m\\ \mu_1 & & \dots & \mu_m\end{pmatrix}.$$

Consider case I. Let $j$ be the first integer such that $2\lambda_j > 2(m-2)+1$. Then the rows of the tableau are given by

$$\begin{array}{ccc}
 & 2\lambda_m & \\
 & \vdots & \\
 & 2\lambda_{j+1} & \\
 & \vdots & \\
 & 2\lambda_j & \\
 & \vdots & \\
 & 2m-3 & \\
\vdots & \vdots & \vdots\\
\lambda_1+1 & 2\lambda_1+1 & \lambda_1\\
\lambda_1 & 2\lambda_1 & \lambda_1\\
\lambda_1-1 & 2\lambda_1-1 & \lambda_1\\
\vdots & \vdots & \vdots\\
2 & 5 & 3\\
1 & 3 & 2\\
0 & 1 & 1
\end{array}$$




We may assume $\lambda_1 > 0$. The shaded area is the $\lambda_1$-place and the $\lambda_j$-place.

We now use induction. The statement is easily checked for $D_4$. Let $\boxed{\phantom{n}\,|\,n}$ be at $\lambda_1$. By induction we can place the domino $\boxed{\phantom{n}\,|\,n-1}$ at any other available place and $\boxed{\phantom{n}\,|\,n-2}$ in between them and perform a hook exchange between $-n$ and $-n+1$. The assertion about the parity of the Weyl group elements follows because we have proved that any two elements with the same left tableau are equivalent by hook exchanges, and hook exchanges preserve parity. Case II follows in the same way.

**Corollary 22.** *Let* $w_1, w_2 \in W_n$ *and* $T(w_1)$, $T(w_2)$ *be the corresponding tableaus. If* $T(w_1)=T(w_2)$, *and* $w_1, w_2$ *have the same parity in case* $D_n$, *then* $w_1 \underset{L}{\sim} w_2$ *with the following exceptions.*

*Type $D_n$.* The tableaus have symbols:

1. $\begin{pmatrix} 0 & 1 & \dots i-1 & i+1 & \\ i+1 & i+3 & \dots & 2i+2 & \end{pmatrix}$ or $\begin{pmatrix} i+2 & i+4 & \dots & & 2i+2 \\ 0 & 1 & \dots & i & i+2 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 1 & \dots & i-1 & i+1 \\ i & i+2 & \dots & & 2i+1 \end{pmatrix}$ or $\begin{pmatrix} i & i+2 & \dots & & 2i+1 \\ 0 & & \dots & i-1 & i+1 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 1 & \dots & i-1 & i+1 & i+2 \\ i & i+2 & \dots & & & 2i+2 \end{pmatrix}$ or $\begin{pmatrix} i+2 & i+4 & \dots & & 2i+2 \\ 0 & 1 & \dots & i-1 & i+2 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 1 & \dots & i-1 & i+1 & i+2 \\ i+1 & i+3 & \dots & & & 2i+3 \end{pmatrix}$ or $\begin{pmatrix} i+1 & i+3 & \dots & & 2i+3 \\ 0 & 1 & \dots & i & i+3 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$

6. $\begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$

*Type $B_n$, $C_n$*

1. $\begin{pmatrix} 0 & 1 & \dots & i & i+2 \\ i+1 & i+3 & \dots & & 2i+2 \end{pmatrix}$ or $\begin{pmatrix} i+2 & i+4 & \dots & & 2i+2 \\ 0 & 1 & \dots & i-1 & i+1 \end{pmatrix}$

2. $\begin{pmatrix} 0 & 1 & \dots & i & i+2 \\ i+1 & i+3 & \dots & & 2i+2 \end{pmatrix}$ or $\begin{pmatrix} i & i+2 & \dots & & 2i+1 \\ 0 & & \dots & i-2 & i \end{pmatrix}$

*Proof*. We first note that the symbols are arranged so that they are transposes of each other. Furthermore, the shapes of the tableaus in Case $B_n$, $C_n$ are the same as Cases 1, 2 in type $D_n$.

The proof proceeds by using the induction assumptions in Theorem 18 and using hook exchanges to place the domino $\boxed{-n \mid n}$ in the same place in both $T(w_1^{-1})$ and $T(w_2^{-1})$. We omit the details.

**Lemma 23.** *Suppose $w \in W_n$. Then there is $w' \in W_n$ such that $w' \approx w$ and $w'$ has a special symbol which is a permutation of the symbol of $w$.*

*Proof*. Again the statement follows by induction. Since we are using all the equivalences in $O(2n)$ for type $D_n$ the proof is the same as for $B_n$, $C_n$.

If the tableau of $w$ doesn't have a symbol as in Cases 1, 2 in Corollary 22 we can use equivalences on the right and left so that

$$w \approx (k, \dots, n, w_1, -n, \dots, -k)$$

or

$$w \approx (-n, \dots, -k, w_1, k, \dots, n).$$

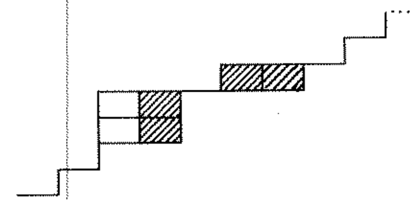
By induction, $w_1$ is equivalent to $w_2$ with special symbol. Then either $(k, \dots, n, w_2, -n, \dots, -k)$ [or $(-n, \dots, -k, w_2, k, \dots, n)$ as the case may be] has special symbol in which case we are done; or else we can find $k'$ and $w_2'$ such that

$$(k, \dots, n, w_2, -n, \dots, -k) \approx (k', \dots, n, w_2', -n, \dots, -k').$$



(Where $w_2'$ doesn't have special symbol. Changing $w_2'$ into an element with special symbol completes the proof. It can be easily checked that the symbol is only being permuted.)

If the symbol of $w$ is as in Cases 1, 2 in Corollary 22 then the shape of the tableau is



and the shaded dominoes are the places for $\boxed{-n \mid n}$ that cannot be interchanged. But then we may assume without loss of generality that $w$ is equivalent to an element of the form

$$(-(n-k-1), \dots, -(n-1), (n-2k-1), n-2, \dots, n-k-2, n, w_1, -n \dots)$$

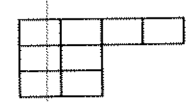
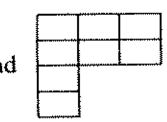
and repeat the argument from before.

The proof is now complete.

This statement would have been much harder to prove for $D_n$ since there are more tableaus that cannot be changed into each other by hook exchanges only.

**Corollary 24.** *If $w_1, w_2 \in W_n$ have the same special symbol $\binom{\lambda}{\mu}$ then $w_1 \approx w_2$.*

*Proof*. Type $B_n$, $C_n$, $O(2n)$: In view of Corollary 22 it is enough to prove the statement for tableaus with shapes

 and 

Since the shapes are transposes of each other, it is enough to see that we can pass from

$$\begin{matrix} 4 & 3 & -2 \\ 2 & 1 & -4 \\ -1 & & \\ -3 & & \end{matrix} \quad \text{to} \quad \begin{matrix} 4 & 1 & -2 \\ 3 & -1 & -3 \\ 2 & & \\ -4 & & \end{matrix}$$

leaving the left tableau unchanged. This can be checked using Propositions 15 and 16.

For type $O(2n)$, $\binom{\lambda}{\mu} \sim \binom{\mu}{\lambda}$ so we also have to show that there are $w_1 \approx w_2$ such that $S(w_1) = \binom{\lambda}{\mu}$, $S(w_2) = \binom{\mu}{\lambda}$. This is proved by induction as in Lemma 23.

**Lemma 25.** *If* $w \in W_n$ *has symbol* $\binom{\lambda}{\mu}$ *then* $\Theta(w\lambda, \lambda)$ *has asymptotic support and wave front set equal to the closure of the nilpotent orbit which has as symbol a permutation of the symbol of* $w$.

*Proof*. Type $B_n$, $C_n$: We show the assertion for induced nilpotents first. Lemma 20 and Corollary 21 show that all elements whose symbols are permutations of a given symbol $\binom{\lambda}{\mu}$ belong to the same double cell. Therefore they have the same wavefront set. Choose an element $w=(k, \ldots n, w', -n, \ldots, \ldots -k)$ in the double cell. Then $V(w\lambda, \lambda)$ is an induced representation from $V_{\mathfrak{m}}(w'\lambda, \lambda)$ on an appropriate parabolic $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$. Let $\pi=V_{\mathfrak{m}}(w'\lambda, \lambda)$ and $\varrho=\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\pi\otimes 1)$. Then by Proposition 4 and Lemma 8

$$AS(\varrho) \subseteq WF(\varrho) \subseteq \mathrm{cl}\,\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(WF(\pi)).$$

By induction, we may assume that $WF(\pi)=AS(\pi)$ is the closure of the nilpotent with symbol equal to a permutation of the symbol of $w'$ (note that the semisimple part of $\mathfrak{m}$ is isomorphic to $B_k$ or $C_k$, respectively). It is clear that $\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(WF(\pi))$ is a special nilpotent and its symbol is a permutation of the symbol of $w$. By Theorem 2, $\dim AS(\varrho)=\dim \mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(WF(\pi))$. So $AS(\varrho)=WF(\varrho)=\mathrm{cl}\,\mathrm{Ind}_{\mathfrak{m}}^{\mathfrak{g}}(WF(\pi))$.

Next we calculate the wavefront sets of the double cells with symbols equal to permutations of $S_p=\begin{pmatrix} 0 & 1 & \ldots & r & r+2 & \ldots & r+p & r+p+1 \\ & 1 & \ldots & r & r+1 & r+2 & \ldots & r+p \end{pmatrix}$.

We show that the wavefront set is the closure of the nilpotent orbit with Jordan cells $(\underbrace{2\ldots2}_{2p}, \underbrace{1\ldots1}_{2r})$. The symbol of this nilpotent is also $S_p$. There are two more types of elements in the double cell of $S_p$, elements with symbol

$$S'_p=\begin{pmatrix} 1 & 2 & \ldots & r+1 & r+2 & \ldots & r+p & r+p+1 \\ 0 & 1 & \ldots & r & r+2 & \ldots & r+p & \end{pmatrix}$$

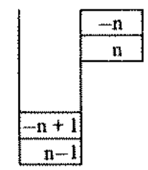
and with symbol

$$S''_p=\begin{pmatrix} 0 & 1 & \ldots & r & r+1 & \ldots & r+p & \\ 1 & & \ldots & r & r+2 & r+p & r+p+1 & \end{pmatrix}.$$

By using the Duflo ordering in $W_n$ (cf. [D2]) and the $\tau$-invariant we can show that the asymptotic wavefront set of the elements corresponding to $S_p$ is contained in the wavefront set for $S_{p+1}$ but not equal. There are several cases. The argument always reduces to constructing two elements, $w_p=(-n, -n+1, w'_p, n-1, n)$ and $w_{p+1}=(-n, -n+1, w'_{p+1}, n-1, n)$ with symbols $S_p$ and $S_{p+1}$.

We first check that the elements with symbol $S_p$ have a wavefront set contained in the closure of the wavefront set of elements with symbol $S_{p+1}$. This is done by finding $w_p$ such that $S(w_p)=S_p$ and $w_{p+1}$, such that $S(w_{p+1})=S_{p+1}$, $w_p$, and $w_{p+1}$ comparable in the Duflo ordering (cf. [D2]) but such that $\tau(w_p) \neq \tau(w_{p+1})$. We use induction. It is enough to find $w_p=(-n, -n+1, w'_p, n-1, n)$ and $w_{p+1}=(-n, -n+1, w'_{p+1}, n-1, n)$ where $w'_p$ and $w'_{p+1}$ satisfy the same conditions as $w_p$ and $w_{p+1}$. This is done by placing the dominoes $\boxed{-\mathrm{n} \mid \mathrm{n}}$ and $\boxed{-\mathrm{n}+1 \mid \mathrm{n}-1}$ in both tableaux for $w$ in the following places






Thus the Weyl group elements with symbol $S_p$ are inequivalent to the elements of symbol $S_q$ for $p \neq q$.

Assume for the moment that the wavefront set of the elements with symbol $S_p$ is the closure of the nilpotent with the same symbol. Let $X=(\underbrace{1\ldots1}_{2p_1}, \underbrace{2\ldots2}_{2p_2}, \underbrace{3\ldots3}_{2p_3}, \ldots)$ be a non-induced special nilpotent. Then $X$ can be characterized by the following inclusions in closures of induced orbits.

$$\mathrm{cl}(\underbrace{1\ldots1}_{2p_1+4p_2}, \underbrace{3\ldots3}_{2p_3}, \ldots) \subsetneq \mathrm{cl}X \subsetneq \mathrm{cl}(\underbrace{2\ldots2}_{p_1+p_2}, \underbrace{3\ldots3}_{2p_3}, \ldots)$$

$$\mathrm{cl}(\underbrace{1\ldots1}_{2p_1+2p_3}, \underbrace{2\ldots2}_{2p_2+2p_3}, 4\ldots4, \ldots) \subsetneq \mathrm{cl}X \subsetneq \mathrm{cl}(\underbrace{2\ldots2}_{2p_2-2p_1}, \underbrace{3\ldots3}_{2p_3+2p_1}, \ldots).$$

Then we can use the argument from before to show that elements with symbols corresponding to $X_{2p_1}$ must have wavefront sets satisfying the same inclusions as the corresponding orbits. The induction step uses elements $w=(-n+2, -n+1, -n, w', n, n-1, n-2)$.

Since we have proved the assertion for induced nilpotents and $X$ is the only nilpotent satisfying the required inclusions, the wavefront set of elements with a certain symbol $S$ is a nilpotent with a special symbol which is a permutation of $S$.

To complete the proof we must determine the wavefront set of the double cell with symbol equal to permutations of $S_p$. The inclusions

$$(\underbrace{1\ldots1}_{2p_1}, \underbrace{2\ldots2}_{2p_2}) \subsetneq \mathrm{cl}(\underbrace{1\ldots1}_{2p_1-2p_2}, \underbrace{3\ldots3}_{2p_2})$$

and

$$(\underbrace{1\ldots1}_{}, \underbrace{2\ldots2}_{2p_2+2}) \nsubseteq \mathrm{cl}(\underbrace{1\ldots1}_{2p_1-2p_2}, \underbrace{3\ldots3}_{2p_2})$$

show that the nilpotent must have symbol $S_p$ or $S'_p$. In addition it follows that the elements in $W_n$ with symbol a permutation of $S_p$ form a double cell. We now show that the representation with symbol $S_p$ occurs in the double cell. This is done by using Proposition 14. We use the parabolic whose semisimple part in the Levi component is $C_{n-1}$. Given $w \in W_n$, the left cell it generates in $W_{n-1}$ can be read off by crossing out $i_1$, $-i_1$ and forming an element in $W_{n-1}$ by identifying the order in the remaining elements with $-n+1, \ldots, +n-1$.

Thus, if we restrict a double cell from $W_n$ to $W_{n-1}$ we get all the cells corresponding to tableaus in the $W_n$-cell where one domino has been deleted and multiplicities corresponding to the number of ways in which a particular tableau can be obtained in this way.

On the other hand, a representation of $W_n$ corresponding to the partition $(\alpha, \beta)$ of $n$ is restricted to $W_{n-1}$ by subtracting a 1 from $\alpha$ or $\beta$ in all possible ways such that we get a partition $(\alpha', \beta')$ of $n-1$ (see [B]). Viewing $(\alpha, \beta)$ as a $2n$-tableau, it follows that the restriction consists of the sum of all $(2n-2)$ tableaus contained from the $2n$-tableau by crossing out a domino. By induction we may assume that $(n-1)$ double cells are formed of Weyl group elements with symbols that are permutations of each other and the representations that occur have the same property.

For $S_p$ the symbols of the Weyl group representations correspond to the following partitions

I. $(\underbrace{0\dots0}_{r+1},\underbrace{1\dots1}_{p})\times(\underbrace{1\dots1}_{r+p})$

II. $(\underbrace{1\dots1}_{r+p+1})\times(0\dots0,\underbrace{1,1,\dots1}_{p})$

III. $(\underbrace{0\dots0}_{r+p+1})\times(\underbrace{1\dots1}_{r},\underbrace{2\dots2}_{p})$.

Now consider $(\underbrace{0\dots0}_{r+1},\underbrace{0}_{1},\underbrace{1\dots1}_{p-1})\times(\underbrace{1\dots1}_{r+p})$. This representation must occur in the restriction. It can only come from one of

a) $(\underbrace{0\dots0}_{r+1},\underbrace{1\dots1}_{p})\times(\underbrace{1\dots1}_{r+p})$

b) $(\underbrace{0\dots0}_{r+1},\underbrace{0}_{1},\underbrace{1\dots1}_{p-2},\underbrace{2}_{1})\times(\underbrace{1\dots1}_{r+p})$

c) $(\underbrace{0\dots0}_{r+1},\underbrace{0}_{1},\underbrace{1\dots1}_{p-1})\times(\underbrace{1\dots2}_{r+p})$.

If b) or c) occur in the double cell for $S_p$, then either $(\underbrace{0\dots0}_{r+2},\underbrace{1\dots1}_{p-1})\times(1\dots2)$ or $(\underbrace{0\dots0}_{r+1},\underbrace{0}_{1},\underbrace{1\dots1}_{p-2},2)\times(\underbrace{0,1\dots1}_{r+p-1})$ would occur in the restriction. But no Weyl group element with symbols I, II, or III could restrict to such a left cell. Thus a) must occur in the double cell and the proof is complete.

*Type $O(2n)$.* The same proof works word for word but substituting the symbols of type $D_n$. The inclusions among the Weyl group elements and the nilpotents are true with the obvious modifications. We note that the proof works only if we allow equivalences $1 \to -1$ and $i_n \to -i_n$. We have also identified the two nilpotents and the two representations for symbols of the type $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$.

**Corollary 26.** *If $S(w)$ is the symbol of an element, we denote by $\{S(w)\}$ the set of the elements in the symbol with multiplicity. If $\{S(w)\} \neq \{S(w')\}$ then the double cells $\Phi(w)$ and $\Phi(w')$ are disjoint. The equivalences in Proposition 15 and 16 permute the symbols.*



*Proof.* Since the wavefront set of $\Theta(w\lambda, \lambda)$ is given by the special nilpotent with symbol a permutation of $S(w)$ the first assertion follows. Since the equivalences in Propositions 15 and 16 preserve double cells, they permute the symbols.

From here on, we will use $D_n$ instead of $O(2n)$.

**Lemma 27.** *Let $w \in W_n$. Then there is $w'$ such that $w \underset{L}{\sim} w'$ and $S(w')$ is special.*

*Proof.* Let $w = (i_n\ w_1\ -i_n)$. By induction, $w_1 \underset{L}{\sim} w_1'$ such that $S(w_1')$ is special. Then $w \underset{L}{\sim} (i_n\ w_1'\ -i_n) = w_2$. If $S(w_2)$ is special we are done. If not we perform some hook exchanges such that $w \underset{L}{\sim} (i_n'\ w_2'\ -i_n')$, $S(w_2')$ not special, and such that when we make $w_2'$ special, $(i_n\ w_2'\ -i_n)$ becomes special. We give details for $D_n$ only.

By the previous discussion we may assume without loss of generality that $S(w) = \begin{pmatrix}\lambda_1, \dots, \lambda_i+1, \dots \lambda_m\\ \mu_1, \dots, \mu_i, \quad \dots \mu_m\end{pmatrix}$ where $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ is special with $\lambda_i \leqq \mu_i$. Since $S(w)$ is not special $\lambda_i = \mu_i = x$. Due to the special form of $w$, $n$ is at $\lambda_i + 1$. If $\lambda_1 \geqq 1$ we can move $n$ from $\lambda_i + 1$ to $\lambda_1$. The resulting element $\bar{w} = (i_n\ \bar{w}_1\ -i_n)$ is such that $S(\bar{w}_1) = \begin{pmatrix}\lambda_1-1 & \lambda_2 & \dots & x+1 & \dots\\ \mu_1 & \mu_2 & \dots & x & \dots\end{pmatrix}$. Since it is not special, $\tilde{w}_1 \underset{L}{\sim} \tilde{w}_2$ such that $S(\tilde{w}_2) = \begin{pmatrix}\lambda_1-1 & \dots & x & \dots\\ \mu_1 & \dots & x+1 & \dots\end{pmatrix}$. But then $S(i_n\ \tilde{w}_2\ -i_n) = \begin{pmatrix}\lambda_1 & \dots & x & \dots\\ \mu_1 & \dots & x+1 & \dots\end{pmatrix}$ and $w \underset{L}{\sim} (i_n, \tilde{w}_2, -i_n)$.

If $\lambda_1 = 0$ then we may assume $\mu_1 \geqq 1$. We can repeat the same argument using $\mu_1$ unless $\mu_1 = 1$ or $\mu_1 = 2$, $\lambda_2 = 1$. If $\mu_1 = 1$ then it may happen that the argument from before produces a $w_1 \underset{L}{\sim} w$ such that $S(w_1) = \begin{pmatrix}1 & \lambda_2 & \dots & \lambda_m\\ 0 & \mu_2 & \dots & \mu_m\end{pmatrix}$ and $\begin{pmatrix}\lambda_2-1 & \dots & \lambda_m-1\\ \mu_2-1 & \dots & \mu_m-1\end{pmatrix}$ is special. Eliminating the cases where we can use the argument from before with $\lambda_2, \mu_2 \dots$, we are left with the symbols

$$\begin{pmatrix}1 & 2 & 3 & \dots\\ 0 & 2 & \dots & \end{pmatrix}, \begin{pmatrix}1 & 2 & 3 & \dots\\ 0 & 3 & \dots & \end{pmatrix} \text{ or } \begin{pmatrix}1 & 2 & 4 & \dots\\ 0 & 3 & \dots & \end{pmatrix}$$

Using Corollary 22 we find $\tilde{w} \underset{L}{\sim} w$ such that $T(\tilde{w}) = T(w)$ and $\tilde{w} = (i_n, i_{n-1}, w_1, -i_{n-1}, -i_n)$. The $i_n, i_{n-1}$, are such that $S(i_{n-1}, i_{n-2}, \tilde{w}_1, -i_{n-2}, -i_{n-1})$ and $S(i_{n-2}, \tilde{w}_1, -i_{n-2})$ have a $\mu_i = \lambda_j$, $i, j \leqq 3$. We write the symbols in the following way.

$$S(\tilde{w}) = \begin{pmatrix}1 & 2 & 3 & \dots\\ 0 & 2 & \dots & \end{pmatrix} \to \begin{pmatrix}1 & 2 & 3 & \dots\\ 0 & 1 & \dots & \end{pmatrix} \to \begin{pmatrix}0 & 2 & 3 & \dots\\ 0 & 1 & \dots & \end{pmatrix} = S(\tilde{w}_1).$$

Then by the induction hypothesis $\tilde{w}_1 \underset{L}{\sim} \tilde{w}_2$ such that $S(\tilde{w}_2) = \begin{pmatrix}0 & 1 & 3 & \dots\\ 0 & 2 & \dots & \end{pmatrix}$ is special. Then due to Corollary 26, $S(i_{n-1}\ \tilde{w}_2\ -i_{n-1})$ and $S(i_n, i_{n-1}, \tilde{w}_2, -i_{n-1}, -i_n)$ must have symbols

$$\begin{pmatrix}0 & 1 & 3 & \dots\\ 0 & 2 & \dots & \end{pmatrix} \to \begin{pmatrix}0 & 1 & 3 & \dots\\ 1 & 2 & \dots & \end{pmatrix} \to \begin{pmatrix}0 & 2 & 3 & \dots\\ 1 & 2 & \dots & \end{pmatrix}$$

and the last one is special. The same argument works for $\mu_1=2$, $\lambda_2=1$ except for the case $\begin{pmatrix}0 & 1 & 3\\ 2 & 3 & 4\end{pmatrix}$ when Corollary 22 doesn't apply ($\boxed{-n \mid n}$ cannot be moved from $\lambda_3$ to $\mu_1$ for one of the parities). This can be treated in the same way as the cases in Lemma 28. We omit the details as well as the fact that if $S(w)=\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$, such that $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ is reducible to $\begin{pmatrix}1\\ 1\end{pmatrix}$ then there is $w' \underset{L}{\sim} w$ such that $S(w')=\begin{pmatrix}\mu\\ \lambda\end{pmatrix}$.

**Lemma 28.** *If $w_1, w_2 \in W_n$ are such that $T(w_1)=T(w_2)$ (and have the same parity for type $D_n$) and $T(w_1)$ has special symbol then $w_1 \underset{L}{\sim} w_2$.*

*Proof*. Type $B_n$, $C_n$: Due to Corollaries 22 and 24 the only special symbols where we cannot use hook exchanges to move $\boxed{-n \mid n}$ to any allowable place in the right tableau are $\begin{pmatrix}1 & & 3\\ & 1 & \end{pmatrix}$ and $\begin{pmatrix}0 & 1 & 3\\ & 1 & 3\end{pmatrix}$. The proof then proceeds as in Corollary 24.

*Type $D_n$*. We consider the symbols such that $\lambda_1 \leqq \mu_1 \dots \leqq \lambda_m \leqq \mu_m$ and $\mu_1 \neq 0$ only, since the argument would be the same in the other case. According to Corollary 22 the only cases we have to check have symbols

A. $\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$, B. $\begin{pmatrix}0 & 3\\ 1 & 3\end{pmatrix}$, C. $\begin{pmatrix}0 & 2\\ 2 & 4\end{pmatrix}$, D. $\begin{pmatrix}0 & 2 & 3\\ 1 & 3 & 4\end{pmatrix}$, E. $\begin{pmatrix}0 & 2\\ 2 & 3\end{pmatrix}, \begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}$.

We note that $\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}$ is in $D_4$ so it is done in Lemma 19. The other cases will all be reduced to calculations in Lemma 19.

A. The shape is $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; & \; \\ \hline \; & \; \\ \cline{1-2} \; & \; \\ \cline{1-2}\end{array}$ and we have to check that $\boxed{-5 \mid 5}$ can be moved

from $\begin{array}{|c|c|c|c|}\hline \; & \; & \; & -5 \\ \hline \; & \; & \; & 5 \\ \hline \; & \; \\ \cline{1-2} \; & \; \\ \cline{1-2}\end{array}$ to $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; & \; \\ \hline \; & -5 \\ \cline{1-2} \; & 5 \\ \cline{1-2}\end{array}$. By induction we can assume that the right

tableau is $\begin{array}{|c|c|c|}\hline \; & \; & -5 \\ \hline \; & \; & 5 \\ \hline -3 & -4 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array}$, otherwise we can make a hook exchange. The tableaus are

$$\begin{array}{ccc} 5 & 4 & 3\\ 2 & 1 & -5\\ -1 & -2 & \\ -3 & -4 & \end{array} \quad \text{or} \quad \begin{array}{ccc} 5 & 4 & 3\\ 2 & -1 & -5\\ 1 & -2 & \\ -3 & -4 & \end{array}$$

In either case, a series of hook exchanges proves the result.



B. The shape of the tableau is $\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; & \; & \; & \; \\ \hline \; \\ \cline{1-1} \; \\ \cline{1-1}\end{array}$ and we have to move $\boxed{-5 \mid 5}$ from $\mu_2$ to $\lambda_2$. Thus we assume $w=(i_5\ w_1\ -i_5)$ such that $S(w)=\begin{pmatrix}0 & 3\\ 1 & 3\end{pmatrix}$, $S(w_1)=\begin{pmatrix}0 & 3\\ 1 & 2\end{pmatrix}$. Since $S(w_1)$ is not special, $w_1 \underset{L}{\sim} w_2$ such that $S(w_2)=\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}$. Then $w \underset{L}{\sim} (i_5\ w_2\ -i_5)$ so $S(i_5\ w_2\ -i_5)=\begin{pmatrix}0 & 3\\ 1 & 3\end{pmatrix}$, but the last 1 was added to $\lambda_2$ so $\boxed{-5 \mid 5}$ is at $\lambda_2$. We have to check that the left tableau is unchanged. This is done in the following way for all cases. Since $w_1 \in W_4$, we can assume that $T(w_1)$ is one of VI, VII, VIII, IX, X, XI from Lemma 18. Then we identify $i_5$ with a half-integer and $i_1 \dots i_4$ with $\pm 1$, $\pm 2$, $\pm 3$, $\pm 4$ in such a way that all order relations and signs between $i_1 \dots i_5$ are preserved. Then we can check that tableau VI cannot give a tableau of symbol $\begin{pmatrix}0 & 3\\ 1 & 3\end{pmatrix}$, and the other tableaus change into special shape by moving the last corner in the first row to the next row. Then we can check easily that adding $i_5$ will produce the original left tableau.

C. The shape of the tableau is $\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; \\ \cline{1-3} \; & \; & \; \\ \cline{1-3} \; \\ \cline{1-1}\end{array}$ and we have to move $\boxed{-6 \mid 6}$ from $\lambda_2$ to $\mu_2$ or $\mu_1$. Since $\begin{pmatrix}0 & 1\\ 2 & 4\end{pmatrix}$ is not reducible to $\begin{pmatrix}1\\ 1\end{pmatrix}$ we assume $\boxed{-5 \mid 5}$ is at $\mu_2$. Then the remaining tableau has symbol $\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}$. We write this as

$$\begin{pmatrix}0 & 2\\ 2 & 4\end{pmatrix} \to \begin{pmatrix}0 & 1\\ 2 & 4\end{pmatrix} \to \begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}.$$

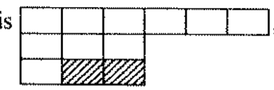
There are 4 elements in $W_4$ with this symbol and 2 tableaus. The Weyl group elements have odd parity. We use Lemma 18 by observing that $1 \leftrightarrow -1$ preserves cell equivalences. The tableaus change as follows (using the same argument as in B).

$$\begin{array}{ccc} 4 & 3 & -2\\ 2 & 1 & \\ -1 & -4 & \\ -3 & & \end{array} \quad \leftrightarrow \quad \begin{array}{cccc} 4 & 3 & -1 & -2\\ 2 & 1 & & \\ -3 & -4 & & \end{array}$$

$$\begin{array}{ccc} 4 & 2 & -3\\ 3 & 1 & \\ -1 & -4 & \\ -2 & & \end{array} \quad \leftrightarrow \quad \begin{array}{cccc} 4 & 2 & -1 & -3\\ 3 & 1 & & \\ -2 & -4 & & \end{array}$$

Thus, $w=(i_6\ i_5\ w_1\ -i_5\ -i_6)$ such that $S(w_1)=\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}$ and $w_1 \underset{L}{\sim} w_2$ such that $S(w_2)$ $=\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}$. Adding $i_5$ and $i_6$ we get symbol

$$\begin{pmatrix}2 & 3\\ 0 & 1\end{pmatrix}\to\begin{pmatrix}2 & 4\\ 0 & 1\end{pmatrix}\to\begin{pmatrix}2 & 4\\ 0 & 2\end{pmatrix}.$$

The shape of the tableau is , where the shaded area is the place of $\boxed{-6\ \ 6}$. It is easy to see that $\boxed{-6\ \ 6}$ can be moved to any other place. Thus assume it is at $\lambda_2$ and repeat the argument from before.

$$\begin{pmatrix}2 & 4\\ 0 & 2\end{pmatrix}\to\begin{pmatrix}2 & 3\\ 0 & 2\end{pmatrix}\to\begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix}\underset{L}{\sim}\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}$$

$$\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}\to\begin{pmatrix}0 & 2\\ 2 & 3\end{pmatrix}\to\begin{pmatrix}0 & 2\\ 2 & 4\end{pmatrix}$$

so the last 1 was added to $\mu_2$. We must check that the left tableau is unchanged.

We only sketch the argument since the calculations are somewhat lengthy. Let $w=(i_6\ i_5\ w_1\ -i_5\ -i_6)$ be such that $T(w_1)=\begin{array}{lll}4 & 3 & -2\\ 2 & 1 & \\ -1 & -4 & \\ -3 & & \end{array}$, $S(i_5\ w_1\ -i_5)$ $=\begin{pmatrix}0 & 1\\ 2 & 4\end{pmatrix}$ and $S(w)=\begin{pmatrix}0 & 2\\ 2 & 4\end{pmatrix}$. We have identified $i_5$ and $i_6$ with half-integers as before. Then we get $i_5>2$ and $i_6<i_5$. Assume $i_6>2$, $3<i_5<4$, the other cases are handled in the exact same way. Then

$$T(w)=\begin{array}{lllll}4 & 3 & 1 & -2 & -i_6\\ i_5 & 2 & -i_5 & & \\ i_6 & -1 & -4 & & \\ -3 & & & & \end{array}$$

Then the tableau with symbol $\begin{pmatrix}2 & 4\\ 0 & 2\end{pmatrix}$ is

$$\begin{array}{llllll}4 & 3 & 1 & -2 & -2 & -i_6\\ i_5 & 2 & -i_5 & & & \\ i_6 & -3 & -4 & & & \end{array}.$$

Then $w \underset{L}{\sim} (i_6\ 2\ w_2\ -2\ -i_6)$ and $T(w_2)=\begin{array}{llll}4 & 3 & -1 & -i_5\\ i_5 & 1 & & \\ -3 & -4 & & \end{array}$ which corresponds to



XII in Lemma 19. It is equivalent on the left to $w_2'$ such that

$$T(w_2')=3\quad 4\quad 3-i_5\quad i_5\quad 1-4-1-3.$$

Then $\tilde{w} \underset{L}{\sim} (i_6, 2, w_2', -2, -i_6)$ and the tableau of this last element coincides with $T(w)$.

D. and E. These cases are done in the same way as C. We omit the details.

**Lemma 29.** *The regular representation of $W_n(C)$, $W_n(B)$ or $W_n(D)$ decomposes into double cells in the same way as the Weyl group elements. In other words, if a representation has symbol $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ then it belongs to the double cell whose elements have tableaus with symbols which are permutations of $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$. The wavefront set is the closure of the nilpotent orbit with special symbol a permutation of $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$. (For type $D_n$ we use the correspondingly defined symbols.)*

*Proof*. According to Proposition 14 and the discussion in Lemma 23 we may assume (by induction) that the restriction of the cell with symbol $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ to $W_{n-1}$ coincides with the restriction of the direct sum of the representations with symbols that are permutations of $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$. We recall that a representation of $W_n$ is given by a pair of partitions $(\alpha,\beta)$ and the symbol is obtained by taking $\lambda_i=\alpha_i+i-1$ and $\mu_i=\beta_i+i-1$. The restriction of a representation with symbol $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ is given by the sum of the representations with symbols that are obtained from $\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ by subtracting a 1 from a $\lambda_i$ or a $\mu_j$ so that the result is still a symbol. We note that for type $D_n$ and symbol $\begin{pmatrix}\lambda\\ \lambda\end{pmatrix}$ there are two cells and since they are both induced, a dimension argument shows that the corresponding induced representations are the only ones occurring. For the other cells the argument is the same as for $C_n$ so we will give details only for this case.

We first consider the case when $\beta=0$. The symbol is $\begin{pmatrix}\lambda_1 & & \dots & \lambda_{m+1}\\ 0 & 1 & \dots & m-1\end{pmatrix}$ and we may assume $\lambda_1\geqq 1$. We want to identify the cell this representation belongs to. We do an induction on the number of $\alpha_i$ which are $\neq 0$ and on the size of the first nonzero $\alpha_i$. Consider the restriction of this representation to $W_{n-1}$. It contains the representation with symbol $\begin{pmatrix}\lambda_1-1, & \lambda_2, & \dots, & \lambda_{m+1}\\ 0, & 1, & \dots, & m-1\end{pmatrix}$. This could come from an element with one of the following symbols:

$$\begin{pmatrix}\lambda_1 & & \dots & \lambda_{m+1}\\ 0 & 1 & \dots & m-1\end{pmatrix};\begin{pmatrix}\lambda_1-1, & & \dots, \lambda_j+1, & \dots & \lambda_{m+1}\\ 0 & 1 & \dots & & m-1\end{pmatrix};\begin{pmatrix}\lambda_1-1, & \lambda_2 & \dots, & \lambda_{m+1}\\ 0 & 1 & \dots & m-2, & m\end{pmatrix}.$$

Due to the induction hypothesis, the second symbol has already been accounted for. We have to exclude the third one. Suppose there is $i>1$ such that $\begin{pmatrix}\lambda_1 & \dots, \lambda_i-1 & \dots & \lambda_{m+1}\\ 0\,1 \dots & & m-1\end{pmatrix}$ is a symbol. Then it must occur in the restriction so it must also come from $\begin{pmatrix}\lambda_1-1, \lambda_2, \dots & \lambda_{m+1}\\ 0, \quad 1 & \dots m-2, m\end{pmatrix}$ or some permutation. The only possibility is when $\lambda_1=m$. But then $\begin{pmatrix} m & \lambda_2 & \dots & \lambda_{m+1}\\ 0 & 1 & \dots & m-1\end{pmatrix}$ and $\begin{pmatrix} m-1, \lambda_2 \dots & \lambda_{m+1}\\ 0, \quad 1 & \dots m-2, m\end{pmatrix}$ are permutations of each other so the representation $\begin{pmatrix}\lambda_1 & \dots \lambda_{m+1}\\ 0\,1 \dots & m-1\end{pmatrix}$ must be in the cell with special symbol a permutation of $\begin{pmatrix}\lambda_1 & \dots \lambda_{m+1}\\ 0 & 1 \dots m-1\end{pmatrix}$ by counting multiplicities in the restriction.

Suppose we cannot subtract a 1 from any of the $\lambda_i$, $i>1$. Then the symbol has to be

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix}=\begin{pmatrix} x & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix}.$$

Its restriction contains $\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix}$. Consider this representation. It can only occur in the restriction of the cells whose symbols are permutations of

$$\begin{pmatrix} x & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix};\ \begin{pmatrix} x-1, & x+1, & \dots, & x+m+1\\ 0, & 1 & \dots, & m-1\end{pmatrix};\ \begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-2, m\end{pmatrix}.$$

Again the occurrence in the middle cell is accounted for. The multiplicity in the restriction of the cell $\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-2, m\end{pmatrix}$ is $\dim\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix}$. However,

$$\dim\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-2, m\end{pmatrix}$$
$$=\dim\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-2\ m-1\end{pmatrix}+\dim\begin{pmatrix} x-1 & x & \dots & x+m\\ 0 & 1 & \dots & m-2, m\end{pmatrix}$$
$$+\dots>\dim\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-2\ m-1\end{pmatrix}=\dim\begin{pmatrix} x & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix}.$$

Thus, if $\begin{pmatrix} x & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix}$ occurs in the cell $\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-2, m\end{pmatrix}$ it cannot account for the occurrence of $\begin{pmatrix} x-1, & x+1 & \dots & x+m\\ 0 & & \dots & m-1\end{pmatrix}$ in the restriction.



Since the occurrence of both $\begin{pmatrix} x & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix}$ and $\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-2, m\end{pmatrix}$ would give too large a multiplicity and since $\begin{pmatrix} x-1 & x+1 & \dots & x+m\\ 0 & & \dots & m-1\end{pmatrix}$ does occur in the restriction of the cell $\begin{pmatrix} x & x+1 & \dots & x+m\\ 0 & 1 & \dots & m-1\end{pmatrix}$, it follows that the representation $\begin{pmatrix} x & x+1 & \dots & x+m\\ & 0 & \dots & m-1\end{pmatrix}$ occurs in the cell with symbol a permutation of $\{0, 1 \dots m-1, x, x+1 \dots x+m\}$.

To complete the proof we do an induction

a) downward on the size of $p=\Sigma\alpha_i$

b) upward on the number of nonzero $\beta_i$

c) upward on the size of the smallest nonzero $\beta_i$.

Consider a representation with symbol $\begin{pmatrix}\lambda_1 & \dots & \lambda_{m+1}\\ \mu_1 & \dots & \mu_m\end{pmatrix}$. It contains $\begin{pmatrix}\lambda_1 & \dots & & \lambda_{m+1}\\ \mu_1 & \dots \mu_i-1 & \dots & \mu_m\end{pmatrix}$ in its restriction, where $i$ is the first integer such that we can subtract a 1 (so $\mu_1=0$, $\mu_2=1$, $\mu_{i-1}=i-2$). This restriction can only come from a cell containing tableaus with symbols

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix},\quad \begin{pmatrix}\lambda_1 \dots & & \lambda_{m+1}\\ \mu_1 \dots \mu_1-1, \dots \mu_j+1 \dots & & \mu_m\end{pmatrix},$$
$$\begin{pmatrix}\lambda_1 \dots \lambda_k+1 \dots \lambda_{m+1}\\ \mu_1 \dots \quad \mu_m\end{pmatrix}\quad \begin{pmatrix}\lambda_1 & & \lambda_{m+1}\\ 0, 1 \dots i-3, i-1, \mu_i-1 \dots & & \mu_m\end{pmatrix}.$$

The induction hypothesis accounts for the second and third possibility. We are left with $\begin{pmatrix}\lambda_1 \dots \lambda_{m+1} & \\ 0, 1 \dots i-3, i-1, \mu_i-1 \dots \mu_m\end{pmatrix}$. By taking into consideration the occurrences of $\begin{pmatrix}\lambda_1 \dots \lambda_k-1 \dots \lambda_{m+1}\\ 0\ 1 \dots i-2, \mu_i \dots \mu_m\end{pmatrix}$ and $\begin{pmatrix}\lambda_1 \dots \lambda_{m+1} & \\ 0\ 1 \dots i-2, \mu_i \dots \mu_j-1 \dots \mu_m\end{pmatrix}$ in the restriction we can reduce to the case

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix}=\begin{pmatrix} x & x+1 & \dots & x+m & \\ 0\ 1 & \dots & i-2, x, x+1 & \dots & x+m-i\end{pmatrix}.$$

This representation restricts to

$$\begin{pmatrix} x-1 & x+1 & \dots & x+m & \\ 0\ 1 & \dots & i-2, x, x+1 & \dots & x+m-i\end{pmatrix}$$
$$+\begin{pmatrix} x & x+1 & \dots & x+m & \\ 0\ 1 & \dots & i-2, x-1, x+1 & \dots & x+m-i\end{pmatrix}.$$

These representations occur in the restrictions of the representations

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix};\ \begin{pmatrix} x-1, & x+1 & \dots & x+m\\ 0, 1 \dots i-3, i-1, x, x+1 & \dots & & x+m-i\end{pmatrix};$$

$$\begin{pmatrix} x & & x+1 & \dots & x+m\\ 0\ 1 \dots i-3, i-1, x-1, x+1 & \dots & & & x+m-i\end{pmatrix}$$

and in representations already accounted for. The cells are $\{\lambda,\mu\}$ and $\{0, 1 \ldots i-3, i-1, \quad x-1, \quad x, \quad x+1, \quad x+1, \ldots x+m-i, \quad x+m-i, \quad x+m-i+1, x+m-i+2, \ldots x+m\}$. But the last two representations contain in their restriction

$$\begin{pmatrix} x-1 & x & \ldots & x+m \\ 0 \quad 1 & \ldots \quad i-3, i-1, x, x+1 & \ldots & x+m-i \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} x & x+1 & \ldots & x+m \\ 0 & 1 & \ldots & i-3, i-1, x-1, x, x+2 \end{pmatrix}$$

which cannot come from $\{\lambda,\mu\}$. Thus the representation $\binom{\lambda}{\mu}$ occurs in the double cell with special symbol a permutation of $\binom{\lambda}{\mu}$.

*Proof of Theorem 18.* Lemmas 19–29 prove all the induction steps except for the following fact. If $w_1, w_2 \in W_n$ are such that $T(w_1) = T(w_2)$ (and have the same parity for $D_n$) then $w_1 \overset{L}{\sim} w_2$. This was proved in Lemma 28 for special symbol.

If the symbol of the tableau is not one in Corollary 22 then we can place $\boxed{-n \mid n}$ in the same place in both right tableaus. Then $w_1 = (i_n\, w_1' \; -i_n)$ and $w_2 = (i_n\, w_2' \; -i_n)$ and $T(w_1') = T(w_2')$. By induction $w_1' \overset{L}{\sim} w_2'$ and we are done.

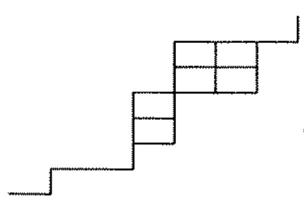
Let now

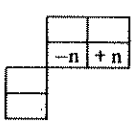
$$S(w_1) = \begin{pmatrix} 0 & 1 & \ldots & i-1 & i+1 \\ i+1 & i+3 & \ldots & \ldots & 2i+2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & \ldots & i-1 & i+1 \\ 1 & i+2 & \ldots & \ldots & 2i+1 \end{pmatrix}$$

or the corresponding $B_n$, $C_n$ symbols

$$\begin{pmatrix} 0 & 1 & \ldots & i & i+2 \\ i+1 & i+3 & \ldots & 2i+2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \ldots & i & i+2 \\ i & i+2 & \ldots & 2i+1 \end{pmatrix}.$$

We fix one of them, say such that the tableau has shape



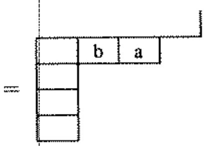
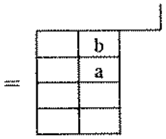
We have to show that, if $w_1$ is an element with the given symbol and $\boxed{-n \mid n}$ is at  in $T(w_1^{-1})$, then there is $w_2$ such that $T(w_1) = T(w_2)$, $\boxed{-n \mid n}$ is in a different position in $T(w_2^{-1})$ and $w_1 \underset{L}{\gtrsim} w_2$.

It is enough to find a $w_2$ which is comparable to $w_1$ in the Duflo ordering. Indeed if this is so, we have an inclusion between the corresponding left annihilators. Since they have equal Gelfand-Kirillov dimension (the tableaus have



the same symbol) they must be equal. By Lemma 27, $w_1 \underset{L}{\gtrsim} w_1'$ and $w_2 \underset{L}{\gtrsim} w_2'$ such that $w_1'$ and $w_2'$ have special symbols. Lemmas 28 and 29 imply that the number of distinct left annihilators with the same asymptotic wavefront set is exactly equal to the number of tableaus with corresponding special symbols. Thus $T(w_1') = T(w_2')$. By Lemma 28, $w_1' \underset{L}{\gtrsim} w_2'$ so $w_1 \underset{L}{\gtrsim} w_2$.

We now construct the elements $w_1$ and $w_2$ with the required properties. Write $w_1 = (i_n\, w_1' \; -i_n)$. It is enough to find $w_2'$ such that $w_1'$ and $w_2'$ are comparable in the Duflo ordering and

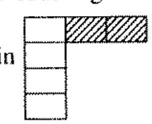
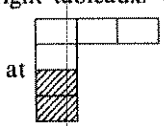
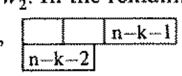
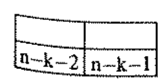
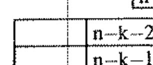
$T(w_1') =$  $\qquad T(w_2') =$ 

Then $w_2 = (i_n\, w_2' \; -i_n)$ satisfies the desired properties. [We leave the details of the facts that $T(w_2')$ satisfies (c) and $T(w_1) = T(w_2)$ to the reader.] We identify $w_1'$ and $w_2'$ with elements in $W_{n-1}$ in the same way as in the proof of Lemma 29.

We observe that $S(w_1')$ is not reducible to $\binom{1}{1}$ so any tableau can be moved to any other tableau by hook exchanges only. Suppose that for a special choice of left tableau for $w_1'$ we can find $w_2'$ such that $T(w_2')$ is obtained from $T(w_1')$ by moving $\boxed{b \mid a}$ to $\boxed{\begin{matrix} b \\ a \end{matrix}}$ and

$$\begin{aligned} w_1' &= (\ldots\, i_m, i_{m-1} \ldots \mid \ldots -i_m, -i_{m-1} \ldots) \\ w_2' &= (\ldots\, i_{m-1}, i_m \ldots \mid \ldots -i_{m-1}, -i_m \ldots). \end{aligned} \tag{$*$}$$

Then, due to the particular shape of the tableaus, any hook exchange we can make in $T(w_1')$ we can also make in $T(w_2')$. This implies that if we perform the same right equivalences (corresponding to hook exchanges) to both $w_1'$ and $w_2'$ the resulting elements must be as in (*) so they will be comparable in the Duflo ordering.

We now choose $w_1'$ in the following way. We place $\boxed{-n+1 \mid n-1}$ in  in both left and right tableaux. We then place the dominoes for $n-2, \ldots, n-k$ vertically starting at  and going down until we reach the end of the tableau. We do the same for $w_2'$. In the remaining tableaux we place the dominoes for $n-k-1$ and $n-k-2$,  in both tableaux for $w_1'$, and  x  in $w_2'$. The rest of the details are straightforward.

## Primitive Ideals with Non-Integral Infinitesimal Charakter

In this section we describe the primitive ideals with regular non-integral infinitesimal character $\lambda$. Recall the notation developed before Proposition 9.

**Theorem 30.** *Let $G$ be an arbitrary complex semisimple Lie group, and $\lambda \in \mathfrak{h}^*$ be regular. Assume that the root system $\Delta_\lambda$ is of classical type. Let $\mathfrak{g}_\lambda$ be a reductive Lie algebra with $\mathfrak{h}$ as a Cartan subalgebra, but having $\Delta_\lambda$ as root system. If $w, w' \in W_\lambda$, then*

$$I^{\mathfrak{g}_\lambda}(w\lambda) = I^{\mathfrak{g}_\lambda}(w'\lambda)$$

*if and only if*

$$I^{\mathfrak{g}}(w\lambda) = I^{\mathfrak{g}}(w'\lambda)\ ;$$

*that is, the primitive ideals for $\mathfrak{g}$ and $\mathfrak{g}_\lambda$ with infinitesimal character $\lambda$ are in a natural one-to-one correspondence. Furthermore, the representations of $W_\lambda$ on double cells are isomorphic.*

The proof is unfortunately not conceptual. Theorem 18 gives a purely combinatorial criterion for $I^{\mathfrak{g}_\lambda}(w\lambda)$ to coincide with $I^{\mathfrak{g}_\lambda}(w'\lambda)$ which is necessary and sufficient; so we only need to establish the same criterion for $\mathfrak{g}$. The sufficiency was a combinatorial consequence of Propositions 15 and 16, which do not assume $\lambda$ to be integral; so that part applies directly to $\mathfrak{g}$. So we need only show that there are as many primitive ideals for $\mathfrak{g}$ as for $\mathfrak{g}_\lambda$ (with infinitesimal character $\lambda$). To do this, it suffices (by [J4, J1]) to show that the decomposition of $L^2(W_\lambda)$ into double cells is the same for $\mathfrak{g}$ and $\mathfrak{g}_\lambda$. This is almost purely combinatorial, and is carried out in Lemma 29. There is only one non-combinatorial element in Lemma 29: the fact that if $I(w)$ and $I(w')$ coincide, then the symbols of $w$ and $w'$ are permutations of each other. This was proved for $\mathfrak{g}_\lambda$ in Lemma 25, using wavefront sets. So what is needed is a way to detect the symbol of $w$ (up to permutation) from combinatorial invariants of $I(w)$. We may as well assume $\Delta_\lambda$ is simple for this purpose. Recall the equivalence relation $\approx$ on $W_\lambda$ defined after Proposition 16; we call its equivalence classes double $J$-cells ($J$ for Joseph).

**Lemma 31.** *With notation as above, suppose*

$$I^{\mathfrak{g}}(w\lambda) = I^{\mathfrak{g}}(w'\lambda).$$

*Then for each $w_1 \approx w$, there is a $w_2 \approx w'$ such that*

$$I^{\mathfrak{g}}(w_1\lambda) = I^{\mathfrak{g}}(w_2\lambda).$$

*Proof*. For the equivalences of Proposition 15, this is proved in [V2]. A similar argument treats those of Proposition 16.

Set

$$GT(w) = \{\tau(w') \mid w' \approx w\}\ ;$$

this is a weak version of the generalized $\tau$ invariant of [V2]. By Lemma 31, $GT(w)$ depends only on $I^{\mathfrak{g}}(w\lambda)$. So to prove Theorem 30, it suffices to show that $GT(w)$ determines the symbol $w$ up to permutation.

**Lemma 32.** *Let $w \in W_n$ and $T(w) = (C_j)_{j=1\ldots m}$ be its tableau where the $C_j$ are the columns in decreasing order of length. Then $\tau(w)$ is given by the following sets*



*Type $B_n$, $C_n$.* $\tau(w) = \{\alpha_i \mid i \neq 1$ when $i \in C_j$, $i+1 \in C_k$ with $k \geqq j$ and $-i \in C_{j'}$, $-i-1 \in C_{k'}$ with $k' \leqq j'$ and $i=1$ when $-1 \in C_j$, $1 \in C_k$ for $k \geqq j\}$. (The Dynkin diagram is labelled $\underset{\alpha_n}{\bullet} - \underset{\alpha_{n-1}}{\bullet} - \cdots \underset{\alpha_2}{\bullet} = \underset{\alpha_1}{\bullet}$.)

*Type $D_n$.* $\tau(w) = \{\alpha_i \mid i \neq 1$ when $i \in C_j$, $i+1 \in C_k$ with $k \geqq j$ and $-i \in C_{j'}$, $-i-1 \in C_{k'}$ with $k' \leqq j'$, and $i=1$ when $\alpha_2 \in \tau(w')$ where $w'$ is obtained from $w$ by changing $1 \leftrightarrow -1\}$.

*Proof*. If we identify the Weyl groups of type $B_n$, $C_n$, $D_n$ with the corresponding subgroups in the Weyl group of $A_{2n-1}$ as before then $\alpha_i \in \tau(w)$ for $i \neq 1$ if and only if the simple roots $\alpha_i \in \tau(w)$ and $\alpha_{-i} \notin \tau(w)$ for $A_{2n-1}$ when we have labelled the simple roots of $A_{2n-1}$ as $\underset{\alpha_n}{\bullet} - \cdots - \underset{\alpha_2}{\bullet} - \bullet - \underset{\alpha_{-2}}{\bullet} \cdots \underset{\alpha_{-n}}{\bullet}$. Then the statements follow from the corresponding statements for $A_{2n-1}$.

**Lemma 33.** *The set $GT(w)$ determines the symbol of $w$ up to permutation.*

*Proof*. We must show that we can recover the shape of the special tableau from the $\tau$-invariants. We use induction on $n$.

*Type $B_n$, $C_n$.* The statement can be checked easily for $C_2$ using Lemma 19. Let

$$M(w_0) = \inf_{w \approx w_0} \{k : \alpha_n, \ldots, \alpha_k \in \tau(w), \alpha_{k-1} \notin \tau(w)\}$$

$$m(w_0) = \inf_{w \approx w_0} \{k : \alpha_n, \ldots, \alpha_k \notin \tau(w), \alpha_{k-1} \in \tau(w)\}.$$

Clearly $M(w_0)$ and $m(w_0)$ only depend on $GT(w)$. Consider the set of tableaux $\{T(w)\}_{w \approx w_0}$ with a given symbol. If $\alpha_n, \ldots, \alpha_k \in \tau(w)$ then by Lemma 32 the domino for $n-1$ must be above the domino for $n$, the domino for $n-2$ above $n-1$ and so on up to $k$. To get $k$ to be as small as possible we must place the domino for $n$ as low as possible, $n-1$ above it horizontally, as low as possible and so on until we reach the first row of the tableau. Then $M(w_0)$ is attained by a $w_M \approx w_0$ of special tableau of the form

$$w_M = (-n, -n+1, \ldots, -n+r, n-r-l \ldots n-r-1, \tilde{w}_M, -n+r+1, \ldots, -n+r+l,\\ n-r, \ldots, n)$$

where $r+l = n-M(w_0)$, $2r$ is the number of rows of length 1, $l$ the number of rows of length larger than 1.

Furthermore, any other element of special tableau attaining $M(w_0)$ is equivalent on the left to an element with the same form as $w_M$.

We note that the length of the first row of $\tilde{w}_M$ is just 2 less than the length of the first row of $w_M$. Similarly, we get

$$w_m = (n \ldots n-s,\ -(n-s-t), \ldots, -(n-s-1),\\ \tilde{w}_m,\ n-s-1, \ldots, n-s-t,\ -(n-s), \ldots, -n).$$

By induction, we may assume that we know the tableaus of $\tilde{w}_m$ and $\tilde{w}_M$. But then we know that the length of the first column of $w_m$ is the length of the first column of $\tilde{w}_m$ plus 2. We get an equation for $2r+l$ so we can solve for $r$ and $l$ so together with the tableau of $\tilde{w}_M$ this determines the tableau of $w_M$.

*Type $D_n$.* By Lemma 19, the statement can be check for $D_4$. Using the outer automorphism of $D_n$ and the identification of the Weyl group with a subset of $W_n$ by $0 \leftrightarrow 1$ or $0 \leftrightarrow -1$ we can reduce the problem to using double cells of type $O(2n)$. The proof is the same as for $B_n$, $C_n$ with the exception that it may happen that the tableau for $w_M$ may have $D_n$-symbol $\binom{\lambda}{\mu}$ while $w_m$ may have $D_n$-symbol $\binom{\mu}{\lambda}$. In this case the first column of $\tilde{w}_m$ may differ by only 1 from the column of $w_m$. The calculation of the symbol in this case gives just a permutation of the symbol one would get by using the same symbol for $w_M$ and $w_m$ so we get the same double cell. The proof is complete, and with it the proof of theorem.

Let $X$ be any special nilpotent element in a semisimple Lie algebra of classical type. We recall that, the Fourier transform of the canonical measure associated to the orbit of $X$ is identified to a harmonic polynomial $p(x)$ on the Cartan subalgebra (cf. Proposition 5). We also recall the identifications $\mathfrak{h}_c \simeq \mathfrak{h} \times \mathfrak{h}$ and $W_c \simeq W \times W$, and $W$ is embedded in $W_c$ as the diagonal.

**Corollary 34.** *Let $O(X)$ be the orbit of a special nilpotent. Then the polynomial $p(x)$ is, up to a constant, the unique $W$-invariant polynomial obtained from the irreducible representation associated to $X$ by the Springer correspondence (normalized as in [L]).*

*Proof.* By Theorem 18 and Proposition 5 the representation of the Weyl group that $p(x)$ is obtained from, is the one with the same symbol as the symbol of $X$. By [L], this is the same as the representation associated to $X$ by the Springer correspondence.

In order to be able to prove Corollary 34 for any nilpotent orbit it is necessary to find the wavefront sets for the primitive ideals with non-integral infinitesimal character as well.

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