

We then deduce that

$$P^n(\text{hit } F_n \text{ before } |z| = r) \geq \frac{C^3 n}{-\log \text{Cap}(F_n)} \quad \text{for all } n \geq N_r^c. \quad \text{Q.E.D.}$$

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The Local Structure of Characters

DAN BARBASCH* AND DAVID A. VOGAN, JR.*

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

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Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the Cartan decomposition and K the maximal compact subgroup with Lie algebra $\mathfrak{k}$. Let Θ be the character of an irreducible representation. Then Θ has an asymptotic expansion at zero (in the sense of Taylor series). As consequences of this expansion we obtain results about the asymptotic directions in which the K -types occur and about the Gelfand-Kirillov dimension of the representation.

1. INTRODUCTION

Let G be a connected semisimple Lie group with finite center. Let π be an irreducible admissible representation of G on a Hilbert space $\mathcal{H}$, and Θ_π its distribution character: if $f \in C_c^\infty(G)$,

$$\Theta_\pi(f) = \text{Tr} \left(\int_G f(g) \pi(g) dg \right).$$

Harish-Chandra has made an exhaustive analysis of the analytic properties of the distribution Θ_π [5, 6]; in particular it is given by integration against a function which is locally L^1 . Our goal is to relate the singularities of Θ_π at the identity to the structure of π . In section 2 and the first part of section 3, we apply Harish-Chandra's results in a simple way to get the following theorem.

THEOREM 1.1. *Let θ_π be the lift of Θ_π to a neighborhood of the identity on $\mathfrak{g} = \text{Lie}(G)$. If $f \in C_c^\infty(\mathfrak{g})$ and $t > 0$, define*

$$f_t(X) = t^{-\dim \mathfrak{g}} f(t^{-1}X).$$

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Then there are an integer r and tempered distributions $\{D_i\}_{i=-r}^{\infty}$ on $\mathfrak{g}$, such that for $f \in C_c^\infty(\mathfrak{g})$,

$$\theta_\pi(f) \sim \sum_{i=-r}^{\infty} t^i D_i(f)$$

as $t \rightarrow 0^+$. (This asymptotic expansion should be understood in the same sense as a Taylor series for a smooth function of t .) Furthermore, the support of $\hat{D}_i$ is a union of nilpotent orbits in $\mathfrak{g}^*$.

We define $AS(\Theta_\pi) \subseteq \mathfrak{g}^*$ to be the union of the supports of the $\hat{D}_i$; $AS(\Theta_\pi)$ is a union of nilpotent orbits. (Probably $AS(\Theta_\pi)$ coincides with the wavefront set of Θ_π at the identity, but we have been unable to prove this.) Let $K \subseteq G$ be a maximal compact subgroup. Then $\pi|_K$ has a distribution character Θ_π^K . Following Kashiwara and Vergne [8], we identify $\hat{K}$ with a subset of the orbits of K in the dual $\mathfrak{k}^*$ of the Lie algebra of K (via the highest weight theory); and we define ([8], Definition 6.1(b))

$$AS(\Theta_\pi^K) = \{\mu \in \mathfrak{k}^* \mid \mu = \lim_{n \rightarrow \infty} t_n \mu_n, \text{ with } \mu_n \in \mathfrak{k}^*,$$

$$t_n \in \mathbb{R}^+, \mu_n \text{ occurs in } \pi|_K, \text{ and } t_n \rightarrow 0\}.$$

THEOREM 1.2 (see Theorem 3.5 and its proof). θ_π^K has an asymptotic expansion $\sum t^i E_i$ in tempered distributions on $\mathfrak{k}$, computable in terms of the D_i . In particular

$$AS(\Theta_\pi^K) = \bigcup_i \text{supp } E_i \subseteq \{\mu \in \mathfrak{k}^* \mid \exists \lambda \in AS(\Theta_\pi) \text{ with } \lambda|_{\mathfrak{k}} = \mu\}.$$

If (for example) π is a discrete series representation, then equality holds.

The last assertion (together with Proposition 3.7, which computes $AS(\Theta_\pi)$ when π is in the discrete series) proves a conjecture of Kashiwara and Vergne ([8], end of Example 6.3). The rest of section 3 is largely devoted to showing that these expansions behave well under parabolic induction (Theorem 3.5).

In section 4 we relate $AS(\Theta_\pi)$ to primitive ideals. Among other things, we show

THEOREM 1.3. The dimension d of $AS(\Theta_\pi)$ is equal to the Gelfand-Kirillov dimension of $U(\mathfrak{g})$ modulo the annihilator of π . Furthermore, if

$$\theta_\pi \sim \sum_{i=-r}^{\infty} t^i D_i$$

and $D_{-r} \neq 0$, then $d = 2r$.

Thus the Gelfand-Kirillov dimension of the annihilator of π measures the

singularity of Θ_π at the identity; this may be regarded as related to the formula

$$\Theta_\pi(1) = \dim \pi$$

for finite dimensional representations.

Finally, we study the behavior of the asymptotic expansions under coherent continuation (Theorem 4.7). This leads to a proof of Conjecture 5.1 of [11], describing the asymptotic distribution of eigenvalues of the Casimir operator for K in π (Corollary 4.8).

2. ASYMPTOTIC EXPANSIONS OF DISTRIBUTIONS

Let Ω be a neighborhood of zero in $\mathbb{R}^n$, and θ a distribution on $C_c^\infty(\Omega)$. For $f \in C_c^\infty(\mathbb{R}^n)$ and $t > 0$, we define $f_t \in C_c^\infty(\mathbb{R}^n)$ by

$$f_t(x) = t^{-n} f(t^{-1}x).$$

We say that θ admits an asymptotic expansion at 0 if there is an integer and a family of distributions $\{D_i \mid r \leq i < \infty\}$ on $\mathbb{R}^n$, such that for $f \in C_c^\infty(\mathbb{R}^n)$

$$\theta(f_t) \sim \sum_{i=r}^{\infty} t^i D_i(f) \quad \text{as } t \rightarrow 0^+$$

in the following sense. For each positive integer N and compact set K there is a constant $C = C_{N,K} > 0$, a positive integer $k = k_{N,K}$, and a number $\epsilon_{N,K} > 0$, such that if $\text{supp } f \subseteq K$, and $0 < t \leq \epsilon$, then $\text{supp } f_t \subseteq \Omega$ (so that $\theta(f_t)$ is defined) and

$$\left| \theta(f_t) - \sum_{i=r}^N t^i D_i(f) \right| \leq C \cdot \sup_{|\alpha| \leq k} |D^\alpha f| t^{N+1}.$$

$$X=0$$

(Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \sum \alpha_i$, and

$$D^\alpha = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n}$$

as usual.) Obviously the D_i are unique if they exist. If $D_r \neq 0$, we call r the order of θ at 0. We write $\theta \sim \sum_{i=r}^{\infty} t^i D_i$.

LEMMA 2.1. Let θ be a distribution on a neighborhood Ω of 0 in $\mathbb{R}^n$, admitting an asymptotic expansion $\theta \sim \sum t^i D_i$. Then D_i is homogeneous of degree i ; that is, if $f \in C_c^\infty(\mathbb{R}^n)$ and $s > 0$,

$$D_i(f_s) = s^i D_i(f).$$

The trivial proof is left to the reader.

We will be interested in the Fourier transforms of asymptotic expansions; so we need

LEMMA 2.2. *Every homogeneous distribution on $\mathbb{R}^n$ is tempered.*

Proof. Let D be a distribution on $C_c^\infty(\mathbb{R}^n)$, homogeneous of degree i . We must show that there are integers k_1, k , $k \geq 0$, and a constant $C > 0$, such that if $f \in C_c^\infty(\mathbb{R}^n)$

$$|D(f)| \leq C \cdot \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leq k_2}} (1 + |x|)^{k_1} |D^\alpha f|.$$

To see this, first choose $C_1 > 0$ and an integer $k_2 \geq 0$ so that if f is supported in the ball of radius 2 about 0, then

$$|D(f)| \leq C_1 \sup_{|\alpha| \leq k_2} |D^\alpha f|.$$

Put $k_1 = \max(0, i + k_2 + n + 1)$. To get the desired estimate, we need a special partition of unity. Choose a positive smooth function φ on say $(-\infty, 1.1)$ satisfying

- (a) $\varphi(x) = 0$ for $x \leq .6$, and $\varphi(x) \leq 1$ always
- (b) $\varphi(x) = 1$ for $.9 \leq x < 1.1$.

Extending φ to all of $\mathbb{R}$ by defining

- (c) $\varphi(x) = 1 - \varphi(x/2)$ for $1.1 \leq x \leq 1.8$
- (d) $\varphi(x) = 0$ for $x > 1.8$.

Put $\varphi_j(x) = \varphi(2^{-j}x)$. Then $\varphi_j \in C_c^\infty(\mathbb{R})$, $\text{supp } \varphi_j \subseteq (2^{j-1}, 2^{j+1})$, and $\sum_{j=0}^\infty \varphi_j(x) = 1$ for $x \geq .9$. Set

$$D_0(f) = D\left(f\left(1 - \sum_{j=0}^\infty \varphi_j(|x|)\right)\right)$$

$$D_1(f) = D\left(f \cdot \sum_{j=0}^\infty \varphi_j(|x|)\right).$$

Obviously D_0 satisfies an estimate of the desired sort; we need only show that D_1 does. Put $f^j(x) = f(x) \varphi_j(|x|)$. If $f \in C_c^\infty(\mathbb{R}^n)$, obviously

$$D_1(f) = \sum_{j=0}^\infty D(f^j).$$

We estimate each term of the sum separately. Fix j , and put $t = 2^{-j}$. Then

$$\begin{aligned} D(f^j) &= t^{-i} D((f^j)_t) \\ &= 2^{ij} D(f_t \cdot \varphi(|x|)). \end{aligned}$$

Now $\text{supp}(f_t \cdot \varphi(|x|))$ is contained in the ball of radius 2 about 0. So

$$|D(f_t \cdot \varphi(|x|))| \leq C_1 \cdot \sup_{|\alpha| \leq k_2} |D^\alpha(f_t \cdot \varphi)|.$$

If we use the rule for differentiating products, and the fact that the first k_2 derivatives of φ are uniformly bounded, we find that this is bounded by

$$C_2 \cdot 2^{j(k_2+n)} \sup_{|\alpha| \leq k_2} \sup_{2^{j-1} < |x| < 2^{j+1}} |D^\alpha(f)|.$$

(The power of two comes from the fact that we are differentiating some dilation of f , and the range of $|x|$ from the support of φ .) So

$$|D(f^j)| \leq C_2 \cdot 2^{j(i+k_2+n)} \sup_{|\alpha| \leq k_2} \sup_{2^{j-1} < |x| < 2^{j+1}} |D^\alpha(f)|.$$

For the values of x in question,

$$\begin{aligned} 2^{j(i+k_2+n)} &\leq (2 \cdot |x|)^{(i+k_2+n+1)} \cdot 2^{-j} \\ &\leq (2 \cdot |x|)^{k_1} \cdot 2^{-j}. \end{aligned}$$

Hence

$$|D(f^j)| \leq C_3 \cdot 2^{-j} \sup_{|\alpha| \leq k_2} \sup_x |x^{k_1} D^\alpha(f)|;$$

so

$$\left| \sum_{j=0}^\infty D(f^j) \right| \leq C_3 \cdot \sup_{|\alpha| \leq k_2} \sup_x |x^{k_1} D^\alpha(f)|.$$

This proves the estimate for D_1 , and hence for D . Q.E.D.

If α is a multi-index, let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

LEMMA 2.3. *Let θ be a distribution on a neighborhood Ω of 0 in $\mathbb{R}^n$, admitting an asymptotic expansion $\theta \sim \sum t^i D_i$. If α and β are multi-indices, then $\theta \circ x^\alpha$ (θ composed with multiplication by x^α) and $\theta \circ D^\beta$ admit asymptotic expansions. More precisely:*

$$\begin{aligned} \theta \circ x^\alpha &\sim \sum t^i (D_{i-|\alpha|} \circ x^\alpha) \\ \theta \circ D^\beta &\sim \sum t^i (D_{i+|\beta|} \circ D^\beta). \end{aligned}$$

Proof. Formally these are obvious; the necessary estimates of remainders are trivial. Q.E.D.

COROLLARY 2.4. Let $D = \sum_{|\alpha| \leq N} g_\alpha D^\alpha$ be a differential operator on Ω , with $g_\alpha \in C^\infty(\Omega)$. Let $g_\alpha \sim \sum_{\beta} g_{\alpha\beta} x^\beta$ be the Taylor expansion of g_α at 0 ($g_{\alpha\beta} \in \mathbb{Y}$). Let θ be a distribution on Ω admitting an asymptotic expansion $\theta \sim \sum_{i=-r}^\infty t^i D_i$ at 0. Then $\theta \circ D$ admits the asymptotic expansion

$$\theta \circ D \sim \sum_{i=r-N}^\infty t^i \sum_{k=r-i-1}^N \left(\sum_{|\alpha|+|\beta|=k} g_{\alpha\beta} D_{i+k} \circ x^\beta D^\alpha \right).$$

Proof. Notice first that the inner sum is finite. So formally the result follows from Lemma 2.3; the necessary estimates are a straightforward consequence of Taylor's theorem with remainder, and are left to the reader. Q.E.D.

DEFINITION 2.5. Let θ be a distribution on a neighborhood Ω of 0 in $\mathbb{R}^n$, admitting an asymptotic expansion $\theta \sim \sum t^i D_i$. The asymptotic support of θ at 0, $AS(\theta)$, is the closure of the union of the supports of the D_i .

It is easy to see that $AS(\theta)$ is contained in the wave front set of θ at zero.

COROLLARY 2.5. In the setting of Corollary 2.4,

$$AS(\theta \circ D) \subseteq AS(\theta).$$

If D is multiplication by a function which does not vanish at zero, then equality holds.

Proof. If T is a tempered distribution on $\mathbb{R}^n$, then

$$(T \circ x^\beta D^\alpha)^\wedge = c_{\alpha\beta} \hat{T} \circ D^\beta x^\alpha,$$

where $c_{\alpha\beta}$ is an appropriate power of i . In particular,

$$\text{supp}(T \circ x^\beta D^\alpha)^\wedge \subseteq \text{supp } \hat{T}.$$

The first statement follows immediately. For the second, note simply that D^{-1} is an operator of the same sort, at least in a neighborhood of zero. Q.E.D.

Of course the second statement can be generalized enormously, using some sort of ellipticity condition on D ; this is standard for the wavefront set. Since we don't need such results, they are omitted.

EXAMPLE. Let $C \subseteq \mathbb{R}^n$ be a closed cone, Ω a neighborhood of 0, and g a function on $C \cap \Omega$ which is smooth on a neighborhood of $C \cap \Omega$. For $f \in C_c^\infty(\Omega)$, put

$$\theta(f) = \int_{C \cap \Omega} f(x) g(x) dx,$$

dx being the usual measure on $\mathbb{R}^n$. Then θ is a distribution on Ω , and has an

asymptotic expansion $\theta \sim \sum_{i=0}^\infty t^i D_i$. Here D_i is defined as follows: let $g \sim \sum g_\alpha x^\alpha$ be the Taylor expansion of g at 0. Then

$$D_i(f) = \int_C f(x) \left(\sum_{|\alpha|=i} g_\alpha x^\alpha \right) dx.$$

This is obvious if $g = 1$, and the general case follows from Corollary 2.4. This example will be used in the next section to obtain asymptotic expansions of characters.

3. ASYMPTOTIC EXPANSIONS OF INVARIANT EIGENDISTRIBUTIONS

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Consider an invariant eigendistribution Θ on G . Let $\mathfrak{z}$ be the center of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$. In this section we will lift the distribution Θ to a distribution on the Lie algebra and show that it has an asymptotic expansion according to the definition in section 2. In particular Θ could be the character of an irreducible representation. In this case we will use the asymptotic expansion to obtain information about the representation.

We start by collecting some of the facts about invariant eigendistributions that we will use later.

THEOREM 3.1. Let

$$\Omega = \{X \in \mathfrak{g}; |\text{Im } \lambda| < \pi \text{ for any eigenvalue } \lambda \text{ of } \text{ad } X\}.$$

Then Ω is a G -invariant neighborhood of 0 in $\mathfrak{g}$.

(1) The exponential map $\exp: \mathfrak{g} \rightarrow G$ is an analytic diffeomorphism when restricted to Ω onto the open set $\mathcal{W} = \exp \Omega$. Put

$$j(X) = \det((\exp(\text{ad } X/2) - \exp(-\text{ad } X/2))/\text{ad } X).$$

Define $\xi(X) = j(X)^{1/2}$, with the square root chosen so that $\xi(0) = 1$. Then the Haar measure dx and the Euclidean measure dX are related by

$$dx = \xi(X)^2 dX.$$

Let $\varphi \in C^\infty(\Omega)$. Define $f_\varphi \in C^\infty(\mathcal{W})$ by

$$f_\varphi(\exp X) = \xi(X)^{-1} \varphi(X).$$

Then

$$\int \varphi_1 \varphi_2 dX = \int f_{\varphi_1} f_{\varphi_2} dx.$$

If Θ is a distribution on $\mathcal{W}$ we can define θ a distribution on Ω by the relation

$$\theta(\varphi) = \Theta(f_\varphi).$$

If Θ is G -invariant then so is θ . From here on assume Θ is invariant.

(2) Let $I(\mathfrak{g}_c)$ be the ring of invariant polynomials on $\mathfrak{g}_c$. Then there is an algebra isomorphism

$$z \mapsto p_z$$

between $\mathfrak{z}$ and $I(\mathfrak{g}_c)$ such that if Θ lifts to θ then $z \cdot \Theta$ lifts to $\partial(p_z)\theta$. In particular, if

$$z \cdot \Theta = \chi(z)\Theta \quad \text{for all } z \in \mathfrak{z}$$

for some character $\chi: \mathfrak{z} \rightarrow \mathbb{C}$, then

$$\partial(p)\theta = \chi(p)\theta$$

where $\chi(p_z) = \chi(z)$.

In other words, if Θ is an eigendistribution for $\mathfrak{z}$ then θ is an eigendistribution for $\partial(I(\mathfrak{g}_c))$.

Proof. In this form these results are due to Harish-Chandra. A summary can be found in [2]. The proofs are in [3], [5], and [6].

We will denote by $\mathcal{N}$ the set of nilpotent elements in $\mathfrak{g}$. By definition

$$\mathcal{N} = \{X \in \mathfrak{g}; \text{ad } X \text{ is nilpotent}\}.$$

We will often identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ by the Cartan-Killing form $B(\cdot, \cdot)$. We define the Fourier transform as

$$\mathcal{F}_{\mathfrak{g}} f(X) = \int_{\mathfrak{g}^*} e^{i\lambda(X)} f(\lambda) d\lambda$$

for any $f \in C_c^\infty(\mathfrak{g}^*)$ or $f \in \mathcal{S}(\mathfrak{g}^*)$ (the Schwartz space). Sometimes when there is no ambiguity in terms of the algebra used we will denote the Fourier transform by f . Via the identification mentioned before we can also write

$$\mathcal{F}_{\mathfrak{g}} f(X) = \int_{\mathfrak{g}} e^{iB(X, Y)} f(Y) dY$$

where $f \in \mathcal{S}(\mathfrak{g})$.

THEOREM 3.2. Let Θ be an invariant eigendistribution, θ the lifting to $C_c^\infty(\Omega)$ according to Theorem 3.1. Then θ has an asymptotic expansion. If $\theta \sim \sum r! D_i$ then the D_i are tempered, invariant and $\text{supp } \hat{D}_i \subseteq \mathcal{N}$.

Proof. According to [5], θ is a function, analytic on $\mathfrak{g}' \cap \Omega$, locally L^1 . Suppose $\mathfrak{h}_1, \dots, \mathfrak{h}_r$ is a complete set of representatives of Cartan subalgebras. Let Δ_i be the set of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_{i,c}$ and P_i a positive system. We define

$$\pi_i(Z) = \prod_{\alpha \in P_i} \alpha(Z).$$

According to Harish-Chandra, for each $f \in C_c^\infty(\mathfrak{g})$ we define

$$\varphi_f(Z) = \overline{\pi_i(Z)} \int_{G \cdot H_i} f(\text{Ad } x \cdot Z) dx$$

where H_i is the Cartan subgroup corresponding to $\mathfrak{h}_i$ and $Z \in \mathfrak{h}_i'$. The main property of φ_f is that it defines a continuous map from $\mathcal{S}(\mathfrak{g})$ to $\mathcal{S}'(\mathfrak{h})$ (see for example [12], vol. 2, section 8.4).

Let $f \in C_c^\infty(\Omega)$. If $\mathfrak{h}_i^j$ are the connected components of the sets $\mathfrak{h}_i'$, Weyl's integral formula implies

$$\theta(f) = \sum_{i,j} c_{ij} \int_{\mathfrak{h}_i^j \cap \Omega} \pi_i(Z) \theta_i(Z) \varphi_f^j(Z) dZ$$

where $\theta_{ij} = \theta|_{\mathfrak{h}_i^j \cap \Omega}$. It is known that $\pi_i \theta_{ij}$ is an analytic function which extends to a neighborhood of the closure of $\mathfrak{h}_i^j \cap \Omega$; in fact it is a linear combination of terms consisting of polynomials multiplied by exponentials. By changing variables,

$$\theta(f_i) = \sum_{i,j} c_{ij} \int_{\mathfrak{h}_i^j \cap \Omega} \pi_i(tZ) \theta_{ij}(tZ) \varphi_f^j(Z) dZ.$$

By the example in section 1 this has an asymptotic expansion. By Lemmas 2.1 and 2.2 the D_i 's are homogeneous and therefore tempered. By the uniqueness of the asymptotic expansion the D_i 's are also invariant. Due to Theorem 3.1,

$$\partial(p)\theta = \chi(p)\theta \quad \text{for } p \in I(\mathfrak{g}_c).$$

Let $\deg p = s$, p homogeneous of positive degree. Then

$$[\partial(p)\theta](f_i) = t^{-s}\theta((\partial(p)f)_i).$$

Thus, by the uniqueness of the expansion,

$$\partial(p)D_i = \chi(p)D_{i-s},$$

so for r large enough,

$$\partial(p^r)D_i = 0.$$

This implies that (letting $I^+(g_0)$ denote the ideal of invariant polynomials with zero constant term)

$$\text{supp } \hat{D}_1 \subseteq \{X: p(X) = 0 \text{ for all } p \in I^+(g_0)\}.$$

On the other hand, it is well known that

$$\mathcal{N} = \{X: p(X) = 0 \text{ for all } p \in I^+(g_0)\}.$$

We now mention some facts about induced representations which will be used later.

Let $P = MAN$ be a parabolic subgroup and π an irreducible representation of M on a Hilbert space $\mathcal{H}$. Let $\nu \in \mathfrak{a}_0^*$ where $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is the Lie algebra of P . Let π_p be the representation of P on $\mathcal{H}$ defined by

$$\pi_P(man) = e^{(\sqrt{-1}\nu - \rho)(\log a)} \pi(m)$$

where $\rho(X) = \frac{1}{2} \text{tr}(\text{ad } X|_{\mathfrak{n}})$.

Let $\pi_p = \text{Ind}_P^G \pi_p$.

LEMMA 3.3. Let $\Theta = \text{tr } \pi$ and $\Theta_p = \text{tr } \pi_p$. If $f \in C_c^\infty(G)$ and $\varphi \in C_c^\infty(P)$, then

$$\begin{aligned} \text{tr } \pi_P(\varphi) &= \int_{MAN} e^{(\rho + \sqrt{-1}\nu)(\log a)} \Theta(m) \varphi(man) dm da dn \\ \Theta_p(f) &= \int_{MA} e^{(\rho + \sqrt{-1}\nu)(\log a)} \Theta(m) \int_{KN'} f(kmank^{-1}) dk dn dm da \\ &= \int_{MA} e^{\sqrt{-1}\nu(\log a)} \Theta(m) D(ma) \int_{G/M_A} f(xmax^{-1}) dx dm da \end{aligned}$$

where $D(\exp(X_m + X_a)) = |\det(e^{\text{ad}(X_m + X_a/2)} - e^{-\text{ad}(X_m + X_a/2)})|_{\mathfrak{n}}$ for $X_m \in \mathfrak{m}$, $X_a \in \mathfrak{a}$. In particular let $f^x(Z) = f(\text{Ad } x \cdot Z)$ and $f|_P$ be the function f restricted to P . Then

$$\Theta_p(f) = \int_{G/P} \text{tr } \pi_P(f^x|_P) dx.$$

Proof. This is well known when P is minimal. The proof carries over with simple modifications. See [13].

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition. Let K be the compact group corresponding to $\mathfrak{k}$.

LEMMA 3.4. Let $\varphi \in C_c^\infty(\mathfrak{s})$, $\varphi \geq 0$, $\int_{\mathfrak{s}} \varphi(X) dX = 1$ and $f \in C_c^\infty(K)$. Define

$$\begin{aligned} \varphi_\epsilon(X) &= e^{-\dim \mathfrak{s}} \varphi(\epsilon^{-1}X) \\ f_\epsilon(\exp Xk) &= \varphi_\epsilon(X) f(k) \end{aligned}$$

where $X \in \mathfrak{s}$, $k \in K$. Let π be an irreducible representation with character Θ . $\pi|_K$ also has a character which we denote by Θ^K . Let $\psi \in C_c^\infty(\Omega \cap \mathfrak{k})$ be such that $\psi(Y) = f(\exp Y)$. Let θ and θ^K be the distributions corresponding to Θ and Θ^K . Let ξ and ξ_K be the functions in Theorem 3.1 corresponding to G and K . Then

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \pi(f_\epsilon) &= \pi_K(f) \quad (\text{weakly}) \\ \lim_{\epsilon \rightarrow 0} \Theta(f_\epsilon) &= \Theta^K(f) \\ \lim_{\epsilon \rightarrow 0} \theta(\psi_\epsilon) &= \theta^K(\xi|_K \psi) \end{aligned}$$

where $\psi_\epsilon(X + Y) = \varphi_\epsilon(X) \psi(Y)$ and $X \in \mathfrak{s}$, $Y \in \mathfrak{k}$.

Proof. We have the relations

$$\begin{aligned} \pi(f_\epsilon) &= \int_G f_\epsilon(x) \pi(x) dx \\ &= \int_{\mathfrak{s}} D(X) \varphi_\epsilon(X) \pi(\exp X) dX \int_K f(k) \pi(k) dk \end{aligned}$$

where $D(X)$ is the Jacobian of the map $(X, k) \rightarrow \exp X \cdot k$. Since $\pi_K(f) = \int_K f(k) \pi(k) dk$ and $\int \varphi(X) dX = 1$, $\pi(f_\epsilon) = \pi_K(f) = \pi_K(f) \int_{\mathfrak{s}} [D(X) \pi(\exp X) - I] \varphi_\epsilon(X) dX$. Then for any $v \in \mathcal{H}$,

$$\|(\pi(f_\epsilon) - \pi_K(f))v\| \leq \|\pi_K(f)\| \int_{\mathfrak{s}} \|D(x) \pi(\exp X)v - v\| \varphi_\epsilon(X) dX.$$

By changing variables,

$$\int_{\mathfrak{s}} \|D(X) \pi(\exp X)v - v\| \varphi_\epsilon(X) dX = \int_{\mathfrak{s}} \|D(\epsilon X) \pi(\exp \epsilon X)v - v\| \varphi(X) dX.$$

Since φ is compactly supported,

$$\|D(\epsilon X) \pi(\exp \epsilon X)v - v\| \leq C$$

for all $X \in \text{supp } \varphi$ and $\epsilon < 1$. Thus we can apply the bounded convergence theorem to obtain

$$\lim_{\epsilon \rightarrow 0} \pi(f_\epsilon)v = \pi_K(f)v.$$

The fact that π_K is of trace class is well known. We need part of the proof of this fact to prove the statements in the lemma. Let

$$\Omega = 1 - (X_1^2 + \cdots + X_r^2) \in U(\mathfrak{k})$$

where $X_1, \dots, X_r$ is an orthonormal basis of $\mathfrak{k}$. Then let $\mathcal{H}$ be the space of the

representation π and $\{v_i\}_{i \in N}$ an orthonormal basis of K -finite vectors for $\mathcal{H}$. Then

$$\begin{aligned}\Theta(f_\epsilon) &= \sum_i (\pi(f_\epsilon)v_i, v_i) \\ \Theta^K(f) &= \sum_i (\pi_K(f)v_i, v_i) \\ \theta(\psi_\epsilon) &= \sum_i (\pi(f_{\psi_\epsilon})v_i, v_i) \\ \theta^K(\psi) &= \sum_i (\pi_K(f_\psi)v_i, v_i).\end{aligned}$$

Similarly to the previous statement,

$$\pi(f_{\psi_\epsilon}) = \int_G \int_{\psi_\epsilon}(x) \pi(x) dx = \int_{\mathfrak{g}} \xi(X) \psi_\epsilon(X) \pi(\exp X) dX$$

converges weakly to $\pi_K(f_{\xi/\xi_K \psi})$.

Thus any finite sum occurring in $\Theta(f_\epsilon)$ or $\theta(\psi_\epsilon)$ converges to the corresponding sum in $\Theta^K(f)$ and $\theta^K(\xi/\xi_K \psi)$ respectively.

Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$ and let $A \subseteq \mathfrak{t}^*$ parametrize $\hat{K}$. For $\delta \in A$, let $\mathcal{H}_\delta$ denote the δ -primary subspace of $\mathcal{H}$; we may assume that each v_i lies in some $\mathcal{H}_\delta$. Then for $h \in C_c^\infty(G)$,

$$\sum_i (\pi(h)v_i, v_i) = \sum_{\delta \in A} c(\delta)^{-s} \sum_{v_i \in \mathcal{H}_\delta} (\pi(\Omega^s h)v_i, v_i)$$

where $c(\delta)$ is the value of Ω on δ . Let $m(\delta)$ be the multiplicity of δ in $\mathcal{H}$ and $d(\delta)$ the dimension of the representation δ . Then if $\|\pi(x)\| < C$ for $x \in \text{supp}(h)$,

$$\sum_{|\delta| \geq N} c(\delta)^{-s} \sum_{v_i \in \mathcal{H}_\delta} |(\pi(\Omega^s h)v_i, v_i)| \leq \|\Omega^s h\|_{1,G} \cdot C \cdot \sum_{|\delta| \geq N} m(\delta) c(\delta)^{-s} d(\delta).$$

It is known that

$$\sum_{\delta} m(\delta) c(\delta)^{-s} d(\delta)$$

converges if s is large enough.

In order to complete the proof we have to show that $\|\Omega^s f_\epsilon\|_{1,G}$ and $\|\Omega^s f_{\psi_\epsilon}\|_{1,G}$ are bounded independently of ϵ . We note

$$\Omega^s f_\epsilon = \varphi_\epsilon \cdot \Omega^s f,$$

so by a change of variables

$$\begin{aligned}\|\Omega^s f_\epsilon\|_{1,G} &\leq \|\Omega^s f\|_{1,K} \\ \Omega^s f_\epsilon(x) &= \left(1 - \sum \frac{\partial^2}{\partial t_i^2}\right)^s \Big|_{t_i=0} (\xi^{-1} \psi_\epsilon)(\log(x \exp(-t_1 X_1 - \dots - t_r X_r))).\end{aligned}$$

By using the Campbell-Baker-Hausdorff formula we can write

$$\begin{aligned}\log(\exp Y \exp X \exp(-t_1 X_1 - \dots - t_r X_r)) \\ = M_t(Y, X, t_r X_r) + M_s(Y, X, t_i X_i)\end{aligned}$$

where $M_t \in \mathfrak{k}$ and $M_s \in \mathfrak{s}$ are absolutely convergent infinite series in Lie brackets of $X, Y, t_i X_i$. We note that each term of M_s contains at least one Y . Then

$$\|\Omega^s f_{\psi_\epsilon}\|_{1,G} = \int_{\mathfrak{t}+\mathfrak{s}} \left(1 - \sum \frac{\partial^2}{\partial t_i^2}\right)^s \Big|_{t_i=0} \xi(M_t + M_s) \varphi_\epsilon(M_t) \psi(M_t) dY dX.$$

By changing variables to $Y = \epsilon Y'$ we get

$$\begin{aligned}\int_{\mathfrak{t}+\mathfrak{s}} \left(1 - \sum \frac{\partial^2}{\partial t_i^2}\right)^s \Big|_{t_i=0} \xi(M_s(\epsilon Y, X, t_i X_i) + M_t(\epsilon Y, X, t_i X_i)) \\ \cdot \varphi(\epsilon^{-1} M_s(\epsilon Y, X, t_i X_i)) \psi(\epsilon Y, X, t_i X_i) dY dX.\end{aligned}$$

Then $\epsilon^{-1} M_s(\epsilon Y, X, t_i X_i)$ is analytic at $\epsilon = t_1 = \dots = t_r = 0$. The rest of the proof is straightforward.

Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ be a parabolic subalgebra and $\mathfrak{p}^* = \mathfrak{m}^* + \mathfrak{a}^* + \mathfrak{n}^*$ be its dual.

THEOREM 3.5. Let $P = MAN$ be a parabolic, $\nu \in \mathfrak{a}_\nu^1$ and $\pi \in \hat{M}$ an irreducible representation of M on a Hilbert space. Let π_ν be the induced representation, Θ and Θ_ν the corresponding characters. Let θ and θ_ν be their lifts to the algebras $\mathfrak{g}$ and $\mathfrak{m}$. Let $\Theta_\mathfrak{p}$ be the character of $\pi_\mathfrak{p}$ defined earlier and $\theta_\mathfrak{p}$ its lifting to $\mathfrak{p}$.

For $\varphi \in C_c^\infty(\mathfrak{g})$ define φ^x to be

$$\varphi^x(Z) = \varphi(\text{Ad } x \cdot Z)$$

and $\varphi|_\mathfrak{p}$ the restriction of φ to $\mathfrak{p}$. Then

$$\theta_\nu(\varphi) = \int_K \theta_\mathfrak{p}(\xi_\mathfrak{p}/\xi \cdot \varphi^x|_\mathfrak{p}) d\mathbf{x}. \quad (1)$$

Let $\theta_\mathfrak{p} \circ \xi_\mathfrak{p}/\xi \sim \sum t^i D_i$ and $\theta_\mathfrak{p} \sim \sum t^i E_i$. Then

$$\theta_\nu \sim \sum t^{\dim \mathfrak{p} - \dim \mathfrak{a} + i} \int_K D_i(\varphi^x|_\mathfrak{p}) d\mathbf{x} \quad (2)$$

and

$$AS(\Theta_\nu) \subseteq \text{Ad } K \cdot \{\lambda \in \mathfrak{g}^*; \lambda|_\mathfrak{p} \in AS(\Theta_\mathfrak{p})\}. \quad (3)$$

Assume that E_i satisfy the following. Let $\varphi \geq 0$. Then if $\text{supp } \varphi \cap \text{supp } E_j = \emptyset$ for $j < r$ and $E_r(\varphi) \neq 0$, then $E_r(\varphi) > 0$. Then equality holds in (3). (This is e.g. true for representations which are unitarily induced from limits of discrete series).

Finally,

$$AS(\theta_p) = \{\lambda \in \mathfrak{p}^*: \lambda|_{\mathfrak{a}+\mathfrak{n}} = 0 \text{ and } \lambda|_{\mathfrak{m}} \in AS(\theta)\}. \quad (4)$$

Proof. We note that

$$\begin{aligned} \xi_p(X) &= |\det_{\mathfrak{p}}(e^{\text{ad } X/2} - e^{-\text{ad } X/2})_{\text{ad } X}|^{1/2} \\ &= \xi_M(X_{\mathfrak{m}}) R(X_{\mathfrak{m}} + X_{\mathfrak{a}})^{1/2} \end{aligned}$$

and

$$\xi(X) = \xi_M(X_{\mathfrak{m}}) \cdot R(X_{\mathfrak{m}} + X_{\mathfrak{a}})$$

where $X = X_{\mathfrak{m}} + X_{\mathfrak{a}} + X_{\mathfrak{n}} \in \mathfrak{p}$ and

$$R(X_{\mathfrak{m}} + X_{\mathfrak{a}}) = |\det_{\mathfrak{n}}((e^{\text{ad } X/2} - e^{-\text{ad } X/2})_{\text{ad } X})|.$$

Then (1) and (2) follows from these formulas and Lemma 3.3. Relation (3) is a simple consequence of these formulas. We note that if the E_i 's satisfy the positivity condition then so do the D_i 's by the formula in Corollary 2.4.

Suppose $\lambda \notin AS(\theta_p)$ but $\lambda \in \text{Ad } K \cdot \{\lambda \in \mathfrak{g}^*: \lambda|_{\mathfrak{p}} \in AS(\theta_p)\}$. Then for any $\varphi \in C_c^\infty(\mathfrak{g}^*)$, $\text{supp } \varphi$ contained in a small enough neighborhood of λ ,

$$\int_K D_i(\varphi^x|_{\mathfrak{p}}) dx = 0$$

for all i . Let r be such that $\lambda \in \text{supp } \hat{D}_r$ but $\text{Ad } K \cdot \lambda \notin \text{supp } \hat{D}_i$ with $i < r$. Then there is $\varphi \in C_c^\infty(\mathfrak{g}^*)$ such that $\varphi \geq 0$, $\varphi(\lambda) \neq 0$ and

$$\hat{D}_i(\varphi) \neq 0$$

while

$$\text{supp}(\varphi^x|_{\mathfrak{p}}) \cap \text{supp } \hat{D}_i = \emptyset, \quad i < r \text{ and all } x \in K.$$

Then, it follows

$$\int_K \hat{D}_i(\varphi^x) dx \neq 0$$

since

$$\hat{D}_i(\varphi^x) \geq 0 \text{ for all } x.$$

We now relate the support of θ_p with the support of θ . From Lemma 3.3 we deduce the relation

$$\theta_p(\varphi) = \theta((\mathcal{F}_{\mathfrak{a}}^{-1}\varphi)_{\mathfrak{n}}(\nu))$$

where $\mathcal{F}_{\mathfrak{a}}$ is the Fourier transform with respect to $\mathfrak{a}$ and

$$\varphi_{\mathfrak{n}}(X) = \int_{\mathfrak{n}} \varphi(X + X_{\mathfrak{n}}) dX_{\mathfrak{n}}, \quad X \in \mathfrak{m} + \mathfrak{a}.$$

Assume

$$\theta(\psi_t) \sim \sum t^i F_i(\psi)$$

for $\psi \in C_0^\infty(\mathfrak{m})$. Then we have the relation

$$\theta_p(\mathcal{F}_{\mathfrak{p}}\varphi)_t = \theta(\mathcal{F}_{\mathfrak{m}}(\varphi(\cdot, t\nu, 0)))_t$$

where $\varphi \in C_c^\infty(\mathfrak{p}^*)$ and the three coordinates refer to the decomposition $\mathfrak{p}^* = \mathfrak{m}^* + \mathfrak{a}^* + \mathfrak{n}^*$.

Let

$$\varphi(\cdot, t\nu, 0) \sim \sum (\partial(p_{s,\nu}))_0 \varphi) t^s$$

be the Taylor expansion of φ in the variable t .

Thus

$$E_j(\mathcal{F}_{\mathfrak{p}}\varphi) = \sum_{s \leq j} \partial(p_{s,\nu})|_{\lambda_0=0} F_{j-s}(\mathcal{F}_{\mathfrak{m}}(\varphi|_{\mathfrak{n}})).$$

This implies relation (4) in the theorem.

Let $\delta \in \hat{K}$ be an irreducible representation of K . Then we can associate to each such δ an orbit of a regular element in $\mathfrak{f}^*$ by the coadjoint action. We identify $\hat{K}$ with this set of orbits.

THEOREM 3.6. *Let π be an irreducible representation of G on a Hilbert space $\mathcal{H}$. Let Θ be its character and Θ^K be the character of $\pi_K = \pi|_K$. Then Θ^K has an asymptotic expansion at 0. Let*

$$K(\pi) = \{\delta \in \mathfrak{f}^*: \text{there is a sequence } t_n \delta_n \rightarrow \delta \text{ such that}$$

$$t_n \rightarrow 0 \text{ and } \delta_n \text{ occurs with positive multiplicity in } \pi_K\}.$$

Then

$$AS(\Theta^K) = K(\pi). \quad (1)$$

Also

$$AS(\Theta^K) \subseteq \{\delta \in \mathfrak{f}^*: \text{there is } \lambda \in AS(\theta) \text{ such that } \lambda|_{\mathfrak{k}} = \delta\}. \quad (2)$$

If π is a discrete series representation then equality holds. (Related results may be found in [8], Theorem 6.2. A weak version of (2) was recently proved by Howe and Wallach (unpublished).)

Proof. We first show that θ^K has an asymptotic expansion at zero when θ is a discrete series character. Thus we assume $\text{rank } \mathfrak{g} = \text{rank } \mathfrak{k}$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Cartan-Killing form. Then

$$\varrho(p)\theta = p(\lambda)\theta \quad \text{for } p \in I(\mathfrak{q}_\theta)$$

and λ a regular elliptic element. By the work of Harish-Chandra we may assume θ is a globally defined (not just on $C_c^\infty(\Omega)$) tempered eigendistribution. Its Fourier transform is a linear combination of measures supported on orbits of regular elliptic elements, in fact by [9] just one such orbit. Let $f \in C_c^\infty(\mathfrak{g}^*)$. Choose a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ and let T be its Cartan subgroup. Then ([9])

$$\theta(\mathcal{F}_\theta f) = \varphi_f^T(\lambda)$$

where φ_f and its properties were explained during the proof of Theorem 3.2. Since

$$(\mathcal{F}_\theta f)_t = t^{-\dim \mathfrak{g}} \mathcal{F}_\theta(f_{t^{-1}}),$$

$$\theta((\mathcal{F}_\theta f)_t) = t^{-\dim \mathfrak{g}} \theta(\mathcal{F}_\theta(f_{t^{-1}})) = t^{-(\dim \mathfrak{g} - \text{rank } \mathfrak{g})/2} \varphi_f(t, f, \lambda).$$

As already mentioned, $\varphi_f \in \mathcal{S}(\mathfrak{t})$ where $\mathfrak{t}$ are the connected components of $\mathfrak{t}^*$. Thus

$$\varphi_f^T(t, \lambda) = \sum_{i \leq N} t^i E_i(f) + t^{N+1} O(f)$$

where $|O(f)| \subseteq \nu_\epsilon(f)$ is a seminorm in the Schwartz topology.

In particular we apply this to ψ_ϵ defined in Lemma 3.4. We recall that

$$\lim_{\epsilon \rightarrow 0} \theta(\psi_\epsilon) = \theta^K(\xi/\xi_K \psi)$$

so that

$$\lim_{\epsilon \rightarrow 0} \varphi_{\mathcal{F}_\theta^{-1}\psi_\epsilon}^T(\lambda) = \theta^K(\xi/\xi_K \psi). \quad (*)$$

For any function $f \in C_c^\infty(\mathfrak{t}^*)$ we define $f_K \in C^\infty(\mathfrak{g}^*)$ by

$$f_K(\lambda) = f(\lambda|_{\mathfrak{t}}).$$

Let $f = \mathcal{F}_\mathfrak{t}^{-1}\psi$ where ψ is as in Lemma 3.4. Then

$$\lim_{\epsilon \rightarrow 0} \mathcal{F}_\mathfrak{g}^{-1}\psi_\epsilon = f_K.$$

We show that f_K coincides with a Schwartz function along the support of

$\varphi_f^T(\lambda)$. Indeed let $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{s}^*$ be the Cartan decomposition. Let $X \in \mathfrak{g}^*$ be such that

$$-\eta < -B(X, X) = |B(X_\mathfrak{t}, X_\mathfrak{t})| - |B(X_\mathfrak{s}, X_\mathfrak{s})| < |\eta|.$$

Then

$$B(X_\mathfrak{s}, X_\mathfrak{s}) < \eta + |B(X_\mathfrak{t}, X_\mathfrak{t})|.$$

Thus, on the set

$$\mathcal{U}_\eta = \{X: |B(X, X)| < \eta\}$$

for any polynomial $p \in S(\mathfrak{g}_\epsilon)$ there is $p_K \in S(\mathfrak{k}_\epsilon)$ such that

$$|p(X)| \leq |p_K(X_\mathfrak{t})| \quad \text{for all } X \in \mathcal{U}_\eta.$$

Let $h_1 \in C_c^\infty(\mathbb{R})$, $h_1 \geq 0$ such that $\text{supp } h_1 \subset (-1, +1)$. Let

$$h(X) = h_1(B(X, X)).$$

Then

$$hf_K \in \mathcal{S}(\mathfrak{g}^*)$$

and

$$hf_K = f_K \text{ on } \mathcal{U}_{1/2}$$

if we assume $h_1 \equiv 1$ on $(-\frac{1}{2}, +\frac{1}{2})$.

Thus $\varphi_{f_K}^T(t, \lambda)$ makes sense for t small enough and λ regular. In addition, for any seminorm ν_ϵ on $\mathcal{S}(\mathfrak{g}^*)$ there is a seminorm $\nu_{\epsilon'}$ on $\mathcal{S}(\mathfrak{k}^*)$ such that

$$\nu_{\epsilon'}(hf_K) \leq \nu_{\epsilon'}(f).$$

The set $\mathcal{U}_{1/2}$ contains the G orbit of $t\lambda$ for t small enough as well as the set of nilpotents $\mathcal{N}$. This implies that

$$\lim_{\epsilon \rightarrow 0} \varphi_{\mathcal{F}_\mathfrak{g}^{-1}\psi_\epsilon}^T(t, \lambda) = \varphi_{h/K}^T(t, \lambda)$$

$$\lim_{\epsilon \rightarrow 0} E_i(\mathcal{F}_\mathfrak{g}^{-1}\psi_\epsilon) = E_i(hf_K), \quad \text{all } i.$$

Finally, since $\psi_\epsilon = \varphi_\epsilon \otimes \psi$ we get

$$\mathcal{F}_\mathfrak{g}^{-1}\psi_\epsilon = \mathcal{F}_\mathfrak{s}^{-1}\varphi_\epsilon \otimes \mathcal{F}_\mathfrak{t}^{-1}\psi.$$

We can find a bound on $O(\psi_\epsilon)$ uniform in $\epsilon < 1$. We conclude

$$\lim_{\epsilon \rightarrow 0} \varphi_{\mathcal{F}_\mathfrak{g}^{-1}\psi_\epsilon}^T(t, \lambda) = \varphi_{f_K}^T(t, \lambda)$$

and

$$\varphi_{f_K}^T(t\lambda) \sim \sum E_i(f_K)t^i.$$

Thus by (*) we get

$$\theta^K(\xi/\xi_K \psi_t) = t^{-(\dim \mathfrak{g} - \text{rank } \mathfrak{g})/2} \varphi_{f_K}^T(t\lambda)$$

so

$$\theta^K(\xi/\xi_K \psi_t) \sim \sum t^{i-(\dim \mathfrak{g} - \text{rank } \mathfrak{g})/2} E_i(f_K)$$

where $\psi = \mathcal{F}_1 f$.

Let $S_i = \text{supp } E_i$. We have to identify the union of the supports of the distributions

$$f \mapsto E_i(f_K).$$

Let

$$K(\theta) = \{\delta \in \mathfrak{t}^*: \text{there is } \lambda \in AS(\theta) \text{ such that } \lambda|_{\mathfrak{t}} = \delta\}.$$

We want to show that $AS(\theta^K) = K(\theta)$. The inclusion $AS(\theta^K) \subseteq K(\theta)$ is clear. Assume $\delta \in K(\theta)$ but $\delta \notin AS(\theta^K)$. Then there is a neighborhood of δ , $V_\delta \subseteq \mathfrak{t}^*$ such that for every $f \in C_c^\infty(\mathfrak{t}^*)$ with $\text{supp } f \subseteq V_\delta$,

$$E_i(f_K) = 0 \quad \text{for all } i.$$

On the other hand, there is λ such that $\lambda|_{\mathfrak{t}} = \delta$ and $\lambda \in \bigcup_i \text{supp } E_i$. Let r be the smallest integer such that $\{\lambda \in \mathfrak{g}^*: \lambda|_{\mathfrak{t}} = \delta\} \cap \text{supp } E_r \neq \emptyset$. Then we can shrink V_δ so that

$$\{\lambda : \lambda|_{\mathfrak{t}} \in V_\delta\} \cap \text{supp } E_i = \emptyset \quad \text{for } i < r.$$

Choose $f, g \in C_c^\infty(V_\delta)$, $f \geq 0$, $g \leq 0$, and a neighborhood U_δ of δ so that $f \geq \epsilon$ and $g \leq -\epsilon$ on U_δ . Let $\lambda_0 \in \text{supp } E_r$ be such that $\lambda_0|_{\mathfrak{t}} \in U_\delta$. Let U_{λ_0} be a neighborhood of λ_0 such that if $\lambda \in U_{\lambda_0}$ then $\lambda|_{\mathfrak{t}} \in U_\delta$. Let $h \in C_c^\infty(U_{\lambda_0})$ be such that $|h| \leq \epsilon$. Then $E_i(f_K - h) = 0$ for $i < r$. Since φ_r^T is positive on positive functions,

$$E_r(f_K - h) \geq 0, \quad E_r(h - g_K) \geq 0.$$

Thus $E_r(g_K) \leq E_r(h) \leq E_r(f_K)$ so $E_r(h) = 0$ for all $h \in C_c^\infty(U_{\lambda_0})$ a contradiction. Thus

$$K(\theta) = AS(\theta^K).$$

The case when θ is the character of a unitarily induced representation from a representation in the limits of the discrete series is identical since in that case θ

is the Fourier transform of a φ_f^L with L a non-compact Cartan subgroup. We now deal with the case when θ is not necessarily unitarily induced. Let $\theta_\nu = \text{Ind}_P^G(\pi \otimes \nu \otimes 1)$. If $P = MAN$ then the same proof as in Lemma 3.3 and Theorem 3.4 shows that, for $\psi \in C_c^\infty(K)$,

$$\text{tr } \pi_K(\psi) = \int_K \text{tr } \pi_{M_K}(\psi^x|_{M_K}) dx.$$

Here $M_K = M \cap K$ and m_K is its Lie algebra. For $\psi \in C_c^\infty(\mathfrak{t} \cap \Omega)$ this formula reads

$$\theta_\nu^K(\psi) = \int_K \theta^{M_K}(\xi_K/\xi_{M_K} \psi^x|_{m_K}) dx \quad (3)$$

where ξ_K and ξ_{M_K} are the functions defined in Theorem 3.1. Thus if θ^{M_K} has an asymptotic expansion at 0, θ_ν^K also has an asymptotic expansion. The formulas mentioned in this theorem plus the ones in Theorem 3.5 imply the following. Suppose π is in the limits of the discrete series. Let

$$\theta_\nu \sim \sum \nu^i E_i$$

and

$$\theta_\nu^K \circ \xi/\xi_K \sim \sum \nu^i F_j.$$

Then for $f \in C_c^\infty(\mathfrak{t}^*)$,

$$F_j(\mathcal{F}_1 f) = E_j(f_K). \quad (4)$$

(This holds for unitary ν by the arguments above, and in general by analytic continuation.)

Since any character is a linear combination of such θ_ν , the inclusion (2) follows from formula (4). Using formula (3) and the statements about $AS(\theta_\nu)$ in Theorem 3.5 and the equality in (2) for discrete series we can conclude equality in (2) for the character of just one induced representation.

Finally we show $K(\pi) = AS(\theta^K)$. Let $\delta \in \tilde{K}$ and χ_δ be its character. Then for $\varphi \in C_c^\infty(K)$,

$$\theta^K(\varphi) = \sum_{\delta \in \tilde{K}} m(\delta) \chi_\delta(\varphi).$$

On the other hand

$$\theta^K(\varphi_1) \sim \sum \nu^i D_i(\varphi) \quad \text{for } \varphi \in C_c^\infty(\mathfrak{t} \cap \Omega).$$

Since the D_i are K -invariant we may assume $\varphi^x = \varphi$ for all $x \in K$. Substituting $\mathcal{F}\varphi$ for φ we get

$$\theta^K(\mathcal{F}_t\varphi) = \sum_{\delta \in K} m(\delta) d(\delta) \varphi(\delta)$$

$$\theta^K(\mathcal{F}_t\varphi_t) = \sum_{\delta \in K} m(\delta) d(\delta) \varphi(t\delta)$$

where $d(\delta)$ is the dimension of the representation δ . Thus

$$\sum_{\delta \in K} m(\delta) d(\delta) \varphi(t\delta) \sim \sum t^i \hat{D}_i(\varphi).$$

Suppose $\mu \in K(\pi)$ but $\mu \notin AS(\theta^K)$. Then there is a neighborhood of V_μ of μ such that $\text{supp } \varphi \subseteq V_\mu$ implies $\hat{D}_i(\varphi) = 0$ for all i . But then

$$\lim_{t \rightarrow 0} \sum_{\delta} m(\delta) d(\delta) \varphi(t\delta) = 0.$$

Let $\varphi \geq 0$. There is a sequence $t_n \delta_n \rightarrow \mu$. Assume $\varphi \equiv 1$ in a small neighborhood of μ . Then

$$\sum_{\delta} m(\delta) d(\delta) \varphi(t_n \delta) \geq m(\delta_n) d(\delta_n) \varphi(t_n \delta_n) \geq 1$$

for n large enough, a contradiction. Conversely, let $\mu \in AS(\theta^K)$ but $\mu \notin K(\pi)$. Then there is a set $B_\mu(\epsilon) = \{\delta \in \mathfrak{t}^*: \|\delta\| = 1, \|\delta - (\mu/\|\mu\|)\| < \epsilon\}$ such that, with at most finitely many exceptions, if $m(\delta) > 0$, then $\delta \notin V_\mu(\epsilon) = \{\gamma: \alpha\gamma \in B_\mu(\epsilon) \text{ for some } \alpha \in \mathbb{R}^+\}$. Let φ be such that $\text{supp } \varphi \subseteq \text{Ad } K \cdot V_\mu(\epsilon)$. Then

$$\sum m(\delta) d(\delta) \varphi(t\delta) \equiv 0 \quad \text{for } t \text{ small}$$

so all $\hat{D}_i(\varphi) = 0$. Thus $\mu \notin AS(\theta^K)$, a contradiction. The proof is now complete.

We now determine the asymptotic K -types for the discrete series more precisely. Consider G such that $\text{rank } G = \text{rank } K$. Assume $\mathfrak{t} \subseteq \mathfrak{k}$ is a Cartan subalgebra. Let $\mathfrak{t}'$ be the set of regular elements and choose compatible orderings for the root systems $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and $\Delta(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Let $\mathfrak{t}^+$ be the set of regular elements in a positive chamber for $\mathfrak{t}$ and $\mathfrak{t}^+ = \bigcup_{j=1}^s \mathfrak{t}'^j$ where $\mathfrak{t}'^j$ are positive chambers for $\mathfrak{g}$.

According to [4] the discrete series is parametrized by a lattice $L \subseteq \mathfrak{t}^*$. Using the identification between $\mathfrak{g}$ and $\mathfrak{g}^*$ given by $B(\cdot, \cdot)$ we can identify any $\lambda \in L$ with a $Z_\lambda \in \mathfrak{t}'$ such that, according to [9], the eigendistribution θ_λ defined before, equals the Fourier transform of $\varphi_\lambda(Z_\lambda)$ (up to a constant). Then as pointed out earlier, it is known that $\varphi_\lambda' \in \mathcal{S}'(\mathfrak{t}')$ so $\varphi_\lambda'(tZ_\lambda) \sim \sum t^i D_{i,\lambda}(f)$ at $t = 0$. Clearly, $\text{m supp } D_{i,\lambda} = AS(\theta_\lambda)$.

PROPOSITION 3.7. *Let*

$$\eta = \{X \in \mathfrak{g}: X = \lim t_i \text{Ad } x_i Z_\lambda \text{ where } t_i \rightarrow 0\}.$$

Then $\eta = AS(\theta_\lambda)$ and η does not depend on λ but only on the chamber $\mathfrak{t}$ which contains Z_λ .

Proof. Consider any $X \in \eta$. Clearly X is nilpotent. We may assume X is such that $X \notin \text{Ad } G \cdot (Y) = \text{Ad } G \cdot (Y)$ for any other $Y \in \eta$. It is enough to show $O(X) = \text{Ad } G \cdot X$ is contained in the support of some $D_{i,\lambda}$. We will show that the G -invariant measure on $O(X)$ is the limit of $t^i \varphi_\lambda'(tZ)$ for some α and a certain class of $f \in C_c^\infty(\mathfrak{g})$ when $t \rightarrow 0^+$. Since $AS(\theta_\lambda) \subseteq \eta$ trivially, this will complete the proof. We will use the following facts without proof.

(The transverse described in Lemma 3.8 was first introduced by Harish-Chandra. As stated here, Lemma 3.8 was first proved by R. Rao in connection with some results (unpublished) on the measures supported on nilpotent orbits.)

LEMMA 3.8. *Let* $q = \dim O(X)$. *Consider the Lie triple* $\{\Lambda, H, Y\}$. *Let* $\mathfrak{z}_Y = \text{Cent}_{\mathfrak{g}} Y$ *and* $\mathcal{W} = X + \mathfrak{z}_Y$. *Then the map* $\psi: G \times \mathcal{W} \rightarrow \mathfrak{g}$ *given by* $\psi(x, v) = \text{Ad } x \cdot v$ *is a submersion. In particular, for any* $v \in \mathcal{W}$, $\mathcal{W}$ *is transverse to* $O(v)$ *and* $\mathcal{W}^G = \text{Ad } G \cdot \mathcal{W}$ *is open. $O(X) \cap \mathcal{W}$ is a finite set. Let* $\beta \in C_c^\infty(G \times \mathcal{W})$. *Then there is* $f_\beta \in C_c^\infty(\mathcal{W}^G)$ *such that* $\beta \mapsto f_\beta$ *is well defined and onto* $C_c^\infty(\mathcal{W}^G)$. *Then*

$$\int_{G/G_0} f_\beta(\text{Ad } x \cdot v) dv = \int_{G \times \mathcal{W} \cap O(v)} \beta(x, v) d\omega(u) dx.$$

For v *semisimple, $O(v) \cap \mathcal{W}$ is closed and* $dE(u)$ *is a measure defined by some form. For* $v = X$ *it is a linear combination of delta functions.*

If $t > 0$ *then*

$$f_\beta(tv) = t^{-(n-\alpha)/2} f_\beta(v)$$

where $\beta_i = \beta \circ \varphi$ *and* $\varphi: G \times \mathcal{W} \rightarrow G \times \mathcal{W}$ *is given by*

$$\varphi(x, X + u) = (xy_t^{-1}, X + u_t), \quad \gamma_t = \exp(-\frac{1}{2} \log tH)$$

and $u_t = ty_t u$.

We get back to the proof of Proposition 3.7.

Assume $Z_0 \in \mathfrak{t}'$ such that $O(tZ_0)$ contains X in its closure. Then we may assume $O(Z) \cap \mathcal{W} \neq \emptyset$. Assume

$$O(Z) \cap \mathcal{W} = \{X + u: u \text{ in some closed submanifold of } \mathfrak{z}_Y\}.$$

Then

$$O(tZ) \cap \mathcal{W} = \{X + t \text{Ad } \gamma_t \cdot u: X + u \in O(Z) \cap \mathcal{W}\}.$$

We claim that $\|u\|$ has to be bounded. Indeed, the set $O(tZ) \cap \mathcal{W}$ can have only elements in its closure such that $\dim O(v) = q$. Otherwise there would be

$Y \in \eta$ such that $\text{cl } O(Y) = O(Y)$ contains X . By dimension considerations and the fact that such a set is algebraic, the boundary of $O(tZ) \cap \mathcal{Q}$ is finite. Assume that $\|u\|$ is unbounded. Let $3r = \sum_{i \geq 0} V_i$ where V_i are eigenspaces for $\text{ad } H$ with eigenvalue $-i$. Then if $u = \sum u_i$,

$$t \text{ Ad } \gamma_t u = \sum t^{1+(i/2)} u_i.$$

If $\|u\|$ is unbounded, we can find sequences u_s and $t_s \rightarrow 0$ such that $\|t_s u_s\| \rightarrow \infty$. This gives a contradiction. It follows that, since $O(Z) \cap \mathcal{Q}$ is also closed, that $O(Z) \cap \mathcal{Q}$ is compact. Thus

$$\begin{aligned} t^{(n-\alpha)/2} \int_{G/G_Z} f_\beta(t \text{ Ad } xZ) dx &= \int_{G/G_Z} f_{\beta_1}(\text{Ad } xZ) dx \\ &= \int_G \int_{\mathcal{Q} \cap O(Z)} \beta_1(x, u) d\omega(u) dx \\ &= \int_G \int_{\mathcal{Q} \cap O(Z)} \beta(x, X + t\gamma_t u) d\omega(u) dx. \end{aligned}$$

Since $\mathcal{Q} \cap O(Z)$ is compact and $X + t\gamma_t u \rightarrow X$ as $t \rightarrow 0$, it follows at once that $O(Z) \cap \mathcal{Q} = \{X\}$ and

$$\lim_{t \rightarrow 0^+} t^{(n-\alpha)/2} \int_{G/G_{Z_0}} f_\beta(t \text{ Ad } xZ_0) dx = c_{Z_0} \int_{G/G_X} f_\beta(\text{Ad } x \cdot X) dx$$

and $c_{Z_0} > 0$. In terms of the function φ_f we can write

$$\lim_{t \rightarrow 0^+} t^{-\alpha} \varphi_f(tZ_0) = c_{Z_0} \int_{G/G_X} f(\text{Ad } x \cdot X) dx$$

if $\text{supp } f \subseteq \mathcal{Q}^G$. Or we can write

$$\varphi_f(0; Z_0^\alpha) = c_{Z_0} \int_{G/G_X} f(\text{Ad } x \cdot X) dx, \quad \text{supp } f \subseteq \mathcal{Q}^G.$$

But now since φ_f is a Schwartz function, it follows that

$$\lim_{t \rightarrow 0^+} \varphi_f(tZ; Z_0^\alpha) = c_{Z_0} \int_{G/G_X} f(\text{Ad } x \cdot X) dx, \quad \text{supp } f \subseteq \mathcal{Q}^G.$$

This is enough to conclude that $X \in \text{cl } O(tZ)$ for any other $Z \in \mathfrak{t}$ as well. These arguments complete the proof.

COROLLARY 3.9. *Let θ be an invariant distribution supported on a set of nilpotent elements η . Suppose θ is homogeneous of degree $n - \alpha$. Let $n - r$ be the largest dimension of an orbit contained in η . Then $\alpha \geq (n - r)/2$. In particular, if $\alpha = (n - r)/2$ then θ is a linear combination of measures.*

Proof. Let $X \in \eta$. It is enough to consider what happens when we consider $\theta|_{\mathcal{Q}^G}$. Let dx be the Haar measure on G , dZ and du the Euclidean measures on $\mathfrak{g}$ and 3γ respectively. We can lift θ to a distribution $dx \otimes \tau_\theta$ where τ_θ is a distribution on $\mathcal{Q}$ by

$$(dx \otimes \tau_\theta)(\beta) = \theta(f_\beta).$$

Then, if θ is supported on $O(X)$, τ_θ is supported on

$$O(X) \cap \mathcal{Q} = \{X\}$$

by the previous argument. Thus $\tau_\theta = D^s \delta_X$ where s is a multiindex and D^s a differential operator on $C^\infty(3\gamma)$. Let $\beta(x, X + u) = \beta_1(x) \beta_2(X + u)$. We have the relations

$$\begin{aligned} \theta(R_t f_\beta) &= t^\alpha \theta(f_\beta) = t^\alpha \int_G \beta_1(x) dx \cdot \tau_\theta(\beta_2), \\ \theta(R_t f_\beta) &= \theta(t^{-(n-r)/2} f_{\beta_1}) = t^{-(n-r)/2} \int_G \beta_1(x\gamma_t) dx \cdot \tau_\theta(\beta_{2,t}) \end{aligned}$$

where

$$\beta_{2,t}(X + u) = \beta_2\left(X + \sum_{i \geq 0} t^{1+(i/2)} u_i\right).$$

Thus

$$\tau_\theta(\beta_{2,t}) = t^{\alpha+(n-r)/2} \tau_\theta(\beta_2)$$

for all $\beta_2 \in C_c^\infty(\mathcal{Q})$. By using the fact that τ_θ is a derivative of the delta function and the relation

$$D^s(\beta_{2,t})|_{u=0} = (t^{1+(i/2)})^{|s|} D^s \beta|_{u=0}$$

for any multiindex s_j and D_j a differential operator on $C^\infty(V)$, we get the relation,

$$\sum |s_j| \left(1 + \frac{j}{2}\right) = \alpha + (n - r)/2.$$

Thus, $\alpha + (n - r)/2 > 0$ if one of the $|s_j| > 0$ and $\alpha + (n - r)/2 = 0$ if and only if $|s_j| = 0$ for all j . This completes the proof.

The set described in Proposition 3.7 is being computed on a case-by-case basis by D. Peterson; his results are complete at least for $SU(p, q)$.

4. RELATIONS WITH PRIMITIVE IDEALS AND COHERENT CONTINUATION

Let π be an irreducible admissible representation of G , Θ_π its character, and θ_π the corresponding distribution near 0 on $\mathfrak{g}$. In this section we will relate the asymptotic expansion of θ_π to the primitive ideal I_π associated to π . Suppose (as we may without changing Θ_π or I_π) that π is realized on a Hilbert space $\mathcal{H}$, and that $\pi|_K$ is unitary. Let $X \subseteq \mathcal{H}$ denote the subspace of K -finite vectors. Then X is a module for the (complexified) enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}$; we denote this action by π also. The ideal $I_\pi \subseteq U(\mathfrak{g}_\mathbb{C})$ is defined to be the annihilator of X . We have a homogeneous ideal $\text{gr}(I_\pi) \subseteq S(\mathfrak{g}_\mathbb{C})$, the symmetric algebra of $\mathfrak{g}_\mathbb{C}$. The associated cone $\mathcal{V}(I_\pi)$ is defined to be the zero variety of $\text{gr}(I_\pi)$ inside $\mathfrak{g}_\mathbb{C}^*$ (which is in a natural way the maximal spectrum of $S(\mathfrak{g}_\mathbb{C})$). The Gelfand-Kirillov dimension $d(I_\pi)$ may be defined to be the (complex) dimension of $\mathcal{V}(I_\pi)$. One knows that $\mathcal{V}(I_\pi)$ is a finite union of nilpotent orbits of the complex adjoint group $G_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. (For all this see [1].) In interpreting the following theorem, one should bear in mind the (easily verified) fact that if $X \in \mathfrak{g}^*$, then

$$\dim_{\mathbb{C}}(G_\mathbb{C} \cdot X) = \dim_{\mathbb{C}}(G \cdot X).$$

THEOREM 4.1. *With notation as above,*

$$AS(\Theta_\pi) \subseteq \mathcal{V}(I_\pi) \cap \mathfrak{g}^*.$$

Furthermore,

$$d(I_\pi) = \dim_{\mathbb{R}} AS(\Theta_\pi) = \dim_{\mathbb{C}} \mathcal{V}(I_\pi) = \dim_{\mathbb{R}} \mathcal{V}(I_\pi) \cap \mathfrak{g}^*.$$

The first term in the asymptotic expansion of the lift θ_π of Θ_π to $\mathfrak{g}$ is a linear combination of Fourier transforms of invariant measures on nilpotent orbits of dimension $d(I_\pi)$, and hence has homogeneity degree $-\frac{1}{2}d(I_\pi)$.

Proof. Write $\theta_\pi \sim \sum_{i=-\infty}^{\infty} i! D_i$. To prove the first assertion, it suffices to show that if $p \in S(\mathfrak{g})$ is a polynomial function on $\mathfrak{g}_\mathbb{C}^*$ vanishing on $\mathcal{V}(I_\pi)$, and $i \geq -r$, then there is some integer $N = N(p, i) \geq 1$ such that

$$\hat{D}_i \circ p^N = 0.$$

Since such a polynomial lies in $\sqrt{\text{gr}(I_\pi)}$ by the Nullstellensatz, we may as well assume $p \in \text{gr}(I_\pi)$. By a standard argument, if this holds for p_1 and p_2 , then it holds for $p_1 + p_2$; so we may assume p is homogeneous, say of degree s . Let $\partial(p)$ be the constant coefficient linear differential operator on $\mathfrak{g}$ associated to p ; then what we must show is

$$D_i \circ \partial(p)^N = 0. \quad (4.2)$$

Let $\{U_j\}_{j=0}^{\infty}$ be the canonical filtration of $U(\mathfrak{g}_\mathbb{C})$, so that $U_j/U_{j-1} \cong S^j(\mathfrak{g}_\mathbb{C})$. Choose an element $u \in I_\pi \cap U_s$ such that $p = u + U_{s-1} \in U_s/U_{s-1} \cong S^s(\mathfrak{g}_\mathbb{C})$. Consider u as a left invariant differential operator on G . If $\mathcal{H}^\infty$ is the space of smooth vectors in $\mathcal{H}$, then $\pi(u)$ is a well defined operator on $\mathcal{H}^\infty$, and if $f \in C_c^\infty(G)$, then

$$\pi(u \cdot f) = \pi(f) \pi(u).$$

Choosing an orthonormal basis $\{v_i\}$ of $\mathcal{H}$ consisting of K -finite vectors, we find

$$\begin{aligned} \Theta_\pi(u \cdot f) &= \sum \langle \pi(u \cdot f)v_i, v_i \rangle \\ &= \sum \langle \pi(f) \pi(u)v_i, v_i \rangle = 0 \end{aligned}$$

since $u \in I_\pi$; so $\Theta_\pi \circ u = 0$. Let $\tilde{u}$ be the lift of the distribution u to $\mathfrak{g}$ defined as in section 3; then

$$\theta_\pi \circ \tilde{u} = 0. \quad (4.3)$$

Clearly $\tilde{u}$ is a differential operator of order s . Write

$$\tilde{u} = \sum_{i=-s}^{\infty} \tilde{u}_i,$$

with

$$\tilde{u}_i = \sum_{\substack{|\beta| - |\alpha| = i \\ |\alpha| \leq 0}} c_{\alpha\beta} X^\alpha D^\beta$$

in terms of some basis $\{X_i\}$ of $\mathfrak{g}$. By the Campbell-Baker-Hausdorff formula, one sees easily that

$$\tilde{u}_{-s} = \partial(p).$$

We now establish (4.2) by induction on i . By (4.3) and Corollary 2.4,

$$0 = \theta_\pi \circ \tilde{u} \sim \sum_{j=-r-s}^i i! E_j,$$

where

$$E_j = \sum_{m+i=j} D_m \circ \tilde{u}_i.$$

Since $E_j = 0$, we have for each i an equation of the form

$$D_i \circ \partial(p) = D_i \circ \tilde{u}_{-s} = - \sum_{j=-r}^{i-1} D_j \circ \tilde{u}_{i-j-s}. \quad (4.4)$$

Suppose then that (4.2) is known for $j < i$. Choose $N = N(p, i)$ so large that for $-r \leq j \leq i-1$,

$$\tilde{u}_{i-j-s} \circ \partial(p)^{N-1} = \partial(p)^{N(p,i)} Q_j,$$

for some differential operator Q_j ; this is possible since $\tilde{u}_{i-j-s}$ has polynomial coefficients. Applying $\partial(p)^{N-1}$ to both sides of (4.4), we get

$$\begin{aligned} D_i \circ \partial(p)^N &= - \sum_{j=-r}^{i-1} D_j \circ \tilde{u}_{i-j-s} \partial(p)^{N-1} \\ &= - \sum_{j=-r}^{i-1} D_j \circ \partial(p)^{N(p,i)} Q_j \\ &= 0 \end{aligned}$$

by induction. This proves the first assertion of the theorem. By the remarks preceding the theorem, it follows that

$$d(I_\pi) = \dim_{\mathbb{C}} \mathcal{V}(I_\pi) \geq \dim_{\mathbb{R}} \mathcal{V}(I_\pi) \cap \mathfrak{g}^* \geq \dim_{\mathbb{R}} AS(\theta_\pi). \quad (4.5)$$

Write $2d' = \dim_{\mathbb{R}} AS(\theta_\pi)$. By Corollary 3.9, $AS(\theta_\pi)$ cannot support a homogeneous invariant distribution of degree less than $-d'$, so the order $-r$ of θ_π at 0 (recall that $\theta_\pi \sim \sum_{i=-r}^{\infty} t^i D_i$; to say that θ_π has order $-r$ means $D_{-r} \neq 0$) satisfies $r \leq d'$. Suppose we can show that $2r \geq d(I_\pi)$. This will force equalities in (4.5), proving the second assertion; and the last will follow from Corollary 3.9. So it is enough to show $2r \geq d(I_\pi)$. Let θ_π^K be the K -character of π , and θ_π^K its lift to $\mathfrak{t}$. By Theorem 3.6,

$$\theta_\pi^K \sim \sum_{i=-s}^{\infty} t^i E_i;$$

we assume $E_{-s} \neq 0$. The proof of Theorem 3.6 shows how to compute the E_i from the D_i ; in particular we have $s \leq r$, so it suffices to show that $2s \geq d(I_\pi)$. Choose $f \in C_c^\infty(\mathfrak{t})$, invariant under $\text{Ad}(K)$, so that $f \geq 0$, and $\int(\mu) \geq 1$ for $|\mu| \leq 1$ (in some norm on $\mathfrak{t}^*$). Then $(f_i)^\wedge \geq 1$ for $|\mu| \leq t^{-1}$. With notation as in the proof of Theorem 3.6, this gives

$$\begin{aligned} \theta_\pi^K(f) &= \sum_{\mu \in \check{K}} m_\mu(f_\alpha)^\wedge(\mu) \cdot \dim(\mu) \\ &\geq \sum_{\substack{\mu \in \check{K} \\ |\mu| \leq t^{-1}}} m_\mu \cdot \dim(\mu). \end{aligned}$$

By Theorem 1.2 of [11], this last term satisfies

$$\sum_{\substack{\mu \in \check{K} \\ |\mu| \leq t^{-1}}} m_\mu \cdot \dim(\mu) \geq A(t^{-1})^{\frac{1}{2}d(I_\pi)}.$$

Thus $-s \leq -\frac{1}{2}d(I_\pi)$ as desired. Q.E.D.

We turn now to the behavior of asymptotic expansions under coherent continuation. For more details of the definitions, see [7], [10], or [11]. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and some translate $A \subseteq \mathfrak{h}^*$ of the lattice of weights of finite dimensional representations of G . Let $\{\theta(\lambda) \mid \lambda \in A\}$ be a coherent family of virtual characters of G . This means that each $\theta(\lambda)$ is an integral combination of irreducible characters of infinitesimal character λ , and if F is a finite dimensional representation of G with weights $A(F) \subseteq \mathfrak{h}^*$ (counted with multiplicity) then

$$\theta(\lambda) \cdot \theta(F) = \sum_{\mu \in A(F)} \theta(\lambda + \mu). \quad (4.6)$$

Fix a system $\Delta^+ \subseteq \Delta(\mathfrak{g}_e, \mathfrak{h}_e)$ of positive roots. We will assume in addition that whenever λ is dominant (i.e. $2\langle \alpha, \lambda \rangle / \langle \alpha, \alpha \rangle$ is not a negative integer for $\alpha \in \Delta^+$) then $\theta(\lambda)$ is either 0 or the character of an irreducible representation $\pi(\lambda)$, and that the latter is the case whenever λ is nonsingular. Any irreducible representation π occurs as some $\pi(\lambda)$ in such a coherent family, which is unique up to obvious equivalences. In this case the non-zero $\pi(\lambda)$ have annihilators $I_{\pi(\lambda)}$ with a common Gelfand-Kirillov dimension $2d$; and every constituent of each virtual character $\theta(\lambda)$ has Gelfand-Kirillov dimension at most $2d$ (cf. [11]). Accordingly we can write

$$\theta_\lambda \sim \sum_{i=-d}^{\infty} t^i D_i(\lambda).$$

Now the $D_i(\lambda)$ were constructed in section 3 from the formulas for θ_λ . Since these formulas depend nicely on λ (cf. [7], [10]), we deduce

PROPOSITION 4.7. *Let $\{\theta_\lambda \mid \lambda \in A\}$ be a coherent family of virtual characters as above, with θ_λ the character of an irreducible representation $\pi(\lambda)$ such that $d(I_{\pi(\lambda)}) = 2d$ whenever λ is dominant and regular. For fixed i , the distributions*

$$\{D_i(\lambda) \mid \lambda \in A\}$$

span a finite dimensional space. Let $\{E_{ij}\}$ be a basis of this space. Then

$$D_i(\lambda) = \sum_j p_{ij}(\lambda) E_{ij},$$

with p_{ij} a polynomial function on $\mathfrak{h}^*$, homogeneous of degree $i + \frac{1}{2} \dim(\mathfrak{g}/\mathfrak{h})$. If $i + j = -d$, then the p_{ij} are harmonic.

Proof. Everything but the last statement follows from the remarks preceding the proposition and straightforward computation. For the last statement, notice that if F is a finite dimensional representation of G , lifting (4.6) to $\mathfrak{g}$ gives

$$\theta(\lambda) \cdot T(F) = \sum_{\mu \in \Delta(F)} \theta(\lambda + \mu);$$

here $T(F)$ is a smooth function near 0 of $\mathfrak{g}$, and

$$T(F)(0) = \dim(F).$$

Applying Corollary 2.4, we get

$$\begin{aligned} D_{-s}(\lambda) \cdot \dim(F) &= \sum_{\mu \in \Delta(F)} D_{-s}(\lambda + \mu), \\ (\dim F) \cdot \sum p_{-s,j}(\lambda) E_{-s,j} &= \sum_{\mu \in \Delta(F)} \sum_j p_{-s,j}(\lambda + \mu) E_{-s,j}. \end{aligned}$$

Since the $E_{-s,j}$ are linearly independent, this gives

$$\dim(F) \cdot p_{-s,j}(\lambda) = \sum_{\mu \in \Delta(F)} p_{-s,j}(\lambda + \mu).$$

By Lemma 4.3 of [11], it follows that $p_{-s,j}$ is harmonic. (Since $p_{-s,j}$ is already known to be a polynomial function, one could give a simple direct proof.) Q.E.D.

COROLLARY 4.8. *With notation as in Proposition 4.7, there is a harmonic polynomial c , homogeneous of degree $\frac{1}{2} \dim(\mathfrak{g}/\mathfrak{h}) - d$, such that for λ dominant,*

$$\lim_{t \rightarrow \infty} t^{-d} \sum_{|\mu| \leq t} \dim \mu \cdot m^*(\mu) = c(\mu).$$

Here $m_\lambda(\mu)$ is the multiplicity of μ in $\pi(\lambda)$.

Proof. Write $\theta^K(\lambda) \sim \sum_{i=0}^{\infty} t^i E_i$. By (4) in the proof of Theorem 3.6, Theorem 4.1, and Proposition 4.7, $\hat{E}_{-d}$ is a linear combination of homogeneous measures on $\mathfrak{h}^*$, with coefficients homogeneous harmonic polynomials in λ of degree $\frac{1}{2} \dim(\mathfrak{g}/\mathfrak{h}) - d$. In particular, if $f \in C_c^\infty(\mathfrak{f})$ is $\text{Ad}(K)$ invariant, there is a harmonic polynomial c_f of the specified degree, such that

$$\lim_{t \rightarrow \infty} t^{-d} \sum_{|\mu| \leq t} \dim \mu \cdot m^*(\mu) f(t^{-1}\mu) = c_f(\lambda).$$

The corollary follows by an obvious approximation argument. Q.E.D.

This corollary establishes Conjecture 5.1 of [11].

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