and $H_2(Z_D, \mathbf{Q})$ has the s-action by f'. By these discussions, we have an s-isomorphism

$$H_2(Z_D, \mathbf{Q}) \xrightarrow{\sim} H_{2d}(Z, \mathbf{Q})$$

sending $[D]$ to $[C]$ and $[\mathbf{P}^1]$ to $[\mathbf{P}^1 \times V_0]$.

On the other hand, it can be seen that under this isomorphism the map $\varpi^* \varpi_* \rho^*$ is preserved. Thus for the proof of Lemma 2, we may finally assume that V is a one-dimensional disk D and $f: D \to N$ satisfies $f^{-1}(0)=0$. For this case, we use the identification of our s-action with Kazhdan-Lusztig's one [9]. Kazhdan-Lusztig's action can be directly computed and turns out to coincide with our formula ([5; Lemma 13 and Appendix]). Thus the proof of Lemma 2 is completed and so is that of Theorem in § 4.

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Holonomic Systems on a Flag Variety Associated to Harish-Chandra Modules and Representations of a Weyl Group

Toshiyuki Tanisaki

Dedicated to Professor Hirosi Nagao on his 60th birthday

§ 1. Introduction

1.1. In [KT] we studied the characteristic cycles of holonomic systems on a flag variety associated to highest weight modules of a complex semisimple Lie algebra, and investigated its relation to the representations of a Weyl group.

In this paper we consider Harish-Chandra modules instead of highest weight modules, and prove a theorem similar to the main theorem of [KT] (Theorem 1 below). The main theorem of [KT] turns out to be a special case of Theorem 1 and this paper gives a generalization of the result of [KT], although the proof is essentially the same as the one in [KT].

1.2. Let G be a connected complex semisimple algebraic group and $G_{\mathbf{R}}$ a real form of G. We assume that $G_{\mathbf{R}}$ is connected for simplicity. We fix a maximal compact subgroup $K_{\mathbf{R}}$ of $G_{\mathbf{R}}$ and denote its complexification in G by K.

We consider the abelian category $\mathscr{H}(\mathfrak{g}, K)$ whose objects are $(\mathfrak{g}, K)$-modules of finite length with trivial central character. Here $\mathfrak{g}$ is the Lie algebra of G. The result of Beilinson-Bernstein [BB] implies that $\mathscr{H}(\mathfrak{g}, K)$ is equivalent to the abelian category $\mathscr{M}(\mathfrak{g}, K)$ consisting of coherent $\mathscr{D}$-Modules on the flag variety X with K-actions. Using the fact that X is the union of finitely many K-orbits (Matsuki [M]), it is easily shown that any irreducible component of the characteristic variety $\mathrm{Ch}(\mathfrak{M})$ of $\mathfrak{M} \in \mathscr{M}(\mathfrak{g}, K)$ is the closure of the conormal bundle $T_O^* X$ of a K-orbit O, and in particular $\mathfrak{M}$ is holonomic (actually regular holonomic). We take the multiplicity of $\mathfrak{M}$ along $\overline{T_O^* X}$ into account and consider the characteristic cycle $\mathbf{Ch}(\mathfrak{M}) \in \bigoplus_O \mathbf{Z}_{\geq 0}[\overline{T_O^* X}]$, where O is running through the K-orbits on

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X. Let $K(\mathcal{M}(\mathfrak{g}, K))$ be the Grothendieck group of $\mathcal{M}(\mathfrak{g}, K)$. By the additivity of **Ch** we have a **Z**-linear homomorphism

$$(*)\quad K(\mathcal{M}(\mathfrak{g}, K)) \xrightarrow{\mathbf{Ch}} \bigoplus_{O} \mathbf{Z}[\overline{T_O^* X}].$$

One can define natural actions of the Weyl group W on $K(\mathcal{M}(\mathfrak{g}, K))$ and $\oplus_O \mathbf{Z}[\overline{T_O^* X}]$ (see Section 3 below). Then our main theorem is the following.

Theorem 1. *The* **Z**-*linear homomorphism*

$$K(\mathcal{M}(\mathfrak{g}, K)) \xrightarrow{\mathbf{Ch}} \bigoplus_{O} \mathbf{Z}[\overline{T_O^* X}]$$

is W-equivariant.

1.3. The contents of this paper are as follows. In Section 2 we summarize the known results concerning the Beilinson-Bernstein theory, the Riemann-Hilbert correspondence, Harish-Chandra modules and K-orbits on the flag variety. In Section 3 we give the definitions of W-actions and prove Theorem 1. Additional remarks are stated in Section 4.

1.4. On this occasion we give a remark on the paper [KT]. After writing it up, we learned that Theorem 6 in [KT] was already conjectured by Joseph [J], and V. Ginsburg informed us that he also proved the same theorem by a different method (letter dated January 25, 1984).

1.5. The author expresses his hearty thanks to Professor M. Kashiwara and Professor R. Hotta for valuable suggestions.

§ 2. Harish-Chandra modules and holonomic systems

2.1. The Beilinson-Bernstein theory

Let G be a connected semisimple algebraic group over the complex number field **C** with Lie algebra $\mathfrak{g}$. We denote the flag variety by X. X is naturally identified with the set of all the Borel subalgebras of $\mathfrak{g}$. We denote the sheaf of regular functions and the sheaf of differential operators on X by $\mathcal{O}_X$ and $\mathcal{D}_X$, respectively. The natural action of G on X induces an algebra homomorphism $U(\mathfrak{g}) \xrightarrow{D} \Gamma(X, \mathcal{D}_X)$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$ and χ_0 the trivial central character of $\mathfrak{z}(\mathfrak{g})$. χ_0 is the algebra homomorphism from $\mathfrak{z}(\mathfrak{g})$ onto **C** given by $\mathfrak{z}(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \to U(\mathfrak{g})/\mathfrak{g}U(\mathfrak{g}) = \mathbf{C}$. A $U(\mathfrak{g})$-module M is said to have the trivial central character if $z.m = \chi_0(z)m$ for all $z \in \mathfrak{z}(\mathfrak{g})$ and $m \in M$.

Theorem 2. (Beilinson-Bernstein [BB])

(i) *D is surjective with* $\operatorname{Ker} D = U(\mathfrak{g}) \operatorname{Ker} \chi_0 = U(\mathfrak{g})(\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}U(\mathfrak{g}))$.

(ii) *The abelian category of finitely generated $U(\mathfrak{g})$-modules M with trivial central character and that of coherent $\mathcal{D}_X$-modules $\mathfrak{M}$ are naturally equivalent to each other. The correspondence is given by $M = \Gamma(X, \mathfrak{M})$ and $\mathfrak{M} = \mathcal{D}_X \otimes_{U(\mathfrak{g})} M$.*

2.2. The Riemann-Hilbert correspondence

Let $\mathcal{O}_{X_{an}}$ be the sheaf of holomorphic functions. We set

$$\mathcal{D}_{X_{an}} = \mathcal{O}_{X_{an}} \otimes_{\mathcal{O}_X} \mathcal{D}_X \quad \text{and} \quad \mathfrak{M}_{an} = \mathcal{D}_{X_{an}} \otimes_{\mathcal{D}_X} \mathfrak{M} = \mathcal{O}_{X_{an}} \otimes_{\mathcal{O}_X} \mathfrak{M}$$

for a $\mathcal{D}_X$-Module $\mathfrak{M}$. A coherent $\mathcal{D}_X$-Module $\mathfrak{M}$ is said to be regular holonomic if $\mathfrak{M}_{an}$ is a holonomic $\mathcal{D}_{X_{an}}$-Module with regular singularity in the sence of [KK]. For a regular holonomic $\mathcal{D}_X$-Module $\mathfrak{M}$, we set $\mathcal{DR}(\mathfrak{M}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_{X_{an}}}(\mathcal{O}_{X_{an}}, \mathfrak{M}_{an})$. $\mathcal{DR}(\mathfrak{M})$ is a bounded complex of $\mathbf{C}_X$-Modules which is an object of the derived category. Furthermore it is known that $\mathcal{DR}(\mathfrak{M})$ is a perverse sheaf, that is, $\mathcal{K} := \mathcal{DR}(\mathfrak{M})$ satisfies the following conditions.

(i) $\mathcal{H}^i(\mathcal{K})$ is constructible for each i.

(ii) $\mathcal{H}^i(\mathcal{K}) = 0$ for $i < 0$.

(iii) $\operatorname{codim}(\operatorname{supp}(\mathcal{H}^i(\mathcal{K}))) \geq i$ for $i \geq 0$.

(iv) $\operatorname{codim}(\operatorname{supp}(\mathcal{H}^i(\mathcal{K}^*))) \geq i$ for $i \geq 0$, where $\mathcal{K}^* = \mathbf{R}\mathcal{H}om_{\mathbf{C}_X}(\mathcal{K}, \mathbf{C}_X)$.

Theorem 3 (Kashiwara, Mebkhout see [Ka]). *$\mathcal{DR}$ gives an equivalence between the abelian category of regular holonomic $\mathcal{D}_X$-Modules and that of perverse sheaves on X.*

2.3. Harish-Chandra modules

Let $G_{\mathbf{R}}$ be a connected real form of G. We fix a maximal compact subgroup $K_{\mathbf{R}}$ of $G_{\mathbf{R}}$ and denote its complexification by K.

Definition. A $\mathfrak{g}$-module M which has also a K-module structure is called a $(\mathfrak{g}, K)$-module if the following conditions hold.

(i) Any $m \in M$ is contained in a finite-dimensional K-invariant subspace M_0 and the induced homomorphism $K \to GL(M_0)$ is a homomorphism of algebraic groups.

(ii) If $\mathfrak{k}$ is the Lie algebra of K, then the $\mathfrak{k}$-module structure on M obtained by differentiating the K-action coincides with the one obtained by restricting the $\mathfrak{g}$-module structure.

(iii) $k.(X.m) = (\operatorname{Ad}(k)X).(k.m)$ for $k \in K$, $X \in \mathfrak{g}$ and $m \in M$.

Let $\mathcal{H}(\mathfrak{g}, K)$ be the abelian category consisting of $(\mathfrak{g}, K)$-modules of finite length which have the trivial central character as $U(\mathfrak{g})$-modules. By the correspondence of Theorem 2 $\mathscr{H}(\mathfrak{g}, K)$ is equivalent to the category $\mathscr{M}(\mathfrak{g}, K)$ consisting of coherent $\mathscr{D}_X$-Modules with K-actions. We say that a coherent $\mathscr{D}_X$-Module $\mathfrak{M}$ has a K-action if an isomorphism $p^*\mathfrak{M} \simeq q^*\mathfrak{M}$ of $\mathscr{D}_{K\times X}$-Modules which satisfies the usual cocycle condition is given, where $K\times X \xrightarrow{q} X$ and $K\times X \xrightarrow{p} X$ are defined by $q(k, x)=k.x$ and $p(k, x)=x$.

In order to investigate $\mathscr{M}(\mathfrak{g}, K)$ we need the following.

Proposition 1 (Matsuki [M], see also Vogan [V] and 2.4 below).

(i) *There exist finitely many K-orbits on X.*

(ii) *For $x \in X$ let K_x be the stabilizer of x in K and $(K_x)_0$ its identity component. Then the order of any element of $K_x/(K_x)_0$ is at most 2. In particular $K_x/(K_x)_0$ is an abelian group.*

We denote the set of the K-orbits on X by $\mathscr{C}$.

Lemma 1. *For $\mathfrak{M} \in \mathscr{M}(\mathfrak{g}, K)$ any irreducible component of the characteristic variety $\mathrm{Ch}(\mathfrak{M})$ is the closure of the conormal bundle $T^*_O X$ of some $O \in \mathscr{C}$. In particular $\mathfrak{M}$ is holonomic.*

Proof. Since $\mathrm{Ch}(\mathfrak{M})$ is an involutive subvariety of T^*X, it is sufficient to show that $\mathrm{Ch}(\mathfrak{M})$ is contained in $\coprod_{O\in\mathscr{C}} T^*_O X$. Set $M=\Gamma(X, \mathfrak{M}) \in \mathscr{M}(\mathfrak{g}, K)$. Take a finite-dimensional K-invariant subspace M_0 of M so that $M=U(\mathfrak{g})M_0$ and set $M_i=U_i(\mathfrak{g})M_0$. Then $\mathrm{gr}\, M=\bigoplus_{i\in\mathbf{Z}}(M_i/M_{i-1})$ is a finitely generated $S(\mathfrak{g})$-module and the support of the associated coherent sheaf $\widetilde{\mathrm{gr}M}$ on $\mathfrak{g}^*$ is contained in $\mathfrak{k}^\perp=\{x \in \mathfrak{g}^* \mid \langle x, \mathfrak{k}\rangle=0\}$. Let $T^*X \xrightarrow{\gamma} \mathfrak{g}^*$ be the natural map. Then we have $\mathrm{Ch}(\mathfrak{M}) \subset \gamma^{-1}(\mathrm{supp}(\widetilde{\mathrm{gr}M})) \subset \gamma^{-1}(\mathfrak{k}^\perp)=\coprod_{O\in\mathscr{C}} T^*_O X$. Here the first inclusion follows from the definition since $\mathfrak{M}=\mathscr{D}_X \otimes_{U(\mathfrak{g})} M$.

Moreover we have the following.

Proposition 2 (Beilinson-Bernstein [BB], see also Vogan [V]). *If $\mathfrak{M} \in \mathscr{M}(\mathfrak{g}, K)$, then $\mathfrak{M}$ is regular holonomic.*

Hence by Theorem 3 $\mathscr{M}(\mathfrak{g}, K)$ is equivalent to the abelian category $\mathscr{F}(\mathfrak{g}, K)$ consisting of the perverse sheaves on X with K-actions. Thus we have the following equivalence of the abelian categories:

$$(*) \quad \mathscr{H}(\mathfrak{g}, K) \simeq \mathscr{M}(\mathfrak{g}, K) \simeq \mathscr{F}(\mathfrak{g}, K).$$

Next we describe the simple objects of these categories. For $O \in \mathscr{C}$ and a one-dimensional local system (locally constant sheaf whose stalks are one-dimensional $\mathbf{C}$-vector spaces) γ on O with a K-action, let ${}^\pi\gamma$ be the DGM-extension of γ to $\bar{O}$. We also use the same notations for the zero extensions of γ and ${}^\pi\gamma$ to X. Then $\gamma[\text{-codim } O]$ and ${}^\pi\gamma[\text{-codim } O]$ are objects of $\mathscr{F}(\mathfrak{g}, K)$ and the latter is a simple object. Furthermore any simple object in $\mathscr{F}(\mathfrak{g}, K)$ is isomorphic to some ${}^\pi\gamma[\text{-codim } O]$. Hence the set of the simple objects is parametrized by

$$\mathscr{S}=\{(O, \gamma) \mid O \in \mathscr{C} \text{ and } \gamma \text{ is a } K\text{-equivariant one-dimensional local system on } O\}.$$

We remark here that the set of K-equivariant one-dimensional local systems on O is parametrized by the set of irreducible (one-dimensional) representations of $K_x/(K_x)_0$ for a fixed $x \in O$.

We denote the objects in $\mathfrak{M}(\mathfrak{g}, K)$ (resp. $\mathscr{H}(\mathfrak{g}, K)$) corresponding to $\gamma[\text{-codim } O]$ and ${}^\pi\gamma[\text{-codim } O]$ under the equivalence (*) by $\mathfrak{M}_{(O,\gamma)}$ and $\mathfrak{L}_{(O,\gamma)}$ (resp. $M_{(O,\gamma)}$ and $L_{(O,\gamma)}$). Then we have the following decomposition of the Grothendieck groups:

$$K(\mathscr{H}(\mathfrak{g}, K))=\bigoplus_{(O,\gamma)} \mathbf{Z}[M_{(O,\gamma)}]=\bigoplus_{(O,\gamma)} \mathbf{Z}[L_{(O,\gamma)}],$$

$$K(\mathscr{M}(\mathfrak{g}, K))=\bigoplus_{(O,\gamma)} \mathbf{Z}[\mathfrak{M}_{(O,\gamma)}]=\bigoplus_{(O,\gamma)} \mathbf{Z}[\mathfrak{L}_{(O,\gamma)}].$$

2.4. K-orbits on X

We give a parametrization of K-orbits on X and other informations for the convenience of the readers. The reader is referred to Matsuki [M] and Vogan [V] for the proofs and other results.

We denote the Lie algebras of $G_\mathbf{R}$, $K_\mathbf{R}$ and K by $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{k}$, respectively. Let $\mathfrak{g}_0=\mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}_0$ and its complexification. Let θ be the involution on $\mathfrak{g}$ defined by $\theta(x+y)=x-y$ ($x \in \mathfrak{k}$, $y \in \mathfrak{p}$). we also denote its restriction to $\mathfrak{g}_0$ by θ.

For a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and a positive root system Δ^+ of $(\mathfrak{g}, \mathfrak{h}_0\otimes_\mathbf{R}\mathbf{C})$ let $\mathfrak{b}(\mathfrak{h}_0, \Delta^+)$ be the corresponding Borel subalgebra of $\mathfrak{g}$.

Proposition 3 (Matsuki [M]). (i) *Any Borel subalgebra of $\mathfrak{g}$ is K-conjugate to $\mathfrak{b}(\mathfrak{h}_0, \Delta^+)$ for some θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and a positive root system Δ^+ of $(\mathfrak{g}, \mathfrak{h}_0\otimes_\mathbf{R}\mathbf{C})$.*

(ii) *Let $\mathfrak{h}_0$ and $\mathfrak{h}_0'$ be θ-stable Cartan subalgebras of $\mathfrak{g}_0$. Let Δ^+ and Δ'^+ be positive root systems of $(\mathfrak{g}, \mathfrak{h}_0\otimes_\mathbf{R}\mathbf{C})$ and $(\mathfrak{g}, \mathfrak{h}_0'\otimes_\mathbf{R}\mathbf{C})$, respectively. Then $\mathfrak{b}(\mathfrak{h}_0, \Delta^+)$ is K-conjugate to $\mathfrak{b}(\mathfrak{h}_0', \Delta'^+)$ if and only if there exists an element $k \in K_\mathbf{R}$ so that $k.\mathfrak{h}_0=\mathfrak{h}_0'$ and $k.\mathfrak{b}(\mathfrak{h}_0, \Delta^+)=\mathfrak{b}(\mathfrak{h}_0', \Delta'^+)$.*

For a θ-stable Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ let $W(\mathfrak{h}_0)$ be the Weyl group of $(\mathfrak{g}, \mathfrak{h}_0\otimes_\mathbf{R}\mathbf{C})$. Set $W(\mathfrak{h}_0, K_\mathbf{R})=(N_G(\mathfrak{h}_0)\cap K_\mathbf{R})/(Z_G(\mathfrak{h}_0)\cap K_\mathbf{R})$ $(\subset W(\mathfrak{h}_0))$. Since the set of $G_\mathbf{R}$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$ and the set of $K_{\mathbf{R}}$-conjugacy classes of θ-stable Cartan subalgebras of $\mathfrak{g}_0$ are in one-to-one correspondence, we have the following.

Corollary (Matsuki [M]). *Let $\{\mathfrak{h}_0^{(i)} \mid i \in I\}$ be a set of representatives of the $G_{\mathbf{R}}$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$ so that each $\mathfrak{h}_0^{(i)}$ is θ-stable. We fix a positive root system $\Delta^{(i)+}$ of $(\mathfrak{g}, \mathfrak{h}_0^{(i)} \otimes_{\mathbf{R}} \mathbf{C})$ for each $i \in I$. Then the set of K-orbits on X (K-conjugacy classes of Borel subalgebras in $\mathfrak{g}$) is parametrized by the set $\coprod_{i \in I} W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}}) \backslash W(\mathfrak{h}_0^{(i)})$, and the K-conjugacy class corresponding to $W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})w$ is the one containing $\mathfrak{b}(\mathfrak{h}_0^{(i)}, w\Delta^{(i)+})$.*

For the classification of the Cartan subalgebras of $\mathfrak{g}_0$ we refer the reader to Sugiura [Su] and Warner [W]. In particular, since the number of the conjugacy classes of Cartan subalgebras is finite, the number of K-orbits on X is finite.

Let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra and Δ^+ a positive root system of $(\mathfrak{g}, \mathfrak{h}_0 \otimes_{\mathbf{R}} \mathbf{C})$. Let O be the K-orbit on X containing $\mathfrak{b} = \mathfrak{b}(\mathfrak{h}_0, \Delta^+)$. We denote the Borel subgroup corresponding to $\mathfrak{b}$ by B. Then O is isomorphic to $K/K_{\mathfrak{b}}$ with $K_{\mathfrak{b}} = \{k \in K \mid k.\mathfrak{b} = \mathfrak{b}\} = K \cap B$. Note that the set of the irreducible K-equivariant local systems on O is in one-to-one correspondence with the set of irreducible representations of the component group $K_{\mathfrak{b}}/(K_{\mathfrak{b}})_0$. This group is described as follows.

Proposition 4 (see Vogan [V]). *In the above notations set $H_{\mathbf{R}} = Z_{G_{\mathbf{R}}}(\mathfrak{h}_0)$ and $H = Z_G(\mathfrak{h}_0 \otimes_{\mathbf{R}} \mathbf{C})$. Then we have:*

$$K_{\mathfrak{b}}/(K_{\mathfrak{b}})_0 = (K \cap B)/(K \cap B)_0 \simeq (K \cap H)/(K \cap H)_0 \simeq (K_{\mathbf{R}} \cap H_{\mathbf{R}})/(K_{\mathbf{R}} \cap H_{\mathbf{R}})_0$$
$$\simeq H_{\mathbf{R}}/(H_{\mathbf{R}})_0 \simeq (\mathbf{Z}/2\mathbf{Z})^N$$

for some non-negative integer N with $0 \leq N \leq \dim_{\mathbf{R}}(\mathfrak{h}_0 \cap \mathfrak{p}_0)$.

§ 3. *W*-module structures

3.1. *W*-module structure on $K(\mathscr{M}(\mathfrak{g}, K))$

Set $G_1 = G \times G$, $\mathfrak{g}_1 = \mathfrak{g} \oplus \mathfrak{g}$ and $K_1 = \Delta G = \{(g, g) \in G_1 \mid g \in G\}$. We first consider $\mathscr{M}(\mathfrak{g}_1, K_1) = \mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)$. The flag variety of G_1 is $X \times X$, where X is the flag variety of G, and its decomposition into ΔG-orbits is given by $X \times X = \coprod_{w \in W} O(w)$, where W is the Weyl group of G and $O(w) = \Delta G.(eB, wB)$. Here we identify X with G/B for a fixed Borel subgroup B. Since each $O(w)$ is simply-connected, we have:

$$K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)) = \bigoplus_{w \in W} \mathbf{Z}[\mathfrak{M}_w] = \bigoplus_{w \in W} \mathbf{Z}[\mathfrak{L}_w],$$

with $\mathfrak{M}_w = \mathfrak{M}_{(O(w),1)}$ and $\mathfrak{L}_w = \mathfrak{L}_{(O(w),1)}$.

Let $X \times X \times X \xrightarrow{p_{ij}} X \times X$ $(1 \leq i < j \leq 3)$ be the natural projection. For $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)$ we have

$$\mathscr{H}^i\Big(\int_{p_{13}} (p_{12}^* \mathfrak{M}_1) \overset{L}{\underset{\mathcal{O}_{X\times X\times X}}{\otimes}} (p_{23}^* \mathfrak{M}_2)\Big) \in \mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)$$

for each i. Hence we can define a multiplication on $K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G))$ by

$$[\mathfrak{M}_1].[\mathfrak{M}_2] = \sum_i (-1)^i \Big[\mathscr{H}^i\Big(\int_{p_{13}} (p_{12}^* \mathfrak{M}_1) \overset{L}{\underset{\mathcal{O}_{X\times X\times X}}{\otimes}} (p_{23}^* \mathfrak{M}_2)\Big)\Big].$$

Proposition 5 (see Lusztig-Vogan [LV] and Springer [Sp]). *The above multiplication defines a ring structure on $K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G))$ so that*

$$\begin{array}{ccc} K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)) & \simeq & \mathbf{Z}[W]. \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ [\mathfrak{M}_w] & \longleftrightarrow & w \end{array}$$

Remark 1. By the Riemann-Hilbert correspondence one can translate this proposition into topological language, and this is actually the approach given in [LV] and [Sp]. Since they consider the Hecke algebra of W, we must specialize the indeterminant q to 1 to get the above result.

Now we define a W-action on $K(\mathscr{M}(\mathfrak{g}, K))$. By Proposition 5 we have only to define an action of the ring $K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G))$ on $K(\mathscr{M}(\mathfrak{g}, K))$. Let $X \times X \xrightarrow{q_i} X$ $(i = 1, 2)$ be the projection onto the i-th factor. For $\mathfrak{M} \in \mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)$ and $\mathfrak{N} \in \mathscr{M}(\mathfrak{g}, K)$ we have

$$\mathscr{H}^i\Big(\int_{q_1} \mathfrak{M} \overset{L}{\underset{\mathcal{O}_{X\times X}}{\otimes}} (q_2^* \mathfrak{N})\Big) \in \mathscr{M}(\mathfrak{g}, K)$$

for each i.

Proposition 6 (Lusztig-Vogan [LV]). *An action of $K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G))$ on $K(\mathscr{M}(\mathfrak{g}, K))$ is defined by*

$$[\mathfrak{M}].[\mathfrak{N}] = \sum_i (-1)^i \Big[\mathscr{H}^i\Big(\int_{q_1} \mathfrak{M} \overset{L}{\underset{\mathcal{O}_{X\times X}}{\otimes}} (q_2^* \mathfrak{N})\Big)\Big],$$

where $\mathfrak{M} \in \mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G)$ and $\mathfrak{N} \in \mathscr{M}(\mathfrak{g}, K)$.

Hence $K(\mathscr{M}(\mathfrak{g}, K))$ is endowed with a W-module structure.

In particular $K(\mathscr{M}(\mathfrak{g} \oplus \mathfrak{g}, \Delta G))$ $(\simeq \mathbf{Z}[W])$ has a $W \times W$-module structure. Note that this action of $W \times W$ coincides with the two-sided regular representation of $W \times W$ on $\mathbf{Z}$[W].

For a simple reflection s of W let X_s be the variety of semisimple-rank 1 parabolic subalgebras of $\mathfrak{g}$ corresponding to s. Write $X \xrightarrow{\pi_s} X_s$ for the natural map.

Proposition 7. *For $\mathfrak{M} \in \mathcal{M}(\mathfrak{g}, K)$ we have:*

$$s.[\mathfrak{M}]=[\mathfrak{M}]+\sum_i (-1)^i\left[\mathscr{H}^i\left(\mathbf{L}\pi_s^* \int_{\pi_s} \mathfrak{M}\right)\right].$$

This is proved by the same method as in the proof of Proposition 5 in [KT], so we omit the proof.

3.2. *W-module structure on* $\bigoplus_O \mathbf{Z}[\overline{T_O^*X}]$

We first review the Springer representations of W. We follow the approach of Lusztig [L] using DGM-extensions (see also Borho-MacPherson [BM]).

Set $\tilde{\mathfrak{g}}=\{(x, \mathfrak{b}) \in \mathfrak{g}\times X \mid x \in \mathfrak{b}\}$ and let $\tilde{\mathfrak{g}}\xrightarrow{p}\mathfrak{g}$ be the natural map. We denote the set of regular semisimple elements (resp. nilpotent elements) in $\mathfrak{g}$ by $\mathfrak{g}_{rs}$ (resp. N) and set $\tilde{\mathfrak{g}}_{rs}=p^{-1}(\mathfrak{g}_{rs})$ (resp. $\tilde{N}=p^{-1}(N)$). Let $\tilde{\mathfrak{g}}_{rs}\xrightarrow{p_{rs}}\mathfrak{g}_{rs}$ and $\tilde{N}\xrightarrow{p_N}N$ be the restrictions of p. Since p_{rs} is a W-principal bundle, we have an action of W on the local system $p_{rs*}(\mathbf{Q}_{\tilde{\mathfrak{g}}_{rs}})$ on $\mathfrak{g}_{rs}$, where $\mathbf{Q}_{\tilde{\mathfrak{g}}_{rs}}$ is the constant sheaf on $\tilde{\mathfrak{g}}_{rs}$ whose stalks are the rational number field $\mathbf{Q}$. By the functoriality of the DGM-extension we have an action of W on ${}^\pi(p_{rs*}(\mathbf{Q}_{\tilde{\mathfrak{g}}_{rs}}))$. Since ${}^\pi(p_{rs*}(\mathbf{Q}_{\tilde{\mathfrak{g}}_{rs}}))$ is isomorphic to $\mathbf{R}p_*(\mathbf{Q}_{\tilde{\mathfrak{g}}})$ as an object in the derived category (Lusztig [L]) and since $\mathbf{R}p_{N*}(\mathbf{Q}_{\tilde{N}})$ is isomorphic to $\mathbf{R}p_*(\mathbf{Q}_{\tilde{\mathfrak{g}}})\mid N$ by the base change theorem, we have an action of W on $\mathbf{R}p_{N*}(\mathbf{Q}_{\tilde{N}})$.

For $x \in N$ set $X_x=p^{-1}(x)=\{\mathfrak{b} \in X \mid x \in \mathfrak{b}\}$. Then the action of W on $\mathbf{R}p_*(\mathbf{Q}_{\tilde{N}})$ induces its action on $H^i(X_x, \mathbf{Q})=R^ip_{N*}(\mathbf{Q}_{\tilde{N}})_x$ for each i. This is the Springer representation of W in the usual sence.

For $O \in \mathscr{C}$ we set $Z_O=\overline{T_O^*X}$ and $Z=\bigcup_{O\in\mathscr{C}} Z_O$. Z is an algebraic variety of pure dimension $d=\dim X$. We identify T^*X with $\tilde{N}$ via the Killing form on $\mathfrak{g}$. Then we have $Z=p_N^{-1}(N(\mathfrak{p}))$ with $N(\mathfrak{p})=N\cap\mathfrak{p}$. Hence we have an action of W on $H_c^i(N(\mathfrak{p}), \mathbf{R}p_{N*}(\mathbf{Q}_{\tilde{N}})\mid N(\mathfrak{p}))=H_c^i(Z, \mathbf{Q})$. Since the dual space of the top cohomology group $H_c^{2d}(Z, \mathbf{Q})$ has a natural basis $\{[\overline{T_O^*X}]\}_{O\in\mathscr{C}}$, we have a W-action on the vector space $(H_c^{2d}(Z, \mathbf{Q}))^*=\bigoplus_{O\in\mathscr{C}} \mathbf{Q}[\overline{T_O^*X}]$.

Remark 2. In order to define a W-action we can use the method of Kazhdan-Lusztig [KL] in place of the above approach. The coincidence of these two approaches is proved in Hotta [H1; Appendix] though it is not exactly of this form.

Next we review a geometric description of the action of simple reflections of W on the space $(H_c^{2d}(Z, \mathbf{Q}))^*$ due to Hotta [H1], [H2]. We fix a simple reflection s. We define natural maps ρ_s, ϖ_s and τ_s by the following commutative diagram:

$$\begin{array}{ccc} T^*X_s\times_{X_s} X & \xrightarrow{\rho_s} T^*X=\tilde{N} \hookrightarrow & X\times N \\ \Big\downarrow{\scriptstyle \varpi_s} & & \Big\downarrow{\scriptstyle \tau_s} \\ T^*X_s & \hookrightarrow & X_s\times N. \end{array}$$

For $O \in \mathscr{C}$ we say that O is s-vertical (resp. s-horizontal) if $\tau_s^{-1}(\tau_s(Z_O))=Z_O$ (resp. $\tau_s^{-1}(\tau_s(Z_O))\supsetneq Z_O$). Let $\mathscr{C}_v^s$ and $\mathscr{C}_h^s$ be the set of s-vertical and s-horizontal K-orbits on X, respectively. Set $O_s=\pi_s(O)$ for $O \in \mathscr{C}$. The following is obvious.

Lemma 2.

(i) *O is s-vertical if and only if $\pi_s^{-1}(\pi_s(x))\cap O$ is open dense in $\pi_s^{-1}(\pi_s(x))$ for any $x \in O$.*

(ii) *$X_s=\bigcup_{O\in\mathscr{C}_v^s} O_s$ (disjoint union).*

(iii) *$\rho_s^{-1}(Z)=\bigcup_{O\in\mathscr{C}_v^s} Z_O$ (irreducible decomposition).*

(iv) *$\varpi_s(Z_O)=\overline{T_{O_s}^*X_s}$ for $O \in \mathscr{C}_v^s$.*

Proposition 8 (Hotta [H1], [H2]). *Let O be a K-orbit on X.*

(i) *If O is s-vertical, we have*

$$s.[Z_O]=-[Z_O].$$

(ii) *If O is s-horizontal, we have*

$$s.[Z_O]=[Z_O]+(\varpi_s^*\circ\varpi_{s*}\circ\rho_s^*)([Z_O]),$$

where ϖ_s^*, ρ_s^* and ϖ_{s*} are pull-back and direct image of algebraic cycles. Furthermore $(\varpi_s^*\circ\varpi_{s*}\circ\rho_s^*)([Z_O]) \in \bigoplus_{O'} \mathbf{Z}_{\geq 0}\,[Z_{O'}]$, where O' is running through $O' \in \mathscr{C}_v^s$ with $O'\subset\overline{\pi_s^{-1}(\pi_s(O))}$.

By the above proposition we see that the $\mathbf{Z}$-lattice $\bigoplus_{O\in\mathscr{C}} \mathbf{Z}[\overline{T_O^*X}]$ in $(H_c^{2d}(Z, \mathbf{Q}))^*$ is W-invariant. Hence we have an action of W on $\bigoplus_{O\in\mathscr{C}} \mathbf{Z}[\overline{T_O^*X}]$.

3.3. We prove the following in this subsection

Theorem 1 (repeated). *The $\mathbf{Z}$-linear homomorphism*

$$K(\mathcal{M}(\mathfrak{g}, K))\xrightarrow{\mathbf{Ch}} \bigoplus_{O\in\mathscr{C}} \mathbf{Z}[\overline{T_O^*X}]$$

is W-equivariant.

Remark 3. Set $\partial O=\bar{O}-O$. Since $\mathfrak{M}_{(O,\gamma)}|X-\partial O$ and $\mathfrak{L}_{(O,\gamma)}|X-\partial O$ are isomorphic to $\mathscr{H}_O^{\operatorname{codim} O}(\mathcal{O}_{X-\partial O})\otimes_{\mathbf{C}_{X-\partial O}}\gamma$, we see that $\mathrm{Ch}(\mathfrak{M}_{(O,\gamma)})$ and $\mathrm{Ch}(\mathfrak{L}_{(O,\gamma)})$ belong to $[\overline{T_O^*X}]+(\bigoplus_{O'\subsetneq\partial O}\mathbf{Z}_{\geq 0}[\overline{T_{O'}^*X}])$. Hence Ch is surjective.

Proof of Theorem 1. It is sufficient to show $\mathrm{Ch}(s.[\mathfrak{M}])=s.\,\mathrm{Ch}(\mathfrak{M})$ for any simple reflection s and $\mathfrak{M}\in\mathscr{M}(\mathfrak{g},K)$. By [Sa] (see [KT; Theorem 7]) there exist integers $m_s(O',O)$ for $O\in\mathscr{C}$ and $O'\in\mathscr{C}_v^s$ so that

$$\mathrm{Ch}(\mathfrak{M})=\sum_{O\in\mathscr{C}}n_O[\overline{T_O^*X}]$$

implies

$$\mathrm{Ch}\Big(\int_{\pi_s}\mathfrak{M}\Big):=\sum_i(-1)^i\,\mathrm{Ch}\Big(\mathscr{H}^i\Big(\int_{\pi_s}\mathfrak{M}\Big)\Big)=\sum_{O\in\mathscr{C}}n_O\Big(\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'}^*X_s}]\Big).$$

Furthermore we have

$$\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'}^*X_s}]=(\varpi_{s*}\circ\rho_s^*)([\overline{T_O^*X}])$$

for $O\in\mathscr{C}_h^s$. Hence we have

$$\mathrm{Ch}\Big(\mathbf{L}\pi_s^*\int_{\pi_s}\mathfrak{M}\Big)=\varpi_s^*\Big(\mathrm{Ch}\Big(\int_{\pi_s}\mathfrak{M}\Big)\Big)=\sum_{O\in\mathscr{C}}n_O\Big(\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'}^*X}]\Big).$$

Thus by Proposition 7

$$\begin{aligned}\mathrm{Ch}(s.[\mathfrak{M}])=&\sum_{O\in\mathscr{C}_v^s}n_O\Big([\overline{T_O^*X}]+\sum_{O'\in\mathscr{C}_v^s}m_s(O',O)[\overline{T_{O'}^*X}]\Big)\\&+\sum_{O\in\mathscr{C}_h^s}n_O\big([\overline{T_O^*X}]+(\varpi_s^*\circ\varpi_{s*}\circ\rho_s^*)([\overline{T_O^*X}])\big).\end{aligned}$$

On the other hand by Proposition 8

$$\begin{aligned}s.\,\mathrm{Ch}([\mathfrak{M}])=&\sum_{O\in\mathscr{C}_v^s}n_O(-[\overline{T_O^*X}])\\&+\sum_{O\in\mathscr{C}_h^s}n_O\big([\overline{T_O^*X}]+(\varpi_s^*\circ\varpi_{s*}\circ\rho_s^*)([\overline{T_O^*X}])\big).\end{aligned}$$

Thus we have only to prove that if O and O' are distinct elements of $\mathscr{C}_v^s$, then $m_s(O',O)=0$ and $m_s(O,O)=-2$.

For $O\in\mathscr{C}_v^s$ set $\hat{O}=\pi_s^{-1}(\pi_s(O))$. If we set $\hat{K}=\langle K,P\rangle$, the decomposition of X into $\hat{K}$-orbit is given by $X=\coprod_{O\in\mathscr{C}_v^s}\hat{O}$, and hence this decomposition satisfies the Whitney condition. For $O\in\mathscr{C}_v^s$ we define $\mathfrak{M}_O\in\mathscr{M}(\mathfrak{g},K)$ by $\mathscr{DR}(\mathfrak{M}_O)=\mathbf{C}_{\hat{O}}[-\operatorname{codim} O]$. Then

$$\mathrm{Ch}(\mathfrak{M}_O)\in[\overline{T_{\hat{O}}^*X}]+\sum_{O'}\mathbf{Z}_{\geq 0}[\overline{T_{O'}^*X}],$$

where O' is running through $O'\in\mathscr{C}_v^s$ with $O'\subset\partial O$. Hence using induction on dim O we see that it is sufficient to show

$$\mathrm{Ch}\Big(\mathbf{L}\pi_s^*\int_{\pi_s}\mathfrak{M}_O\Big)=-2\mathrm{Ch}(\mathfrak{M}_O)$$

for $O\in\mathscr{C}_v^s$. By the Riemann-Hilbert correspondence we have

$$\begin{aligned}\mathscr{DR}\Big(\mathbf{L}\pi_s^*\int_{\pi_s}\mathfrak{M}_O\Big)&=\pi_s^{-1}(\mathbf{R}\pi_{s*}(\mathscr{DR}(\mathfrak{M}_O)))[1]\\&=\pi_s^{-1}(\mathbf{R}\pi_{s*}(\mathbf{C}_{\hat{O}}))[-\operatorname{codim} O+1].\end{aligned}$$

Since π_s is a $\mathbf{P}^1$-bundle, we have the following triangle

$$\begin{tikzcd} & \mathbf{C}_{\hat{O}} \arrow[dl, "+1"'] & \\ \mathbf{C}_{\hat{O}}[-2] \arrow[rr] & & \pi_s^{-1}(\mathbf{R}\pi_s^*(\mathbf{C}_{\hat{O}})). \arrow[ul] \end{tikzcd}$$

By the Riemann-Hilbert correspondence we have

$$\begin{tikzcd} & \mathfrak{M}_O[1] \arrow[dl, "+1"'] & \\ \mathfrak{M}_O[-1] \arrow[rr] & & \mathbf{L}\pi_s^*\int_{\pi_s}\mathfrak{M}_O. \arrow[ul] \end{tikzcd}$$

Hence

$$\mathrm{Ch}\Big(\mathbf{L}\pi_s^*\int_{\pi_s}\mathfrak{M}_O\Big)=\mathrm{Ch}(\mathfrak{M}_O[1])+\mathrm{Ch}(\mathfrak{M}_O[-1])=-2\mathrm{Ch}(\mathfrak{M}_O)$$

and we are done.

§4. Complements

4.1. We describe the W-module structure of $K(\mathscr{M}(\mathfrak{g},K))$ and $(H_c^{2d}(Z,\mathbf{Q}))^*$ more explicitly.

The following is a generalization of a result of Kazhdan-Lusztig [KL; (6.1) and (6.2)]. Since the proof is the same as that of [KL], we omit it.

Proposition 9. *As a W-module, we have*

$$(H_c^{2d}(Z,\mathbf{Q}))^*\simeq\bigoplus_{x\in K\backslash N(\mathfrak{p})}(H_{2d_x}(X_x,\mathbf{Q}))^{C_K(x)},$$

where $d_x=\dim X_x$ and $C_K(x)=Z_K(x)/(Z_K(x))_0$.

Hence suitable information on the K-conjugacy classes of nilpotent elements in $\mathfrak{p}$ and the Springer correspondence of the group G give the irreducible decomposition of $(H_c^{2d}(Z, \mathbf{Q}))^*$ as a W-module.

Next we consider $K(\mathcal{M}(\mathfrak{g}, K))$. The explicit description of the action of a simple reflection with respect to the basis $\{[\mathfrak{M}_{(O,\gamma)}] \mid (O, \gamma) \in \mathscr{S}\}$ is given in [LV] (see also [V]). We include it here for the convenience of the readers.

Lemma 3 (Lusztig-Vogan [LV]). Fix a simple reflection s, $(O, \gamma) \in \mathscr{S}$ and $x \in O$. Set $\hat{O}=\pi_s^{-1}(\pi_s(O))$ and $L_x^s=\pi_s^{-1}(\pi_s(x))$.

(a) If $L_x^s \subset O$, then $s.[\mathfrak{M}_{(O,\gamma)}]=-[\mathfrak{M}_{(O,\gamma)}]$.

(b1) If $L_x^s \cap O=\{x\}$ and $O'=\hat{O}-O$ is a single K-orbit, then there exists a unique locally constant extension $\hat{\gamma}$ of γ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}]=[\mathfrak{M}_{(O',\gamma')}]$ with $\gamma'=\hat{\gamma}\mid O'$.

(b2) If $L_x^s \cap O=L_x^s-\{\text{point}\}$, then $O'=\hat{O}-O$ is a single K-orbit, there exists a unique locally constant extension $\hat{\gamma}$ of γ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}]=[\mathfrak{M}_{(O',\gamma')}]$ with $\gamma'=\hat{\gamma}\mid O'$.

(c1) If $L_x^s \cap O=\{x, y\}$, then $O'=\hat{O}-O$ is a single K-orbit, γ has two distinct extension $\hat{\gamma}_1$, $\hat{\gamma}_2$ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}]=-[\mathfrak{M}_{(O,\gamma)}]+[\mathfrak{M}_{(O',\gamma_1')}]+[\mathfrak{M}_{(O',\gamma_2')}]$ with $\gamma_i'=\hat{\gamma}_i\mid O'$.

(c2) If $L_x^s \cap O=L_x^s-\{\text{two points in one } K\text{-orbit}\}$, then γ has at most one extension to $\hat{O}$. In the case γ has an extension $\hat{\gamma}$ to $\hat{O}$, $\hat{\gamma}\mid\hat{O}-O$ has a unique extension $\hat{\gamma}_*$ to $\hat{O}$ different from $\hat{\gamma}$ and $s.[\mathfrak{M}_{(O,\gamma)}]=[\mathfrak{M}_{(O,\gamma_*)}]$ with $\gamma_*=\hat{\gamma}_*\mid O$. In the case γ does not have an extension, $s.[\mathfrak{M}_{(O,\gamma)}]=[\mathfrak{M}_{(O,\gamma)}]$.

(d1) If $L_x^s \cap O=\{x\}$ and $\hat{O}-O$ is a union of two orbits O' and O'' with $\dim O=\dim O''=\dim O'-1$, then γ has a unique extension $\hat{\gamma}$ to $\hat{O}$ and $s.[\mathfrak{M}_{(O,\gamma)}]=[\mathfrak{M}_{(O',\gamma')}]-[\mathfrak{M}_{(O'',\gamma'')}]$ with $\gamma'=\hat{\gamma}\mid O'$ and $\gamma''=\hat{\gamma}\mid O''$.

(d2) If $L_x^s \cap O=L_x^s-\{\text{two points in two } K\text{-orbits}\}$, then $s.[\mathfrak{M}_{(O,\gamma)}]=[\mathfrak{M}_{(O,\gamma)}]$.

Now we give an alternative description of the W-module $(H_c^{2d}(Z, \mathbf{Q}))^* \simeq (K(\mathcal{M}(\mathfrak{g}, K))/\mathrm{Ker}\,\mathbf{Ch}) \otimes_{\mathbf{Z}} \mathbf{Q}$ different from that of Proposition 9.

Proposition 10. *Let $\{\mathfrak{h}_0^{(i)}\}_{i\in I}$ be a set of representatives of the conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$ so that $\theta(\mathfrak{h}_0^{(i)})=\mathfrak{h}_0^{(i)}$. We fix for each $i \in I$ a positive root system $\Delta^{(i)+}$ of the root system $\Delta^{(i)}$ of $(\mathfrak{g}, \mathfrak{h}_0^{(i)}\otimes_{\mathbf{R}}\mathbf{C})$. We identify Cartan subalgebras $\mathfrak{h}_0^{(i)}\otimes_{\mathbf{R}}\mathbf{C}$ via the Borel subalgebras $\mathfrak{b}(\mathfrak{h}_0^{(i)}, \Delta^{(i)+})$ and regard W as its Weyl group. Then there exist linear characters $W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})\xrightarrow{\chi_i}\{\pm 1\}$ so that*

$$(K(\mathcal{M}(\mathfrak{g}, K))/\mathrm{Ker}\,\mathbf{Ch})\underset{\mathbf{Z}}{\otimes}\mathbf{Q} \simeq \bigoplus_{i\in I} \mathrm{Ind}_{W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})}^{W}(\chi_i).$$

Remark 4. If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras of $\mathfrak{g}$ contained in Borel subalgebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively, then there exists $g \in G$ so that $\mathrm{Ad}(g)\mathfrak{h}_1=\mathfrak{h}_2$ and $\mathrm{Ad}(g)\mathfrak{b}_1=\mathfrak{b}_2$. Moreover $\mathrm{Ad}(g)\mid\mathfrak{h}_1: \mathfrak{h}_1\to\mathfrak{h}_2$ is uniquely determined by $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

(Sketch of the proof of Proposition 10)

For $O \in \mathscr{C}$ set $A_O=\sum_{(O,\gamma)\in\mathscr{S}}[\mathfrak{M}_{(O,\gamma)}]$. Then we see from Lemma 3 that $V=\bigoplus_{O\in\mathscr{C}}\mathbf{Q}A_O$ is W-invariant. By Remark 3 V is isomorphic to

$$(K(\mathcal{M}(\mathfrak{g}, K))/\mathrm{Ker}\,\mathbf{Ch})\underset{\mathbf{Z}}{\otimes}\mathbf{Q}\simeq(H_c^{2d}(Z,\mathbf{Q}))^*.$$

Let $\mathscr{C}_i$ be the set of $O \in \mathscr{C}$ such that $\mathfrak{b}(\mathfrak{h}_0^{(i)}, w\Delta^+) \in O$ for some $w \in W$. For $i, j \in I$ we write $i \leq j$ if $i=j$ or $\dim(\mathfrak{h}_0^{(j)}\cap\mathfrak{p}_0)<\dim(\mathfrak{h}_0^{(i)}\cap\mathfrak{p}_0)$. Then

$$\tilde{V}^{(i)}=\bigoplus_{j\leq i}\Big(\bigoplus_{O\in\mathscr{C}_j}\mathbf{Z}A_O\Big)$$

is W-invariant and if we set

$$V^{(i)}=\tilde{V}^{(i)}\Big/\sum_{j\leq i,\, j\neq i}\tilde{V}^{(j)},$$

we have $V\simeq\bigoplus_{i\in I}V^{(i)}$ as a W-module. We fix $i \in I$. For $w \in W$ let a_w be the class of A_O in $V^{(i)}$, where O is the K-orbit on X containing $\mathfrak{b}(\mathfrak{h}_0^{(i)}, w\Delta^{(i)+})$. Then we have $V^{(i)}=\bigoplus_{w\in W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})\backslash W}\mathbf{Q}a_w$. By Lemma 3 and the arguments as in [V] we see that $s.a_w=\pm a_{ws}$. Thus

$$V^{(i)}\simeq\mathrm{Ind}_{W(\mathfrak{h}_0, K_{\mathbf{R}})}^{W}(\chi_i)$$

for some χ_i and we are done.

Remark 5. χ_i is uniquely determined by $\chi_i(w)a_e=w.a_e$ for $w \in W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})$. In particular, if $\mathfrak{h}_0^{(i)}$ is a fundamental Cartan subalgebra (i.e. $\dim(\mathfrak{h}_0^{(i)}\cap\mathfrak{k}_0)$ is maximal) we see that χ_i coincides with the restriction of the sign representation sgn of W to $W(\mathfrak{h}_0^{(i)}, K_{\mathbf{R}})=W(K)=$(the Weyl group of K).

4.2. As an example we treat the case when $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebras. By Propositions 9 and 10 we have the following.

Proposition 11. *If $\mathfrak{g}_0$ has a unique conjugacy class of Cartan subalgebras, then we have*

$$\mathrm{Ind}_{W(K)}^{W}(1)\otimes\mathrm{sgn}\simeq\bigoplus_{x\in K\backslash N(\mathfrak{p})}H_{2d_x}(X_x, \mathbf{Q})^{C_K(x)}.$$

In the case $\mathfrak{g}_0$ is a complex semisimple Lie algebra viewed as a real semisimple Lie algebra, the above formula is just

$$\mathbf{Q}[W] \simeq \bigoplus_{x \in G \backslash N} (H_{2d_x}(X_x, \mathbf{Q}) \otimes H_{2d_x}(X_x, \mathbf{Q}))^{C_G(x)}$$

with $C_G(x) = Z_G(x)/(Z_G(x))_0$, from which the completeness theorem of Springer is obtained (see [KL]).

We see by [Su] that if $\mathfrak{g}_0$ is a non-compact real form of a complex simple Lie algebra, $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebra if and only if the pair $(\mathfrak{g}, \mathfrak{k})$ is one of the following three types:

(I) $\mathfrak{g} = A_{2n-1}$, $\mathfrak{k} = C_n$ $(n \geq 2)$,
(II) $\mathfrak{g} = D_n$, $\mathfrak{k} = B_{n-1}$ $(n \geq 4)$,
(III) $\mathfrak{g} = E_6$, $\mathfrak{k} = F_4$.

Remark 6. Note that in the cases (I), (II) and (III) the automorphism θ of $\mathfrak{g}$ is obtained by extending the symmetry of the Dynkin diagram of $\mathfrak{g}$ using the usual presentation of $\mathfrak{g}$ by generators and relations.

In order to write down the formula in Proposition 11 explicitly, we need a classification of K-conjugacy classes of nilpotent elements in $\mathfrak{p}$. Consider the natural map $K \backslash N(\mathfrak{p}) \xrightarrow{\Phi} G \backslash N$. Then we have the following.

Proposition 12 (see [Se]). *For $e \in N$ the following conditions are equivalent.*

(i) *e is conjugate to an element of $N(\mathfrak{p})$ under the action of G.*

(ii) *Let n_i be the number attached to the vertex i of the weighted Dynkin diagram of e ($n_i = 0$, 1 or 2). Then in the Satake diagram of $\mathfrak{g}_0$ we have:*

$$n_i = n_j \quad \text{if } \underset{i}{\circ} \curvearrowright \underset{j}{\circ},$$
$$n_i = 0 \quad \text{if } \underset{i}{\bullet}.$$

Proposition 13. *If $\mathfrak{g}_0$ has a unique conjugacy class of Cartan subalgebras, then the map Φ is injective.*

Proposition 12 is stated in [Se] as a theorem of Antonyon and the proof is given for the case $\mathfrak{g}_0$ is a normal real form. But the proof in [Se] for normal forms applies to the general case under some modification. Proposition 13 is shown by using the results of [Ko] and [KR] if we note Remark 6.

Remark 7. We can prove more generally that the natural maps $K \backslash \mathfrak{p} \rightarrow G \backslash \mathfrak{g}$ and $K \backslash \mathfrak{k} \rightarrow G \backslash \mathfrak{g}$ are injective if $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebras.

Using Propositions 12, 13 and the Springer correspondence (see [Sh] and [ALS]) we can write down the formula in Proposition 11 explicitly.

In the case (I) we have $\mathrm{Ind}_{W(C_n)}^{S_{2n}}(1) = \bigoplus \chi_\sigma$, where σ is running through the partitions of 2n whose parts are even and χ_σ is the irreducible representation corresponding to σ. A direct proof of this formula is given in [T]. I understand that this was originally conjectured by N. Iwahori.

In the case (II) we have $\mathrm{Ind}_{W(B_{n-1})}^{W(D_n)}(1) = 1 \oplus \chi$, where χ is the irreducible representation of $W(D_n)$ corresponding to the pair of partitions $((n-1>1), \emptyset)$ in the usual conventions. But this is trivial.

In the case (III) we have $\mathrm{Ind}_{W(F_4)}^{W(E_6)}(1) = 1_p \oplus 20_p \oplus 24_p$.

Remark 8. Using the recent result of Matsuki describing the closure relations of K-orbits on X, we can construct for each K-orbit O a non-singular variety Y with a K-action and a K-equivariant proper surjective map $Y \xrightarrow{f} \bar{O}$ which is generically finite. This is an obvious generalization of the desingularization of Schubert varieties given in [D]. If $\mathfrak{g}_0$ has only one conjugacy class of Cartan subalgebras, the above f is birational and in this case we can calculate the dimension of each stalk of the intersection cohomology sheaf of the closure of any K-orbit by the method given in [Sp] (see [LV]).

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The Dimensions of the Spaces of All Class Functions Invariant under the Twisting Operators of Finite Classical Groups

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§ 1. Introduction

In [1]–[3], the author has determined the action of the twisting operator on the unipotent class functions of finite split reductive groups in terms of the Fourier transform of Lusztig (cf. [9]). Although the close relation between the twisting operator and the Lusztig Fourier transform should be expected for the non-unipotent class functions, there seem to be some difficulties for the direct approach. So, we do not take up the problem straightforwardly in this paper; instead we are to determine the dimensions of the spaces of all class functions invariant under the twisting operator of finite classical groups. The formulae are presented so as to give us the distinct features concerning the twisting operators and also to be a possible key for the future investigations.

Now, let us introduce several notations to show our results. Let H be a connected reductive group over a finite field $\mathbf{F}_q$ with q-elements and F be the Frobenius mapping. Let t_1^* be the twisting operator on the space $C(H^F)$ of all class functions of H^F associated with the twisting t_1 of the conjugacy classes of H^F (cf. [3, Introduction]),

$$t_1\colon H^F/\sim \to H^F/\sim .$$

The twisting t_1 maps unipotent conjugacy classes of H^F to unipotent ones, and thus decomposes unipotent conjugacy classes $H^F_{\mathrm{unip}}/\sim$ into equivalence classes:

$$(H^F_{\mathrm{unip}}/\sim)/\underset{t_1}{\sim}.$$

Now, the adjoint group H_{ad} of H is written as a direct product

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