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THE BRUHAT ORDER ON SYMMETRIC VARIETIES

To Jacques Tits on the occasion of his sixtieth birthday

ABSTRACT. Let G be a connected reductive linear algebraic group over an algebraically closed field of characteristic not 2. Let θ be an automorphism of order 2 of the algebraic group G. Denote by K the fixed point group of θ and by B a Borel group of G.

It is known that the number of double cosets BgK is finite. This paper gives a combinatorial description of the inclusion relations between the Zariski-closures of such double cosets. The description can be viewed as a generalization of Chevalley's description of the inclusion relations between the closures of double cosets BgB, which uses the Bruhat order of the corresponding Weyl group.

0. INTRODUCTION

If an algebraic group B acts on an algebraic variety X, then the set V of B-orbits on X has a natural partial order: if $v, v' \in V$, then $v' \leqslant v$ if v' is contained in the Zariski closure of V. In particular, if the set of orbits is finite then the ordered set V is a finite combinatorial object associated to the B-action.

An instance of such a partially ordered set is quite familiar from the basic theory of linear algebraic groups. Let G be a connected and reductive linear algebraic group and let B be a Borel subgroup of G. Then B, acting by left translations, has a finite number of orbits on the flag variety $X = G/B$. Let T be a maximal torus of G contained in B, let N be the normalizer of T in G and let $W = N/T$ be the corresponding Weyl group. The Borel subgroup B defines a set S of involutorial generators of W such that (W, S) is a Coxeter group (in the sense of [3]). Bruhat's lemma says that the double cosets BgB are represented by the elements of N, which implies that there is a bijection of the set of orbits V onto the Weyl group W. Moreover, the order on V corresponds to the combinatorially defined Bruhat order on W. The latter order is defined as follows: Let $w \in W$ and take a shortest expression $w = s_1 \ldots s_p$ of w as a product of elements of S. Then $w' \leqslant w$ if and only if we can write $w' = s_{i_1} \ldots s_{i_q}$, with $1 \leqslant i_1 < \cdots < i_q \leqslant p$. This combinatorial description of the geometrically defined order on V goes back to Chevalley. The first published proof seems to be in Borel and Tits [2]. The combinatorial Bruhat order exists on any Coxeter group (W, S). It is studied in [6]. That paper contains several characterizations of the Bruhat order.

In this paper, we study another situation where a Borel subgroup of a

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connected, reductive linear algebraic group G operates with finitely many orbits. Assume that G is defined over an algebraically closed field F of characteristic $\neq 2$ and let θ be an automorphism of G of order 2. We assume that B and T (as above) are θ-stable and let K denote the fixed point subgroup of θ. One knows that K is a (not necessarily connected) reductive group. We denote by X the quotient variety G/K and call it the symmetric variety defined by (G, θ). Then B, acting by left translations, has finitely many orbits in X. We study here the ordered set V of orbits. We call the order on V defined above the *Bruhat order on* V. The example above of B-orbits on G/B is a special case (see Section 10).

Rather than working with B-orbits on G/K, it is sometimes more convenient to work with either K-orbits on G/B, or with double cosets BxK, which are orbits for the obvious action of $B \times K$ on G. It is clear that there are canonical closure-preserving bijections between these three kinds of orbits. We shall usually work with the $(B \times K)$-orbits.

If the base field is $\mathbb{C}$, we can view G as the complexification of a reductive real Lie group $\mathbf{G}$ such that θ is the complexification of a Cartan involution of $\mathbf{G}$ (so that $\mathbf{G}$ is non-compact). Now G/K is the complexification of the symmetric space defined by $\mathbf{G}$ and the Cartan involution. For the finiteness of V in this situation see Matsuki [12]. The K-action on G/B here appears in connection with the (infinite-dimensional) representation theory of the Lie group $\mathbf{G}$. In fact, the classification of Harish–Chandra modules for $\mathbf{G}$ is intimately related to the geometry of K-orbits on G/B (as one can see from [19]).

We now sketch the contents of this paper. Our discussion of the set V of orbits is based on another description of V which was given in [15]. This description (also recalled in Section 1) goes as follows: Let the notation be as above and let $\mathscr{V} = \{x \in G \mid x\theta(x)^{-1} \in N\}$. Then T (resp. K) acts on $\mathscr{V}$ by left (resp. right) translations and we can identify V with the orbit set $T\backslash\mathscr{V}/K$ (which is finite).

The map $\mathscr{V} \to W$ which sends $x \in \mathscr{V}$ to the image in W of $x\theta(x)^{-1}$ induces a map $\varphi: V \to W$ whose image lies in the set $\mathscr{I} = \{w \in W \mid \theta(w) = w^{-1}\}$ of twisted involutions of W. This map φ plays an important role in our analysis of the Bruhat order. In [15] some properties of twisted involutions were developed. Section 3 of this paper gives some further properties.

If $x \in \mathscr{V}$, then $T_1 = x^{-1}Tx$ is a θ-stable maximal torus of G and any such torus is of this form. The K-conjugacy class of T_1 is uniquely determined by the image of x in V, so that we obtain a map of V onto the set Σ of K-conjugacy classes of θ-stable maximal tori.

It is clear that N acts on $\mathscr{V}$ by left multiplication, whence an action of W on V. This action is discussed in some detail in Section 2, where it is also shown that the set $W\backslash V$ of W-orbits on V is in bijective correspondence with the set Σ. We discuss the classification of K-classes of θ-stable maximal tori in Section 9.

If $x \in \mathscr{V}$ and if $v \in V$ denotes the image of x, we let $\mathcal{O}_v$ denote the corresponding $(B \times K)$-orbit BxK. To study the orbit closures, we use a familiar argument involving minimal parabolic subgroups. If $s \in S$, let P_s be the corresponding (minimal) parabolic subgroup of semisimple rank 1 containing B. In Section 4 we consider the product map $P_s \times \mathcal{O}_v \to G$ and describe the set of orbits $\mathcal{O}_{v'}$ in its image. Arguments of this sort are quite familiar. For example, they play an essential role in Tits' theory of (B, N) pairs and in the study of the usual Bruhat order in [2]. To study the image of the product map above, we have to distinguish several cases, according to the 'position of s relative to v'. In analogy with the case of real semisimple groups, we can speak of s as being real, complex, compact imaginary, or non-compact imaginary with respect to v. These matters were discussed in [15] and the relevant facts are recalled in Section 1. The discussion of Section 4 is really a collection of results already given in [11]. For the convenience of the reader, we have given complete proofs.

In order to formulate the properties of the Bruhat order on V, we have found it convenient to use the monoid $M = M(W)$ generated by elements $m(s)$ ($s \in S$) with defining relations $m(s)^2 = m(s)$ and the 'braid relations' $m(s)m(t)m(s)\ldots = m(t)m(s)m(t)\ldots$ $(s, t \in S, s \neq t)$ as in (W, S). It is introduced in Section 3. If $s \in S$ and $v \in V$, we define $m(s) \cdot v \in V$ by $P_s \operatorname{cl}(\mathcal{O}_v) = \operatorname{cl}(\mathcal{O}_{m(s)\cdot v})$, where 'cl' denotes Zariski closure. This determines an action of M on V. Our set V has a length function l defined by $l(v) = \dim \mathcal{O}_v - d$, where d is the common dimension of the closed $(B \times K)$-orbits. The action of M on V has the following properties: (i) $v \leqslant m(s) \cdot v$; (ii) if $v' \leqslant v$, then $m(s) \cdot v' \leqslant m(s) \cdot v$; (iii) l is strictly monotonic for the Bruhat order; and (iv) if $v \neq m(s) \cdot v$, then $l(m(s) \cdot v) = l(v) + 1$. In Sections 5 and 6, we consider partially ordered sets X with an M-action and a length function such that (i)–(iv) above hold. We have the notion of a *reduced decomposition* of an element of X; this is a generalization of a reduced decomposition of an element of W. We show the equivalence of a number of properties of such an M-set; these properties are generalizations of familiar properties of the Bruhat order on W, such as the exchange property and the property $Z(s, w, w')$ of [6]. As a consequence, we obtain a combinatorial characterization of the Bruhat order on V. It is the weakest partial order on V which satisfies conditions (i)–(iv) above. We give several explicit combinatorial descriptions of this weakest order.

In Section 3, devoted to twisted involutions, we also introduce a length function and an action of $M(W)$ on the set $\mathscr{I}$ of twisted involutions. The weakest partial order on $\mathscr{I}$ which has properties (i)–(iv) above is the Bruhat order on $\mathscr{I}$. (This is usually not the order on $\mathscr{I}$ induced by the Bruhat order on W.) We discuss the Bruhat order on $\mathscr{I}$ in some detail in Section 8. The map $\varphi: V \to \mathscr{I}$ defined above is compatible with the respective Bruhat orders. If θ acts trivially on T, then $\mathscr{I}$ is just the set of involutions in the Weyl group W. In this case, we obtain a combinatorial Bruhat order on the set of involutions in W, which perhaps merits some further study (for example, in particular cases such as $W = S_n$). Let $\theta_1 = -w_0\theta$, where w_0 is the longest element of W and let $\mathscr{I}_1$ be the set of twisted involutions for θ_1. Then the map $a \to aw_0$ is a bijection of $\mathscr{I}$ onto $\mathscr{I}_1$ which reverses the Bruhat orders. Thus we have a sort of 'duality' for twisted involutions.

Another problem involving $\mathscr{I}$ is that of the description of the image of the map $\varphi: V \to \mathscr{I}$. This problem is relevant for the description of K-conjugacy classes of θ-stable maximal tori (and for the description of conjugacy classes of Cartan subalgebras in a real semisimple Lie algebra). We give two descriptions of image (φ). The first description, given in Theorem 7.13, uses the 'weak order' on V, which is the order generated by the order relations $v \leqslant m(s)\cdot v$ $(v \in V, s \in S)$. It states that $a \in \mathrm{image}(\varphi)$ if and only if a is dominated in the weak order by $a_{\max} = \varphi(v_{\max})$, where $v_{\max}$ is the maximal element of $V(a_{\max}$ can be read off from the Satake diagram of $(G, \theta))$. The other description, given in Proposition 7.16, is in terms of eigenspaces associated to a and $a_{\max}$ and is more in the spirit of results about the classification of Cartan subalgebras of a real semisimple Lie algebra (see [10]).

In Section 10, we have worked out some concrete examples of the Bruhat order on V. We mention in particular the following example. Let $G = \mathrm{GL}_n(F)$ and define the involution θ by $\theta(g) = {}^tg^{-1}$. Then the map $\varphi: V \to \mathscr{I}$ is a bijection which preserves the Bruhat order. Using the duality for twisted involutions mentioned above, we obtain an order-reversing bijection of V onto the set $\mathscr{I}_n$ of involutions in the symmetric group S_n (here $\mathscr{I}_n$ is given the Bruhat order for involutions discussed above).

Let G and B be as above. A G-variety X is *spherical* if B has finitely many orbits on X. Some of our results (see, for example, 4.2) carry over to general spherical varieties, but we have not made a systematic study of the order relation on B-orbits in the general case. We hope to come back to this question.

The ideas from this paper can be used to give a completely geometric construction of the Hecke algebra representations introduced in [11] (in loc. cit. results from representation theory are used). This will be dealt with in a paper by one of us (T.A.S.).

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1. Preliminaries

All algebraic groups and algebraic varieties are taken over an algebraically closed base field F of characteristic $\neq 2$. Throughout the paper G will denote a connected reductive algebraic group and θ is an automorphism of G of order 2. The fixed point subgroup K of θ is a reductive group. If G is semisimple and simply connected, then K is connected, but in general K is not necessarily connected.

1.1. Let T be a θ-stable maximal torus of G with normalizer $N(T)$. We let $W = W(T) = N(T)/T$ be the Weyl group of (G, T) and $\Phi = \Phi(T)$ the corresponding root system. The involution θ acts on W and Φ.

For $\alpha \in \Phi$, let x_α denote a one-parameter subgroup of G associated to α. This is an isomorphism of the additive group F onto a closed unipotent subgroup U_α of G, which is normalized by T, such that

$$tx_\alpha(\xi)t^{-1} = x_\alpha(\alpha(t)\xi) \quad (t \in T, \xi \in F).$$

The subgroup G_α of G generated by U_α and $U_{-\alpha}$ is semisimple of rank 1; it is isomorphic to either $\mathrm{SL}_2(F)$ or $\mathrm{PGL}_2(F)$.

We choose the isomorphisms x_α such that, for every $\alpha \in \Phi$, $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ lies in $N(T)$. Then $s_\alpha = n_\alpha T$ is the reflection in W defined by α. Moreover $n_\alpha^2 = \alpha^\vee(-1) = t_\alpha$ and $n_{-\alpha} = t_\alpha n_\alpha = n_\alpha t_\alpha$, where $\alpha^\vee$ denotes the coroot (or multiplicative one-parameter subgroup) defined by α. There exists $c_\alpha \in F^*$ such that $\theta(x_\alpha(\xi)) = x_{\theta(\alpha)}(c_\alpha\xi)$ $(\xi \in F)$. There are several cases.

(a) $\theta(\alpha) \neq \pm\alpha$. Then α is *complex* (relative to θ). We may then assume that $c_\alpha = c_{\theta(\alpha)} = 1$ and $\theta(n_\alpha) = n_{\theta(\alpha)}$, $\theta(n_{-\alpha}) = n_{-\theta(\alpha)}$.

(b) $\theta(\alpha) = -\alpha$. Then α is *real*. We may assume that $c_\alpha = c_{-\alpha} = 1$ and $\theta(n_\alpha) = n_{-\alpha}$.

If $\theta(\alpha) = \alpha$, then α is *imaginary*. There are two subcases:

(c) $c_\alpha = 1$. Then α is *compact imaginary*. Now $\theta(n_\alpha) = n_\alpha$.

(d) $c_\alpha = -1$. Then α is *non-compact imaginary*. We have $\theta(n_\alpha) = n_{-\alpha}$.

We refer to [15, 2.4] for more details.

If α is either real or imaginary, then G_α is θ-stable. If α is compact imaginary, then G_α is contained in K. If α is real or non-compact imaginary, then $(G_\alpha \cap K)^0$, the identity component of $G_\alpha \cap K$, is a maximal torus of G_α.

1.2. We now assume that the θ-stable maximal torus T is contained in a θ-stable Borel subgroup B, so that (B, T) is a *standard pair* in the sense of [15, 2.3]. We let $W = W(T)$. The Borel subgroup B defines a system of positive roots Φ^+ in Φ and a basis Δ of Φ, both stabilized by θ. Let $S = \{s_\alpha \mid \alpha \in \Delta\}$, so that (W, S) is a Coxeter group. The corresponding length function is denoted by l. Let U denote the unipotent radical of B. The group U is generated by $\{U_\alpha \mid \alpha \in \Phi^+\}$. It is a θ-stable subgroup.

We define a morphism $\tau: G \to G$ by $\tau(x) = x\theta(x^{-1})$ $(x \in G)$. Then $\tau(G)$ is a closed subvariety of G and $\tau(x) = \tau(y)$ if and only if $k = x^{-1}y \in K$. The morphism τ induces an isomorphism of the coset space G/K onto $\tau(G)$ [15, 2.2]. We note that $\theta(\tau(x)) = \tau(x)^{-1}$ for $x \in G$.

We set $\mathscr{V} = \{x \in G \mid \tau(x) \in N(T)\}$. Note that $\tau(\mathscr{V}) \subset \{n \in N(T) \mid \theta(n) = n^{-1}\}$. The group $T \times K$ acts on $\mathscr{V}$ by $(t, k) \cdot x = txk^{-1}$. Let $V = V(G)$ be the set of $(T \times K)$-orbits on $\mathscr{V}$. If $v \in V$, we let $x(v) \in \mathscr{V}$ be a representative of the orbit v in $\mathscr{V}$. We also have an action of the group $B \times K$ on G given by $(b, k) \cdot x = bxk^{-1}$ and the orbits are the (B, K) double cosets BxK. The inclusion map $\mathscr{V} \to G$ maps $(T \times K)$-orbits into $(B \times K)$-orbits. The following result is proved in [15, §4]:

1.3 THEOREM. (i) *V is finite.* (ii) *The inclusion map $\mathscr{V} \to G$ induces a bijection of the set V of $(T \times K)$-orbits on $\mathscr{V}$ onto the set of $(B \times K)$-orbits on G. Thus G is the disjoint union of the double cosets $Bx(v)K$ $(v \in V)$.*

1.4 REMARKS. (a) Define a (left) action of G on (the set) G by: $g * x = gx\theta(g)^{-1}(g, x \in G)$. We call this the *twisted action* of G on G. For $x \in G$, let $G * x = \{g * x \mid g \in G\}$ denote the twisted orbit of x. Thus $\tau(g) = g * 1$ and $\tau(G) = G * 1$, the twisted orbit of $1 \in G$. It is clear that $\tau(\mathscr{V}) = \tau(G) \cap N(T)$. It is immediate that $\tau(G) \cap N(T)$ is stable under the twisted action of $N(T)$ and that the restriction $\tau|_{\mathscr{V}}: \mathscr{V} \to \tau(G) \cap N(T)$ maps the set V of $(T \times K)$-orbits on $\mathscr{V}$ bijectively onto the set of twisted T-orbits on $\tau(G) \cap N(T)$. In working with explicit examples, this is usually the easiest model to use in order to get an explicit description of the set V.

(b) Let $\mathscr{B}$ denote the variety of Borel subgroups of G and let $\mathscr{S}$ be the set of all pairs (B_1, T_1) where $B_1 \in \mathscr{B}$ and $T_1 \subset B_1$ is a θ-stable maximal torus. Then K acts on $\mathscr{B}$ and $\mathscr{S}$ by conjugation. Define $\rho: \mathscr{S} \to \mathscr{B}$ by $\rho(B_1, T_1) = B_1$. It is an easy consequence of 1.3 (see [15, Cor. 4.4]) that ρ induces a bijection $\mathscr{S}/K \to \mathscr{B}/K$ of the sets of K-orbits. In particular, if $B_1 \in \mathscr{B}$ and T_1 and T_2 are θ-stable maximal tori contained in B_1, then there exists $k \in B_1 \cap K$ such that $kT_1k^{-1} = T_2$.

1.5 THE BRUHAT ORDER. If $v \in V$, we denote by $\mathscr{O}_v$ the double coset $Bx(v)K$ which corresponds to v. Clearly each double coset $\mathscr{O}_v$ is a smooth subvariety of G and the closure $\mathrm{cl}(\mathscr{O}_v)$ is a union of double cosets. The *Bruhat order* on V is the (partial) order on V defined by the closure relations on the double cosets $\mathscr{O}_v = Bx(v)K$. Thus $v' \leqslant v$ if and only if $\mathscr{O}_{v'} \subset \mathrm{cl}(\mathscr{O}_v)$. In Section 10 we show that the Bruhat order on the Weyl group W is a special case of the Bruhat order on V. In [15] some properties of the Bruhat order on V were established. In this paper, we shall make a more detailed study of this order.

Let $p: G \to G/K$ and $\pi: G \to G/B$ denote the canonical projections. Then p (resp. π) determines a bijection of the set $\{\mathscr{O}_v \mid v \in V\}$ of $(B \times K)$-orbits on G onto the set of B-orbits on G/K (resp. K-orbits on G/B). For $v \in V$, we let $\mathscr{K}_v = p(\mathscr{O}_v)$ and $\mathscr{B}_v = \pi(\mathscr{O}_v)$. Then G/K is the disjoint union of the orbits $\mathscr{K}_v$ $(v \in V)$ and G/B is the disjoint union of the orbits $\mathscr{B}_v$ $(v \in V)$. It is clear that the following conditions on $v, v' \in V$ are equivalent: (i) $v' \leqslant v$; (ii) $\mathscr{O}_{v'} \subset \mathrm{cl}(\mathscr{O}_v)$; (iii) $\mathscr{K}_{v'} \subset \mathrm{cl}(\mathscr{K}_v)$; and (iv) $\mathscr{B}_{v'} \subset \mathrm{cl}(\mathscr{B}_v)$. If K is not connected, then the orbits $\mathscr{O}_v$ and $\mathscr{B}_v$ are not necessarily irreducible varieties. The orbits $\mathscr{K}_v$ are always irreducible varieties.

1.6. If $v \in V$, then $T_1 = x(v)^{-1}Tx(v)$ is a θ-stable maximal torus and it is easily seen that any θ-stable maximal torus is conjugate to one of this form. Fix $v \in V$, let $x = x(v)$ and let $T_1 = x^{-1}Tx$. Then the inner automorphism $\mathrm{Int}(x^{-1})$ defines an isomorphism f_v of Φ onto the root system $\Phi_1 = \Phi(T_1)$. If $\alpha \in \Phi$ and if x_α is as in 1.1, then $\mathrm{Int}(x^{-1}) \circ x_\alpha$ is a one-parameter subgroup associated to the root $\alpha_1 = f_v(\alpha)$ of Φ_1. We say that α is complex, real,... for v if $\alpha_1 = f_v(\alpha)$ is complex, real,... in the sense of 1.1 for T_1. If α is complex, real,... for v, then we also say that the reflection $s_\alpha \in W$ is complex, real,... for v.

We recall from [15, §3] that an element $a \in W$ is a *twisted involution* if $\theta(a) = a^{-1}$. Let $\mathscr{I} = \mathscr{I}(W, \theta)$ be the set of twisted involutions in W. If $v \in V$, then $\varphi(v) = \tau(x(v))T \in W$ is a twisted involution. (The element $\varphi(v) \in \mathscr{I}$ is independent of the choice of representative $x(v) \in \mathscr{V}$ for v.) Thus we have defined a map $\varphi: V \to \mathscr{I}$. This map φ plays an important role in our study of the Bruhat order on V.

We let $\mathscr{R} = \{s_\alpha \mid \alpha \in \Phi\}$ denote the set of reflections in W.

Let $v \in V$ and let $a = \varphi(v)$. Then $\alpha \in \Phi$ is complex (for v) if $a\theta(\alpha) \neq \pm\alpha$, real if $a\theta(\alpha) = -\alpha$, and imaginary if $a\theta(\alpha) = \alpha$. We let $C(v)$ be the set of reflections $s = s_\alpha$ in $\mathscr{R}$ which are complex for v and let $S(v)'$ (resp. $S(v)''$) denote the set of $s \in S \cap C(v)$ such that $l(sa\theta(s)) = l(a) - 2$ (resp. $l(sa\theta(s)) = l(a) + 2$). We let $R(v)$ (resp. $I(v)_n$, $I(v)_c$) be the set of $s \in \mathscr{R}$ such that s is real (resp. non-compact imaginary, compact) for v. We set $I(v)=I(v)_n \cup I(v)_c$. We set $S(v)_r = S \cap R(v)$, $S(v)_n = S \cap I(v)_n$, $S(v)_c = S \cap I(v)_c$ and $S(v)_i = S \cap I(v)$. (These notations are slightly different from those in [15, 4.7].)

1.7 LEMMA. *Let u, $v \in V$ be such that $\varphi(u) = \varphi(v)$. Then we have*: (i) $S(u)' = S(v)'$; (ii) $S(u)'' = S(v)''$; (iii) $R(u) = R(v)$; *and* (iv) $I(u) = I(v)$.

This follows immediately from the remarks above. If u, v are as in Lemma 1.7, it is not necessarily the case that $I(u)_c = I(v)_c$ or that $I(u)_n = I(v)_n$.

1.8 REMARK. The definition of the map $\varphi\colon V \to \mathscr{I}$ depends on a number of choices. We shall show that φ is essentially independent of these choices. Let $\mathscr{W}$ be the *canonical Weyl group* of G (see [4]). The underlying set of $\mathscr{W}$ is the set of G-orbits on $\mathscr{B} \times \mathscr{B}$. Let $\sigma\colon \mathscr{B} \times \mathscr{B} \to \mathscr{W}$ denote the canonical map. We define a map $\phi\colon \mathscr{B} \to \mathscr{W}$ by $\phi(B') = \sigma(B', \theta(B'))$. The map ϕ is constant on K orbits and hence induces a map $\phi'\colon \mathscr{B}/K \to \mathscr{W}$ of the set of orbits $\mathscr{B}/K$. Associated to the pair (B, T), there is a canonical isomorphism $\eta = \eta_{B,T}$: $W(T) \to \mathscr{W}$ defined as follows: if $w \in W(T)$, then $\eta(w) = \sigma(B, {}^wB)$. If $v \in V$ and if $\pi(\mathscr{O}_v)$ is the corresponding K-orbit on $\mathscr{B}$, then a straightforward argument shows that $\phi'(\pi(\mathscr{O}_v)) = \eta(\varphi(v))$. This shows that the map $\varphi\colon W \to \mathscr{I}$ is canonical.

1.9 REDUCTIONS. Let $V^0 = V^0(G)$ denote the set of $(T \times K^0)$-orbits on $\mathscr{V}$. It follows from 1.3 that V^0 is a finite set which parametrizes the double cosets BxK^0. The finite group K/K^0 acts on V^0 in the obvious way. It is known that K/K^0 is an elementary abelian 2-group. See, e.g., [20, Prop. 7]. (The result in loc. cit. is stated for characteristic zero, but exactly the same proof goes through in arbitrary characteristic $\neq 2$.)

1.10 LEMMA. (i) *V is the set of K/K^0 orbits on V^0.* (ii) *Let G' be the derived group of G. The inclusion $G' \to G$ induces a bijection $V^0(G') \to V^0(G)$.* (iii) *If ι: $\tilde{G} \to G$ is a central isogeny of algebraic groups with involution, then ι induces a bijection $V^0(\tilde{G}) \to V^0(G)$.*

We omit the proofs, which are straightforward.

1.11. Now let G be as before and let $\iota\colon \tilde{G} \to G'$ be a simply connected covering of its derived group. The automorphism of G' induced by θ can be lifted to an involution of $\tilde{G}$ [17, 9.16]. Moreover, since $\tilde{G}$ is semisimple and simply connected, the fixed point subgroup of the lifting of θ to $\tilde{G}$ is connected, by a theorem of Steinberg [17, 8.2], so that $V^0(\tilde{G}) = V(\tilde{G})$. We conclude that $V(G)$ can be canonically identified with the set of orbits of the elementary abelian 2-group K/K^0 acting on $V(\tilde{G})$.

The preceding remarks give a reduction, in a sense, of the study of the

2. The W-action on V

We fix a standard pair (B, T) and continue with the notation of Section 1. In particular, the map $\varphi\colon W \to \mathscr{I}$ is defined as in 1.4. It will be convenient to define a (left) action of W on (the set) W, the *twisted action*, as follows: if w, $w_1 \in W$, then $w * w_1 = ww_1\theta(w)^{-1}$. If $w_1 \in W$, then $W * w_1 = \{w * w_1 \mid w \in W\}$ is the *twisted W-orbit* of w_1. An easy argument shows that the set $\mathscr{I}$ of twisted involutions is stable under the twisted action, so that we get a twisted action of W on $\mathscr{I}$.

Let $v \in V$ and let $x = x(v)$. If $n \in N(T)$, then $nx \in \mathscr{V}$ and its image in V depends only on the image of n in W. We thus obtain a (left) action of W on V, denoted by $(w, v) \to w \cdot v$ ($w \in W$, $v \in V$).

2.1 LEMMA. *Let $w \in W$ and $v \in V$. Then $\varphi(w \cdot v) = w * \varphi(v)$.*

The proof is trivial.

Thus we see that φ is equivariant with respect to the action of W on V and the twisted action of W on $\mathscr{I}$. In particular the image of φ is a union of twisted orbits.

We recall that a torus T_0 in G is *θ-split* if $\theta(t) = t^{-1}$ for every $t \in T_0$. If T_0 is a θ-split torus, it is clear that $\tau(T_0) = T_0$. The following result is proved in [13, 5.1]:

2.2 PROPOSITION. *Let t be a semisimple element of $\tau(G)$. Then t is contained in a θ-split torus.*

We shall establish a connection between W-orbits in V and K-conjugacy classes of θ-stable maximal tori using Proposition 2.3 below.

2.3 PROPOSITION. *Let T' be a θ-stable maximal torus of G and let $x \in G$ be such that $\tau(x) \in T'$. Then there exists $y \in N(T')$ such that $\tau(x) = \tau(y)$.*

For the proof, we need the following lemma. Let Z denote the center of G.

2.4 LEMMA. *Assume that there exists $t' \in T'$ such that $t'\tau(T') \subset Z$. Then T' is contained in a θ-stable Borel subgroup.*

Proof. By passing to the quotient group G/Z, we may assume for the proof that Z is trivial. The assumption then implies that θ is trivial on T', so that T' is a maximal torus of K; hence rank K = rank G. It is known that $K^0 \cap T$ is a maximal torus of K^0 [13, 5.1], so that $T = K^0 \cap T$. Hence T' and $T = K^0 \cap T$ are conjugate in K^0. Thus there exists $k \in K^0$ such that $kTk^{-1} = T'$. Thus kBk^{-1} is a θ-stable Borel subgroup containing T'.

Proof of Proposition 2.3. First we treat the case in which T' is contained in a θ-stable Borel subgroup B'. Let $U' = R_u(B')$. Write the element x of the proposition in the form $x = uyu'$, with u, u' in U' and y in $N(T')$, according to Bruhat's lemma. Put $\tau(x) = t$. Then

$$x = uyu' = t\theta(x) = (t\theta(u)t^{-1})(t\theta(y))\theta(u').$$

The uniqueness part of Bruhat's lemma implies that $y = t\theta(y)$, whence $\tau(y) = t = \tau(x)$.

Next assume that T' is not contained in a θ-stable Borel subgroup. Put $\tau(x) = t \in T'$. By the previous lemma, there exists $t_1 \in T'$ such that $t\tau(t_1)$ is non-central. Replacing x by $t_1^{-1}x$ (which is permissible), we may assume that t is non-central. Let H be the connected centralizer of t; H is a proper, connected reductive subgroup of G. Since $\theta(t) = t^{-1}$, we see that H is θ-stable and it is clear that $T' \subset H$. By 2.2, $t \in \tau(H)$. It now follows by induction that there exists $y \in N(T') \cap H$ with $\tau(y) = t = \tau(x)$.

We now return to the action of W on V.

2.5 PROPOSITION. *Let $v, v' \in V$. If $\varphi(v) = \varphi(v')$, then v and v' lie in the same W-orbit.*

Proof. Let $x = x(v)$ and $x' = x(v')$. Then $\tau(x') = t\tau(x)$ for some $t \in T$. Put $T' = x^{-1}Tx$. This is a θ-stable maximal torus. If $y = x^{-1}x'$, then $\tau(y) = x^{-1}tx \in T'$. Applying 2.3, we conclude that there exists $n \in N(T)$ such that $\tau(y) = \tau(x^{-1}nx)$, which is equivalent to $\tau(x') = \tau(nx)$. But this implies that there exists $k \in K$ such that $x'k = nx$. If we put $w = nT$, this implies that $v' = w \cdot v$, which proves the proposition.

2.6. Let $\mathcal{T}$ be the variety of maximal tori of G. This is an affine variety, isomorphic to $G/N(T)$, on which θ acts. Let $\mathcal{T}^\theta$ be the fixed point set of θ, i.e. the set of θ-stable maximal tori. It is an affine variety on which K acts by conjugation. If $v \in V$, then $x(v)^{-1}Tx(v) \in \mathcal{T}^\theta$. This determines a map of V to the orbit set $\mathcal{T}^\theta/K$; it is easy to check that this map is independent of the choice of representative $x(v)$ for v and is constant on W-orbits. Thus we get a map of orbit sets γ: $V/W \to \mathcal{T}^\theta/K$. Since φ: $V \to \mathcal{I}$ is W-equivariant (with respect to the twisted action of W on $\mathcal{I}$), we also get a map ϕ: $V/W \to \mathcal{I}/W$.

2.7 PROPOSITION. (i) *γ: $V/W \to \mathcal{T}^\theta/K$ is bijective.* (ii) *ϕ: $V/W \to \mathcal{I}/W$ is injective.* (iii) *There is a bijection of* image$(\phi)/W$ *onto $\mathcal{T}^\theta/K$.*

Proof. (i) follows easily from the definitions and (ii) is a consequence of 2.5. The assertion of (iii) is a consequence of (i) and (ii).

If T' is a θ-stable maximal torus, we let $W_K(T')$ denote the image of $N_K(T') = N(T') \cap K$ in $W(T')$. Let $v \in V$, let $x = x(v)$ and let $T' = x^{-1}Tx$; clearly T' is θ-stable. Let f_v: $W(T) \to W(T')$ be the automorphism determined by $\mathrm{Int}(x^{-1})$. We set $W_K(v) = f_v^{-1}(W_K(T'))$.

2.8 PROPOSITION. *Let $v \in V$.* (i) *The stabilizer $W_v = \{w \in W \mid w \cdot v = v\}$ is equal to $W_K(v)$.* (ii) *Let $w \in W$, let $a = \varphi(v)$ and let $W_a = \{w \in W \mid w * a = a\}$. Then $\varphi(w \cdot v) = \varphi(v)$ if and only if $w \in W_a$.*

Proof. (i) It is clear that $W_K(v) \subset W_v$. Let $B' = x(v)^{-1}Bx(v)$ and let $T' = x(v)^{-1}Tx(v)$. In order to show that $W_v \subset W_K(v)$, it suffices to show that if $n \in N(T')$ and if $nB'n^{-1}$ is K-conjugate to B', then $nT' \in W_K(T')$. This can be shown using 1.4(b). We omit the details. (ii) The proof follows from 2.1.

The following result has been observed by several mathematicians:

2.9 COROLLARY. *The number of elements in V is given by*

$$\mathrm{Card}(V) = \sum \mathrm{Card}\,(W(T')/W_K(T'),$$

where the sum is taken over a set of representatives T' for $\mathcal{T}^\theta/K$.

2.10 REMARKS. See [9] or Section 9 for the classification of K-conjugacy classes of θ-stable maximal tori. The classification is essentially the same as the classification of conjugacy classes of Cartan subalgebras of a real semisimple Lie algebra given by Kostant [10]. If $T' \in \mathcal{T}^\theta$, it seems to be hard to give a nice description of $W_K(T')$ in terms of combinatorial data involving only Φ, θ, and the (Satake) diagram associated to (G, θ). (See [16] or [8, p. 532] for these diagrams. The diagrams in [16] contain slightly more information than the usual Satake diagrams.)

2.11. In order to tie up the results of this section with those of the next section, we need to introduce some more notation. Let $X^*(T)$ (resp. $X_*(T)$) be the free $\mathbb{Z}$-module of characters (resp. multiplicative one-parameter subgroups) of T and let $E = E(T) = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Then W and θ act on the real vector space E. We give E a positive definite inner product invariant under W and θ. We identify the root system Φ with a subset of E by means of the inner product and the duality between $X^*(T)$ and $X_*(T)$. Thus Φ is a reduced root system in the real Euclidean space E.

3. More on twisted involutions

A number of properties of twisted involutions were developed in [15, §3]. In this section, we develop further properties which will be used in the sequel. Many of the results of this section follow directly from the definitions and from results of loc. cit., so we shall often state results without giving proofs.

3.1. We follow the set-up of Section 3 of loc. cit. Thus Φ is a reduced root system in a real Euclidean space E, $W = W(\Phi)$ is the Weyl group, Φ^+ is a set of positive roots for Φ and Δ is the corresponding basis of Φ. (We do not assume that Φ spans E.) We let θ be an orthogonal linear transformation of E such that $\theta^2 = \mathrm{Id}_E$ and such that Φ, Φ^+ and Δ are θ-stable. We let $\mathscr{R} = \{s_\alpha \mid \alpha \in \Phi\}$ be the set of reflections in W and let $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of simple reflections. Let $\mathscr{I}$ be the set of twisted involutions in W and let $\mathscr{J} = \{w \in W \mid w^2 = 1\}$ be the set of involutions in W. The group W acts on (the set) W by twisted conjugation: $w * x = wx\theta(w)^{-1} (w, x \in W)$. It is clear that $\mathscr{I}$ is stable under this W-action.

If $r \in \mathscr{J}$, we define a map $\eta(r)\colon \mathscr{I} \to \mathscr{I}$ as follows: Let $a \in \mathscr{I}$. If $r * a = a$, we set $\eta(r)(a) = ra$; if $r * a \neq a$, we set $\eta(r)(a) = r * a$. Clearly the composition $\eta(r) \circ \eta(r)$ is the identity map of $\mathscr{I}$. Thus $\eta(r)$ is a bijection of $\mathscr{I}$ of period 2 and, if $r \neq 1$, then $\eta(r)$ does not have any fixed points. We shall often denote $\eta(r)(a)$ by $r \circ a$ (thus $r \circ a$ does *not* denote the product of r and a as elements of W). If $r_1, \dots, r_k$ are elements of $\mathscr{J}$ and $a \in \mathscr{I}$, then we frequently write $r_1 \circ r_2 \circ \cdots \circ r_k \circ a$ for $r_1 \circ (r_2 \circ (\cdots \circ (r_k \circ a) \cdots))$.

We carry over to twisted involutions the notations and terminology of Section 1 regarding real, complex and imaginary roots and reflections. Thus, for example, if $a \in \mathscr{I}$, then

$$C(a) = \{s_\alpha \in \mathscr{R} \mid a\theta(\alpha) \neq \pm\alpha\}$$

and

$$S(a)'' = \{s \in C(a) \cap S \mid l(sa\theta(s)) = l(a) + 2\}.$$

We define the sets $I(a)$, $R(a)$, $S(a)_i$, $S(a)_r$, and $S(a)'$ in a similar fashion.

3.2 LEMMA. *Let $r \in \mathscr{R}$ and let $a \in \mathscr{I}$. (i) If $r \in R(a) \cup I(a)$, then $r \circ a = ra$. If $r \in C(a)$, then $r \circ a = r * a$. (ii) Let $s \in S$. If $sa < a$ (resp. $sa > a$), then $s \circ a < a$ (resp. $s \circ a > a$). If $s \circ a = sa$, then $l(s \circ a) = l(a) \pm 1$. If $s \circ a = s * a$, then $l(s \circ a) = l(a) \pm 2$.*

Proof. The last statement of (ii) follows from [15, 3.2]. The other proofs are easy.

If $\mathbf{s} = (s_1, \dots, s_k)$ is a sequence in S, we define by induction a sequence $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$ in $\mathscr{I}$ as follows: $a_0 = 1$ and if $i \in [1, k]$, then $a_i = s_i \circ a_{i-1}$. We set $\omega(\mathbf{s}) = a_k$. We say that k is the *length* of the sequence and we write $l(\mathbf{s}) = k$.

3.3 DEFINITION. Let $\mathbf{s} = (s_1, \dots, s_k)$ be a sequence in S and let $\mathbf{a}(\mathbf{s}) = (a_0, \dots, a_k)$. We say that the sequence is an *admissible sequence* if $0 = l(a_0) < l(a_1) < \cdots < l(a_k)$. If $\mathbf{s}$ is an admissible sequence and $a = \omega(\mathbf{s})$, we say that $\mathbf{s}$ is an admissible sequence for a.

We let w_0 denote the longest element of W. Since $\theta(\Phi^+) = \Phi^+$, we see that

w_0 is a twisted involution. If $J \subset \Delta$, let W_J denote the subgroup of W generated by $\{s_\alpha \mid \alpha \in J\}$ and let w_J be the longest element of W_J. We let Φ_J be the root subsystem of Φ spanned by J.

3.4 LEMMA. *For every $a \in \mathscr{I}$, there exists an admissible sequence $\mathbf{s}$ for a.*

Proof. The proof is by induction on $\mathrm{rank}(\Phi) = |\Delta|$. If $\mathrm{rank}(\Phi) = 0$, the result is clear. Assume that $\mathrm{rank}(\Phi) > 0$ and that the result holds for root systems of smaller rank. Let $a \in \mathscr{I}$. By [15, 3.3], there exists $J \subset \Delta$ and a sequence $(s_1, \dots, s_k)$ in S such that: (i) J is θ-stable and $w_J\theta(\alpha) = -\alpha$ for every $\alpha \in J$; and (ii) $a = s_k * \cdots * s_1 * w_J$ and $l(a) = 2k + l(w_J)$. Let E_J be the subspace of E spanned by J and let θ_J denote the restriction of θ to E_J. If $J \neq \Delta$ it follows from the inductive hypothesis, applied to (Φ_J, θ_J), that there exists an admissible sequence $(t_1, \dots, t_p)$ for w_J. Thus $(t_1, \dots, t_p, s_1, \dots, s_k)$ is an admissible sequence for a. If $J = \Delta$, then we must have $a = w_0$. In this case, let $\mathbf{s} = (s_1, \dots, s_k)$ be an admissible sequence in S of maximal length and let $b = \omega(\mathbf{s})$. If $b \neq w_0$, then there exists $s \in S$ such that $sb > b$. It then follows from 3.2 that $(s_1, \dots, s_k, s)$ is an admissible sequence, which gives a contradiction. Thus $b = w_0$ and we are done.

Now let $\mathbf{s} = (s_1, \dots, s_k)$ be an admissible sequence in S and let $\mathbf{a}(\mathbf{s}) = (a_0, a_1, \dots, a_k)$. Let $J_1(\mathbf{s}) = \{i \in [1, k] \mid s_i \circ a_{i-1} = s_i a_{i-1}\}$ and $J_2(\mathbf{s}) = \{i \in [1, k] \mid s_i \circ a_{i-1} = s_i * a_{i-1}\}$; set $\sigma_1(\mathbf{s}) = |J_1(\mathbf{s})|$ and $\sigma_2(\mathbf{s}) = |J_2(\mathbf{s})|$.

3.5 LEMMA. *Let the notation be as above and let $a = a_k = \omega(\mathbf{s})$. Then $l(a) = \sigma_1(\mathbf{s}) + 2\sigma_2(\mathbf{s})$.*

If $c \in \mathrm{End}_{\mathbb{R}}(E)$ and $\lambda \in \mathbb{R}$, we let $E(c, \lambda) = \{x \in E \mid c(x) = \lambda x\}$ denote the λ-eigenspace of c on E. If $a \in \mathscr{I}$ is a twisted involution, we set $E_-(a) = E(a\theta, -1)$ and $E_+(a) = E(a\theta, +1)$.

3.6 LEMMA. *Let $s \in \mathscr{R}$, let $w \in W$ and let $a \in \mathscr{I}$. Then $w \cdot E_-(a) = E_-(w * a)$, so that $\dim E_-(a) = \dim E_-(w * a)$. If $s \in I(a)$ (resp. $s \in R(a)$), then $\dim E_-(sa) = \dim E_-(a) + 1$ (resp. $\dim E_-(sa) = \dim E_-(a) - 1$).*

3.7 LEMMA. *Let $\mathbf{s}$ be an admissible sequence in S and let $a = \omega(\mathbf{s})$. Then $\sigma_1(\mathbf{s}) = \dim E_-(a) - \dim E_-(1)$.*

Thus we see that $\sigma_1(\mathbf{s})$ depends only on $a = \omega(\mathbf{s})$.

3.8 DEFINITION. If $a \in \mathscr{I}$, we define an integer $\lambda(a)$ by $\lambda(a) = \dim E_-(a) - \dim E_-(1)$.

3.9 PROPOSITION. *Let $a \in \mathscr{I}$ and set $L(a) = \frac{1}{2}[l(a) + \lambda(a)]$. Then $L(a)$ is an integer. If $\mathbf{s} = (s_1, \dots, s_k)$ is an admissible sequence with $a = \omega(\mathbf{s})$, then $k = L(a)$.*

Proof. Let $\mathbf{s}$ be an admissible sequence such that $a = \omega(\mathbf{s})$ and let k be the length of $\mathbf{s}$. It follows from 3.5 and 3.7 that $l(a) = 2k - \lambda(a)$, so that $k = \frac{1}{2}[l(a) + \lambda(a)]$. This proves 3.9.

We say that the integer $L(a)$ defined in Proposition 3.9 is the *length of a as a twisted involution*. It is clear that $L(a) \leqslant l(a)$.

3.10 THE MONOID $M(W)$. At this stage, it is convenient to introduce a certain monoid $M = M(W)$ which is canonically associated with the Coxeter group $W = (W, S)$. The monoid M is generated by elements $m(s)$ $(s \in S)$ which satisfy the following relations:

(i) $m(s)^2 = m(s)$ $(s \in S)$;

(ii) ('braid relations') Let $s, t \in S$, with $s \neq t$ and let p be the order of st:

(1) If p is even, $p = 2k$, then $(m(s)m(t))^k = (m(t)m(s))^k$.

(2) If p is odd, $p = 2k + 1$, then $(m(s)m(t))^k m(s) = (m(t)m(s))^k m(t)$.

The monoid M is a well-known object (see [5, p. 87] and [3, Ch. IV, §2, Ex. 23, p. 55]). If $w \in W$ and if $\mathbf{s} = (s_1, \ldots, s_k)$ is a reduced decomposition of w, then the element $m(w) = m(s_1)m(s_2)\ldots m(s_k)$ depends only on w and is independent of choice of reduced decomposition $\mathbf{s}$ of w. Moreover, $M = \{m(w) \mid w \in W\}$ and $m(w) \neq m(w')$ if $w \neq w'$. If $s \in S$ and $w \in W$, then $m(s)m(w) = m(sw)$ if $sw > w$ and $m(s)m(w) = m(w)$ if $sw < w$. For the moment, we shall let $\sigma\colon W \to M$ denote the bijection $w \to m(w)$. We define the left and right actions of M on W by $m' \cdot w = \sigma^{-1}(m'm(w))$ and $w \cdot m' = \sigma^{-1}(m(w)m')$ $(m' \in M,\ w \in W)$. We define $\kappa\colon M \to M$ by $\kappa(m(w)) = m(\theta(w)^{-1})$.

3.11 LEMMA. (i) *κ is an anti-automorphism of M (that is, an isomorphism of M onto the opposite monoid M^{op}).* (ii) *Let $\mathscr{I}(M) = \{m(a) \mid a \in \mathscr{I}\}$. Then $\mathscr{I}(M)$ is the fixed point set of κ.*

We define the twisted action of the monoid M on (the set) M as follows: If $m, m' \in M$, then $m * m' = mm'\kappa(m)$. It follows from the definitions that $\mathscr{I}(M)$, the fixed point set of κ, is M-stable; i.e. if $m \in M$ and $m' \in \mathscr{I}(M)$, then $m * m' \in \mathscr{I}(M)$. If $m' \in M$, then we let $M * m' = \{m * m' \mid m \in M\}$ denote the *twisted M-orbit of m'*. We transfer the twisted action of M on M to an action of M on W by means of the bijection σ: if $w \in W$ and $m' \in M$, then $m' * w = \sigma^{-1}(m' * m(w))$, so that we have $m(m' * w) = m' * m(w)$. We call this action the twisted action of M on W. By 3.11, the set $\mathscr{I}$ of twisted involutions is stable under the twisted action of M, so that we get an induced twisted action of M on $\mathscr{I}$.

3.12 LEMMA. *Let $s \in S$ and $a \in \mathscr{I}$. If $sa < a$, then $m(s) * a = a$. If $sa > a$, then $m(s) * a = s \circ a$.*

This follows from the definitions, from 3.2 and from [15, 3.2].

3.13 LEMMA. *$\mathscr{I}$ is equal to $M * 1$, the twisted M-orbit of 1.*

3.14 LEMMA. *Let $w \in W$, let $\mathbf{s} = (s_1, \ldots, s_k)$ be a reduced decomposition of w^{-1} and let $a = m(w) * 1$. Then $l(w) \geqslant L(a)$. If $l(w) = L(a)$, then $\mathbf{s}$ is an admissible sequence and $\omega(\mathbf{s}) = a$.*

3.15 DEFINITION. We define a subset $\Gamma = \Gamma(W, \theta)$ of W by $\Gamma = \{w \in W \mid l(w) = L(m(w) * 1)\}$. For each $a \in \mathscr{I}$, we set $\Gamma(a) = \{w \in \Gamma \mid m(w) * 1 = a\}$.

Thus Γ is the disjoint union of the sets $\Gamma(a)$ $(a \in \mathscr{I})$.

3.16 LEMMA. (i) *If $a \in \mathscr{I}$, then $\Gamma(a)$ is non-empty.* (ii) *Let $w \in \Gamma$ and let $\mathbf{s} = (s_1, \ldots, s_k)$ be a reduced decomposition of w^{-1}. Then $\mathbf{s}$ is an admissible sequence.* (iii) *Let $\mathbf{s} = (s_1, \ldots, s_k)$ be admissible and let $w = s_k \ldots s_2 s_1$. Then $w \in \Gamma$ and $\mathbf{s}$ is a reduced decomposition of w^{-1}.*

3.17. *The weak order on M.* For later use, it will be convenient to introduce a (partial) order, denoted by $\vdash$, on the set $\mathscr{I}$ of twisted involutions. (This order on $\mathscr{I}$ is not intended to be an analogue of the Bruhat order on W.)

Let $a, b \in \mathscr{I}$. We write $a \mapsto b$ if $a \neq b$ and if there exists $s \in S$ such that $b = m(s) * a$; we also write $s\colon a \mapsto b$ if we want to indicate the role of s. We note that it is possible to have $s\colon a \mapsto b$ and $t\colon a \mapsto b$ without having $s = t$. We define a relation $\vdash$ on $\mathscr{I}$ as follows: Let $a, b \in \mathscr{I}$; then $a \vdash b$ if there exists a sequence $a_0, \ldots, a_k$ in $\mathscr{I}$ such that we have

$$a = a_0 \mapsto a_1 \mapsto \cdots \mapsto a_k = b.$$

It is clear that $\vdash$ is a (partial) order on $\mathscr{I}$. We say that $\vdash$ is the weak order on $\mathscr{I}$.

3.18 LEMMA. *Let $a, b \in \mathscr{I}$.* (i) *Assume that $a \mapsto b$. Then $a < b$ (Bruhat order) and $L(b) = L(a) + 1$. Either $l(b) = l(a) + 1$ or $l(b) = l(a) + 2$.* (ii) *Assume that $a \vdash b$ and $a \neq b$. Then $a < b$, $L(a) < L(b)$ and $l(a) < l(b)$.* (iii) *$a \vdash w_0$. In particular, if $a \neq w_0$, then $L(a) < L(w_0)$.*

3.19 LEMMA. *Let $a, b \in \mathscr{I}$.* (i) *Let $w_1 \in \Gamma(a)$. Then $a \vdash b$ if and only if there exists $w_2 \in W$ such that $w_1 w_2 \in \Gamma(b)$ and $l(w_1 w_2) = l(w_1) + l(w_2)$.* (ii) *Assume $a \vdash b$ and let $k = L(a)$, $l = L(b)$. Then there exists a sequence $\mathbf{s} = (s_1, \ldots, s_l)$ in S such that $a = s_k \circ \cdots \circ s_1 \circ 1$ and $b = s_l \circ \cdots \circ s_1 \circ 1$.*

3.20 LEMMA. (i) *Let $a, b \in \mathscr{I}$, with $E_-(a) \subset E_-(b)$. Let $c = ba^{-1}$. Then c is an involution and $E_-(b) = E_-(a) \oplus E(c, -1)$ (orthogonal direct sum).* (ii) *Let $b \in \mathscr{I}$ and let $c \in W$ be an involution such that $E(c, -1) \subset E_-(b)$. Then $a = cb$ is a twisted involution and $E_-(a) \subset E_-(b)$.*

It is well known that if $c \in W$ is an involution, then there exists a family $\{\beta_1, \ldots, \beta_j\}$ of pairwise orthogonal roots such that $c = s_{\beta_1} \ldots s_{\beta_j}$. In fact, one can assume that the family $\{\beta_1, \ldots, \beta_j\}$ consists of pairwise strongly orthogonal roots [3, Ch. VI, §1, Exer. 15].

3.21 LEMMA. *Let $a, b \in \mathcal{I}$ with $E_-(a) \subset E_-(b)$. Then there exists a family $\{\beta_1, \ldots, \beta_j\}$ of pairwise orthogonal roots such that we have an orthogonal direct sum decomposition $E_-(b) = E_-(a) \oplus \mathbb{R}\beta_1 \oplus \cdots \oplus \mathbb{R}\beta_j$.*

We consider the following condition on an element $z \in \mathcal{I}$:

3.22 CONDITION. *If $b \in \mathcal{I}$ and if $b \vdash z$, then for every $c \in W * b$, we have $c \vdash z$.*

3.23 REMARK. We return to the notation of Section 2. Let $v_{\max} \in V$ correspond to the dense $(B \times K)$-orbit on G and let $a_{\max} = \varphi(v_{\max})$. We shall show in Section 7 that $a_{\max}$ satisfies Condition 3.22.

3.24 PROPOSITION. *Let $z \in \mathcal{I}$ satisfy Condition 3.22. Then the following two conditions on $a \in \mathcal{I}$ are equivalent*: (i) $a \vdash z$; *and* (ii) *there exists $b \in W * a$ such that $E_-(b) \subset E_-(z)$.*

Proof. (i) $\Rightarrow$ (ii). The proof is by induction on $L(z) - L(a)$. If $L(z) - L(a) = 0$, then $z = a$ by 3.18. Assume that $k = L(z) - L(a) > 0$. Then there exists $c \in \mathcal{I}$ with $a \mapsto c$ and $c \vdash z$. Clearly $L(z) - L(c) = k - 1$, so by induction there exists $w \in W$ such that $E_-(w * c) \subset E_-(z)$. Since $a \mapsto c$, there exists $s \in (S(a)'' \cup S(a)_i)$ such that $c = s \circ a$. If $s \in S(a)''$, then $a = s * c \in W * c$ and we are done. Assume that $s \in S(a)_i$ and let $s = s_\alpha$, with $\alpha \in (\Delta \cap E_+(a))$. Then we have an orthogonal direct sum decomposition $E_-(c) = E_-(a) \oplus \mathbb{R}\alpha$, so that $E_-(a) \subset E_-(c)$. Thus

$$E_-(w * a) = w \cdot E_-(a) \subset w \cdot E_-(c) = E_-(w * c) \subset E_-(z).$$

(ii) $\Rightarrow$ (i). The proof is by induction on $d(z, a) = \dim E_-(z) - \dim E_-(a)$. If $d(z, a) = 0$, then it follows from the hypothesis that there exists $w \in W$ such that $E_-(w * a) = E_-(z)$, which implies that $w * a = z$. By Condition 3.22, this implies that $a \vdash z$. Assume now that $k = d(z, a) > 0$. By hypothesis, there exists $b \in W * a$ with $E_-(b)$ properly contained in $E_-(z)$. It follows from 3.21 that there exists $\beta \in (\Phi \cap E_-(z) \cap E_+(b))$. By a standard property of root systems, there exists $w \in W$ such that $\alpha = w * \beta \in \Delta$. Let $s = s_\alpha$ and let $c = w * b$. Then $\alpha \in E_+(c) \cap E_-(w * z)$ and $E_-(c) \subset E_-(w * z)$. Let $d = s \circ c$. Since $\alpha \in E_+(c)$, we have $c \mapsto d$ and

$$E_-(c) \subset E_-(d) = E_-(c) \oplus \mathbb{R}\alpha \subset E_-(w * z),$$

so that $E_-(w^{-1} * c) \subset E_-(z)$. Since $d(z, d) = k - 1$, the inductive hypothesis implies that $d \vdash z$. Thus $c \mapsto d \vdash z$. Since $a \in W * c$, it follows from Condition

4. Product of a Minimal Parabolic and a Double Coset

Let α be a simple root and let $s = s_\alpha$ be the corresponding reflection in W. We let $P_s = P_\alpha$ be the 'standard' parabolic subgroup of G associated to α; P_α is the parabolic subgroup with Lie algebra $L(P_\alpha) = L(U_{-\alpha}) \oplus L(B)$. It is a minimal parabolic subgroup. Let $v \in V$, let $x = x(v)$ and let $\mathcal{O}_v = BxK$ be the corresponding double coset. Clearly $P_s\mathcal{O}_v = P_sxK$ is a union of $(B \times K)$-orbits. In this section, we analyze the orbit structure of $P_s\mathcal{O}_v$ in some detail. It turns out that $P_s\mathcal{O}_v$ is the union of either one, two or three orbits. Most of our results can be obtained from analyzing the action of $K \cap P_s$ on the coset space $x^{-1}P_sx/x^{-1}Bx$, which is isomorphic as a variety to the projective line $\mathbb{P}^1 = \mathbb{P}^1(F)$. All of the results of this section are elementary, but they are complicated to state, since we have to consider a number of cases. Most of the results here are sketched in [11] and [15, 6.7], but very little detail is given.

4.1 ORBITS OF A SUBGROUP OF Aut($\mathbb{P}^1$). The group Aut($\mathbb{P}^1$) of automorphisms of $\mathbb{P}^1$ is isomorphic to $PGL_2(F)$ (see [1, III, 10.8]). Let H be an algebraic subgroup of Aut($\mathbb{P}^1$) and assume that H has a dense orbit on $\mathbb{P}^1$. There are four possible cases:

Case I. Either H^0 is unipotent or $\dim H = 2$. In this case there are two orbits, one dense orbit and one fixed point.

Case II. $H = $ Aut($\mathbb{P}^1$). Clearly $\mathbb{P}^1$ is the only orbit in this case.

Case III. H is a torus. There are three orbits, one dense orbit and two fixed points.

Case IV. H^0 is a torus and H/H^0 is of order 2. There are two orbits. The dense H^0-orbit is H-stable and H permutes the two fixed points of H^0.

4.2 ORBITS OF $B \times K$ ON $P_s\mathcal{O}_v$. Let $v \in V$ and let $x = x(v)$. Thus $\mathcal{O}_v = BxK$. Let $B_1 = x^{-1}Bx$ and $P_1 = x^{-1}P_sx$. Let $\mathcal{P} = \mathcal{P}_s$ denote the set of G-conjugates of P_s and let $\lambda: \mathcal{B} \to \mathcal{P}$ be the canonical projection. To simplify notation, we often denote $\mathcal{B}$ (resp. $\mathcal{P}$) by $\mathbf{X}$ (resp. $\mathbf{Y}$), when we consider $\mathcal{B}$ (resp. $\mathcal{P}$) as a projective variety with G-action. Let $\mathbf{x}$ (resp. $\mathbf{y}$) denote B_1 (resp. P_1) considered as a point of the variety $\mathbf{X}$ (resp. $\mathbf{Y}$). Define $\pi: G \to \mathcal{B} = \mathbf{X}$ by $\pi(g) = g^{-1}Bg$. We note that λ and π are locally trivial fiberings. It follows from the definitions that $\pi^{-1}(K \cdot \mathbf{x}) = \mathcal{O}_v$ and that $(\lambda \circ \pi)^{-1}(K \cdot \mathbf{y}) = P_s\mathcal{O}_v$. As an easy consequence of these observations, we have:

4.2.1. *Let $v, v' \in V$. If $\mathcal{O}_{v'} \subset P_s\mathcal{O}_v$, then $P_s\mathcal{O}_{v'} = P_s\mathcal{O}_v$.*

It is obvious that $\lambda^{-1}(\mathbf{y}) = P_1 \cdot \mathbf{x}$ and that $\lambda^{-1}(K \cdot \mathbf{y})$ is the K-orbit of $P_1 \cdot \mathbf{x}$. Since $P_s\mathcal{O}_v = \pi^{-1}(\lambda^{-1}(K \cdot \mathbf{y}))$, we see that $P_s\mathcal{O}_v$ is a smooth subvariety of G. By 1.5, K-orbits on $\lambda^{-1}(K \cdot \mathbf{y}) = K \cdot (P_1 \cdot \mathbf{x})$ correspond bijectively to $(B \times K)$-
orbits on $P_s\mathcal{O}_v$. Let $\mathbf{x}_1 \in P_1 \cdot \mathbf{x}$ and assume that $\mathbf{x}_1 = k \cdot \mathbf{x}$ for some $k \in K$

Then an easy argument shows that $k \in K \cap P_1$. Now $P_1 \cdot \mathbf{x}$ is isomorphic as a variety to $\mathbb{P}^1$ and the action of P_1 on $P_1 \cdot \mathbf{x}$ defines a morphism of algebraic groups $h\colon P_1 \to \mathrm{Aut}(P_1 \cdot \mathbf{x}) \cong \mathrm{Aut}(\mathbb{P}^1)$. Let $H = h(K \cap P_1)$.

It follows from the remarks above that there is a canonical bijective correspondence between K-orbits on $\lambda^{-1}(K \cdot \mathbf{y})$ and H-orbits on $P_1 \cdot \mathbf{x}$. Since K has only a finite number of orbits on $\mathbf{X}$, we see that H has a dense orbit on $P_1 \cdot \mathbf{x}$. Thus we can use the analysis of 4.1 to get information on $(B \times K)$-orbits on $P_s\mathcal{O}_v$.

Case I. Either H^0 is unipotent or $\dim H = 2$. Then $P_s\mathcal{O}_v$ is the union of two orbits, one dense and one of codimension 1.

Case II. $H = \mathrm{Aut}(P_1 \cdot \mathbf{x})$. Then $P_s\mathcal{O}_v = \mathcal{O}_v$.

Case III. H is a torus. There are three orbits, one dense orbit and two orbits of codimension 1.

Case IV. H^0 is a torus and H/H^0 is of order 2. There are two orbits, one dense and one of codimension 1.

For each of Cases I, III, and IV above, there are two possibilities, depending on whether $\mathcal{O}_v$ is dense in $P_s\mathcal{O}_v$ or of codimension 1. We now analyze cases according to whether s is real, complex, ... for v. Some of our results depend on results from [15, in particular 5.1 and 6.7].

4.3 CASE ANALYSIS

4.3.1. *Case A*: *s is complex for v.* This corresponds to Case I above. We have $P_s\mathcal{O}_v = \mathcal{O}_v \cup \mathcal{O}_{s \cdot v}$. If $s \in S(v)'$ (resp. $s \in S(v)''$), then $\mathcal{O}_v$ (resp. $\mathcal{O}_{s \cdot v}$) is dense and $\mathcal{O}_{s \cdot v}$ (resp. $\mathcal{O}_v$) is of codimension 1.

4.3.2. *Case B*: *s is compact imaginary for v.* We are in Case II. We have $P_s\mathcal{O}_v = \mathcal{O}_v$ and $s \cdot v = v$.

4.3.3. *Case C*: *s is real for v.* We are in either Case III or Case IV. In both cases $\mathcal{O}_v$ is dense in $P_s\mathcal{O}_v$ and there exists $v' \in V$ such that $P_s\mathcal{O}_v = \mathcal{O}_v \cup \mathcal{O}_{v'} \cup \mathcal{O}_{s \cdot v'}$. In Case III (resp. Case IV) we have $s \cdot v' \neq v'$ (resp. $s \cdot v' = v'$).

4.3.4. *Case D*: *s is non-compact imaginary for v.* We are in Case III or Case IV. In both cases, there exists $v' \in V$ such that $P_s\mathcal{O}_v = \mathcal{O}_{v'} \cup \mathcal{O}_v \cup \mathcal{O}_{s \cdot v}$, with $\mathcal{O}_{v'}$ dense and $\mathcal{O}_v$ and $\mathcal{O}_{s \cdot v}$ of codimension 1. In Case III (resp. Case IV), $s \cdot v \neq v$ (resp. $s \cdot v = v$).

4.4 LEMMA. *Let $v \in V$, let $a = \varphi(v)$ and let $s \in S$ be such that $sa < a$. Then there exists $v' \in V$ such that*: (i) $P_s\mathcal{O}_v = P_s\mathcal{O}_{v'}$; and (ii) *$\mathcal{O}_v$ is dense in $P_s\mathcal{O}_v$ and $\mathcal{O}_{v'}$ is of codimension* 1. *Moreover $\varphi(v') = s \circ a < a$. If $s \in S(v)'$, then $v' = s \cdot v$ and $\varphi(v') = s * a$. If $s \in S(v)_r$, then $\varphi(v') = sa$.*

Proof. Since $sa < a$, either $s \in S(v)'$ or $s \in S(v)_r$ so that we are in either Case A or Case C. The result now follows from the analysis of cases above and from [15, 5.1].

4.5 LEMMA. *If $v \in V$ and $s \in S$, then $P_s\,\mathrm{cl}(\mathcal{O}_v)$ is closed.*

This follows from [18, p. 68, Lemma 2].

The following theorem plays an important role in our discussion of the Bruhat order on V.

4.6 THEOREM. *Let $v \in V$. Then there exists a closed orbit $\mathcal{O}_{v_0}$ and a sequence $\mathbf{s} = (s_1, \ldots, s_k)$ in S such that $\mathrm{cl}(\mathcal{O}_v) = P_{s_k} \ldots P_{s_1}\mathcal{O}_{v_0}$ and $\dim \mathcal{O}_v = k + \dim \mathcal{O}_{v_0}$. Let $a = \varphi(v)$. Then $\mathbf{s}$ is an admissible sequence for a and $L(a) = k = \dim \mathcal{O}_v - \dim \mathcal{O}_{v_0}$.*

Proof. The proof is by induction on $L(a)$. If $L(a) = 0$, then $a = 1$ and $\mathcal{O}_v$ is closed [15, 6.6]. In this case, we may take $v = v_0$ and $k = 0$. If $a \neq 1$, choose $s \in S$ such that $sa < s$. By 4.3 and 4.4, there exists $v' \in V$ such that: (a) $\mathrm{cl}(\mathcal{O}_v) = P_s\,\mathrm{cl}(\mathcal{O}_{v'})$; (b) $\dim(\mathcal{O}_v) = \dim(\mathcal{O}_{v'}) + 1$; and (c) $\varphi(v') = s \circ a < a$, so that $L(\varphi(v')) = L(\varphi(v)) - 1$. The proof now follows easily if we apply the inductive hypothesis to v'.

4.7 ACTION OF $M(W)$ ON V. We use the above construction to define an action of the monoid $M = M(W)$ on V. Let $v \in V$ and $s \in S$. It follows easily from the analysis of this section (or from the irreducibility of $\mathcal{X}_v$ – see 1.5) that there is a unique dense $(B \times K)$-orbit in $P_s\mathcal{O}_v$. We define $m(s) \cdot v = v'$, where $\mathcal{O}_{v'}$ is the unique dense orbit in $P_s\mathcal{O}_v$. We note that if $sts\ldots = tst\ldots$ is a braid relation, then $P_sP_tP_s \ldots = P_tP_sP_t \ldots$. This follows from standard results on the multiplication of Bruhat cells (see [3, Ch. IV, §2]). Thus the above definition defines an action of M on V.

5. Partial orders on M-sets

Most of the basic properties of the Bruhat order on V now follow from elementary combinatorial arguments involving the action of the monoid $M(W)$ on V. In Sections 5 and 6, we will give these combinatorial arguments. We shall work in a fairly general setting, since similar arguments apply in related situations of interest, for example the 'Bruhat order' on the set of twisted involutions and the 'Bruhat order' on the set of B-orbits for a homogeneous spherical variety. Surprisingly, the braid relations on $M(W)$ are never used in our arguments.

In Sections 5 and 6, S is a finite set and M is a finite monoid. We assume that we are given a mapping assigning to each $s \in S$ an element $m(s) \in M$. We assume further that the set $\{m(s) \mid s \in S\}$ generates M and that $m(s)^2 = m(s)$ for $s \in S$. We let X be a finite M-set. We let $m \cdot x$ denote the action of $m \in M$ on $x \in X$ and we let $M \cdot x = \{m \cdot x \mid m \in M\}$ denote the M-orbit of x. If $\mathbf{s} = (s_1, \ldots, s_k)$ is a finite sequence in S, we let $k = l(\mathbf{s})$ and we let $m(\mathbf{s}) = m(s_k) \ldots m(s_1)$ (note the reversal in order). Thus every $m \in M$ is of the form $m(\mathbf{s})$ for some sequence $\mathbf{s}$; the identity element e of M corresponds to the empty sequence.

5.1 THE LENGTH FUNCTION AND THE WEAK ORDER. We say that $x \in X$ is a *minimal element* of X if $x \notin M \cdot y$ for every $y \in X$, $y \neq x$. We let X_0 be the set of minimal elements. We assume from now on that we are given a function $l\colon X \to \mathbb{N}$ with the following properties: (1) $l(x) = 0$ if and only if $x \in X_0$; and (2) if $s \in S$ and $x \in X$ are such that $x \neq m(s) \cdot x$, then $l(m(s) \cdot x) = l(x) + 1$. We say that l is the *length function* on X. It is clear that if a length function on X exists, then it is unique.

Let $x, y \in X$. We write $y \mapsto x$ if $x \neq y$ and there exists $s \in S$ such that $x = m(s) \cdot y$. We define a relation $\vdash$ by: $y \vdash x$ if there exists a sequence $(x_0, \ldots, x_k)$ such that $y = x_0 \mapsto x_1 \mapsto \cdots \mapsto x_k = x$. It follows from the existence of the length function that $\vdash$ is a partial order on X. We call this order the *weak order* on X. It is clear that $y \vdash x$ if and only if $x \in M \cdot y$ and that X_0 is the set of minimal elements of X with respect to the weak order.

5.2 THE STANDARD ORDER ON X. If $y, x \in X$, we write $y \to x$ if there exist $z \in X$, $t \in S$ and a sequence $\mathbf{s}$ in S such that the following conditions hold:

(i) $y = m(\mathbf{s}) \cdot z$ and $l(y) = l(z) + l(\mathbf{s})$; and
(ii) $x = m(\mathbf{s}) \cdot m(t) \cdot z$ and $l(x) = l(z) + l(\mathbf{s}) + 1$.

We define a relation $\preceq$ on X as follows: $y \preceq x$ if there exists a sequence $\mathbf{x} = (x_0, \ldots, x_k)$ in X such that $y = x_0 \to x_1 \to \cdots \to x_k = x$. It is clear that $\preceq$ is a partial order on X. Note that $y \to x$ if and only if $y \preceq x$ and $l(y) + 1 = l(x)$. We say that $\preceq$ is the *standard order* on the M-set X. Clearly the standard order on X has the following properties:

(iii) If $x \in X$ and $s \in S$, then $x \preceq m(s) \cdot x$; and
(iv) Let $y, x \in X$. If $y \preceq x$ and $l(x) \leqslant l(y)$, then $y = x$.

We write $y \prec x$ if $y \preceq x$ and $y \neq x$.

5.3 DEFINITION. A partial order $\leqslant$ on X is *compatible with the action of* M on X if the following three conditions are satisfied for all $y, x \in X$ and $s \in S$:

(i) $x \leqslant m(s) \cdot x$; (ii) if $y \leqslant x$, then $m(s) \cdot y \leqslant m(s) \cdot x$; and (iii) if $y \leqslant x$ and $l(x) \leqslant l(y)$, then $y = x$.

We observe that the standard order $\preceq$ on X is compatible with the M-action if and only if condition 5.3(ii) holds for $\preceq$.

5.4 EXAMPLE. Let the notation be as in 3.10. With a little work, one can show that the Bruhat order on W is compatible with the left action of $M(W)$ on W.

We consider the following property of an M-set X with length function l as above:

PROPERTY 5.5 (Weak Exchange Property). *Let $s, t \in S$, let $z \in X$ and let $\mathbf{s}$ be a sequence in S. Assume that:*

(i) *$l(m(\mathbf{s}) \cdot m(t) \cdot z) = l(z) + l(\mathbf{s}) + 1$ and $m(s) \cdot m(\mathbf{s}) \cdot m(t) \cdot z = m(\mathbf{s}) \cdot m(t) \cdot z$; and*
(ii) *$l(m(\mathbf{s}) \cdot z) = l(z) + l(\mathbf{s})$ and $l(m(s) \cdot m(\mathbf{s}) \cdot z) = l(z) + l(\mathbf{s}) + 1$.*

Then $m(\mathbf{s}) \cdot m(t) \cdot z = m(s) \cdot m(\mathbf{s}) \cdot z$.

5.6 PROPOSITION. (i) *Assume that there exists a partial order on X compatible with the M-action. Then X has the weak exchange property.* (ii) *Assume that X has the weak exchange property. Then the standard order on X is compatible with the M-action. Furthermore the standard order on X is the weakest partial order on X compatible with the M-action.*

Proof. (i) Let $\leqslant$ be a partial order on X compatible with the M-action. Let $s, t, \mathbf{s}$ and z be as in 5.5 and assume that 5.5(i) and 5.5(ii) hold. Then $z \leqslant m(t) \cdot z$ and consequently

$$m(s) \cdot m(\mathbf{s}) \cdot z \leqslant m(s) \cdot m(\mathbf{s}) \cdot m(t) \cdot z = m(\mathbf{s}) \cdot m(t) \cdot z.$$

Since $l(m(s) \cdot m(\mathbf{s}) \cdot z) = l(z) + l(\mathbf{s}) + 1 = l(m(\mathbf{s}) \cdot m(t) \cdot z)$, it follows from 5.3(iii) that $m(s) \cdot m(\mathbf{s}) \cdot z = m(\mathbf{s}) \cdot m(t) \cdot z$, so that the weak exchange property holds.

(ii) Assume that the weak exchange property holds and let the notation be as in 5.2. We need to show that if $y \preceq x$ and if $s \in S$, then $m(s) \cdot y \preceq m(s) \cdot x$. By definition of the standard order, it will suffice to show that if $y \to x$, then $m(s) \cdot y \preceq m(s) \cdot x$. There are four cases to consider:

Case 1. $y = m(s) \cdot y$ and $x = m(s) \cdot x$.
Case 2. $y = m(s) \cdot y$ and $x \mapsto m(s) \cdot x$.
Case 3. $y \mapsto m(s) \cdot y$ and $x \mapsto m(s) \cdot x$.
Case 4. $y \mapsto m(s) \cdot y$ and $x = m(s) \cdot x$.

In Cases 1 and 2 the argument is trivial and in Case 3 we only need the definitions. For Case 4, we must use the weak exchange property. By the definition of $y \to x$, there exist $z \in X$, $t \in S$ and a sequence $\mathbf{s}$ such that $y = m(\mathbf{s}) \cdot z$, $x = m(\mathbf{s}) \cdot m(t) \cdot z$ and (assuming Case 4) such that Conditions 5.5(i) and 5.5(ii) are satisfied. It then follows from the weak exchange property that $y = x$. Thus $\preceq$ is compatible with the M-action.

Now let $\leqslant$ be a partial order on X compatible with the M-action. We need to show that if $y \preceq x$, then $y \leqslant x$. It will suffice to show that if $y \to x$, then $y \leqslant x$. Assume that $y \to x$ and let z, t and $\mathbf{s}$ be such that 5.2(i) and 5.2(ii) hold. By compatibility of $\leqslant$ with the M-action, we have $z \leqslant m(t) \cdot z$, and consequently $y = m(\mathbf{s}) \cdot z \leqslant m(\mathbf{s}) \cdot m(t) \cdot z = x$. This completes the proof of 5.6.

5.7 DEFINITION. Let $x \in X$. A *reduced decomposition* of x is a pair $(\mathbf{x}, \mathbf{s})$, where $\mathbf{x} = (x_0, \ldots, x_k)$ is a sequence in X and $\mathbf{s} = (s_1, \ldots, s_k)$ is a sequence in S, which satisfies the following conditions: (1) $x_0 \in X_0$; and (2) for each $i \in [1, k]$, we have $x_{i-1} \mapsto m(s_i) \cdot x_{i-1} = x_i$. We say that k is the length of the reduced decomposition $(\mathbf{x}, \mathbf{s})$.

Let $(\mathbf{x}, \mathbf{s})$ be as above. It is clear that $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_k$ and that $l(x_i) = i$ for $i \in [0, k]$. It is also clear that x_0 and $\mathbf{s}$ determine $(\mathbf{x}, \mathbf{s})$. It is not necessarily the case that $\mathbf{x}$ determines $\mathbf{s}$.

5.8 DEFINITION. Let $(\mathbf{x} = (x_0, \ldots, x_k), \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of $x \in X$. A sequence $\mathbf{y} = (y_0, \ldots, y_k)$ in X is a *subexpression* of $(\mathbf{x}, \mathbf{s})$ if $x_0 = y_0$ and, for each $i \in [1, k]$, one of the following three alternatives holds: (α) $y_{i-1} = y_i$; (β) $y_{i-1} \mapsto m(s_i) \cdot y_{i-1}$, $y_i \mapsto m(s_i) \cdot y_i$, and $m(s_i) \cdot y_{i-1} = m(s_i) \cdot y_i$; or ($\gamma$) $y_{i-1} \mapsto m(s_i) \cdot y_{i-1} = y_i$. We say that y_k is the *final term* of the subexpression $\mathbf{y}$.

5.9 LEMMA. *Let $x \in X$ and let $k = l(x)$. Then there exists a reduced decomposition of x and every reduced decomposition of x has length k.*

The proof is trivial.

5.10 LEMMA. *Let $\leqslant$ be a partial order on X which is compatible with the M-action. Let $(\mathbf{x} = (x_0, \ldots, x_k), \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of X and let $\mathbf{y} = (y_0, \ldots, y_k)$ be a subexpression of $(\mathbf{x}, \mathbf{s})$. Then $y_i \leqslant x_i$ for $i \in [0, k]$.*

Proof. The proof is by induction on i. For $i = 0$, the result is clear. Assume $i > 0$ and that $y_j \leqslant x_j$ for $j < i$. One of the alternatives (α), (β), (γ) of Definition 5.8 holds for y_i. We treat the three cases separately.

(α) $y_i = y_{i-1} \leqslant x_{i-1} < x_i$.

(β) We have $y_{i-1} \leqslant x_{i-1}$, so that $m(s_i) \cdot y_{i-1} \leqslant m(s_i) \cdot x_{i-1} = x_i$. Hence $y_i \leqslant m(s_i) \cdot y_i = m(s_i) \cdot y_{i-1} \leqslant x_i$.

(γ) $y_i = m(s_i) \cdot y_{i-1} \leqslant m(s_i) \cdot x_{i-1} = x_i$.

This proves 5.10.

5.11 THE CHAIN CONDITION. Let $\leqslant$ be a partial order on X. We say that $\leqslant$ satisfies the *chain condition* if the following conditions on $y, x \in X$ are equivalent: (i) $y \leqslant x$; and (ii) there exists a sequence $\mathbf{x} = (x_0, \ldots, x_k)$ in X such that $y = x_0 < x_1 < \cdots < x_k = x$ and $l(x_i) = l(x_{i-1}) + 1$ for every $i \in [1, k]$. It is clear that the weak order on X and the standard order on X satisfy the chain condition.

5.12 SOME PROPERTIES OF PARTIAL ORDERS ON X. Let $\leqslant$ be a partial order on X which is compatible with the M-action. We consider a number of possible properties of the partial order $\leqslant$.

PROPERTY 5.12(a). (Subexpression Property.) *Let $(\mathbf{x}, \mathbf{s})$ be a reduced decomposition of $x \in X$. Then $y \leqslant x$ if and only if there exists a subexpression $\mathbf{y}$ of $(\mathbf{x}, \mathbf{s})$ with final term y.*

If $z \in X$ and $s \in S$, we set $\mathscr{E}(s, z) = \{x \in X \mid x \mapsto m(s) \cdot x = z\}$.

PROPERTY 5.12(b). *Let y, $x \in X$ and $s \in S$ be such that $x \mapsto m(s) \cdot x$ and $y < m(s) \cdot x$. Then one of the following three conditions holds*: (i) $y \leqslant x$; (ii) *$y \mapsto m(s) \cdot y$ and there exists $y' \in \mathscr{E}(s, m(s) \cdot y)$ such that $y' \leqslant x$; or* (iii) *there exists $y'' < x$ such that $y'' \mapsto m(s) \cdot y'' = y$.*

For each $s \in S$ and $x \in X$, we define a subset $p(s, x)$ of X by:

$$p(s, x) = \{m(s) \cdot x\} \cup \mathscr{E}(s, m(s) \cdot x).$$

We set $X_{\leqslant}(x) = \{y \in X \mid y \leqslant x\}$.

PROPERTY 5.12(c). (One-step Property.) *Let $s \in S$ and $x \in X$ be such that $x \mapsto m(s) \cdot x$. Then $X_{\leqslant}(m(s) \cdot x) = \bigcup_{y \leqslant x} p(s, y)$.*

Thus, the one-step property gives a description of the behaviour of $X_{\leqslant}(x)$ as we go up one step from x to $m(s) \cdot x$. This property clearly amounts to an inductive description of the partial order $\leqslant$.

PROPERTY 5.12(d). (Property $Z(s, x, y)$.) *Let y, $x \in X$ and $s \in S$ be such that $x \mapsto m(s) \cdot x$ and $y \mapsto m(s) \cdot y$. Then the following three properties are equivalent*: (i) *either $y \leqslant x$ or there exists $y' \in \mathscr{E}(s, m(s) \cdot y)$ such that $y' \leqslant x$*; (ii) *$m(s) \cdot y \leqslant m(s) \cdot x$; and* (iii) *$y \leqslant m(s) \cdot x$.*

PROPERTY 5.12(e). (Exchange property.) *Let $(\mathbf{x} = (x_0, \ldots, x_k)$, $\mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of $x \in X$. Let $s \in S$ and $y \in X$ be such that $y \mapsto m(s) \cdot y = x$. Then there exists $i \in [1, k]$ and a reduced decomposition $(\mathbf{y} = (y_0, \ldots, y_k), \mathbf{s}')$ of x such that $y_{k-1} = y$ and $\mathbf{s}' = (s_1, \ldots, \hat{s}_i, \ldots, s_k, s)$.*

5.13 REMARK. Except for Property 5.12(c), the Properties 5.12(a)–5.12(e) are all analogues of standard properties of the Bruhat order on W (where $M = M(W)$ and $M(W)$ acts on W by the left action). In particular, the exchange property is an analogue of the usual exchange property and the property $Z(s, x, y)$ is an analogue of the property $Z(s, w, w')$ of [6]. Our use of subexpressions was suggested by the paper of Deodhar [7], although our definition of subexpression is somewhat different from his.

6. Partial orders compatible with the M-action

We continue with the notation of Section 5. We assume throughout Section 6 that $\leqslant$ is a partial order of X compatible with the M-action. In this section, we shall show that Properties 5.12(a)–(c) are equivalent and that these properties imply Property 5.12(d). With an additional assumption, they also imply Property 5.12(e). Properties 5.12(a)–(c) also imply that the partial order $\leqslant$ agrees with the standard order $\preceq$. This implies that $\leqslant$ satisfies the chain condition.

6.1 LEMMA. *Assume that the partial order $\leqslant$ has the subexpression property. Then $\leqslant$ is equal to the standard order $\preceq$. In particular, $\leqslant$ satisfies the chain condition.*

Proof. It follows from 5.6 that the standard order $\preceq$ is weaker than $\leqslant$ and that $\preceq$ is compatible with the M-action. Assume that $y \leqslant x$ and let $(\mathbf{x}, \mathbf{s})$ be a reduced decomposition of $\mathbf{x}$. By the subexpression property, there exists a subexpression $\mathbf{y}$ of $(\mathbf{x}, \mathbf{s})$ with final term y. By 5.10, we have $y \preceq x$. Hence the partial orders $\leqslant$ and $\preceq$ are equal.

6.2 PROPOSITION. *The subexpression property implies Property $Z(s, x, y)$.*

Proof. Assume the subexpression property and let $y, x \in X$ and $s \in S$ be such that $x \mapsto m(s)\cdot x$ and $y \mapsto m(s)\cdot y$. We want to show that the three conditions (i)–(iii) of 5.12(d) are equivalent. It follows from the compatibility of $\leqslant$ with the M-action that (ii) and (iii) are equivalent and that (i) implies (ii). We need to show that (ii) implies (i). Assume that $m(s)\cdot y \leqslant m(s)\cdot x$. Since $x \mapsto m(s)\cdot x$, there exists a reduced decomposition $(\mathbf{x} = (x_0, \ldots, x_{k+1})$, $\mathbf{s} = (s_1, \ldots, s_{k+1}))$ of $m(s)\cdot x$, such that $x_k = x$ and $s_{k+1} = s$. By the subexpression property, there is a subexpression $\mathbf{y} = (y_0, \ldots, y_{k+1})$ of $(\mathbf{x}, \mathbf{s})$ with $y_{k+1} = m(s)\cdot y$. One of the alternatives (α), (β), (γ) of Definition 5.8 holds for y_{k+1}. We treat the three cases separately.

(α) $y_k = y_{k+1}$. Then $y \leqslant m(s)\cdot y = y_{k+1} = y_k \leqslant x_k = x$.

(β) Since $y_{k+1} = m(s)\cdot y$, condition (β) cannot occur.

(γ) We have $y_k \mapsto m(s)\cdot y_k = y_{k+1} = m(s)\cdot y$. It follows that $\{y, y_k\} \subset \mathscr{E}(s, m(s)\cdot y)$ and $y_k \leqslant x_k = x$. This proves 6.2.

6.3 PROPOSITION. *The subexpression property implies Property* 5.12(b).

Proof. Assume that $x \mapsto m(s)\cdot x$ and that $y < m(s)\cdot x$. Let $(\mathbf{x} = (x_0, \ldots, x_{k+1}), \mathbf{s} = (s_1, \ldots, s_{k+1}))$ be a reduced decomposition of $m(s)\cdot x$ with $x_k = x$ and $s_{k+1} = s$. Then there exists a subexpression $(y_0, \ldots, y_{k+1})$ of $(\mathbf{x}, \mathbf{s})$ with $y_{k+1} = y$. One of the alternatives (α), (β), (γ) holds for y_{k+1}. If (α) holds, then Condition (i) of 5.12(b) holds, if (β) holds, then 5.12(b)(ii) holds and if (γ) holds, then 5.12(b)(iii) holds.

6.4 PROPOSITION. *Property* 5.12(b) *implies the subexpression property.*

Proof. Assume Property 5.12(b). We want to prove the following result:

(6.4.a) *Let $z \in X$ and let $(\mathbf{z}, \mathbf{s})$ be a reduced decomposition of z. If $y \leqslant z$, then there exists a subexpression $\mathbf{y}$ of $(\mathbf{z}, \mathbf{s})$ with final term y.*

The proof of (6.4a) is by induction on $l(z)$. The result is obvious for $l(z) = 0$. Assume the result holds for length $\leqslant k$ and let $l(z) = k + 1$. Let $\mathbf{z} = (z_0, \ldots, z_{k+1})$ and let $\mathbf{s} = (s_1, \ldots, s_{k+1})$. Set $z_k = x$ and $s_{k+1} = s$. Then $x \mapsto m(s)\cdot x = z$ and $y \leqslant z = m(s)\cdot x$. If $y = z$, we are done, so we may assume that $y < z$. Thus we are in the situation of 5.12(b). If 5.12(b)(i) holds, then $y \leqslant x = z_k$ and the proof follows by induction. In case 5.12(b)(ii), we have $y \mapsto m(s)\cdot y$ and there exists $y' \in \mathscr{E}(s, m(s)\cdot y)$ with $y' \leqslant x$. By induction, there exists a subexpression $\mathbf{y}' = (y'_0, \ldots, y'_k)$ of $(\mathbf{z}' = (z_0, \ldots, z_k), \mathbf{s}' = (s_0, \ldots, s_k))$ with $y'_k = y'$. Hence, $\mathbf{y} = (y'_0, \ldots, y'_k, y)$ is a subexpression of $(\mathbf{z}, \mathbf{s})$. In case 5.12(b)(iii), there exists $y'' \leqslant x$ with $y'' \mapsto m(s)\cdot y'' = y$. By induction, there exists a subexpression $\mathbf{y}'' = (y''_0, \ldots, y''_k)$ of $(\mathbf{z}', \mathbf{s}')$ with $y''_k = y''$. Hence $(y''_0, \ldots, y''_k, y)$ is a subexpression of $(\mathbf{z}, \mathbf{s})$. This proves 6.4.

6.5 PROPOSITION. *Property* 5.12(b) *is equivalent to the one-step property.*

Proof. (a) *Property* 5.12(b) $\Rightarrow$ *One-step property.* Assume Property 5.12(b) and let $x \in X$ and $s \in S$ be such that $x \mapsto m(s)\cdot x$. Let $X(s, x) = \bigcup_{v \leqslant x} p(s, v)$. We want to prove that $X_{\leqslant}(m(s)\cdot x) = X(s, x)$. Clearly $m(s)\cdot x \in X(s, x)$. Assume $y < m(s)\cdot x$. By 5.12(b) there are three cases to consider.

Case (i). $y \leqslant x$. Then $y \in p(s, x) \subset X(s, x)$.

Case (ii). There exists $y' \leqslant x$ such that $y' \mapsto m(s)\cdot y' = m(s)\cdot y$ and $y \mapsto m(s)\cdot y$. Then $y \in p(s, y') \subset X(s, x)$.

Case (iii). There exists $y'' < x$ such that $y'' \mapsto m(s)\cdot y'' = y$. Then $y \in p(s, y'') \subset X(s, x)$. Thus $X_{\leqslant}(m(s)\cdot x) \subset X(s, x)$.

Now let $y \in X(s, x)$. Then there exists $v \leqslant x$ with $y \in p(s, v)$.

Case (1). $m(s)\cdot v = v$. Then $y \vdash v$ and $v \leqslant x$, so that $y \leqslant x < m(s)\cdot x$.

Case (2). $v \mapsto m(s)\cdot v$. Then $y \vdash m(s)\cdot v \leqslant m(s)\cdot x$.

Thus $X(s, x) \subset X_{\leqslant}(m(s)\cdot x)$. Consequently $X(s, x) = X_{\leqslant}(m(s)\cdot x)$.

(b) *One-step property* $\Rightarrow$ *Property* 5.12(b). Assume $y < m(s)\cdot x$ and

Case (1). $m(s)\cdot v = v$. Then $y \vdash v \leqslant x$ and 5.12(b)(i) holds.

Case (2). $v \mapsto m(s)\cdot v$. Then either $y = m(s)\cdot v$ and 5.12(b)(iii) holds or $y \mapsto m(s)\cdot y = m(s)\cdot v$ and 5.12(b)(ii) holds. This proves (b).

We need an additional assumption to prove that the subexpression property implies the exchange property. We consider the following property of the M-set X.

PROPERTY 6.6. *Let $u, v, z \in X$ and $s \in S$ be such that $\{u, v\} \subset \mathscr{E}(s, z)$. Let* $(\mathbf{u}, \mathbf{s})$ *be a reduced decomposition of u and let $k = l(u) = l(v)$. Then there exists a sequence $\mathbf{v} = (v_0, \ldots, v_k)$ in X such that $(\mathbf{v}, \mathbf{s})$ is a reduced decomposition of v.*

6.7 PROPOSITION. *Assume that (M, X) has Property 6.6. Then the subexpression property implies the exchange property.*

Proof. Let x, $y \in X$ and $s \in S$ be such that $y \mapsto m(s)\cdot y = x$. Let $(\mathbf{x} = (x_0, \ldots, x_k), \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of x. Then there exists a subexpression $\mathbf{y} = (y_0, \ldots, y_k)$ of $(\mathbf{x}, \mathbf{s})$ with $y_k = y$. Since $l(y) + 1 = k$, an easy argument shows that there exists a unique $i \in [1, k]$ such that $y_j = x_j$ for $j = 1, \ldots, i-1$ and either (α) $y_i = y_{i-1}$ or (β) $\{y_i, x_{i-1}\} \subset \mathscr{E}(s_i, x_i)$. In either case, a straightforward argument using Property 6.6 shows that there exists a sequence $\mathbf{z} = (z_0, \ldots, z_k)$, with $z_{k-1} = y$ and $z_k = x$, such that $(\mathbf{z}, (s_1, \ldots, \hat{s}_i, \ldots, s_k, s))$ is a reduced decomposition of x. Thus the exchange property holds.

Summarizing our results, we have the following implications between the properties discussed in 5.12:

Subexpression property $\Leftrightarrow$ Property 5.12(b) $\Leftrightarrow$ One-step property.

Subexpression property $\Rightarrow$ Property $Z(s, x, y)$ + '$\leqslant$ is weakest partial order compatible with M-action' + '$\leqslant$ agrees with $\preceq$' + chain condition.

Subexpression property + Property 6.6 $\Rightarrow$ Exchange property.

7. The Bruhat order on V

We return to the notation of Section 4. In particular M denotes $M(W)$. In this section we shall use the combinatorial results of Sections 5 and 6 to derive properties of the Bruhat order on V. First we need several lemmas.

7.1 LEMMA. *All closed orbits $\mathcal{O}_v$ have the same dimension d.*

Proof. Let $\mathcal{O}_v$ be closed, let $x = x(v)$ and let $B' = x^{-1}Bx$. Then $B'K = x^{-1}BxK$ is closed and hence B' is a θ-stable Borel subgroup of G [15, proof of 6.6]. An easy argument shows that $\dim \mathcal{O}_v = \dim B' + \dim K - \dim B' \cap K$. But $B' \cap K$ is a Borel subgroup of K [13, 5.1] so that the result follows.

If $v \in V$, we define its length $l(v)$ by $l(v) = \dim \mathcal{O}_v - d$. Then l is a strictly monotonic function (with respect to the Bruhat order) from V to $\mathbb{N}$. Moreover $l(v) = 0$ if and only if $\mathcal{O}_v$ is closed. Let $\vdash$ be the 'weak order' on the M-set V defined as in Section 5. Let $V_0 = \{v \in V \mid l(v) = 0\}$.

7.2 LEMMA. (i) *If $v \in V$ and $s \in S$ are such that $m(s)\cdot v \neq v$, then $l(m(s)\cdot v) = l(v) + 1$.* (ii) *$V_0$ is the set of minimal elements of V with respect to the weak order.* (iii) *The Bruhat order on V is compatible with the M-action and the length function l.* (iv) *Let $v \in V$ and let $a = \varphi(v)$. Then $l(v) = L(a)$.*

Proof. The proof of (i) follows from the results of Section 4 and the proofs of (ii) and (iv) follow from Theorem 4.6. The proof of (iii) follows easily from the definitions. The only tricky point to check is Condition 5.3(iii). This follows from the fact that the orbits $\mathscr{X}_v$ are irreducible varieties (see 1.5).

7.3 CASE ANALYSIS. In this subsection, we reformulate the case analysis of 4.3 in terms of the action of M on V. Let $s \in S$, let $v \in V$ and let $a = \varphi(v)$. Following Section 5, we set

$$\mathscr{E}(s, v) = \{v' \in V \mid v' \mapsto m(s)\cdot v' = v\}.$$

7.3.1. *Case A: s is complex for v.* If $s \in S(v)''$, then $v < m(s)\cdot v = s\cdot v$. If $s \in S(v)'$, then $s\cdot v < m(s)\cdot(s\cdot v) = v$ and $\mathscr{E}(s, v) = \{s\cdot v\}$.

7.3.2. *Case B*: $s \in S(v)_c$. Then $m(s)\cdot v = v$ and $\mathscr{E}(s, v)$ is empty.

7.3.3. *Case C*: $s \in S(v)_r$. Then $m(s)\cdot v = v$ and there exists v' such that $\mathscr{E}(s, v) = \{v', s\cdot v'\}$. Moreover $s \in S(v')_n$ and $\varphi(v') = \varphi(s\cdot v') = s \circ a < a$.

7.3.4. *Case D*: $s \in S(v)_n$. Then $v < m(s)\cdot v$ and $\mathscr{E}(s, m(s)\cdot v) = \{v, s\cdot v\}$.

In Case D (resp. Case C), one sometimes has $s\cdot v = v$ (resp. $s\cdot v' = v'$).

7.4 LEMMA. *Let $v \in V$, let $s \in S$ and let $a = \varphi(v)$.* (i) *If $v < m(s)\cdot v$, then $\varphi(m(s)\cdot v) = s \circ a$.* (ii) *Assume that $sa < a$. Then there exists $v' \in V$ such that $v' < m(s)\cdot v' = v$ and $\varphi(v') = s \circ a$.*

Proof. The proof of (i) follows from [15, 5.1] and from 7.3. The proof of (ii) follows from 4.4 and from (i) above.

7.5 LEMMA. *Let v, $v' \in V$ and $s \in S$ be such that $v' \mapsto m(s)\cdot v' = v$. Then* $\mathrm{cl}(\mathcal{O}_v) = \bigcup_{u \leqslant v'} P_s \mathcal{O}_u$.

Proof. By 4.5, $P_s\,\mathrm{cl}(\mathcal{O}_{v'})$ is closed. Since $m(s)\cdot v' = v$, it follows that $P_s\,\mathrm{cl}(\mathcal{O}_{v'}) = \mathrm{cl}(\mathcal{O}_v)$. It is clear that $\mathrm{cl}(\mathcal{O}_{v'}) = \bigcup_{u \leqslant v'} \mathcal{O}_u$. The conclusion of 7.5 now follows.

Let $s \in S$ and $v \in V$. Following Section 5 we let $n(s, v) =$

$\{m(s)\cdot v\} \cup \mathscr{E}(s, m(s)\cdot v)\}$. Note that $v \in p(s, v)$. For $v \in V$, let $V_{\leqslant}(v) = \{v' \in V \mid v' \leqslant v\}$.

7.6 LEMMA. *Let $s \in S$ and $v \in V$. Then $P_s \mathcal{O}_v = \bigcup_{v' \in p(s,v)} \mathcal{O}_{v'}$.*
The proof follows immediately from the definitions and from 4.2.1.

7.7 PROPOSITION. *Let $s \in S$ and $v \in V$ be such that $v \mapsto m(s)\cdot v$. Then $V_{\leqslant}(m(s)\cdot v) = \bigcup_{u \leqslant v} p(s, u)$.*
The proof follows immediately from 7.5 and 7.6.

We see from 7.7 that the Bruhat order on V satisfies the one-step property of 5.12. We now adapt the terminology of Section 5 for the M-set V. In particular, reduced decompositions and subexpressions are defined as in Section 5. It is immediate from 7.2(ii) that every element of V has a reduced decomposition.

7.8 PROPOSITION. *Let $(\mathbf{v}, \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of $v \in V$ and let $a = \varphi(v)$. Then $a = s_k \circ \cdots \circ s_1 \circ 1$.*
The proof follows from 7.4 by induction on $l(v)$.

7.9 PROPOSITION. *Let $v \in V$ and let $a = \varphi(v)$. (i) Let $w \in \Gamma(a)$ (notation as in Section 3) and let $\mathbf{s} = (s_1, \ldots, s_k)$ be a reduced decomposition of w^{-1}. Then there exists a reduced decomposition $(\mathbf{v}, \mathbf{s}')$ of v with $\mathbf{s}' = \mathbf{s}$. (ii) Conversely, let $(\mathbf{v}, \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of v and let $w = s_k s_{k-1} \ldots s_1$. Then $w \in \Gamma(a)$ and $\mathbf{s}$ is a reduced decomposition of w^{-1}.*

Proof. (i) The proof is by induction on $l(v)$. If $l(v) = 0$, then the result is obvious. Assume that $k = l(v) > 0$. We have $a = s_k \circ s_{k-1} \circ \cdots \circ 1$, so that $s_k a < a$. By 7.4(ii), there exists $v_{k-1} \in V$ such that $v_{k-1} < m(s_k)\cdot v_{k-1} = v$ and $b = \varphi(v_{k-1}) = s_{k-1} \circ \cdots \circ s_1 \circ 1$. Let $w' = s_{k-1}s_{k-2}\ldots s_1$. It is clear that $l(v_{k-1}) = k-1$, that $w' \in \Gamma(b)$ and that $(s_1, \ldots, s_{k-1})$ is a reduced decomposition of w'^{-1}. By induction there exists a sequence $\mathbf{v}' = (v_0, \ldots, v_{k-1})$ such that $(\mathbf{v}', (s_1, \ldots, s_{k-1}))$ is a reduced decomposition of v. The proof of (i) now follows easily. (ii) Since $(\mathbf{v}, \mathbf{s})$ is a reduced decomposition of v, we have $a = s_k \circ \cdots \circ s_1 \circ 1$ and $L(a) = k = l(v)$. Thus $(s_1, \ldots, s_k)$ is a reduced decomposition of w^{-1} and $w \in \Gamma(a)$.

7.10 COROLLARY. *Let $v, v' \in V$ be such that $\varphi(v) = \varphi(v')$ and let $(\mathbf{v}, \mathbf{s}')$ be a reduced decomposition of v. Then there exists a reduced decomposition $(\mathbf{v}', \mathbf{s}')$ of v' such that $\mathbf{s}' = \mathbf{s}$.*

It follows from 7.10 that the Bruhat order on V has Property 6.6. It now follows from 7.7 and 7.10 and from the results of Section 6 that the Bruhat order on V has Properties 5.10(a)–(e) of Section 5. Summarizing these results,

7.11 THEOREM. (Main Theorem.) *The Bruhat order on V has the following properties:*

(i) *The one-step property.*
(ii) *The subexpression property.*
(iii) *Property* 5.10b.
(iv) *Property $Z(s, v', v)$.*
(v) *The exchange property.*
(vi) *It satisfies the chain condition.*
(vii) *It is the weakest partial order on V which is compatible with the M-action.*

Furthermore, the Bruhat order on V agrees with the standard order on the M-set V.

We let $v_{\max}$ denote the unique maximal element of the poset V; thus $v_{\max}$ corresponds to the dense $(B \times K)$-orbit on G. We let $a_{\max} = \varphi(v_{\max})$. We have the following characterization of $a_{\max}$ (see [15, 5.2]): Let $\Pi = \{\alpha \in \Delta \mid a_{\max}\theta(\alpha) = \alpha\}$. Then $a_{\max} = w_\Pi w_0$. We note that Π is the set of simple roots which are marked by a black dot in the diagram of (G, θ).

7.12 LEMMA. *Let $v \in V$, let $(\mathbf{v}' = (v'_0, \ldots, v'_k), \mathbf{s}' = (s'_0, \ldots, s'_k))$ be a reduced decomposition of v and let $l = l(v_{\max})$. Then there exists a reduced decomposition $(\mathbf{v} = (v_0, \ldots, v_l), \mathbf{s} = (s_1, \ldots, s_l))$ of $v_{\max}$ with $v'_j = v_j$ for $j \in [0, k]$ and $s'_j = s_j$ for $j \in [1, k]$. Thus every reduced decomposition of v can be extended to a reduced decomposition of $v_{\max}$.*
This follows immediately from [15, 5.2] and from the case analysis of 7.3.

The following proposition characterizes the image of the map $\varphi: V \to \mathscr{I}$ in terms of the weak order $\vdash$ on the M-set $\mathscr{I}$.

7.13 THEOREM. *Let $a \in \mathscr{I}$. Then $a \in \operatorname{image}(\varphi)$ if and only if $a \vdash a_{\max}$.*

Proof. Let $k = L(a)$ and let $l = L(a_{\max})$. Assume that $a = \varphi(v)$ for some $v \in V$. Let $(\mathbf{v}' = (v_0, \ldots, v_k), \mathbf{s}' = (s_1, \ldots, s_k))$ be a reduced decomposition of v. Extend $(\mathbf{v}', \mathbf{s}')$ to a reduced decomposition $(\mathbf{v} = (v_0, \ldots, v_l), \mathbf{s} = (s_1, \ldots, s_l))$ of $v_{\max}$. Then $a_{\max} = s_l \circ \cdots \circ s_{k+1} \circ a$ and $L(a_{\max}) = (l-k) + L(a)$. It follows from the definition of the weak order that $a \vdash a_{\max}$. Conversely, assume that $a \vdash a_{\max}$. By 3.19, there exists a sequence $\mathbf{s} = (s_1, \ldots, s_l)$ such that $a = s_k \circ \cdots \circ s_1 \circ 1$ and $a_{\max} = s_l \circ \cdots \circ s_1 \circ 1$. Let $w = s_l s_{l-1} \cdots s_1$. Then $w \in \Gamma(a_{\max})$ and $\mathbf{s}$ is a reduced decomposition of $(a_{\max})^{-1}$. By 7.9, there exists a sequence $\mathbf{v} = (v_0, \ldots, v_l)$ such that $(\mathbf{v}, \mathbf{s})$ is a reduced decomposition of $v_{\max}$. Thus we have

$$v_k = (m(s_k)m(s_{k-1})\ldots m(s_1))\cdot v_0 \quad \text{and} \quad \varphi(v_k) = s_k \circ \cdots \circ s_1 \circ 1 = a,$$

7.14 COROLLARY. image(φ) $= \mathscr{I}$ *if and only if* $a_{\max} = w_0$.
This follows from 3.18(iii).

7.15 COROLLARY. $a_{\max}$ *satisfies Condition* 3.22.
Proof. It follows from 2.1 that image(φ) is a union of twisted W-orbits. The proof now follows from 7.13.

7.16 PROPOSITION. *The following conditions on* $a \in \mathscr{I}$ *are equivalent*: (i) $a \in \text{image}(\varphi)$; *and* (ii) *there exists* $b \in W * a$ *such that* $E_-(b) \subset E_-(a_{\max})$.
The proof follows from 3.24, 7.13 and 7.15.

7.17 COROLLARY. *The following two conditions are equivalent*: (i) $a_{\max} \in W * 1$; *and* (ii) $\varphi(V) = W * 1$.

8. The Bruhat order on twisted involutions

Let the notation be as in Section 3. In particular $M = M(W)$. We define the Bruhat order on the set $\mathscr{I}$ of twisted involutions to be the standard order $\preceq$ on the M-set $\mathscr{I}$, as defined in Section 5. We show that the Bruhat order on $\mathscr{I}$ is compatible with the (twisted) M-action, so that it is the weakest partial order on $\mathscr{I}$ compatible with the M-action. We also show that the Bruhat order on $\mathscr{I}$ has properties analogous to all of the usual properties of the Bruhat order on the Weyl group W. Our proofs are somewhat unsatisfactory, since at one stage we have used the classification of involutions of semisimple groups, properties of the Bruhat order on V and properties of the map $\varphi: V \to \mathscr{I}$.

We let $\leqslant$ denote the usual Bruhat order on W. It follows from Section 4 that the function $L: \mathscr{I} \to \mathbb{N}$ is the length function on the M-set $\mathscr{I}$. Let the left action, the right action, and the twisted action of M on W be defined as in 3.10 and 3.11. If $w, w' \in W$ and $s \in S$, then $w \leqslant w'$ implies that $m(s) \cdot w \leqslant m(s) \cdot w'$ and $w \cdot m(s) \leqslant w' \cdot m(s)$.

8.1 LEMMA. *The partial order on* $\mathscr{I}$ *induced by the Bruhat order on* W *is compatible with the twisted* M*-action on* $\mathscr{I}$ *(with respect to the length function* L *on* $\mathscr{I}$*).*

Proof. Let $a, b \in \mathscr{I}$ and $s \in S$. Then $a \leqslant m(s) \cdot a \leqslant m(s) \cdot a \cdot m(\theta(s)) = m(s) * a$. If $a \leqslant b$, then $m(s) \cdot a \leqslant m(s) \cdot b$ and hence

$$m(s) * a = (m(s) \cdot a) \cdot m(\theta(s)) \leqslant (m(s) \cdot b) \cdot m(\theta(s)) = m(s) * b$$

Assume now that $a \leqslant b$ and $L(a) \geqslant L(b)$. By 3.18(iii), there exists a sequence $(s_1, \ldots, s_k)$ in S such that $s_k \circ \cdots \circ s_1 \circ a = w_0$ and $L(a) + k = L(w_0)$. This implies that

Thus we obtain

$$w_0 = m(s_k) * \cdots * m(s_1) * a \leqslant m(s_k) * \cdots * m(s_1) * b \leqslant w_0,$$

so that $w_0 = m(s_k) * \cdots m(s_1) * b$. This gives $L(w_0) \leqslant L(b) + k \leqslant L(a) + k = L(w_0)$, which shows that $L(b) + k = L(w_0)$. An easy argument now shows that $s_k \circ \cdots \circ s_1 \circ b = w_0$, so that we have $s_k \circ \cdots \circ s_1 \circ a = s_k \circ \cdots \circ s_1 \circ b$. Since, for $s \in S$, the map $\eta(s): \mathscr{I} \to \mathscr{I}$ given by $\eta(s)(c) = s \circ c$ is a bijection, we see that $a = b$. This proves 8.1.

8.2 COROLLARY. *The Bruhat order on* $\mathscr{I}$ *is compatible with the* M*-action.*
This follows from 5.6.

At this stage we shall need to use some results concerning the Bruhat order on V. Let the notation be as in Section 2. We recall that a parabolic subgroup P of G is θ-split if P and $\theta(P)$ are opposite parabolic subgroups, i.e. if $P \cap \theta(P)$ is a Levi subgroup of both P and $\theta(P)$. We say that (G, θ) is *quasi-split* if there exists a θ-split Borel subgroup. It is known [16] that (G, θ) is quasi-split if and only if $\Pi = \{\alpha \in \Delta \mid a_{\max}\theta(\alpha) = \alpha\}$ is the empty set. Hence we have:

8.3 LEMMA. *The following conditions on* (G, θ) *are equivalent*: (i) (G, θ) *is quasi-split*; (ii) $a_{\max} = w_0$; *and* (iii) $\varphi: V \to \mathscr{I}$ *is surjective.*
The proof follows from 7.14.

We return now to the notation of Section 3. We let G denote the simply connected algebraic group with root system Φ.

8.4 LEMMA. *There exists an involution* θ_1 *of* G *such that* (G, θ_1) *is quasi-split and the action of* θ_1 *on* Φ *agrees with* θ.

Proof. This follows from an inspection of the tables in [16]. In [loc. cit., no. 6] there is an *a priori* proof for the case of inner involutions. The general case follows by taking the product with a 'diagram automorphism'.

Denote the involution θ_1 of G given by 8.4 by θ. Let the notation for (G, θ) be as in earlier sections. Then we may identify W, $\mathscr{I}$, etc. in this section with the corresponding $W = W(T)$, $\mathscr{I}$, etc. of earlier sections.

8.5 THEOREM. (Exchange Property.) *Let* $b \in \mathscr{I}$, *let* $\mathbf{s} = (s_1, \ldots, s_k)$ *be an admissible sequence for* b *and let* $s \in S$ *be such that* $s \circ b \mapsto b$. *Then there exists* $i \in [1, k]$ *such that* $s \circ b = s_k \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_1 \circ 1$.

Proof. Let $v \in V$ be such that $\varphi(v) = b$. By 7.9, there exists a reduced decomposition $(\mathbf{v}, \mathbf{s})$ of s with $\mathbf{s}$ an admissible sequence for b. Since $s \circ b \mapsto b$, it follows from 7.4 that there exists $v' \in V$ such that $v' \mapsto m(s) \cdot v' = v$ and $\varphi(v') = s \circ b$. The conclusion of Theorem 8.5 now follows from the exchange property (Property 5.12(e)) for (V, M), applied to (v', v).

on $\mathcal{I}$ has the subexpression property. The other properties of the Bruhat order on $\mathcal{I}$ will then follow from the results of Section 6. Our proofs will make strong use of the fact that the maps $\eta(s)\colon \mathcal{I} \to \mathcal{I}$ are bijective. Arguments of this type are not possible for the Bruhat order on V.

8.6 LEMMA. *Let $a, b \in \mathcal{I}$ with $L(b) = L(a) + 1$. Let $(s_1, \ldots, s_k)$ be an admissible sequence for b. Assume that there exists $s \in S$ and a sequence $(s_{k+1}, \ldots, s_{k+r})$ in S such that $c = s \circ s_{k+r} \circ \cdots \circ s_{k+1} \circ a$ is equal to $s_{k+r} \circ \cdots \circ s_{k+1} \circ b$ and $L(c) = r + L(b)$. Then there exists $i \in [1, k]$ such that $a = s_k \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_1 \circ 1$.*

Proof. The proof is by induction on r. Let $r = 0$. Then $a = s \circ b \mapsto b$, and by the exchange property there exists $i \in [1, k]$ such that $a = s_k \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_1 \circ 1$. Now let $r > 0$. We have $s \circ c \mapsto c$ and $(s_1, \ldots, s_{k+r})$ is an admissible sequence for c. By the exchange property, there exists $i \in [1, k + r]$ such that

$$s_{k+r} \circ \cdots \circ s_{k+1} \circ a = s \circ c = s_{k+r} \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_1 \circ 1.$$

There are two cases to consider:

Case 1. $i \leqslant k$. Then

$$s_{k+r} \circ \cdots \circ s_{k+1} \circ a = s_{k+r} \circ \cdots \circ s_{k+1} \circ s_k \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_1 \circ 1.$$

Since the maps $\eta(s)$ are bijective, this implies that $a = s_k \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_1 \circ 1$.

Case 2. $i > k$. Then

$$s_{k+r} \circ \cdots \circ s_{k+1} \circ a = s_{k+r} \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_{k+1} \circ b.$$

Cancelling, we get

$$s_i \circ s_{i-1} \circ \cdots \circ s_{k+1} \circ a = s_{i-1} \circ \cdots \circ s_{k+1} \circ b.$$

Since $i - 1 \leqslant k + r - 1$, we see by induction that there exists $j \in [1, k]$ such that $a = s_k \circ \cdots \circ \hat{s}_j \circ \cdots \circ s_1 \circ 1$. This proves 8.6.

We recall that $a \to b$ if and only if $a \preceq b$ and $l(a) + 1 = l(b)$.

8.7 THEOREM. (Strong Exchange Property.) *Assume that $a \to b$ and let $(s_1, \ldots, s_k)$ be an admissible sequence for b. Then there exists $i \in [1, k]$ such that*

$$a = s_k \circ \cdots \circ \hat{s}_i \circ \cdots \circ s_1 \circ 1.$$

Proof. By 3.18, there exists a sequence $(t_1, \ldots, t_p)$ in S such that $w_0 = t_p \circ \cdots \circ t_1 \circ a$ and $L(w_0) = p + L(a)$. The compatibility of the M-action with $\preceq$ implies that

$$w_0 = m(t_p) * \cdots * m(t_1) * a \preceq m(t_p) * \cdots * m(t_1) * b \preceq w_0,$$

so that $w_0 = m(t_p) * \cdots * m(t_1) * b$. Since $L(b) + p = L(w_0) + 1$, there exists a unique integer $r \in [1, p]$ such that

$$d = m(t_r) * m(t_{r-1}) * \cdots * m(t_1) * b = m(t_{r-1}) * \cdots * m(t_1) * b.$$

Thus we have $c = m(t_r) * \cdots * m(t_1) * a \preceq d$. Since $L(c) = r + L(a) = r - 1 + L(b) \geqslant L(d)$, it follows from the compatibility of $\preceq$ with the M-action that

$$t_r \circ \cdots \circ t_1 \circ a = c = d = t_{r-1} \circ \cdots \circ t_1 \circ b.$$

The conclusion of Theorem 8.7 now follows from 8.6.

We are now in a position to apply the results of Section 6 to the case of twisted involutions. First we make the following observation:

8.8 OBSERVATION. *Let $s \in S$ and let $a \in \mathcal{I}$. Then $p(s, a) = \{a, s \circ a\}$. If $sa < a$, then $\mathcal{E}(s, a) = \{s \circ a\}$ and if $sa > a$, then $\mathcal{E}(s, a)$ is empty.*

8.9 REMARK. It follows from 8.8 that in the definition of a subexpression for the M-set $\mathcal{I}$, the alternative (β) of Definition 5.8 does not occur.

8.10 COROLLARY. *The subexpression property holds for the Bruhat order on $\mathcal{I}$.*

Proof. It follows from the definition that the Bruhat order on $\mathcal{I}$ satisfies the chain condition. The subexpression property now follows easily from 8.7 by induction.

It follows from 8.10 and the results of Section 6 that the Bruhat order on $\mathcal{I}$ has the Properties 5.10(a)–(e). The results 8.11–8.14 below are either reformulations of these properties in the framework of twisted involutions or else follow easily from such reformulations.

8.11 PROPOSITION. *Let $a, b \in \mathcal{I}$ and let $w \in \Gamma(b)$. Then $a \preceq b$ if and only if there exists $w_1 \leqslant w$ such that $a = m(w_1) * 1$. If $a \preceq b$, then there exists $w_1 \leqslant w$ such that $w_1 \in \Gamma(a)$.*

For $b \in \mathcal{I}$, let $\mathcal{I}_{\preceq}(b) = \{a \in \mathcal{I} \mid a \preceq b\}$.

8.12 PROPOSITION. (One-Step Property.) *Let $s \in S$ and $b \in \mathcal{I}$ be such that $b \mapsto s \circ b$. Then $\mathcal{I}_{\preceq}(s \circ b) = \bigcup_{a \preceq b} \{a, s \circ a\}$.*

Note that Proposition 8.12 gives an easy inductive definition of the Bruhat order on $\mathcal{I}$.

8.13 PROPOSITION. (Property $Z(s, a, b)$.) *Let $a, b \in \mathcal{I}$ and $s \in S$ be such that $a \mapsto s \circ a$ and $b \mapsto s \circ b$. Then the following three conditions are equivalent*: (i) $a \preceq b$; (ii) $s \circ a \preceq s \circ b$; *and* (iii) $a \preceq s \circ b$.

8.14 PROPOSITION. *Let* $a, b \in \mathscr{I}$ *and* $s \in S$ *be such that* $a \prec s \circ b$ *and* $b \prec s \circ b$. *Then one of the following two conditions holds*: (i) $a \preceq b$; *or* (ii) *there exists* $c \prec b$ *such that* $c \mapsto s \circ c = a$.

8.15 DUALITY FOR TWISTED INVOLUTIONS. Let $\iota = -w_0$ denote the opposition involution of the root system Φ. Set $\theta_1 = \iota\theta = -w_0\theta = -\theta w_0$. Then θ_1 is an involution of the root system Φ. We will show that there exists a natural duality between the twisted involutions for θ and θ_1 which reverses the respective Bruhat orders. First a few words about notation. We carry over the notation of Sections 3 and 8 from the case (Φ, W, θ) to the case (Φ, W, θ_1), always using a subscript '1' to denote that we are dealing with θ_1 rather than θ. Thus, for example: (i) $\mathscr{I}_1 = \{c \in W \mid \theta_1(c) = c^{-1}\}$ is the set of twisted involutions for θ_1; (ii) if $s \in S$, then $m(s) *_1 c$ denotes the twisted action of $m(s)$ on $c \in \mathscr{I}_1$; (iii) if $c \in \mathscr{I}_1$, then $I_1(c)$ is the set of reflections which are imaginary with respect to c and θ_1; and (iv) $\preceq_1$ denotes the Bruhat order on $\mathscr{I}_1$.

8.16 LEMMA. *Define* $\delta\colon W \to W$ *by* $\delta(w) = ww_0$. *Then* δ *maps* $\mathscr{I}$ *bijectively onto* $\mathscr{I}_1$.

We omit the proof.

Let $a \in \mathscr{I}$. Then $\delta(a)\theta_1 = -a\theta$. Thus we see that $I(a) = R_1(\delta(a))$, $R(a) = I_1(\delta(a))$, $S''(a) = S'_1(\delta(a))$, and $S'(a) = S''_1(\delta(a))$. As a consequence of these observations, we obtain:

8.17 LEMMA. *Let* $s \in S$ *and* $a \in \mathscr{I}$. *Then* $\delta(s \circ a) = s \circ_1 \delta(a)$. *Moreover we have*: (i) $a \mapsto m(s) * a \Leftrightarrow m(s) *_1 \delta(a) = \delta(a)$; *and* (ii) $a = m(s) * a \Leftrightarrow \delta(a) \mapsto m(s) *_1 \delta(a) = \delta(s \circ a)$.

8.18 LEMMA. *If* $a \in \mathscr{I}$, *then* $L_1(\delta(a)) = L(w_0) - L(a)$.

Proof. Let $\mathbf{s} = (s_1, \ldots, s_k)$ be an admissible sequence for a. Then $k = L(a)$ and $a = s_k \circ \cdots \circ s_1 \circ 1$. Using 8.17, we obtain $w_0 = \delta(1) = s_1 \circ_1 \cdots \circ s_k \circ_1 \delta(a)$ and $L(w_0) = k + L_1(\delta(a))$.

We define a partial order $\preceq_2$ on $\mathscr{I}_1$. Let $a, b \in \mathscr{I}$. Then $\delta(b) \preceq_2 \delta(a)$ if and only if $a \preceq b$. We wish to show that the partial orders $\preceq_1$ and $\preceq_2$ are equal. The main step is the following proposition:

8.19 PROPOSITION. *The partial order* $\preceq_2$ *on* $\mathscr{I}_1$ *is compatible with the* M-*action.*

Proof. Let $a, b \in \mathscr{I}$ and let $s \in S$. First we want to show that $\delta(a) \preceq_2 m(s) *_1 \delta(a)$. If $a \mapsto m(s) * a$, then $m(s) *_1 \delta(a) = \delta(\alpha)$ and we are done. Assume that $a = m(s) * a$. Then $s \circ a \mapsto a$ and consequently $\delta(a) \preceq_2 \delta(s \circ a) = m(s) *_1 \delta(a)$. Next we want to show that if $a \preceq b$, then

(*) $m(s) *_1 \delta(b) \preceq m(s) *_1 \delta(a)$.

It will suffice to show that (i) holds when $a \to b$. There are the usual four cases to consider:

Case 1. $a = m(s) * a$ and $b = m(s) * b$.
Case 2. $a = m(s) * a$ and $b \mapsto m(s) * b$.
Case 3. $a \mapsto m(s) * a$ and $b \mapsto m(s) * b$.
Case 4. $a \mapsto m(s) * a$ and $b = m(s) * b$.

In Case 1, we have: (a) $s \circ a \mapsto a$; (b) $s \circ b \mapsto b$; (c) $\delta(a) \mapsto m(s) *_1 \delta(a) = \delta(s \circ a)$; and (d) $\delta(b) \mapsto m(s) *_1 \delta(b) = \delta(s \circ b)$. Since $a \to b$, it follows from Property 5.10(b) that $s \circ a \to s \circ b$, so that $m(s) *_1 \delta(b) \preceq_2 m(s) *_1 \delta(a)$.

In Case 4, we have $a \mapsto s \circ a = m(s) * a \preceq m(s) * b = b$. Since $L(s \circ a) = L(b)$, we have $s \circ a = b$, which implies that $b = s \circ a$. Thus $m(s) *_1 \delta(b) = \delta(s \circ b) = \delta(a) = m(s) *_1 \delta(a)$.

The proofs in Cases 2 and 3 are similar (but easier) and will be left to the reader. Thus (*) holds in all four cases.

Assume now that $\delta(b) \preceq_2 \delta(a)$ and $L_1(\delta(a)) \leqslant L_1(\delta(b))$. Then $a \preceq b$ and it follows from 8.18 that $L(b) \leqslant L(a)$. This implies that $a = b$ by the compatibility of $\preceq$ with the M-action. Thus $\delta(b) = \delta(a)$. This proves 8.19.

8.20 THEOREM. *Let* $a, b \in \mathscr{I}$. *Then* $a \preceq b$ *if and only if* $\delta(b) \preceq_1 \delta(a)$.

Proof. We need to show that the partial orders $\preceq_1$ and $\preceq_2$ on $\mathscr{I}_1$ are equal. It follows from 8.19 and 5.6 that $\preceq_1$ is weaker than $\preceq_2$. Let $Q_0 = \{(a, b) \in \mathscr{I} \times \mathscr{I} \mid a \preceq b\}$, let $Q_1 = \{(c, d) \in \mathscr{I}_1 \times \mathscr{I}_1 \mid c \preceq_1 d\}$ and let $Q_2 = \{(c, d) \in \mathscr{I} \times \mathscr{I} \mid c \preceq_2 d\}$. Then $|Q_0| = |Q_2|$ and $Q_1 \subset Q_2$, so that $|Q_1| \leqslant |Q_0|$. If we reverse the roles of θ and θ_1, we see that $|Q_0| \leqslant |Q_1|$ so that $|Q_0| = |Q_1|$. Thus $|Q_1| = |Q_2|$, which implies that $Q_1 = Q_2$.

8.21 REMARK. We note that elements of minimal length in twisted W-orbits on $\mathscr{I}$ correspond to elements of maximal length in twisted W-orbits on $\mathscr{I}_1$. Thus Propositions 3.3 and 3.5 of [15] are equivalent.

The following result is an interesting consequence of 8.3 and the classification of involutions.

8.22 PROPOSITION. *Let* G *be a simple group of adjoint type, let* T' *be a maximal torus of* G *and let* $c \in W(T')$ *be an involution. Then there exists an involution* $n \in N(T')$ *which represents* c.

Proof. By [16, No. 4] there exists an inner automorphism $\theta = \mathrm{Int}(a)$ of G such that (G, θ) is quasi-split. Since G is adjoint, $a^2 = 1$. Let (B, T) be a standard pair for (G, θ) and let the notation be as usual. Then $\mathscr{I}$ is the set of involutions in W and $\omega\colon V \to \mathscr{I}$ is surjective. An easy argument shows that

$a \in T$. It suffices to prove 8.22 for $T' = T$. Let $C(a)$ denote the conjugacy class of a in G. Then $\tau(G) = C(a)a^{-1}$. Thus $\mathscr{I} = \varphi(V) = (N(T) \cap \tau(G)) \bmod T = C(a) \bmod T$. Since $C(a)$ consists of involutions, we are done.

9. Classification of θ-stable maximal tori

In Section 2 we showed that there is a bijective correspondence between the set $\mathscr{T}^\theta/K$ of K-classes of θ-stable maximal tori and the set image$(\varphi)/W$ of twisted W-orbits in the image of φ. However, this correspondence is still not very clear. In this section we shall clarify this correspondence. We obtain as a consequence a classification of K-classes of θ-stable maximal tori which is an exact analogue of the classification of Kostant [10] of conjugacy classes of Cartan subalgebras of a real semisimple Lie algebra. A similar classification of θ-stable maximal tori has been given by Helminck [9], although his approach is different from ours. He also treats the much more difficult problem of classifying classes of maximal k-rational, θ-stable tori, where k is a field of definition for (G, θ). His results give an explanation of the similarity between our results and those of Kostant.

9.1. If D is a subtorus of an algebraic torus C, then $E(D) = X_*(D) \otimes_{\mathbb{Z}} \mathbb{R}$ is a linear subspace of $E(C) = X_*(C) \otimes_{\mathbb{Z}} \mathbb{R}$. Assume now that C is a θ-stable subtorus of G. We set $C_+ = (K \cap C)^0$ and $C_- = \{c \in C \mid \theta(c) = c^{-1}\}^0$. We set $E_+(C) = E(C_+)$ and $E_-(C) = E(C_-)$. Clearly $E_+(C)$ (resp. $E_-(C)$) is the $+1$ eigenspace (resp. -1 eigenspace) of θ on $E(C)$ and $E(C) = E_+(C) \oplus E_-(C)$. We let (B, T) be as in earlier sections and we let $E = E(T)$.

9.2 LEMMA. *Let C and D be θ-stable maximal tori of G. Then C and D are K-conjugate if and only if C_- and D_- are K-conjugate.*

Proof. It is clear that if C and D are K-conjugate, then C_- and D_- are K-conjugate. Assume now that C_- and D_- are K-conjugate. After conjugating by an element of K, we may assume that $C_- = D_-$. Let $H = Z_G(C_-)$, the centralizer of C_- in G. Then H is a θ-stable connected reductive group and C_+ and D_+ are maximal tori of $(K \cap H)^0 = K_1$. Thus C_+ and D_+ are K_1-conjugate, so that C and D are K-conjugate.

We continue with the notations of earlier sections. Let $y \in \mathscr{V}$ be a representative of the maximal element $v_{\max}$ of V and let $T_0 = y^{-1}Ty$. It is known [16, §1] that $A = (T_0)_-$ is a maximal θ-split torus of G. Let $\zeta\colon E(T_0) \to E(T) = E$ be the linear isomorphism determined by Int(y). We pull back the inner product on E to $E(T_0)$ by means of ζ; thus ζ is an isometry. The isometry ζ determines an isomorphism, again denoted by ζ, of $W(T_0)$ onto $W = W(T)$. The involution θ acts on $E(T_0)$ and E by isometries. We have the following result, whose proof will be omitted:

9.3 LEMMA. (i) *If $x \in E(T_0)$, then $\zeta(\theta(x)) = a_{\max}\theta(\zeta(x))$.* (ii) *$\zeta$ maps $E_-(T_0) = E(A)$ onto $E_-(a_{\max})$.*

We set $E_- = E_-(a_{\max})$. Let $W(A) = N_G(A)/Z_G(A)$, let

$$W_1(T_0) = \{w \in W(T_0) \mid w \cdot A = A\}$$

and

$$W_2(T_0) = \{w \in W_1(T_0) \mid w(a) = a (a \in A)\}.$$

The following results are proved in [13, §4]:

9.4 PROPOSITION. (i) *The canonical map $N_K(A)/Z_K(A) \to W(A)$ is an isomorphism.* (ii) *The canonical map $W_1(T_0)/W_2(T_0) \to W(A)$ is an isomorphism.*

Let $\zeta\colon W(T_0) \to W$ be as above and let $W_i = \zeta(W_i(T_0))$, $i = 1, 2$. Let $W_0 = W_1/W_2$. It follows easily from the definitions that W_1 is the centralizer of $a_{\max}\theta$ in W. We consider W_0 as a group of isometries of E_-; the group W_0 is the group of isometries of E_- which are induced by the elements of W_1.

9.5 REMARK. Let $\Phi(A) \subset X^*(A)$ denote the set of restrictions to A of the elements of $\Phi(T_0)$. We may identify $\Phi(A)$ with a subset of $E(A)$ by means of the duality between $X_*(A)$ and $X^*(A)$ and the inner product on $E(A)$ (induced by the inner product on $E(T_0)$). It is shown in [13, §4] that $\Phi(A)$ is a (not necessarily reduced) root system in $E(A)$ and that $W(A)$ is the corresponding Weyl group. The group $W(A)$ is the 'little Weyl group' associated to (G, θ). Let Φ_0 denote the image $\zeta(\Phi(A)) \subset E_-$. Then Φ_0 is a root system in E_- and W_0 is the corresponding Weyl group. Let $\Psi_0 = \Phi(T_0) \cap E(A)$ and let $\Psi = \Phi \cap E_-$. Then $\Psi = \zeta(\Psi_0)$. We also have $\Psi = \{\alpha \in \Phi \mid a_{\max}\theta(\alpha) = -\alpha\}$. We note that Ψ is a subset of both Φ and Φ_0; however, in general, Φ_0 is not a subset of Φ. In particular, we can (and shall) consider Ψ as a root subsystem of Φ_0. An easy argument shows that Ψ is W_0-stable.

9.6 LEMMA. (i) *Let C be a θ-stable maximal torus of G. Then C_- is K-conjugate to a subtorus of A.* (ii) *Let A_1 and A_2 be subtori of A. Then A_1 and A_2 are K-conjugate if and only if they are $W(A)$-conjugate.*

The proof of (i) uses the K-conjugacy of maximal θ-split tori [20] and (ii) follows from [13, 11.1].

9.7 DEFINITION. A subtorus A_1 of A is *admissible* if there exists a θ-stable maximal torus C of G with $C_- = A_1$. A linear subspace E_1 of E_- is *admissible* if there exists an admissible subtorus A_1 of A such that $E_1 = \zeta(E(A_1))$.

We let $\mathscr{A}(E_-)$ denote the subset of admissible linear subspace of E_-. The action of W_0 on E_- induces an action of W_0 on $\mathscr{A}(E_-)$.

9.8 CLASSIFICATIONS OF θ-STABLE MAXIMAL TORI–FIRST STEP. Using 9.2 and 9.6, we can set up a natural bijection of the set $\mathscr{T}^\theta/K$ onto the set $\mathscr{A}(E_-)/W_0$ of W_0-orbits on $\mathscr{A}(E_-)$. This goes as follows. Let C be a θ-stable maximal torus. Then C_- is K-conjugate to an admissible subtorus A_1 of A. We let $\mu(C)\in\mathscr{A}(E_-)/W_0$ denote the W_0-orbit of the admissible subspace $\zeta(E(A_1))$. Then it follows from 9.2 and 9.6 that $\mu(C)$ is independent of the choices made and the map μ induces a bijection μ_0: $\mathscr{T}^\theta/K\to\mathscr{A}(E_-)/W_0$. Next we need a better characterization of the admissible subspaces of E_-. This is given in Proposition 9.11 below. First we need two technical lemmas.

9.9 LEMMA. *Assume that (G,θ) is θ-split, so that A is a maximal torus of G. Then the following two conditions are equivalent*: (i) $-1\in W(A)$; *and* (ii) *θ is an inner automorphism of G.*

Proof. (ii) $\Rightarrow$ (i). Assume that $\theta=\mathrm{Int}(h)$ for some $h\in G$. Then $h\in N(A)$ and it is clear that the image of h in $W(A)$ is equal to -1. (i) $\Rightarrow$ (ii). Let $g\in N(A)$ represent $-1\in W(A)$. Then $\theta\circ\mathrm{Int}(g)$ acts trivially on the maximal torus A. It then follows from a standard result [1, 14.9] that $\theta\circ\mathrm{Int}(g)$ is an inner automorphism, so that θ is an inner automorphism.

9.10 LEMMA. *Let $v\in V$, let $x\in\mathscr{V}$ be a representative of v, let $a=\varphi(v)$ and let $T_1=x^{-1}Tx$. Then $E_-(a)$ is equal to the subspace $E(\mathrm{Int}(x)\cdot E(T_1)_-)$ of $E(T)=E$.*

We omit the proof, which follows by a straightforward argument.

The following proposition characterizes admissible subspaces of E_- in terms of twisted involutions:

9.11 PROPOSITION. *Let E_1 be a linear subspace of E_-. Then the following three conditions are equivalent*:

(i) *E_1 is an admissible subspace of E_-*;
(ii) *there exists $a\in\mathscr{I}$ such that $E_1=E_-(a)$; and*
(iii) *there exists an involution c in W such that E_- is the orthogonal direct sum of E_1 and $E(c,-1)$.*

Proof. (ii) $\Leftrightarrow$ (iii). This follows from 3.20.

(i) $\Rightarrow$ (ii). Let A_1 be an admissible subtorus of A with $\zeta(E(A_1))=E_1$ and let $C\in\mathscr{T}^\theta$ be such that $C_-=A_1$. An argument similar to the proof of 9.2 shows that there exists $h\in Z_G(A_1)$ such that $\mathrm{Int}(h)(C)=T_0$. Thus $\mathrm{Int}(yh)(C)=T$, which implies that $yh\in\mathscr{V}$. Let v denote the image of yh in V and let $a=\varphi(v)$. It follows from 9.10 that $\mathrm{Int}(yh)$: $E(C)\to E(T)$ maps $E(A_1)$ onto $E_-(a)$. But since $h\in Z_G(A_1)$, we see that $\mathrm{Int}(yh)(E(A_1))$ is equal to $\mathrm{Int}(y)(E(A_1))=\zeta(E(A_1))=E_1$.

(iii) $\Rightarrow$ (i). Let $d=\mathrm{Int}(y^{-1})(c)$, where we consider $\mathrm{Int}(y^{-1})$ as an isomorphism of $W(T)$ onto $W(T_0)$. Let $\Phi_d(T_0)=\{\alpha\in\Phi(T_0)\mid d(\alpha)=-\alpha\}$. For each $\alpha\in\Phi_d(T_0)$, let the root subgroup U_α be defined as in 1.1. Let H be the subgroup of G generated by the set of root subgroups $\{U_\alpha\mid\alpha\in\Phi_d\}$. Then H is a θ-stable connected semisimple subgroup of G. Let θ' denote $\theta|_H$ and let $A_2=(A\cap H)^0$. Then A_2 is a maximal torus of H so that H is θ'-split. Clearly the root system $\Phi(H,A_2)$ can be identified with $\Phi_d(T_0)$. It follows from the facts recalled after 3.20 that $-1\in W(A_2)$ and it then follows from 9.9 that θ' is an inner automorphism. Thus there exists a maximal torus C_2 of H which is contained in $H\cap K$. Let $A_1=\{t\in A\mid\alpha(t)=1\ (\alpha\in\Phi_d(T_0))\}^0$. It is clear that $\zeta(E(A_1))=E_1$. Moreover, $C=(T_0)_+A_1C_2$ is a θ-stable maximal torus of G and an easy argument shows that $C_-=A_1$, so that A_1 is an admissible subtorus. Therefore E_1 is an admissible subspace.

9.12 CLASSIFICATION OF θ-STABLE MAXIMAL TORI. If we combine the results of 9.8 and 9.11, we can get a reasonably simple description of the set $\Sigma=\mathscr{T}^\theta/K$ of K-conjugacy classes of θ-stable maximal tori of G. Let $\Psi\subset\Phi_0$ be as in 9.5. For each $\alpha\in\Phi_0$, let r_α denote the restriction of s_α to E_-. Let $W(\Psi)$ denote the subgroup of W_0 generated by the set of reflections $\{r_\alpha\mid\alpha\in\Psi\}$. Then Ψ is a root subsystem of Φ_0 and $W(\Psi)$ is the corresponding subgroup of W_0. (Warning: It is not necessarily the case that $W(\Psi)$ is a parabolic subgroup of the Weyl group W_0 of Φ_0.) Let $\mathscr{I}(\Psi)$ denote the set of involutions in $W(\Psi)$. It is clear that $\mathscr{I}(\Psi)$ is stable under conjugation by elements of W_0, so that W_0 acts on $\mathscr{I}(\Psi)$ by conjugation. If $c\in\mathscr{I}(\Psi)$, let $\sigma(c)$ denote the subspace $E_-(c,+1)$ (the $+1$ eigenspace of c on E_-) of E_-. Then it follows from 9.11 that $\sigma(c)$ is an admissible subspace of E_- and that σ: $\mathscr{I}(\Psi)\to\mathscr{A}(E_-)$ is a bijection. Clearly σ is W_0-equivariant, so that we get an induced bijection σ_0: $\mathscr{I}(\Psi)/W_0\to\mathscr{A}(E_-)/W_0$. Combining this with the results of 9.8, we obtain:

9.13 PROPOSITION. *Let the notation be as above. Then there is a canonical bijection from the set $\Sigma=\mathscr{T}^\theta/K$ to the set $\mathscr{I}(\Psi)/W_0$ of W_0-conjugacy classes of involutions in $W(\Psi)$. This bijection is given by the composition*

$$\mathscr{T}^\theta/K\xrightarrow{\mu}\mathscr{A}(E_-)/W_0\xrightarrow{\sigma_0}\mathscr{I}(\Psi)/W_0.$$

The following proposition will be useful in working out concrete examples of the Bruhat order on V.

9.14 PROPOSITION. *Assume that φ: $V\to\mathscr{I}$ is injective. Let* image(φ) *be* *given the order induced by the Bruhat order on $\mathcal{I}$. Then the mapping of V onto* image(φ) *given by φ is an isomorphism of ordered sets.*

Proof. If $v', v \in V$, we write $v' \to v$ if $v' < v$ and $l(v') + 1 = l(v)$, similarly for $\mathcal{I}$. Since both V and $\mathcal{I}$ satisfy the chain condition, it will suffice to prove the following statement:

(a) *The following conditions on $v', v \in V$ are equivalent*: (i) $v' \to v$; *and* (ii) $\varphi(v') \to \varphi(v)$.

Proof of (a). (i) $\Rightarrow$ (ii). Assume that $v' \to v$. Since the Bruhat order on V is the standard order on the M-set V, it follows that there exist $z \in V$, $t \in S$ and a sequence $\mathbf{s}$ in S such that:

(1) $v' = m(\mathbf{s}) \cdot z$ and $l(v') = l(z) + l(\mathbf{s})$; and
(2) $v = m(\mathbf{s}) \cdot m(t) \cdot z$ and $l(v) = l(z) + l(\mathbf{s}) + 1$.

It now follows from 7.4 and the definition of the standard order on $\mathcal{I}$ that $\varphi(v') \to \varphi(v)$.

(ii) $\Rightarrow$ (i). Let $a = \varphi(v')$ and $b = \varphi(v)$ and assume that $a \to b$. Then there exist $c \in \mathcal{I}$, $t \in S$ and a sequence $\mathbf{s}$ in S such that:

(1) $a = m(\mathbf{s}) * c$ and $L(a) = L(c) + l(\mathbf{s})$; and
(2) $b = m(\mathbf{s}) * m(t) * c$ and $L(b) = L(c) + l(\mathbf{s}) + 1$.

An easy inductive argument using 7.4(ii) shows that there exist $z_1, z_2 \in \mathcal{I}$ such that:

(3) $\varphi(z_1) = c$, $v' = m(\mathbf{s}) \cdot z_1$ and $l(v') = l(z_1) + l(\mathbf{s})$; and
(4) $\varphi(z_2) = c$, $v = m(\mathbf{s}) \cdot m(t) \cdot z_2$ and $l(v) = l(z_2) + l(\mathbf{s}) + 1$.

Since φ is injective, we see that $z_1 = z_2$. It now follows from (3) and (4) that $v' < v$.

9.15 COROLLARY. *φ is injective if and only if there exists a unique closed $(B \times K)$-orbit on G. This follows by using 7.10.*

9.16 PROPOSITION. (a) *The following four conditions are equivalent*:

(i) $\Psi = \varnothing$;
(ii) *there is exactly one K-conjugacy class of θ-stable maximal tori*;
(iii) $\varphi(V) = W * 1$;
(iv) $a_{\max} \in W * 1$.

(b) *Assume that the equivalent conditions* (i)–(iv) *above hold. Then φ is injective and induces an isomorphism of ordered sets from V to $W * 1$ ($W * 1$ is given the partial order induced by the Bruhat order on $\mathcal{I}$).*

Proof. (a) The equivalence of (i) and (ii) follows from 9.13, the equivalence of (ii) and (iii) follows from 2.7 and the equivalence of (iii) and (iv) follows from 7.17.

(b) It follows from 7.11, 7.12 and 7.13 that $\varphi^{-1}(\varphi(v_{\max})) = v_{\max}$. By 2.1 and (a) above, φ is a W-equivariant map from V to $W * 1$. Thus $\varphi^{-1}(\varphi(v)) = v$ for every $v \in V$, which shows that φ is injective. The final conclusion now follows from 9.14.

9.17 REMARKS. (i) See [14] for the classification of involutions in a Weyl group. The paper loc. cit. gives an easy algorithm which allows one to read off the conjugacy classes of involutions from the Dynkin diagram.

(ii) The classification given in 9.13 is an exact analogue of the classification of conjugacy classes on Cartan subalgebras of a real semisimple Lie algebra given by Kostant [10]. (See [21, 1.3.1, pp. 88–96] for a nice exposition of Kostant's results.) To make the connection between 9.13 and [10], one needs to use the facts mentioned after 3.20.

(iii) One can get a precise description of the root subsystem Ψ of Φ_0 from the tables in Helgason [8, pp. 532–534] (using the obvious correspondence between the Satake diagrams in [8] and the diagrams in [16]). One needs to use the fact that Ψ is W_0-stable and the fact that (in the notation of [8]) if $\lambda \in \Phi_0$, then $\lambda \in \Psi$ if and only if m_λ is odd.

(iv) We can consider $W(\Psi)$ as a subgroup of both W and of W_0. (However, one cannot in general identify W_0 with a subgroup of W.) By using the classification of involutions of G, it is not too difficult to show that two involutions in $W(\Psi)$ are conjugate in W_0 if and only if they are conjugate in W.

[*Sketch of the argument*: If θ is an inner involution, then the result follows from 2.7 and 7.16. If $\theta = \imath\,(= -w_0)$, then the result is straightforward. Using the classification, one only needs to directly check a few cases when G is of type D_n.]

10. Examples

In this section we study several concrete examples of the Bruhat order on symmetric varieties.

10.1 EXAMPLE. Let $G = G_1 \times G_1$, where G_1 is a connected reductive group and define the involutive automorphism $\theta: G \to G$ by $\theta(x, y) = (y, x)$. Then K, the fixed point subgroup of θ, is the diagonal subgroup of G. Let B_1 (resp. T_1) be a Borel subgroup (resp. maximal torus of G_1) with $T_1 \subset B_1$. If $B = B_1 \times B_1$ and $T = T_1 \times T_1$, then (B, T) is a standard pair for (G, θ). We follow the notation of earlier sections. Let $W_1 = W(T_1)$. Then

$W = W(T) = W_1 \times W_1$. The set $\mathscr{I}$ of twisted involutions of W is equal to $\{(w_1, w_1^{-1}) \mid w_1 \in W_1\}$. Let S_1 be the set of simple reflections for W_1 determined by B_1 and let $S = \{(s_1, 1) \mid s_1 \in S_1\} \cup \{(1, s_2) \mid s_2 \in S_1\}$. Then S is the set of simple reflections for W determined by B. If $s_1 \in S_1$ and $w_1 \in W_1$, then a simple computation shows that $(s_1, 1) * (w_1, w_1^{-1}) = (s_1 w_1, w_1^{-1} s_1)$. It follows from this that $\mathscr{I} = W * (1, 1)$. It is clear that every θ-stable maximal torus T' of G is of the form $T' = T_1' \times T_1'$, where T_1' is a maximal torus of G_1. Thus, there is exactly one K-conjugacy class of θ-stable maximal tori in G. By 9.14, this implies that $\varphi: V \to \mathscr{I}$ is an isomorphism of ordered sets.

10.1.1 LEMMA. *Define* $\rho: W_1 \to \mathscr{I}$ *by* $\rho(w_1) = (w_1, w_1^{-1})$. *Then* ρ *is an isomorphism of ordered sets.*

Proof. It is clear that ρ is a bijection and an easy argument shows that $l(w_1) = L(\rho(w_1))$ for $w_1 \in W_1$. Let $w_1, w_2 \in W_1$ and let $\mathbf{s} = (s_1, \ldots, s_k)$ be a sequence in S_1 which is a reduced decomposition for w_1. Then $((s_k, 1), \ldots, (s_1, 1))$ is an admissible sequence for $\rho(w_1)$. Assume that $w_2 \leqslant w_1$. Then there exists $1 \leqslant i_1 < \cdots < i_q \leqslant k$ such that $w_2 = s_{i_1} \ldots s_{i_q}$. It follows easily that

$$(10.1.2) \quad \rho(w_2) = (w_2, w_2^{-1}) = (s_{i_1}, 1) * \cdots * (s_{i_q}, 1) * (1, 1).$$

A straightforward argument using 8.11 shows that $\rho(w_2) \preceq \rho(w_1)$.

Assume now that $\rho(w_2) \preceq \rho(w_1)$. It follows from the subexpression property for $\preceq$ that there exists a sequence $1 \leqslant i_1 < \cdots < i_q \leqslant k$ such that (10.1.2) holds. But this implies that $w_2 = s_{i_1} \ldots s_{i_q}$, so that $w_2 \leqslant w_1$. Thus ρ is an isomorphism of ordered sets.

We have shown there exist canonical isomorphisms of ordered sets $\varphi: V \to \mathscr{I}$ and $\rho: W_1 \to \mathscr{I}$. Hence $\varphi^{-1} \circ \rho: W_1 \to V$ is an isomorphism of ordered sets. Consequently the Bruhat order on the Weyl group W_1 occurs as a special case of the Bruhat order on the set of orbits on a symmetric variety and also as a special case of the Bruhat order on the set of twisted involutions of a Weyl group.

10.1.3 REMARK. The isomorphism of ordered sets $\varphi^{-1} \circ \rho: W_1 \to V$ is, of course, well known. It is essentially the same as the map of W_1 into the 'canonical Weyl group' $\mathscr{W}_1 = \mathscr{W}(G_1)$ discussed in 1.8. (One works with K_1-orbits on $\mathscr{B}(G_1)$ rather than $(B_1 \times K_1)$-orbits on G_1 in order to make the correspondence.)

10.2 EXAMPLE. Let $G = \mathrm{GL}_n(F)$ and define an involutive automorphism $\theta_1: G \to G$ by $\theta_1(g) = {}^t g^{-1}$. The fixed point subgroup K_1 of θ_1 is the orthogonal group $\mathrm{O}_n(F)$. We will study the Bruhat order for Borel subgroup

orbits on the symmetric variety G/K_1. In order to make explicit matrix computations, it will be convenient to replace θ_1 by another involutive automorphism θ which is conjugate to θ_1 by an inner automorphism. Let $d_0 \in G$ be defined by $d_0(e_j) = e_{n+1-j}$, $j = 1, \ldots, n$, where $(e_1, \ldots, e_n)$ is the standard basis of F^n. Define an involutive automorphism θ of G by $\theta = \mathrm{Int}(d_0) \circ \theta_1$. We note that $d_0 \in K_1$, so that $\mathrm{Int}(d_0) \circ \theta_1 = \theta_1 \circ \mathrm{Int}(d_0)$.

Let B (resp. T) be the group of all upper triangular (resp. diagonal) matrices in G. Then (B, T) is a standard pair for (G, θ). We adapt the notation of earlier sections for (G, θ, B, T). Note that $N = N(T)$ is the group of all monomial matrices (matrices with exactly one non-zero entry in each row and each column). Let $\pi: N \to W = W(T)$ be the canonical projection. We identify the symmetric group S_n with the group of all $n \times n$ permutation matrices; if $\sigma \in S_n$, then $\sigma(e_j) = e_{\sigma(j)}$, $j = 1, \ldots, n$. Then $S_n \subset N$ and π maps S_n isomorphically onto W. For the present, we distinguish carefully between $S_n \subset N$ and $W = N/T$. Note that $d_0 \in S_n$ and that $\pi(d_0)$ is equal to w_0, the longest element of W. The action of θ on $E = E(T)$ is given by $\theta(x) = -w_0(x) = \imath(x)$. Thus θ acts on W by $\theta(w) = w_0 w w_0^{-1}$. We let $\mathscr{J}_n$ (resp. $\mathscr{J}$) denote the set of all involutions in S_n (resp. in W). Note that $\mathscr{I} = \mathscr{J} w_0$.

We define a left action of G on (the set) G by $g \cdot x = g x \theta_1(g)^{-1}$(this is the twisted action of G on G corresponding to θ_1). Define $\tau_1: G \to G$ by $\tau_1(g) = g \cdot 1$. It is a classical result that $\tau_1(G)$ is the set P of all symmetric matrices in G. If $g \in G$, then $\tau(g) = g\theta(g)^{-1} = (g \cdot d_0)d_0$. Thus $\tau(G) = (G \cdot d_0)d_0$. Since d_0 is a symmetric matrix, we see that $G \cdot d_0 = G \cdot 1 = P$, and hence $\tau(G) = Pd_0$. Thus $N \cap \tau(G) = N \cap (Pd_0)$. Since $d_0 \in N$, we obtain $N \cap \tau(G) = (N \cap P)d_0$. Now $N \cap P$ is the set of all symmetric monomial matrices and an elementary matrix calculation shows that $S_n \cap P = \mathscr{J}_n$. It follows easily that $\varphi(V) = (\tau(G) \cap N) \bmod T$ is equal to $\mathscr{J} w_0 = \mathscr{I}$.

By 1.4, τ induces a bijection of V onto the set of twisted T-orbits on $N \cap \tau(G) = (N \cap P)d_0$. We need to describe the twisted T-action on $(N \cap P)d_0$. Let $x \in N \cap P$. Then

$$t * (x d_0) = t x d_0 (d_0 \theta_1(t)^{-1} d_0) = (t \cdot x) d_0.$$

An easy calculation shows that the orbit $T \cdot x$ meets $\mathscr{J}_n$. This implies that $\mathscr{J}_n d_0$ is a set of representatives for the twisted T-orbits on $N \cap \tau(G)$. This in turn shows that $\varphi: V \to \mathscr{I}$ is a bijection. We see from 9.14 that φ is an isomorphism of ordered sets. Define $\delta: \mathscr{I} \to \mathscr{J}$ by $\delta(a) = a w_0$. Then, by 8.20, $\delta \circ r: V \to \mathscr{J}$ is an order reversing bijection (more precisely, for $v', v \in V$, we have $v' \leqslant v$ if and only if $\delta(\varphi(v)) \preceq \delta(\varphi(v'))$). Thus the ordered set V is isomorphic to the ordered set $\mathscr{J}$ of all involutions in W, with the opposite order of the Bruhat order.

10.2.1 LEMMA. *There exists $h \in G$ such that $\theta = \mathrm{Int}(h) \circ \theta_1 \circ \mathrm{Int}(h)^{-1}$.*

Since d_0 is a semisimple element of P, it follows from [13, 6.3] (and is easy to prove by a direct argument) that there exists $h \in G$ such that $\theta_1(h) = h^{-1}$ and $h^2 = d_0$. Thus $\theta_1(d_0 h) = d_0 h^{-1} = h$. Let $g \in G$. Then

$$\theta(hgh^{-1}) = \theta_1(d_0 hgh^{-1} d_0^{-1}) = h\theta_1(g)h^{-1}.$$

Thus $\mathrm{Int}(h) \circ \theta_1 \circ \mathrm{Int}(h)^{-1} = \theta$.

Since θ and θ_1 are conjugate by an element of $\mathrm{Int}(G)$, it follows that all of the above results for (G, θ) carry over to (G, θ_1).

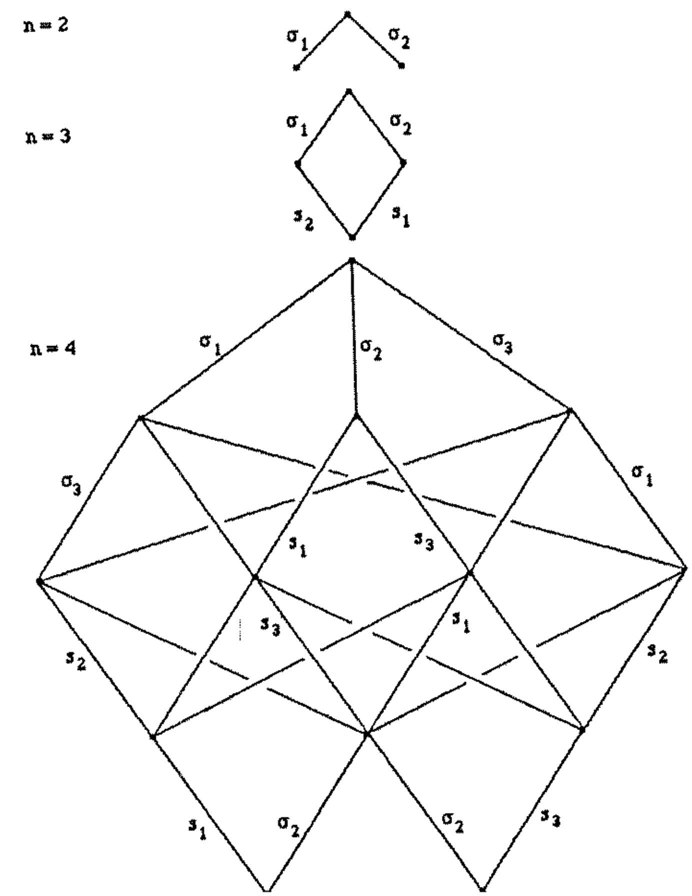
10.3 EXAMPLE. Let $G = \mathrm{SL}_n(F)$ and define the involutive automorphism $\theta_1: G \to G$ by $\theta_1(g) = {}^t g^{-1}$. Then K_1, the fixed point subgroup on θ_1, is the special orthogonal group $\mathrm{SO}_n(F)$. We wish to study the Bruhat order corresponding to the symmetric variety G/K_1. As in 10.2, it is convenient to replace θ_1 by θ, where $\theta(g) = d_0 \theta_1(g) d_0^{-1}$, and d_0 as in 10.2. Let λ be a square root of -1 in F, let $\gamma = \lambda^{n-1}$ and let $d = \gamma d_0$. Then $d \in G$ and $\mathrm{Int}(d) = \mathrm{Int}(d_0)$, so that $\theta = \mathrm{Int}(d) \circ \theta_1$. An argument similar to the proof of Lemma 10.2.1 shows that θ_1 and θ are conjugate by an element of $\mathrm{Int}(G)$.

Let B (resp. T) be the group for all upper triangular (resp. diagonal) matrices in G. Then (B, T) is a standard pair for (G, θ). Let the notation be as usual for (G, θ, B, T). Then $N = N(T)$ is the group of all monomial matrices in G and $W = N(T)/T$ is isomorphic to the symmetric group S_n. We will identify W with S_n in the obvious manner and we identify S_n with the group of $n \times n$ permutation matrices as in 10.2. The group S_n of permutation matrices is not contained in G. The automorphism θ acts on W by $\theta(w) = w_0 w w_0^{-1}$ and $\mathscr{I} = \mathscr{J} w_0$, where $\mathscr{J} = \mathscr{J}_n$ is the set of involutions in W. Let P be the set of all symmetric matrices in G. Then $\tau(G) = P$ and an argument similar to that given in 10.2 shows that $N \cap \tau(G) = (N \cap P)d^{-1}$. Since G is θ-split, $\varphi: V \to \mathscr{I}$ is surjective. As in 10.2, in order to describe the set V or orbits we need to describe the set of twisted T-orbits on $N \cap \tau(G) = (N \cap P)d^{-1}$. The calculations are straightforward matrix computations, but they are more complicated than similar computations in 10.2 because of the restriction that all matrices involved must be of determinant 1. We state the results without giving details of the computations.

10.3.1 LEMMA. *Let $a \in \mathscr{J}$. If a has a fixed point on $[1, n]$, then $|\varphi^{-1}(aw_0)| = 1$. If a has no fixed points on $[1, n]$, then $|\varphi^{-1}(aw_0)| = 2$. Thus $|V| = 2|\mathscr{J}_0| + |\mathscr{J}_1|$, where $\mathscr{J}_0$ is the set of fixed point free involutions in $W = S_n$ and $\mathscr{J}_1$ is the set of involutions in W which have a fixed point. If n is odd, then $\mathscr{J}_0 = \varnothing$ and $|V| = |\mathscr{I}|$; in this case $\varphi: V \to \mathscr{I}$ is an isomorphism of ordered sets.*

the canonical map. Let $\theta_0: G_0 \to G_0$ be defined by $\theta_0(g) = {}^t g^{-1}$ and let $\theta': G' \to G'$ be the automorphism of G' induced by θ_0. Then p determines an isomorphism of ordered sets $V(G_0, \theta_0) \to V(G', \theta')$. Thus we may identify $V(G_0, \theta_0)$ with $V(G', \theta')$. Now $G = \mathrm{SL}_n(F)$ is the simply connected covering group of G' and the automorphism θ' is induced by the automorphism θ_1 of G. Thus we are in the situation of 1.10. Let K' denote the fixed point subgroup of θ' on G'. Then $\Gamma = K'/K'^0$ is of order 2 and $V(G', \theta') \cong \mathscr{J}$ can be identified with the quotient of $V = V(G, \theta_1)$ by Γ. Clearly Γ acts trivially on $\varphi^{-1}(\mathscr{J}_1)$. If $a \in \mathscr{J}_0$, then Γ permutes the two elements of $\varphi^{-1}(a)$.

We picture the ordered set V as a graph, whose vertices are the elements of V. If $v' \leqslant v$ and $l(v') + 1 = l(v)$ then the vertices v' and v are joined by an edge. Since $\leqslant$ satisfies the chain condition, the graph completely describes the Bruhat order. In Figure 1 we have the graphs in the cases $n = 2, 3, 4$. The



lengths of the elements of V decrease from top to bottom. If $v' \mapsto v$, then we label the edge joining them by s_i (resp. σ_i) where $s_i \in S$ is such that $v' \mapsto m(s_i) \cdot v' = v$ and $\varphi(v) = s_i * \varphi(v')$ (resp. $\varphi(v) = s_i \varphi(v')$). (Note: There may be several such elements s_i; however, we give each such edge only one label.) To determine the graphs, one starts from the bottom elements and works upwards. The vertices are described by 10.3.1. For each involution in S_n, one determines an admissible sequence. Using this, one can determine for each $v \in V$ a reduced decomposition. The order relations can then be determined using the subexpression property. We leave the details to the reader.

10.4 EXAMPLE. Let $G = \mathrm{SL}_{2n}(F)$ and let $J \in G$ be given by: $J(e_i) = -e_{n+i}$ and $J(e_{n+i}) = e_i$, $i = 1, \ldots, n$. Define an involutive automorphism $\theta\colon G \to G$ by $\theta(g) = J({}^tg)^{-1}J^{-1}$. Then the fixed point subgroup K is the symplectic group $Sp_{2n}(F)$. Let (B, T) be as in 10.3. We identify $W = W(T)$ with the symmetric group S_{2n}. The involution θ acts on W by $\theta(w) = w_0 w w_0^{-1}$. The set $\mathscr{I}$ of twisted involutions is equal to $\mathscr{J} w_0$, where $\mathscr{J} = \mathscr{J}_{2n}$ is the set of all involutions in S_{2n}. Let $a \in \mathscr{J}$ and let $w \in W$. Then $w * (aw_0) = (waw^{-1})w_0$. Thus the orbit $W * (aw_0)$ is equal to $C(a)w_0$, where $C(a)$ is the (ordinary) conjugacy class of a in W. In particular, $W * 1 = C(w_0)w_0$. The Satake diagram of (G, θ) is ●—○—●—○—···○—●. (See [8] or [16].) It follows from this that $a_{\max} = cw_0$, where $c = (1, 2)(3, 4) \cdots (2n-1, 2n)$ (here we use the usual representation of an element of S_{2n} as a product of disjoint cycles). We have $w_0 = (1, 2n)(2, 2n-1) \cdots (n, n+1)$. It is clear that $C(c)$ is equal to $\mathscr{J}_0$, the set of all fixed point free involutions in S_{2n}, and that $C(w_0) = C(c)$. Thus $1 \in W * a_{\max}$. We let $\mathscr{J}_0 w_0$ be given the order induced by the Bruhat order on $\mathscr{I}$ and let $\mathscr{J}_0$ be given the order induced by the Bruhat order on $\mathscr{J}$.

10.4.1 PROPOSITION. *The image of φ is $\mathscr{J}_0 w_0$ and $\varphi\colon V \to \mathscr{J}_0 w_0$ is an isomorphism of ordered sets. If we define $\psi\colon V \to \mathscr{J}_0$ by $\psi(v) = \varphi(v)w_0$, then ψ is an order-reversing bijection.*

The proof follows from 9.16.

10.5 EXAMPLE. Finally, we briefly discuss the case where $G = \mathrm{SL}_n(F)$ and $\theta(g) = zgz^{-1}$, where $z = \mathrm{diag}(-\zeta, \zeta, \ldots, \zeta)$, with $\zeta^n = -1$. The groups B, T are as before.

Let $v \in V$ and let $x = x(v)$. Then $y = xzx^{-1}$ lies in N and has $n-1$ eigenvalues ζ and one eigenvalue $-\zeta$. The image $\varphi(v) = yT$ is an element of $W = S_n$ of order $\leqslant 2$.

There are n elements v_i, $i = 1, \ldots, n$, of V with $\varphi(v_i) = 1$, numbered such that the corresponding element y is of the form $\mathrm{diag}(\zeta, \ldots, -\zeta, \ldots, \zeta)$, with $-\zeta$ in the ith place. We have $S(v_1)_n = \{s_1\}$, $S(v_i)_n = \{s_{i-1}, s_i\}$ for $1 < i < n$, and

$S(v_n)_n = \{s_{n-1}\}$, where $S = \{s_1, \ldots, s_{n-1}\}$ is the usual set of simple reflections for S_n.

If $\varphi(v) \neq 1$, then it is a transposition (i, j) in S_n. For $i, j \in [1, \ldots, n]$, $i < j$, there is a unique $v_{ij} \in V$ with $\varphi(v_{ij}) = (i, j)$.

(a) We have $(i, j) = s_i(i+1, j)s_i > (i+1, j)$, which implies that $s_i\colon v_{i+1,j} \mapsto v_{ij}$ and $\varphi(v_{ij}) = s_i * \varphi(v_{i+1,j})$.
(b) Similarly we have $s_j\colon v_{ij} \mapsto v_{i,j+1}$ and $\varphi(v_{i,j+1}) = s_j * \varphi(v_{ij})$.
(c) We also have $s_j\colon v_j \mapsto v_{j,j+1}$, $\varphi(v_{j,j+1}) = s_j\varphi(v_j)$ and $s_j\colon v_{j+1} \mapsto v_{j,j+1}$, $\varphi(v_{j,j+1}) = s_j\varphi(v_{j+1})$.

Finally, we note that $l(v_{ij}) = j - i$.

The relations above allow us to give a reduced decomposition for each $v \in V$. We can then show by the subexpression property that all order relations of the form $v' \leqslant v$ with $l(v') + 1 = l(v)$ are of one of the forms (a)–(c) above. So the above relations completely determine the Bruhat order. Note that in this case, the weak order is equal to the Bruhat order.

In Figure 2, we give the picture of the ordered set V for $n = 4$, using the conventions of 10.3. The vertices are marked by the indices of the corresponding elements of V.

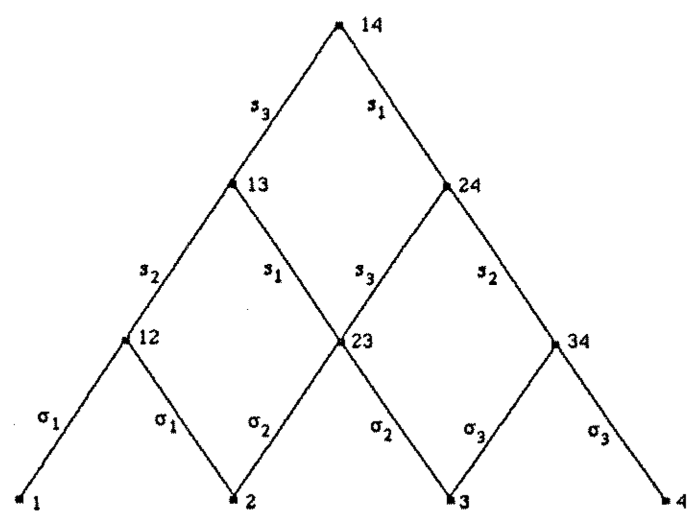


Fig. 2. Graph of the Bruhat order for Example 10.5 ($n = 4$).

REFERENCES

1. Borel, A., *Linear Algebraic Groups*, Benjamin, New York, 1969.
2. Borel, A. and Tits, J., 'Compléments a l'article "Groupes réductifs"', *Publ. Math. I.H.E.S.* **41** (1972), 253–276.
3. Bourbaki, N., *Groupes et algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.
4. Deligne, P. and Lusztig, G., 'Representations of reductive groups over finite fields', *Ann. of Math.* **103** (1976), 103–161.
5. Demazure, M., 'Désingularisation des variétés de Schubert généralisées', *Ann. Ec. Norm. Sup.* **6** (1974), 53–88.
6. Deodhar, V. V., 'Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function', *Invent. Math.* **39** (1977), 187–198.
7. Deodhar, V. V., 'On some geometric aspects of Bruhat orderings, I. A finer decomposition of Bruhat cells', *Invent. Math.* **79** (1985), 499–511.
8. Helgason, S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York, San Francisco, London, 1978.
9. Helminck, A., 'Tori invariant under an involutorial automorphism' (to appear, *Adv. in Math.*).
10. Kostant, B., 'On the conjugacy of real Cartan subalgebras', *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 967–970.
11. Lusztig, G. and Vogan, D. A., Jr, 'Singularities of closures of K-orbits on flag manifolds', *Invent. Math.* **71** (1983), 365–379.
12. Matsuki, T., 'The orbits of affine symmetric spaces under the action of a minimal parabolic subgroup', *J. Math. Soc. Japan* **31** (1979), 331–357.
13. Richardson, R. W., 'Orbits, invariants and representations associated to involutions of reductive groups', *Invent. Math.* **66** (1982), 287–312.
14. Richardson, R. W., 'Conjugacy classes of involutions in Coxeter groups', *Bull. Austral. Math. Soc* **26** (1982), 1–17.
15. Springer, T. A., 'Some results on algebraic groups with involutions', *Advanced Studies in Pure Math.*, Vol. 6, pp. 525–543, Kinokuniya/North-Holland, 1985.
16. Springer, T. A., 'The classification of involutions of simple algebraic groups', *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** (1987), 655–670.
17. Steinberg, R., 'Endomorphisms of linear algebraic groups', *Mem. Amer. Math. Soc.* **80** (1968).
18. Steinberg, R., 'Conjugacy classes in algebraic groups', *Lecture Notes in Math.* **366**, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
19. Vogan, D. A. Jr, 'Irreducible characters of semisimple Lie groups III. Proof of the Kazhdan–Lusztig conjecture in the integral case', *Invent. Math.* **71** (1983), 381–417.
20. Vust, T., 'Opération de groupes réductifs dans un type de cônes presque homogènes', *Bull. Math. Soc. France* **102** (1974), 317–334.
21. Warner, G., *Harmonic Analysis on Semisimple Lie Groups*, I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

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