A combinatorial result on K-orbits on a flag manifold

by

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Introduction

Recently, Carrell and Peterson [1] found an elementary necessary condition for smoothness of certain projective varieties with torus action, which applies in particular in the case of Schubert varieties. Along the way, they obtained a proof of an inequality conjectured by Deodhar.

In the present note we shall apply the results of [1] to another type of varieties, namely the closures of orbits in the flag manifold of the fixed point group K of an inner automorphism of order two of the underlying algebraic group. We obtain an analogue of Deodhar's inequality, in the context of the combinatories of [5]. We do not discuss here analogues of the results of [1] on rational smoothness of Schubert varieties.

I am grateful to J. Carrell for explaining to me his work with Peterson.

1 Preliminaries

1.1 G will denote a connected reductive linear algebraic group over the algebraically closed field k. We fix a Borel group B of G and a maximal torus T contained in B. Let N be the normalizer of T and $W = N/T$ the Weyl group of (G, T). The corresponding root system is denoted by Φ and Φ^+ is the system of positive roots defined by B. We denote by $\Delta \subset \Phi^+$ the basis of Φ and by S the corresponding set of simple reflections. If $\alpha \in \Phi$ let $s_\alpha \in W$ be the reflection defined by it. So $S = \{s_\alpha \in \Delta\}$. The set of all reflections is denoted by R.

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Let x_α denote the one parameter subgroup defined by $\alpha \in \Phi$, an isomorphism of the additive group onto a closed subgroup U_α of G. Denote by G_α the subgroup of G generated by U_α and $U_{-\alpha}$. It is isomorphic to SL_2 or PSL_2. If $r = s_\alpha$ $(= s_{-\alpha})$ is the reflection defined by α we also write $G_\alpha = G_r$. We assume the x_α normalized such that $n_\alpha x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1) \in N$. Then $n_\alpha T = s_\alpha$. If $w \in W$ we denote by $\dot{w} \in N$ a representative.

Let $X = B\backslash G$ be the flag manifold. (We prefer to view it as a manifold with right G-action.) It is a projective variety. The T-fixed points of X are the points $e_w = B\dot{w}$ $(w \in W)$.

1.2 Lemma *Let C be a T-stable irreducible closed curve in X.*

(i) *There is a pair $(w, r) \in W \times R$ such that $C = B\dot{w}G_r$;*

(ii) *The curve of* (i) *contains two T-fixed points, viz. e_w and e_{wr};*

(iii) *$B\dot{w}G_r = B\dot{w}'G_{r'}$ if and only if $r' = r$, $w' = w$ or $r' = r$, $w' = rw$.*

For these facts see [1]. It is easy to see that $B\dot{w}G_r$ is a T-stable curve passing through e_w and that the number of them, for fixed n, equals $\dim X$. One can then apply 1.4 to see that every T-stable curve must be of the form of 1.2(i).

1.3 Denote by $C(w, r)$ the curve of 1.2(i). One can describe it in a slightly different manner. Let $r = s_\alpha$ and take α such that $w\alpha \notin \Phi^+$. Then (cl denoting Zariski closure)

$$C(w, r) = cl(e_w U_\alpha).$$

This readily follows by an application of Bruhat's lemma in G_r.

Let Y be a closed T-stable subvariety of X. Then Y contains T-fixed points. The following result is due to Carrell and Peterson [1].

1.4 Proposition *Let $y \in Y$ be fixed by T.*

(i) *The number of T-stable irreducible closed curves in Y passing through y is at least* $\dim Y$;

(ii) *If Y is smooth in y this number equals* $\dim Y$.

In [loc.cit.] the proposition is applied in the case that Y is a Schubert variety $cl(B\backslash BwB)$. One then obtains the following result (conjectured by Deodhar). Let l be the length function on W defined by S and $\leq$ the Bruhat order. Than if $x \leq w$ the number of $r \in R$ with $x < xr \leq w$ is at least $l(w) - l(x)$.

In the present note we shall apply the proposition in another case. To prepare for this application we review some results of [6] and [5].

2 Groups with involution

2.1 From now on we assume that $char\ k \neq 2$. Let θ be an automorphism of order two of G. We assume that $\theta B = B$, $\theta T = T$. (Given θ, such B and T exist.) Then θ operates on Φ and W and $\theta\Phi^+ = \Phi^+$. We assume, moreover, that θ is *inner* and the fixed point group $G^\theta = K$ is *connected.*

We then have $\theta = Int(a)$, with $a \in T$, a^2 central in G. The assumption about K is harmless. It can always be achieved by replacing G by an isogeneous group (which does not change X).

It is known that K has finitely many orbits in X. We shall apply 1.4 to the closure of such an orbit.

2.2 Parametrization

Let $\mathcal{V} = \{x \in G \mid x(\theta x)^{-1} \in N\}$. Then T acts on $\mathcal{V}$ on the left and K on the right. Put $V = T\backslash\mathcal{V}/K$. If $v \in V$, let $x(v)$ be a representative. Then the map $v \mapsto Bx(v)K$ defines a bijection of V onto the set of K-orbits in X (see [6, §4]). If $v \in V$ denote by $\mathcal{O}_v$ the corresponding orbit.

Define an order on V by: $v' \leq v$ if and only if $\mathcal{O}_{v'} \subset cl(\mathcal{O}_v)$. This "Bruhat order" is studied in [5].

2.3 Relation with W

$\mathcal{V}$ is stable under left multiplication by N and has an induced (left) W-action on V.

Also, we have a map $\varphi : V \mapsto W$ with $\varphi(v) = x(v)\theta(x(v))^{-1}T$. The image of φ lies in the set of all elements of order ≤ 2 of W (see [5, 1.6], notice that because θ is inner, it operates trivially on W).

The orbit $\mathcal{O}_v$ is closed if and only $\varphi(v) = e$ ([6, p. 541]). If $v_0 = BK$, then v_0 is closed and the closed orbits are the $\mathcal{O}_{w\cdot v_0}$ ($w \in W$). See [5, 2.5]. Let N_K be the normalizer of T in K and let W_K be the image of N_K in W. Then $w \cdot v_0 = v_0$ if and only if $w \in W_K$ ([loc.cit., 2.8]). So the number of closed orbits is $|W/W_K|$. If $v \in V$ we put $l(v) = \dim \mathcal{O}_v - \dim \mathcal{O}_{v_0}$.

2.4 Roots

We recall some notions (see [5, 1.1]). Let $\alpha \in \Phi$. Then θ stabilizes G_α. There are two cases:

(a) $G_\alpha \subset K$. Then α is *compact imaginary*,

(b) $G_\alpha \not\subset K$. Now α is *non-compact imaginary*.

In the second case $(G_\alpha,\ \theta|_{G_\alpha})$ is isomorphic to (SL_2, ψ) (resp. (PSL_2, ψ)), with $\psi = Int\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ (resp. the automorphism induced by this).

We also say that the corresponding reflection $r = s_\alpha$ is compact imaginary (resp. non-compact imaginary). Denote the set of such reflections by R_c resp. R_n. We need an auxiliary result.

2.5 Lemma *Let $r = s_\alpha \in R_n$.*

(i) *There exists a unique element $v(r) \in V$ with $\varphi(v(r)) = r$, represented by an element $x \in G_r \cap V$ with $x\theta(x)^{-1} = n_\alpha$;*

(ii) *We have $U_\alpha - \{e\} \subset TU_{-\alpha}xK$.*

It suffices to prove this in the case (SL_2, ψ) (as above). In this case, one may assume $n_\alpha = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and then one can take $x = 2^{-\frac{1}{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. The uniqueness of $v(r)$ is easily checked.

To prove (ii) use the following identity of 2×2-matrices:

$$\begin{pmatrix} 1 & \xi^2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \xi^{-1} & 0 \\ 0 & \xi \end{pmatrix} \quad (\xi \in k^*).$$

3 T-stable curves and K-orbits

Let $C(w,r)$ be as in 1.3. There is a unique $v = v(w,r)$ in V such that $\mathcal{O}_v$ intersects $C(w,r)$ in an open dense subset.

3.1 Lemma *We have $v(w,r) = w \cdot v_0$ if r is compact imaginary and $v(w,r) = w \cdot v(r)$ if r is non-compact imaginary.*

If r is compact imaginary we have $G_r \subset K$, hence $C(w,r) = B\dot{w}G_r \subset B\dot{w}K = \mathcal{O}_{w\cdot v_0}$.

Now let r be non-compact imaginary and assume $r = s_\alpha$, with $w\alpha \notin \Phi^+$. Then $B\dot{w}U_\alpha$ is open dense in $C(w,r)$ (1.3). It follows from 2.5(ii) that $B\dot{w}(U_\alpha - \{e\}) \subset B\dot{w}U_{-\alpha}xK$, where x is as in 2.5 (ii). Since $\dot{w}U_{-\alpha}\dot{w}^{-1} \subset B$, we can conclude that $B\dot{w}(U_\alpha - \{e\}) \subset B\dot{w}xK$. As $\dot{w}x$ represents the element $w \cdot v(r)$ of V, we conclude that $v(w,r) = w \cdot v(r)$, as asserted.

We come now to the main result of this note. The first part is formulated in terms of the Bruhat order of V. It may be viewed as an analogue for this case of Deodhar's inequality, mentioned in no. 1

3.2 Theorem *Let $v \in V$, $w \in W$ be such that $w \cdot v_0 \leq v$.*

(i) *The number of non-compact imaginary $r \in R$ with $w \cdot v(r) \leq v$ is at least $l(v)$;*

(ii) *Equality holds in* (i) *if $cl(\mathcal{O}_v)$ is smooth in e_w.*

We shall apply 1.4 to $Y = cl(\mathcal{O}_v)$ (which is a T-stable subvariety of X). Since $w \cdot v_0 \leq v$ we have $\mathcal{O}_{w\cdot v_0} \subset cl(\mathcal{O}_v) = Y$. Clearly $e_w \in \mathcal{O}_{w\cdot v_0}$. We take $y = e_w$ in 1.4. The T-stable irreducible closed curves in Y through y are the $C(w,r)$ of 1.3, with $r \in R$, lying in Y. By the definition of $v(w,r)$ we have $C(w,r) \subset Y$ if and only if $v(w,r) \leq v$. Let a_c (a_n) be the number of compact imaginary (resp. compact non-imaginary) reflections r with this property. By 3.1, a_c equals the number of all compact imaginary reflections, which is the number of positive roots of the root system of (K,T) (note that T is a maximal torus of K). Also, $\mathcal{O}_{v_0} = BK$ is isomorphic to the flag manifold of K. Hence $a_c = \dim \mathcal{O}_{v_0}$ (see [5, 7.1] for these facts).

By 1.4(i), we have $a_c + a_n \geq \dim \mathcal{O}_v = l(v) + \dim \mathcal{O}_{v_0}$. It follows that $a_c \geq l(v)$, which proves (i). Then (ii) follows from 1.4(ii).

3.3 Remarks

(a) $cl(\mathcal{O}_v)$ is smooth if and only if it is smooth in all points e_w with $w \cdot v_0 \leq v$. In fact, the non smooth points from a closed, T-stable subvariety of Y which, if non-empty, must contain such a e_w.

(b) In [1] is it shown for the case of a Schubert variety Y, that equality in 1.4(i) is already implied by rational smoothness. This is also true in our case, but we shall not give the proof here (which requires the Hecke algebra module introduced by Lusztig-Vogan [3]).

(c) In our application of 1.4 to the case of K-orbit closures in X, where K is the fixed point group of an involution of G, we have to restrict ourselves to the case of an inner automorphism θ.

In the case of an outer automorphism one has $rank\ K < rank\ G$. In that case one should replace T by a maximal torus S of K. But the number of S-stable curves in X is infinite. The application of the results of Carrell and Peterson requires finiteness of this number.

4 An example

4.1 As an illustration, we consider the case that $G = GL_4$ and $\theta = Int\ a$, where $a = diag(1, 1, -1, -1)$. Now $\mathcal{V} = \{x \in G \mid xax^{-1} = N\}$ and V is the set of T-orbits in $\mathcal{V}$, for conjugation action.

The Weyl group is S_4. The map $\varphi : V \to S_4$ is induced by the map $\mathcal{V} \to S_4$ sending x to $xax^{-1}T$. A straightforward check shows that the elememts $v \in V$ can be described as follows.

(a) $\varphi(v) = e$. Then v is represented by a diagonal element of $\mathcal{V}$. These v are labeled v_{ij} $(1 \leq i, j \leq 4)$, represented by a diagonal matrix with i, j-entries 1 and other entries -1.

(b) $\varphi(v)$ is a reflection in S_4. If $r = (ij)$ $(i < j)$ is such a reflection there are two elements $v \in \mathcal{V}$ with $\varphi(v) = r$. We label them $v_{ij,+}$ resp. $v_{ij,-}$, according as the diagonal element of a representing element of $\mathcal{V}$ with the highest label outside $\{i, j\}$ is 1 resp. -1.

(c) $\varphi(v)$ is a product of two commuting reflections. On the set of these $v \in V$, the map φ is bijective. We label them (in an obvious manner) $v_{12,34}$, $v_{13,24}$, $v_{14,23}$.

We see that the number of elements of V is 21. See also [2] and [4, 4.1] (where the example occurs in a more general setting).

Taking, as usual, the roots to be $\varepsilon_i - \varepsilon_j$ $(1 \leq i, j \leq 4,\ i \neq j)$ and the positive ones those with $i < j$, the non-compact imaginary positive roots are $\varepsilon_1 - \varepsilon_3,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_4,\ \varepsilon_2 - \varepsilon_4$. The corresponding elements $v(r)$ of V of 2.5 are

$$v_{13,-},\ v_{23,-},\ v_{14,-},\ v_{24,-}.$$

4.2 We recall a basic general construction of [5, no. 4]. Let $s \in S$ (the set of simple reflections), $v \in V$ and denote by $P_s \supset B$ the parabolic subgroup of semi-simple rank one associated to s. Then $P_s\mathcal{O}_v$ contains a unique open dense orbit $\mathcal{O}_{m(s)v}$. The map $(s, v) \mapsto m(s)v$ of $S \times V$ to V is studied in [5]. In particular, $l(v) \leq l(m(s)v) \leq l(v) + 1$. We have the notion of a *reduced decomposition* of $v \in V$: this is a pair $(\mathbf{v} = (v_0, \ldots, v_k), \mathbf{s} = (s_1, \ldots, s_k))$ of sequences in V resp. S, such that $l(v_0) = 0$, $v_k = v$ and $v_i = m(s_i)v_{i-1}$, $l(v_i) = l(v_{i-1}) + 1$ for $i \in [1, k]$. See [loc.cit., 5.7].

4.3 We now return to our example. Take $v = v_{14,-}$. It has a reduced decomposition $((v_0,\ v_1,\ v_2,\ v_3), (s_2,\ s_1,\ s_3))$, where $v_0 = v_{12}$ (this notation is consistent with that of 2.3), $v_1 = v_{23,-}$, $v_2 = v_{13,-}$, $v_3 = v$. Moreover, $s_1,\ s_2,\ s_3$ are the simple reflections defined by the roots $\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_4$, respectively.

For $x \in V$ put $X_x = \{y \in V \mid y \leq x\}$. Using the "one-step property" of [loc.cit., 5.12(c)] one checks that

$$\begin{aligned} X_{v_1} &= (v_{23,-},\ v_{12},\ v_{13}), \\ X_{v_2} &= (v_{13,-},\ v_{12,-},\ v_{23,-},\ v_{12},\ v_{13},\ v_{23}), \\ X_v &= (v_{14,-},\ v_{12,34},\ v_{13,-},\ v_{24,-},\ v_{12,+},\ v_{12,-}, \\ &\qquad v_{23,-},\ v_{34,+},\ v_{34,-},\ v_{12},\ v_{13},\ v_{14},\ v_{23},\ v_{24}). \end{aligned}$$

We see that all $v(r)$ with r non-compact imaginary are $\leq v$. Since $l(v) = 3$, we conclude from 3.2(ii) that $cl(\mathcal{O}_v)$ is not smooth. However for $w \cdot v_0 \neq v_0$ with $w \cdot v_o \leq v$, the number of $v(r)$ with $w \cdot v(r) \leq v$ equals 3, as one sees from the following list of elements $w \cdot v(r)$, with $w \cdot v_0 \leq v$.

$$
\begin{array}{lll}
w = s_2, & w \cdot v_o = v_{13} & : \ v_{12,-},\ v_{23,-},\ v_{14,+},\ v_{34,-}, \\
w = s_3 s_2, & w \cdot v_o = v_{14} & : \ v_{12,+},\ v_{24,-},\ v_{13,+},\ v_{34,-}, \\
w = s_1 s_2, & w \cdot v_o = v_{23} & : \ v_{12,-},\ v_{13,-},\ v_{24,+},\ v_{34,+}, \\
w = s_1 s_3 s_2, & w \cdot v_o = v_{24} & : \ v_{12,+},\ v_{14,-},\ v_{23,+},\ v_{34,+}
\end{array}
$$

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Program

List of participants

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