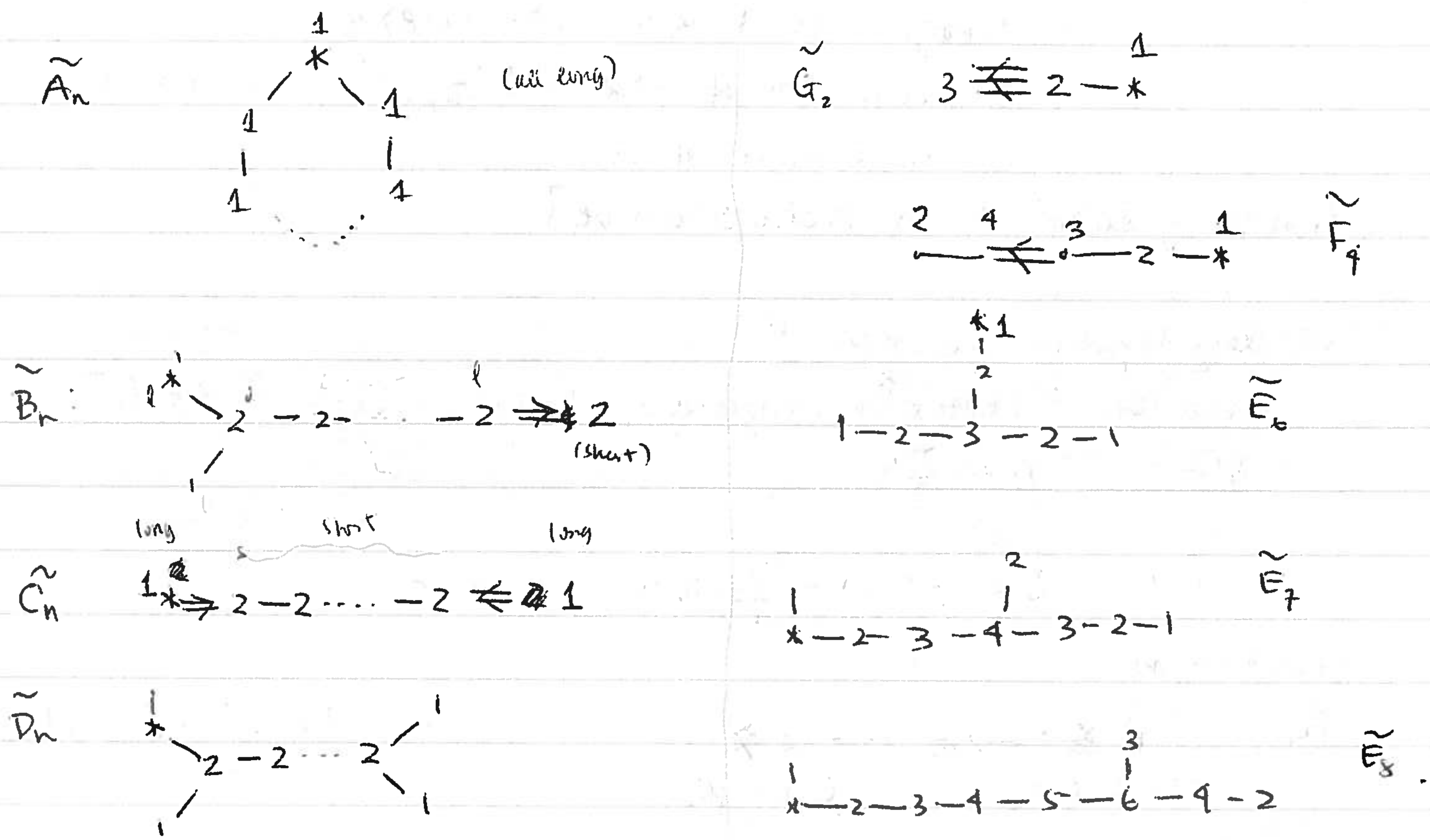
Vogan "classification of simple root systems" 4/9/97
(Lie Groups Seminar)



$R \subset V \cong (\mathbb{R}^n, \langle\, ,\rangle)$; R finite, spanning, all elements $\neq 0$.

$\alpha \in R \rightsquigarrow s_\alpha : V \to V$

[Require, eg, $s_\alpha(R) \subset R$.]

Set $W = W(R)$ = group generated by s_α $\subset O(V)$, $\mathrm{Perm}(R)$.

Choose $\Pi \subset R^+ \subset R$ (simple $\subset$ pos $\subset R$)

Facts:

- Π basis for V
- $R^+ \subseteq \mathbb{N}\Pi$
- only elt of W preserving R^+ is 1.
- α long $\Rightarrow$ its label is $\frac{1}{2}$ the sum of adjacent labels.

Dynkin diagram: • vertices Π

• edges: pairs (α, β) with $\langle \alpha, \beta \rangle < 0$.

• multiplicity of edge $= \left(\frac{\langle \alpha, \alpha \rangle}{\langle \beta, \beta \rangle} \right)^{\pm 1}$ (1, 2, 3)

• points toward shorter.

[Everything except * in pictures above.]

Extended Dynkin diagram:

• additional vertex * corresponding to lowest root $-\beta \in R^-$.

Write $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$

or $0 = 1(-\beta) + \sum_{\alpha \in \Pi} n_\alpha \alpha$.

Non reduced example:

$\widetilde{BC}_n$

$$\begin{array}{cccccccc} \overset{1}{*} & \Rightarrow & 2 & - & 2 & - \cdots - & 2 & \Rightarrow & 2 \\ 2e_1 & & e_1 - e_2 & & & \cdots & e_{n-1} - e_n & & e_n \end{array}$$

roots: $\pm e_i$, $\pm 2e_i$, $\pm e_i \pm e_j$

[Numbers in the diagrams above are the numbers appearing in expression $0 = 1(-\beta) + \sum_{\alpha \in \Pi} n_\alpha \alpha$.]

$n_\alpha \geq 1$.

* Graph: • Γ connected ($\Leftrightarrow$ R simple)

• vertices labelled with integers ≥ 1

• Special vertex * labelled with 1.

assume R reduced. • each vertex has a length "long" or "short" which differ by an integer factor p (2 or 3).

• Special vertex is long; short vertices may be empty.

- correspondingly: γ short $\Rightarrow$ ~~γ~~

$$2 \cdot \mathrm{label}(\gamma) = p \cdot \sum_{\substack{\alpha \text{ adj } \gamma \\ \alpha \text{ long}}} \mathrm{label}(\alpha) + \sum_{\substack{\delta \text{ adj } \gamma \\ \delta \text{ short}}} \mathrm{label}(\delta)$$

<u>Pf</u>. Look at $0 = \sum_\alpha \mathrm{label}(\alpha) \cdot \alpha$ (equation inside V)

Form $2\langle \cdot, \gamma\rangle / \langle \gamma, \gamma \rangle$ to both sides ...

$$0 = \mathrm{label}(\gamma) \cdot 2 + \sum_{\gamma \text{ adj } \alpha} \mathrm{label}(\alpha) \underbrace{\frac{2\langle \alpha, \gamma\rangle}{\langle \gamma, \gamma\rangle}}$$

$= -1$ α, γ both short or long

$= -p$ if α long γ short.

- Final property: α short, $\mathrm{label}(\alpha)$ is divisible by p

[<u>Pf</u>: induct on ht δ to get

if $\delta = \sum_{\alpha \text{ simple}} m_\alpha \alpha$, and δ long, then all m_{short} divisible by p.]

<u>CLAIM</u>. All graphs with these properties are listed in table above. Any such graph comes from a (unique) root system.

<u>Subclaim</u>: Such a graph with 2 root lengths arises from one with one root length, and an automorphism of order p (fixing $*$ & at least one other point).

($\Leftarrow$ "reduced").

Proof of subclaim. Given Γ, extended diagram w/ 2 root lengths.

Define new graph Γ_0:

vertices: long vertices of Γ plus (short vertices of $\Gamma \times \mathbb{Z}/p\mathbb{Z}$).

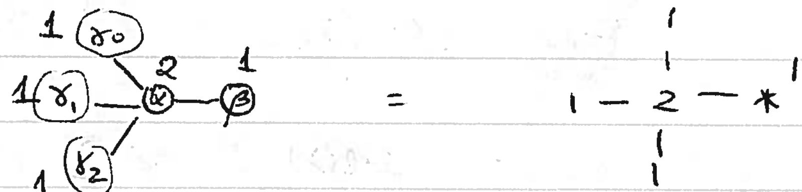
edges: "same"

labels: same on long vertices, same/p on short ones.

Ex. Γ: $\underset{\gamma}{\overset{s}{3}} \Rrightarrow \underset{\alpha}{\overset{l}{2}} - \underset{-\beta}{\overset{l}{1}}$

new vertices: $-\beta, \alpha, \gamma_0, \gamma_1, \gamma_2$.

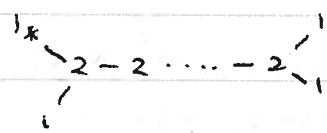
"same edges"



(Has $\mathbb{Z}/3$ action: fixed points are $\tilde{G}_2$.)

Lemma. In this way get Γ_0 with same properties above with all roots long.

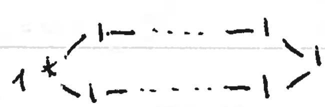
[more examples: $\tilde{B}_n \rightsquigarrow$



$\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{A}_n, \tilde{D}_n$ (one root length)

$\tilde{G}_2$ above

$\tilde{C}_n$:



$\tilde{F}_4 \rightsquigarrow \tilde{E}_6$

Reason that all graphs give root systems

- Case of one root length (enough by automorphism consideration: see G_2 example) look at $\tilde{V}$: basis $\leftrightsquigarrow$ vertices of graph.

Quadratic form on $\tilde{V}$ $\leftrightsquigarrow$ symmetric matrix : 2's on diag

-1 if $i - j$ adjacent.

<u>Claim</u> This form is positive semidefinite with radical $\mathbb{R}\left(\sum_{\alpha\in\Gamma} n_\alpha \alpha\right)$.

Then take $V = \tilde{V}/\text{radical}$

$R =$ elements of length 2 in $\mathbb{Z}\Gamma$

Then verifying axioms of root system are then routine.

<u>Proof of Claim</u>: Bourbaki Groups et algèbres de Lie

Ch V: Exercise for V.2.

(Use property of Γ)

<u>Remark</u>. Classification in 1 root length case is EASY.

So that "finishes" the proof.

SUGGESTS: construct simple Lie algebra with 1 root length from 2 root lengths.

Ex. $\mathfrak{g}$ type G_2; E irrep of $\mathfrak{g}_2$ with short highest weight.

$$\mathfrak{g} \oplus \underbrace{E_\omega \oplus E_{\omega^2}}_{\text{two copies}}$$

dimension $14 \quad 7 \quad 7$

ω, ω^2 : primitive 3rd = cube roots of 1.

Make into a Lie algebra with automorphism of order 3 with the given eigenspaces. Lie bracket obvious except for

$$[E_\omega, E_\omega] \longrightarrow \{E_{\omega^2}\}$$
$$[E_\omega, E_{\omega^2}] \longrightarrow \mathfrak{g}_2$$
$$[E_{\omega^2}, E_{\omega^2}] \longrightarrow E_\omega$$

They exist, but a little harder to do

Get $\quad D_4$: $so(8) \cong \mathfrak{g}_2 \oplus E_\omega \oplus E_{\omega^2}$.