# LANGLANDS' PHILOSOPHY AND KOSZUL DUALITY

WOLFGANG SOERGEL

Contents



## 1. Introduction

In this article I will formulate conjectures relating representation theory to geometry and prove them in some cases. I am trying to extend work of Adams-Barbasch-Vogan [ABV92] and some joint work of myself with Beilinson and Ginzburg [BGS96]. All this should be regarded as a contribution to (the local case of) Langlands' philosophy.

The localization of Beilinson-Bernstein also establishes a relation between representation theory and geometry. This is however very different from the relation to be investigated in this article: Whereas localization leads to geometry on the group itself (or its flag manifold), the results in [ABV92] to be extended in this article lead to geometry on the Langlands dual group.

1.1. **The basic conjecture.** Let $G$ be (only for a short moment) a real reductive Zariski-connected algebraic group. We want to study admissible representations of the associated Lie group $G(\mathbb{R})$ of real points of $G$. More precisely, we want to study the category $\mathcal{M}(G(\mathbb{R}))$ of smooth admissible representations of $G(\mathbb{R})$. Remark that by a theorem of Casselman-Wallach this is nothing else but the category of Harish-Chandra modules for $G(\mathbb{R})$, modulo the choice of a maximal compact subgroup $K \subset G(\mathbb{R})$. Let $\mathfrak{g}$ be the complexification of the Lie algebra of $G(\mathbb{R})$, $U - \mathrm{mod} - U = U - \mathrm{mod} - U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$ and $Z \subset U - \mathrm{mod} - U$ the center of $U - \mathrm{mod} - U$. Then $\mathcal{M}(G(\mathbb{R}))$ decomposes according to infinitesimal character into

$$\mathcal{M}(G(\mathbb{R})) = \bigoplus_{\chi} \mathcal{M}(G(\mathbb{R}))_{\chi}$$

where $\chi$ runs over the set $\mathrm{Max} Z$ of maximal ideals of $Z$.

There are several classifications of the simple objects in $\mathcal{M}(G(\mathbb{R}))$. The most popular ones are by Langlands [Lan73], Vogan-Zuckerman [Vog81], and Beilinson-Bernstein [BB81]. In this article I will exploit a version of the first classification scheme, due to Langlands and Shelstad, and more precisely the form given to it by Adams-Barbasch-Vogan [ABV92].

Let me roughly explain its structure. First of all, we have to forget about the real algebraic group $G$ and just remember the inner class of real forms for $G \times_{\mathbb{R}} \mathbb{C}$ given by $G$. In other words and other notations, we start out with a complex connected reductive algebraic group $G$ endowed with an inner class of real forms. To simplify the exposition let us furthermore choose a finite central subgroup $F$ of $G$. To these data one associates in a more or less unique way (and to detail this "more or less" takes most of the time):

1. a finite family $S_F$ of antiholomorphic involutions $\delta : G \to G$, all from the fixed inner class.
2. a finite connected algebraic covering $G^\vee_F$ of the dual group $G^\vee$ (i.e. of the complex connected reductive algebraic group $G^\vee$ whose root system is dual to the root system of $G$).
3. a collection $\{X(\chi)\}_{\chi \in \mathrm{Max} Z}$ of complex algebraic $G^\vee-$varieties.

I will give details in section 1.3. For every antiholomorphic involution $\delta : G \to G$ let $G(\mathbb{R}, \delta)$ denote the fixed point set of the induced map $\delta : G(\mathbb{C}) \to G(\mathbb{C})$, and for any $\chi \in \mathrm{Max}Z$ let $G(\mathbb{R}, \delta)^{\wedge}_{\chi}$ denote the set of (isomorphism classes of) simple objects in $\mathcal{M}(G(\mathbb{R}, \delta))_{\chi}$. Then Adams-Barbasch-Vogan [ABV92] establish a bijection (depending on additional choices) between

- **(a):** the set $\dot{\bigcup}_{\delta \in S_F} G(\mathbb{R}, \delta)^{\wedge}_{\chi}$ and
- **(b):** the set of (isomorphism classes of) pairs $(Y, \tau)$, where $Y$ is a $G^{\vee}$-orbit on $X(\chi)$ and $\tau$ an irreducible $G^{\vee}_F$-equivariant local system on $Y$.

Strictly speaking this classifies the representations of a whole collection of real forms simultaneously rather then representations of a single group at a time. To explain which geometric parameters $(Y, \tau)$ belong to which real form $\delta \in S_F$ wouldn't make good reading for this introduction (compare [Vog93], 4.16). Let me mention however that any real form in the given inner class is equivalent to $G(\mathbb{R}, \delta)$ with $\delta \in S_F$ if we choose $F$ big enough, hence the above classification captures all real forms.

One point of setting up the above classification scheme is the very beautiful and miraculous fact that the KL-data on the representation theoretic side are mirrored by the IC-data on the geometric side. This is the essence of Vogan's character duality [Vog82]. I conjecture that this duality lifts to the level of categories. To give more details, I have to introduce more notation.

Let $H$ be a complex algebraic group and $X$ a complex algebraic $H$-variety. Suppose $H$ decomposes $X$ into finitely many orbits. Then we can form a graded $\mathbb{C}$-algebra $\mathrm{Ext}_H(X)$ as follows: Take the equivariant derived category $\mathcal{D}_H(X)$ as defined in [BL92]. This is a triangulated $\mathbb{C}$-category containing the category $\mathcal{P}_H(X)$ of equivariant perverse sheaves as a full abelian subcategory. For $\mathcal{F}, \mathcal{G} \in \mathcal{D}_H(X)$ let us denote by $\mathrm{Hom}_H(\mathcal{F}, \mathcal{G})$ the space of homomorphisms in $\mathcal{D}_H(X)$ and put $\mathrm{Hom}^n_H(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_H(\mathcal{F}, \mathcal{G}[n])$. Since $H$ was supposed to act with finitely many orbits, there are (up to isomorphism) only finitely many simple objects in $\mathcal{P}_H(X)$. Let $\mathcal{L} \in \mathcal{P}_H(X)$ denote the direct sum of "all" simple objects (i.e. take one from each isomorphism class). Then put

$$\mathrm{Ext}_H(X) = \oplus_{n \geq 0} \mathrm{Hom}^n_H(\mathcal{L}, \mathcal{L}) = \mathrm{End}^{\bullet}_H(\mathcal{L}).$$

There is our graded $\mathbb{C}$-algebra. We will refer to it as the "geometric extension algebra".

For any ring $R$ let $R-Mod$ be the category of all left $R$-modules. For a graded $\mathbb{C}$-algebra $A = \oplus_{n \geq 0} A^n$ define the full subcategory $A - Nil \subset$

$A - Mod$ by

$$A - Nil = \{M \in A - Mod \mid \dim_{\mathbb{C}} M < \infty, A^n M = 0 \text{ for } n \gg 0\}.$$

So objects in $A - Nil$ are not assumed to be graded, but only "continous along the graduation". Recall now the classification scheme from [ABV92]. The main objective of this article is to formulate, motivate and discuss the following

**Basic conjecture 1.** *There exists an equivalence of categories*

$$\bigoplus_{\delta \in S_F} \mathcal{M}(G(\mathbb{R}, \delta))_\chi \cong \mathrm{Ext}_{G_F^\vee}(X(\chi)) - Nil.$$

*Remarks:*

1. Take again our algebraic group $H$ acting on our variety $X$ with finitely many orbits. Then the simple objects of $\mathrm{Ext}_H(X) - Nil$ are in an obvious bijection with the simple objects of $\mathcal{P}_H(X)$, and these in turn are parametrized by pairs $(Y, \tau)$ with $Y$ an $H$-orbit on $X$ and $\tau$ an irreducible $H$-equivariant local system on $Y$, c.f. [BL92]. Thus we see that the simple objects on the left and on the right of our conjectured equivalence are parametrized by the same set. Certainly our equivalence should identify simple objects which have he same parameter.
2. In this article I will check the conjecture in the following cases: (1) if $G$ is a torus, (2) for arbitrary $G$ and "almost all" central characters, under the additional assumption $F = 1$, (3) for $G = SL_2$, (4) for complex groups, more precisely when $G = G_c \times G_c$ for a semisimple simply connected complex algebraic group $G_c$ and $F = 1$ and the inner class of real forms is such that $S_F$ consists just of one $\delta$ and $G(\mathbb{R}, \delta) = G_c(\mathbb{C})$, under the additional assumption that the central character $\chi$ is integral regular.
3. I will explain in the last section 4 how I expect this conjecture to be related to Vogan's character duality formulas [Vog82, ABV92].
4. It seems reasonable to make analogous conjectures for nonarchimedean fields as well. In these cases $X(\chi)$ should be replaced by spaces of quasiadmissible morphisms of the Weil-Deligne group to the Langlands dual group. I expect that one may identify in special situations the geometric extension algebras with Lusztig's graded affine Hecke algebras.

1.2. **The easiest example.** To give the reader a more concrete idea of what is going on, let me treat the easiest example. Let us take $G = \mathbb{C}^\times$ endowed with the inner class of real forms consisting of the split form $\delta$ so that $G(\mathbb{R}, \delta) = \mathbb{R}^\times$, and choose $F = 1$.

Then $S_F$ will consist of the single element $\delta$, the covering $G_F^\vee \to G^\vee$ will be the identity $\mathbb{C}^\times \to \mathbb{C}^\times$, and $X(\chi)$ will consist just of two points, for every $\chi \in \mathrm{Max}Z$. We find that

$$\begin{aligned} \mathrm{Ext}_{G_F^\vee}(X(\chi)) &= \mathrm{Ext}_{\mathbb{C}^\times}(\text{two points}) \\ &= \mathrm{Ext}_{\mathbb{C}^\times}(\text{point}) \times \mathrm{Ext}_{\mathbb{C}^\times}(\text{point}) \\ &= H^\bullet(B\mathbb{C}^\times;\mathbb{C}) \times H^\bullet(B\mathbb{C}^\times;\mathbb{C}) \\ &= H^\bullet(\mathbb{P}^\infty\mathbb{C};\mathbb{C}) \times H^\bullet(\mathbb{P}^\infty\mathbb{C};\mathbb{C}) \\ &= \mathbb{C}[t] \times \mathbb{C}[t] \end{aligned}$$

where $B\mathbb{C}^\times = \mathbb{P}^\infty\mathbb{C}$ is the classifying space of $\mathbb{C}^\times$ and $deg(t) = 2$. On the other hand, $\mathcal{M}(\mathbb{R}^\times)$ is just the category of continuous finite dimensional representations of $\mathbb{R}^\times$. I leave it to the reader to check for every $\chi \in \mathrm{Max}Z$ the equivalence

$$\mathcal{M}(\mathbb{R}^\times)_\chi \cong (\mathbb{C}[t] \times \mathbb{C}[t]) - Nil.$$

This proves our conjecture in this special case.

1.3. **The geometry of Adams-Barbasch-Vogan.** I promised at the beginning that given a complex connected reductive algebraic group $G$, endowed with an inner class of real forms and a finite central subgroup $F$, I would construct (1) a multiset $S_F$ of real forms, (2) a finite connected covering $G_F^\vee$ of the dual group and (3) a collection $\{X(\chi)\}_{\chi\in\mathrm{Max}Z}$ of $G^\vee$-varieties.

To fullfill this promise I have to make some choices. In [ABV92] the authors investigate in detail how these choices affect the final outcome.

1. Let us choose an antiholomorphic involution $\gamma : G \to G$ in the given inner class that stabilizes a Borel subgroup. Then form the (real) Lie group ${}^\Gamma G = G \rtimes \Gamma (= G(\mathbb{C}) \rtimes \Gamma$ if we wouldn't abuse notation), where $\Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$ acts on $G$ via this antiholomorphic involution $\gamma$. For any $\delta \in {}^\Gamma G - G$ with $\delta^2 \in Z(G)$ the conjugation $int(\delta) : G \to G$ is an antiholomorphic involution, inner to $\gamma$. Let $S_F \subset {}^\Gamma G - G$ be a system of representatives for the $G$-conjugacy classes of $\delta \in {}^\Gamma G - G$ with $\delta^2 \in F$. As explained we may consider $S_F$ as a multiset of antiholomorphic involutions of $G$. For example $S_1 = H^1(\Gamma; G)$.

   I warn the reader that for a complex algebraic group I quite often write just $G$ for the complex points of $G$, to be denoted $G(\mathbb{C})$ in more careful notation. I hope this does not lead to confusion.
2. Let us choose in addition $T \subset B \subset G$ a $\gamma$-stable maximal torus and Borel subgroup of $G$. This gives rise to a based root system $(\mathfrak{X}, R, \Delta; \mathfrak{X}^\vee, R^\vee, \Delta^\vee)$ with an involution $\gamma s : \mathfrak{X} \to \mathfrak{X}$ stabilizing $R$ and $\Delta$. In more detail, $\mathfrak{X} = \mathfrak{X}(T) = \{$homomorphisms of complex algebraic groups $\lambda : T \to \mathbb{C}^\times\}$ and $\gamma s : \mathfrak{X} \to \mathfrak{X}$ is characterized

by the formula $(\gamma s(\lambda))(t) = \overline{\lambda(\gamma(t))}\ \ \forall t \in T$. (The notation $\gamma s$ is explained in more detail in 3.1). Then choose a triple $(G^\vee, T^\vee, \phi^\vee)$ consisting of a connected reductive complex algebraic group $G^\vee$, a maximal torus $T^\vee \subset G^\vee$ and an isomorphism $\phi^\vee : \mathfrak{X}^\vee \to \mathfrak{X}(T^\vee)$ that identifies the root system of $G^\vee$ with the dual of the root system of $G$.

Now the finite central subgroups $F$ of $G$ are in one-to-one correspondence with finite quotients of $\mathfrak{X}/\mathbb{Z}R$, via $F \mapsto \mathfrak{X}(F)$. But the finite connected coverings $\pi : H \to G^\vee$ of $G^\vee$ are also in one-to-one correspondence with finite quotients of $\mathfrak{X}/\mathbb{Z}R$, via $H \mapsto \mathfrak{X}/\mathfrak{X}^\vee(\pi^{-1}(T^\vee))$. In this way any finite central subgroup $F$ of $G$ gives rise to a finite connected covering $G_F^\vee$ of $G^\vee$.

3. This is the most subtle part. We start introducing some notations.

Let $H$ be any complex connected reductive algebraic group, $\mathfrak{h} = LieH$ its Lie algebra and $\lambda \in \mathfrak{h}$ a semisimple element. Put

$$\begin{array}{lcl} \mathfrak{h}(\lambda)_n & = & \{\mu \in \mathfrak{h} \mid [\lambda, \mu] = n\mu\} \\ \mathfrak{n}(\lambda) & = & \oplus_{n=1,2,\ldots}\mathfrak{h}(\lambda)_n \\ e(\lambda) & = & exp(2\pi i\lambda) \in H. \end{array}$$

The "canonical flat through $\lambda$" is the affine subspace $\mathcal{F}(\lambda) = \lambda + \mathfrak{n}(\lambda) \subset \mathfrak{h}$. One proves (see [ABV92]) that the canonical flats partition the set $\mathfrak{h}_s$ of semisimple elements of $\mathfrak{h}$ and even partition each conjugacy class of semisimples. Furthermore the function $e$ is constant on each canonical flat. Let $\mathcal{F}(\mathfrak{h})$ denote the space of all canonical flats (through semisimple elements of $\mathfrak{h}$) and for a semisimple orbit $\mathcal{O} \subset \mathfrak{h}$ put $\mathcal{F}(\mathcal{O}) = \{\Lambda \in \mathcal{F}(\mathfrak{h}) \mid \Lambda \subset \mathcal{O}\}$. Then $e$ is a function $e : \mathcal{F}(\mathfrak{h}) \to H$.

Now recall $G^\vee$ and put $\mathfrak{g}^\vee = LieG^\vee$. We may and will identify $Lie\mathbb{C}^\times = \mathbb{C}$ in such a way that the exponential map of Lie groups corresponds to the usual exponential map. This determines an embedding $\mathfrak{X} = \mathrm{Hom}(T, \mathbb{C}^\times) \to \mathrm{Hom}(LieT, Lie\mathbb{C}^\times) = (LieT)^*$, hence an isomorphism $\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{C} = (LieT)^*$. On the other hand $\phi^\vee$ determines an embedding $\mathfrak{X} = \mathrm{Hom}(\mathbb{C}^\times, T^\vee) \to \mathrm{Hom}(Lie\mathbb{C}^\times, LieT^\vee) = LieT^\vee$, hence an isomorphism $\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{C} = LieT^\vee$.

Let $\mathcal{W}$ be the Weyl group. We have bijections

$$\mathrm{Max}Z = (LieT)^*/\mathcal{W} = (\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{C})/\mathcal{W} = (LieT^\vee)/\mathcal{W} = \mathfrak{g}_s^\vee/G^\vee$$

induced respectively by the Harish-Chandra homomorphism, our above isomorphisms and the embedding $LieT^\vee \subset \mathfrak{g}_s^\vee$. Denote by $\chi \mapsto \mathcal{O}(\chi)$ their composition.

Now choose a collection $\{x_{\alpha^\vee}\}_{\alpha \in \Delta}$ of isomorphisms $x_{\alpha^\vee} : \mathbb{C} \to U_{\phi^\vee(\alpha^\vee)}$ of the additive group $\mathbb{C}$ onto the simple root subgroups of

$G^\vee$. The involution $\gamma s : \mathfrak{X} \to \mathfrak{X}$ has an adjoint $(\gamma s)^\vee : \mathfrak{X}^\vee \to \mathfrak{X}^\vee$. We will abreviate $(\gamma s)^\vee = \gamma : \mathfrak{X}^\vee \to \mathfrak{X}^\vee$. This gives rise to an involutive automorphism $\gamma : G^\vee \to G^\vee$ of the complex algebraic group $G^\vee$ characterized by $\gamma(T^\vee) = T^\vee$, $\phi^\vee\gamma = \gamma\phi^\vee : \mathfrak{X}^\vee \to \mathfrak{X}(T^\vee)$ and $\gamma \circ x_{\alpha^\vee} = x_{\gamma(\alpha^\vee)} \forall \alpha \in \Delta$. Thus we may form the semidirect product ${}^\Gamma G^\vee = G^\vee \rtimes \Gamma$, the so-called $L$-group.

Now for any $\chi \in \mathrm{Max}Z$ we define

$$X(\chi) = \{(y, \Lambda) \mid y \in {}^\Gamma G^\vee - G^\vee, \Lambda \in \mathcal{F}(\mathcal{O}(\chi)) \text{ such that } y^2 = e(\Lambda)\}.$$

This is in a natural way a complex algebraic variety on which $G^\vee$ acts via conjugation. For more details see [ABV92]. This fullfills my promise.

It is useful to remark, that by [ABV92], (12.11)(e) the component groups of isotropy groups of $G_F^\vee$ acting on $X(\chi)$ are all commutative. Hence all local systems $\tau$ appearing in our geometric parameters are of rank one.

1.4. **Thanks.** I thank David Vogan, Victor Ginsburg, and Valery Lunts for very helpful discussions. I thank the DFG, MPI, and CNRS for financial support, and the latter two for providing a stimulating research atmosphere.

## 2. Notations and conventions

2.1. **Groups and varieties.** Let $G, H$ be complex algebraic groups. A holomorphic (resp. antiholomorphic) morphism from $G$ to $H$ is a morphism of group schemes $\phi : G \to H$ such that the diagram

$$\begin{array}{ccc} G & \stackrel{\phi}{\to} & H \\ \downarrow & & \downarrow \\ Spec\mathbb{C} & \to & Spec\mathbb{C} \end{array}$$

with the lower horizontal the identity (resp. the complex conjugation) commutes.

Let $G$ be a complex algebraic group. Two antiholomorphic involutions $\delta, \delta'$ of $G$ are said to belong to the same inner class if and only if $\delta\delta'$ is an inner automorphism of $G$, i.e. $\delta = \delta' int(x)$ with $x \in G$. Two antiholomorphic involutions $\delta, \delta'$ of $G$ are said to be equivalent if and only if they are conjugate by an inner automorphism of $G$, i.e. $\delta = int(x^{-1})\delta' int(x)$ with $x \in G$. Any two equivalent antiholomorphic involutions belong to the same inner class.

Let $A, B$ be groups and $\sigma : B \to \mathrm{Aut}A$ a group homomorphism. Let us write $(\sigma(b))(a) = a^b \;\; \forall a \in A, b \in B$. We define a new group $A \rtimes_\sigma B =$

$A \rtimes B$ as follows: As a set $A \rtimes B = A \times B$ and the multiplication is given by $(a, b)(a_1, b_1) = (a(a_1^b), bb_1)$.

Let $G$ be a complex algebraic group. Then we denote by $LieG$ its Lie algebra. This is a Lie algebra over $\mathbb{C}$. Let $H$ be a Lie group. Then we denote by $Lie_{\mathbb{R}}H$ the Lie algebra of $H$. This is a Lie algebra over $\mathbb{R}$. We denote its complexification by $Lie_{\mathbb{C}}H = \mathbb{C} \otimes_{\mathbb{R}} Lie_{\mathbb{R}}H$. Let $G$ be a complex algebraic group. Then $LieG = Lie_{\mathbb{R}}(G(\mathbb{C}))$ canonically. This is an isomorphism of Lie algebras over $\mathbb{R}$, where on the left we quietly restrict scalars from $\mathbb{C}$ to $\mathbb{R}$. Let $\delta$ be an antiholomorphic involution on $G$. Then $LieG = Lie_{\mathbb{C}}(G(\mathbb{R}; \delta))$ canonically.

Let $X$ be a complex algebraic variety. We often abuse notation and write $X$ instead of $X(\mathbb{C})$. Sometimes we also abuse notation in the other direction and write for example $\mathbb{C}^\times, \mathbb{C}$ instead of $\mathbb{G}_{m,\mathbb{C}}, \mathbb{G}_{a,\mathbb{C}}$.

2.2. **Grothendieck groups.** Let $\mathcal{A}$ be an abelian (resp. triangulated) category. Then we let $[\mathcal{A}]$ be the Grothendieck group of $\mathcal{A}$, generated by the objects with the usual relations for all short exact sequences (resp. distinguished triangles). Any object $A \in \mathcal{A}$ gives an element $[A] \in [\mathcal{A}]$.

Let $\mathcal{A}$ be an additive category. Then we let $\langle \mathcal{A} \rangle$ be the *split* Grothendieck group of $\mathcal{A}$, generated by the objects with the usual relations for all *split* short exact sequences. Any object $A \in \mathcal{A}$ gives an element $\langle A \rangle \in \langle \mathcal{A} \rangle$.

2.3. **Equivariant derived categories.** My basic reference for what follows is [BL92]. Let $G$ be a complex algebraic group and $X$ a complex algebraic $G$-variety. To $X$ we may associate a triangulated $\mathbb{C}$-category $\mathcal{D}_G(X)$, the bounded constructible $G$-equivariant derived category, called $D^b_{G,c}(X)$ in [BL92], 4.1. We note $\mathrm{Hom}_G$ the morphisms in $\mathcal{D}_G(X)$, put $\mathrm{Hom}^\bullet_G = \oplus_n \mathrm{Hom}^n_G$ and use analogous notation $\mathrm{End}_G$, $\mathrm{End}^\bullet_G$ for endomorphisms. If $G = 1$ we come back to our usual triangulated categories $\mathcal{D}_1(X) = D^b_c(X) = \mathcal{D}(X)$ and since $\mathrm{Hom}_1$ looks so stupid, we write $\mathrm{Hom}_{\mathcal{D}}$ in this case.

I cannot repeat the details of the definition of $\mathcal{D}_G(X)$. To give the reader an idea of what $\mathcal{D}_G(X)$ is like, let me just remark that it embeds as a full triangulated subcategory in the bounded derived category $D^b(Sh(EG \times_G X))$ of sheaves of $\mathbb{C}$-vectorspaces on $EG \times_G X$, where $EG$ is a contractible space with a free right $G$-action, e.g. the total space of the universal bundle on the classifying space $BG$ of $G$.

The usual formalism of the six operations of Grothendieck extends to this equivariant situation: For $\mathcal{F}, \mathcal{G} \in \mathcal{D}_G(X)$ we may form $\mathcal{F} \otimes \mathcal{G}, \mathcal{R}Hom(\mathcal{F}, \mathcal{G}) \in \mathcal{D}_G(X)$ and any equivariant morphism $f : X \to Y$ of $G$-varieties gives rise to adjoint pairs of triangulated functors $(f^*, f_*)$

and $(f_!, f^!)$ relating $\mathcal{D}_G(X)$ and $\mathcal{D}_G(Y)$. The categories $\mathcal{D}_G(X)$ also come equipped with a (perverse) t-structure, and we let $\mathcal{P}_G(X)$ be its core, i.e. the $G$-equivariant perverse sheaves for the middle perversity, and $H^i : \mathcal{D}_G(X) \to \mathcal{P}_G(X)$ the perverse cohomology functors. Let $\underline{pt} \in \mathcal{P}_G(pt)$ be the constant object $\mathbb{C}$, and put $\underline{X} = p^*\underline{pt} \in \mathcal{D}_G(X)$ for any $X$, where $p : X \to pt$ is the constant map.

The irreducible objects of $\mathcal{P}_G(X)$ are parametrized by pairs $(Y, \tau)$ with $Y$ a $G$-stable locally closed subset of $X$ and $\tau$ an irreducible $G$-equivariant local system on $Y$. Namely to such a pair one associates the intersection cohomology complex with twisted coefficients $\mathcal{IC}(Y, \tau) \in \mathcal{P}_G(X)$.

Let $\phi : H \to G$ be a morphism of algebraic groups. Then we may define the restriction functors $Res : \mathcal{D}_G(X) \to \mathcal{D}_H(X)$. They are triangulated and compatible with all our "internal" operations. If $\phi$ is either (1) a finite covering or (2) the inclusion of a reductive Levi factor, then $Res$ induces isomorphisms $\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_H(Res\mathcal{F}, Res\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{D}_G(X)$.

2.4. **Geometric extension algebras.** Assume now again that $X$ is a $H$-variety decomposing into finitely many orbits. Remember we defined the graded $\mathbb{C}$-algebra $\mathrm{Ext}_H(X) = \oplus_{n\geq 0}\mathrm{Hom}^n_H(\mathcal{L}, \mathcal{L})$ where $\mathcal{L} = \oplus_{(Y,\tau)}\mathcal{IC}(Y, \tau)$ is the sum over all simple objects of $\mathcal{P}_H(X)$, so that $Y$ runs over all $H$-orbits in $X$ and $\tau$ over all irreducible $H$-equivariant local systems on $Y$.

Suppose $H \subset G$ is a closed subgroup. Then the "induction equivalence" gives an isomorphism $\mathrm{Ext}_H(X) = \mathrm{Ext}_G(G \times_H X)$.

Suppose $H$ is connected. Then $\mathrm{Ext}_H(pt) = H^\bullet(BH; \mathbb{C})$ is just the cohomology ring of the classifying space of $H$. If in particular $T$ is a torus, then $\mathrm{Ext}_T(pt) = S((LieT)^*)$ is just the ring of regular functions on its Lie algebra, graded in such a way that $deg(LieT)^* = 2$. More generally, if $H$ is any connected algebraic group, then $\mathrm{Ext}_H(pt) = H^\bullet(BH; \mathbb{C}) = S((LieT)^*)^{\mathcal{W}}$ consists of the invariants of the Weyl group in the regular functions on the Lie algebra of a maximal torus $T \subset G$.

Let $N$ be a complex algebraic group, $H \subset N$ its identity component and $\mathcal{W} = N/H$ the quotient. Then $N$ acts on $H$ by conjugation, hence it acts on $H^\bullet(BH; \mathbb{C})$ and since $H$ is connected this $N$-action factorizes to give an action of $\mathcal{W}$ on $H^\bullet(BH; \mathbb{C})$. Let us denote this action by $\sigma : \mathcal{W} \to \mathrm{Aut}_{\mathbb{C}}(H^\bullet(BH; \mathbb{C}))$.

Now whenever we are given a $\mathbb{C}$-algebra $A$ together with an action $\sigma : \mathcal{W} \to \mathrm{Aut}_{\mathbb{C}} A$ of a group $\mathcal{W}$ on $A$ by algebra automorphisms, we may form the "twisted group ring" $A^\sigma[\mathcal{W}] = A \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{W}]$ with multiplication $(a \otimes w)(b \otimes v) = a(\sigma(w)(b)) \otimes wv$. In the above notations, we prove:

**Theorem 2.4.1.** *Let $N$ be a complex algebraic group, $H \subset N$ its identity component and $\mathcal{W} = N/H$ the quotient. If $\mathcal{W}$ is commutative, then*

$$\mathrm{Ext}_N(pt) = H^\bullet(BH;\mathbb{C})^\sigma[\mathcal{W}].$$

*If not these two rings are still Morita-equivalent.*

In fact we will prove even a more general theorem to prepare for the calculations concerning $SL_2$. Namely let $H \hookrightarrow N \twoheadrightarrow \mathcal{W}$ be as above, let $X$ be an $N$-variety and $\mathcal{L} \in \mathcal{D}_N(X)$ an object in the equivariant derived category. We have the restriction functor $Res : \mathcal{D}_N(X) \to \mathcal{D}_H(X)$. Its right and left adjoints coincide in our situation, so we denote them just by $Ind : \mathcal{D}_H(X) \to \mathcal{D}_N(X)$. See [BL92], 3.7 for the definitions of $Ind_*$, $Ind_!$. Let me first give the statement I want, before I explain and prove it.

**Theorem 2.4.2.** $\mathrm{End}^\bullet_N(Ind\ Res\mathcal{L}) = \mathrm{End}^\bullet_H(Res\mathcal{L})^\sigma[\mathcal{W}]$.

*Explanation:* I ought to explain the action $\sigma$ of $\mathcal{W}$ on $\mathrm{End}^\bullet_H(Res\mathcal{L})$. Let $E = EN$ denote the universal bundle over the classifying space of $N$. Let $\pi : E \times_H X \to E \times_N X$ be the projection. We have the following two commutative squares

$$\begin{array}{ccc} \mathcal{D}_N(X) & \hookrightarrow & D^+(Sh(E \times_N X)) \\ {\scriptstyle Res} \downarrow\uparrow {\scriptstyle Ind} & & {\scriptstyle \pi^*} \downarrow\uparrow {\scriptstyle \pi_*} \\ \mathcal{D}_H(X) & \hookrightarrow & D^+(Sh(E \times_H X)) \end{array}$$

where $\pi^*, \pi_*$ stand for the functors usually denoted $L\pi^{-1}, R\pi_*$. The horizontal arrows are fully faithful embeddings. In fact there is a right $\mathcal{W}$-action on on $E \times_H X$ given by $[w] : E \times_H X \to E \times_H X$, $(e,x) \mapsto (en, n^{-1}x)$ for any $n \in N$ representing $w \in \mathcal{W}$. Our $\pi : E \times_H X \to E \times_N X$ is just the quotient by this right $\mathcal{W}$-action.

Let the image of $\mathcal{L}$ by the upper horizontal in the above diagram be denoted $\tilde{\mathcal{L}}$. Thus $\mathrm{End}^\bullet_H(Res\mathcal{L}) = \mathrm{End}^\bullet(\pi^*\tilde{\mathcal{L}})$. But any $w \in \mathcal{W}$ gives

$$\sigma(w) : \mathrm{End}^\bullet(\pi^*\tilde{\mathcal{L}}) \to \mathrm{End}^\bullet([w]^*\pi^*\tilde{\mathcal{L}}) \to \mathrm{End}^\bullet(\pi^*\tilde{\mathcal{L}})$$

where the first map is induced by $[w]^*$, the second by $\pi \circ [w] = \pi$. There is our $\mathcal{W}$-action on $\mathrm{End}^\bullet(\pi^*\tilde{\mathcal{L}})$.

*Proof:* We procede in four steps.

1. Define a morphism $\mathrm{End}^\bullet_H(Res\mathcal{L}) \to \mathrm{End}^\bullet_N(Ind\ Res\mathcal{L})$.
2. Define a morphism $\mathbb{C}[\mathcal{W}] \to \mathrm{End}^\bullet_N(Ind\ Res\mathcal{L})$.
3. Prove that they are "$\sigma$-compatible", i.e. fit together to define a morphism
$$\mathrm{End}^\bullet_H(Res\mathcal{L})^\sigma[\mathcal{W}] \to \mathrm{End}^\bullet_N(Ind\ Res\mathcal{L}).$$
4. Prove that this morphism is an isomorphism.

So let's go to work!

(1) The first step is clear by functoriality of $Ind$. Translating into a different picture, the morphism is $\pi_* : \mathrm{End}^\bullet(\pi^*\tilde{\mathcal{L}}) \to \mathrm{End}^\bullet(\pi_*\pi^*\tilde{\mathcal{L}})$.

(2) Translating via our diagram we see that we need a morphism $\mathbb{C}[\mathcal{W}] \to \mathrm{End}^\bullet(\pi_*\pi^*\tilde{\mathcal{L}})$. Indeed any $w \in \mathcal{W}$ defines a morphism $\langle w\rangle : \pi_*\pi^*\tilde{\mathcal{L}} \to \pi_*\pi^*\tilde{\mathcal{L}}$ as the composition $\pi_*\pi^*\tilde{\mathcal{L}} \to \pi_*[w]_*[w]^*\pi^*\tilde{\mathcal{L}} \to \pi_*\pi^*\tilde{\mathcal{L}}$ where the first map comes from the adjointness transformation $in \to [w]_*[w]^*$ and the second from $\pi \circ [w] = \pi$. One may check that $\langle w\rangle\langle v\rangle = \langle wv\rangle$. Thus we defined a morphism $\mathbb{C}[\mathcal{W}] \to \mathrm{End}(\pi_*\pi^*\tilde{\mathcal{L}})$.

(3) Let $f \in \mathrm{End}^\bullet(\pi^*\tilde{\mathcal{L}})$ and $w \in \mathcal{W}$. We have to establish in $\mathrm{End}^\bullet(\pi_*\pi^*\tilde{\mathcal{L}})$ the equation $\langle w\rangle(\pi_* f) = (\pi_*(\sigma(w)f))\langle w\rangle$. For this we consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_*\pi^*\tilde{\mathcal{L}} & \to & \pi_*[w]_*[w]^*\pi^*\tilde{\mathcal{L}} & \to & \pi_*[w]^*\pi^*\tilde{\mathcal{L}} & \to & \pi_*\pi^*\tilde{\mathcal{L}} \\
{\scriptstyle \pi_* f} \downarrow & & {\scriptstyle \pi_*[w]_*[w]^* f} \downarrow & & {\scriptstyle \pi_*[w]^* f} \downarrow & & {\scriptstyle \pi_*\sigma(w) f} \downarrow \\
\pi_*\pi^*\tilde{\mathcal{L}} & \to & \pi_*[w]_*[w]^*\pi^*\tilde{\mathcal{L}} & \to & \pi_*[w]^*\pi^*\tilde{\mathcal{L}} & \to & \pi_*\pi^*\tilde{\mathcal{L}}
\end{array}$$

which gives us the required equation.

(4) We have to prove that the product

$$\mathrm{End}^\bullet(\pi^*\tilde{\mathcal{L}}) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{W}] \to \mathrm{End}^\bullet(\pi_*\pi^*\tilde{\mathcal{L}}), \quad f \otimes w \mapsto \pi_* \circ \mathrm{id} \circ \langle w\rangle$$

is an isomorphism or, equivalently, that the product

$$\mathrm{Hom}^\bullet(\pi^*\tilde{\mathcal{L}}, \pi^*\tilde{\mathcal{L}}) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{W}] \to \mathrm{Hom}^\bullet(\pi^*\pi_*\pi^*\tilde{\mathcal{L}}, \pi^*\tilde{\mathcal{L}}), \quad f \otimes w \mapsto \pi_* \circ adj \circ \pi^*\langle w\rangle$$

is an isomorphism where $adj$ is the map corresponding to the identity id via the adjointness $(\pi^*, \pi_*)$. But $\pi^*\pi_*\pi^*\tilde{\mathcal{L}} \cong \oplus_{w\in\mathcal{W}}\pi^*\tilde{\mathcal{L}}$ and under this isomorphism the $\mathcal{W}$-action on the left by the $\pi^*\langle w\rangle$ corresponds to the "regular action" on the right. So indeed we get an isomorphism. *q.e.d.*

*Proof of the first theorem:* Apply the second theorem to $X = pt$, $\mathcal{L} = \underline{pt}$. *q.e.d.*

## 3. Examples

3.1. **Tori.** To establish the conjecture for $G$ a torus $T$ is not to hard. We begin with some definitions. As is well known there is a contravariant equivalence of abelian categories

$$\left\{\begin{array}{c}\text{diagonalizable}\\ \text{complex algebraic}\\ \text{groups}\end{array}\right\} \begin{array}{c}\mathfrak{X}\\ \to\\ \leftarrow\\ Diag\end{array} \left\{\begin{array}{c}\text{finitely generated}\\ \text{commutative}\\ \text{groups}\end{array}\right\}$$

For morphisms $\gamma$ of the left (resp. right) category we will often write just $\gamma$ instead of $\mathfrak{X}(\gamma)$ (resp. $Diag(\gamma)$). The reason that this does not lead us into a catastrophe is that we only have to work with pairwise commuting involutions.

Any diagonalizable complex algebraic group $G$ admits a unique antiholomorphic involution $s = s_G : G \to G$ such that $\phi(s(g)) = \overline{\phi(g)} \ \ \forall g \in G, \phi \in \mathfrak{X}(G)$. This antiholomorphic involution gives rise to what one calls the split real form of $G$. For every (holomorphic or antiholomorphic) morphism of complex diagonalizable groups $f : G \to H$ we have $f \circ s_G = s_H \circ f$.

Both the holomorphic and antiholomorphic homomorphisms from one complex diagonalizable group to another form an abelian group. We write these groups additively. For example, if $f : G \to H$ is holomorphic (or antiholomorphic), we have $(-f)(g) = f(g)^{-1} \ \ \forall g \in G$.

Now let us get at our torus $T$. It comes with an inner class of real forms, i.e. with an antiholomorphic involution $\gamma$. This gives us an involution $\gamma s = \mathfrak{X}(\gamma s)$ on $\mathfrak{X} = \mathfrak{X}(T)$ and thus an involution $\gamma = (\gamma s)^\vee$ on the dual lattice $\mathfrak{X}^\vee$. Let us describe the varieties $X(\chi)$. They are either empty or isomorphic to $X(0)$, which we henceforth abreviate to $X$. The isotropy group $I$ is the same for all points of $X$ and is in fact the kernel of $(\mathrm{id} - \gamma) : T^\vee \to T^\vee$. Applying our exact contravariant functor $\mathfrak{X}$ we get $\mathfrak{X}(I) = \mathfrak{X}^\vee / (\mathrm{id} - \gamma)\mathfrak{X}^\vee$. Thus $(Lie I)^* = \mathfrak{X}(I) \otimes_{\mathbb{Z}} \mathbb{C} = (\mathfrak{X}^\vee \otimes_{\mathbb{Z}} \mathbb{C})^\gamma$ and $\mathrm{Ext}_{T_F^\vee} X$ is just a cartesian product of say $n_g$ copies of the symmetric algebra $S = S((\mathfrak{X}^\vee \otimes_{\mathbb{Z}} \mathbb{C})^\gamma)$ over the dual of the Lie algebra of the isotropy group.

On the other hand, $T(\mathbb{R}, \gamma)$ admits a Cartan decomposition $T(\mathbb{R}, \gamma) = K \times A$ with $K$ compact and $A$ a vector group. In fact, $A$ is just the one-component of $T(\mathbb{R}, \gamma) \cap T(\mathbb{R}, s)$, and this intersection in turn consists just of all fixed points for some antiholomorphic involution on the kernel of $(\mathrm{id} - s\gamma) : T \to T$. Thus $Lie_{\mathbb{C}} A = Lie(\ker(\mathrm{id} - s\gamma)) = (\mathfrak{X}^\vee \otimes_{\mathbb{Z}} \mathbb{C})^\gamma$. In other words, the $S$ from above is also the enveloping algebra of $Lie_{\mathbb{C}} A$. So for each $\chi \in \mathrm{Max} Z$ either

$$\bigoplus_{\delta \in S_F} \mathcal{M}(T(\mathbb{R}, \delta))_\chi \cong \bigoplus_{n_r \ \ copies} S - Nil$$

for some $n_r$, or the left hand side vanishes. So to establish the conjecture we just have to show that $X(\chi) \neq \emptyset \Leftrightarrow \mathcal{M}(T(\mathbb{R}, \gamma))_\chi \neq 0$ and that if both sides do not vanish, then $n_g = n_r$. But this is calculated in detail in [ABV92] and essentially even in [Lan73] (for $F = 1$).

3.2. **Generic central character.** For most $\chi \in \mathrm{Max} Z$ the situation is very simple. Namely for $\chi$ outside a countable union of proper closed subvarieties of $\mathrm{Max} Z$ only the split real form admits representations with infinitesimal character $\chi$, and the irreducible ones are just the principal series induced from a Borel. I now test the conjecture on this example.

Let $(LieT)^*_{ni} = \{\lambda \in (LieT)^* \mid \langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z} \;\; \forall \alpha \in R\}$ be the set of all "not at all integral" weights and $(\mathrm{Max}Z)_{ni} \subset \mathrm{Max}Z$ its image under the Harish-Chandra homomorphism.

**Lemma 3.2.1.** *If $G$ is endowed with the inner class of real forms containing the split form and $\chi \in (\mathrm{Max}Z)_{ni}$, then $X(\chi) = G^\vee/T^\vee \times \{t \in T^\vee \mid t^2 = 1\}$ as a $G^\vee$-variety.*

*Proof:* Since $G$ is endowed with the split inner class, ${}^\Gamma G^\vee = G^\vee \times \Gamma$. For $\chi \in (\mathrm{Max}Z)_{ni}$ each element of $\mathcal{O}(\chi)$ is its own canonical flat, hence $X(\chi) = \{(y, \lambda) \mid y \in G^\vee, \lambda \in \mathcal{O}(\chi)$ such that $y^2 = exp(2\pi i \lambda)\}$. Let us determine for one $\lambda_0 \in \mathcal{O}(\chi)$ all possible $y$. Assume to simplify $\lambda_0 \in LieT^\vee$. Since $\chi$ was supposed to be not at all integral, the centralizer of $exp(2\pi i \lambda_0)$ in $G^\vee$ is just $T^\vee$ and we find that necessarily $y \in T^\vee$. Thus a bijection $\{t \in T^\vee \mid t^2 = 1\} \to \{y \in G^\vee \mid y^2 = exp(2\pi i \lambda_0)\}$, $t \mapsto t\; exp(\pi i \lambda_0)$. But every $\chi \in (\mathrm{Max}Z)_{ni}$ is in particular regular, so $\mathcal{O}(\chi) = G^\vee/T^\vee$ and we find indeed $X(\chi) = G^\vee/T^\vee \times \{t \in T^\vee \mid t^2 = 1\}$ with $G^\vee$ acting only on the first factor. *q.e.d.*

Let us now assume for simplicity that $F = 1$.

**Lemma 3.2.2.** *Suppose $G$ is endowed with the inner class of real forms containing the split form. Then only one of the $\delta \in S_1$ gives a split form of $G$.*

*Proof:* If we identify $S_1 = H^1(\Gamma; G)$, then $\delta \in S_1$ gives the split form if and only if it maps to the identity element in $H^1(\Gamma; G/Z(G))$. However consider the exact sequence

$$H^1(\Gamma; Z(G)) \to H^1(\Gamma; G) \to H^1(\Gamma; G/Z(G))$$

of nonabelian cohomology. Since the first map factors over the cohomology of a split torus $T \subset G$ and $H^1(\Gamma; T) = 1$, the preimage of the identity element in $H^1(\Gamma; G/Z(G))$ is just the identity element in $H^1(\Gamma; G)$. *q.e.d.*

Let $s \in S_1$ represent the split form. For $\chi \in (\mathrm{Max}Z)_{ni}$ we know that $\mathcal{M}(G(\mathbb{R}, \delta))_\chi = 0$ if $\delta \in S_1$, $\delta \neq s$. So we have for $\chi \in (\mathrm{Max}Z)_{ni}$ just to establish an equivalene of categories

$$\mathcal{M}(G(\mathbb{R}, s))_\chi \cong \mathrm{Ext}_{G^\vee} X(\chi) - Nil.$$

But both sides are just $Nil$-modules over a Cartesian product of $2^{rankG}$ copies of the symmetric algebra $S = S(LieT)$ of the Lie algebra of the maximal torus. For the right hand side this follows from $\mathrm{Ext}_{G^\vee}(G^\vee/T^\vee) = \mathrm{Ext}_{T^\vee}(pt) = H^\bullet(BT^\vee; \mathbb{C}) = S(LieT)$ where the grading is determined by $deg(LieT) = 2$. For the left hand side it follows from the fact that for any $\lambda \in (LieT)^*_{ni}$ the functor

$$\mathcal{M}(T(\mathbb{R}, s))_\lambda \to \mathcal{M}(G(\mathbb{R}, s))_{\xi(\lambda)}$$

of extending trivially to a Borel and inducing (with the shift that ensures that unitary representations go to unitary ones) is an equivalence of categories.

3.3. **The case $G = SL_2$.** There is only one inner class of real forms. Certainly $G^\vee = PGL(2;\mathbb{C})$. Let us compute the $X(\chi)$. We distinguish three cases.

- $\chi$ **not integral.:** Then $X(\chi) = G^\vee/T^\vee \times \{\text{two points}\}$ as we showed already more generally.
- $\chi$ **singular.:** Then $\mathcal{O}(\chi) = \{0\}$ and $X(\chi) = \{y \in G^\vee \mid y^2 = 1\}$. But any element of finite order is semisimple, hence contained in a maximal torus and in a torus there are precisely two roots of one. The stabilizers in $G^\vee$ of these two are $G^\vee$ and $N_{G^\vee}(T^\vee)$, which we will henceforth abreviate $N$. Thus $X(\chi) = pt \,\dot\cup\, (G^\vee/N)$.
- $\chi$ **regular integral.:** Then $exp(2\pi i\lambda) = 1$ for all $\lambda \in \mathcal{O}(\chi)$, hence $X(\chi)$ is just the Cartesian product of $\mathcal{F}(\mathcal{O}(\chi))$ with $\{y \in G^\vee \mid y^2 = 1\}$. But if $\chi \in \mathrm{Max}Z$ is regular integral, we have quite generally $\mathcal{F}(\mathcal{O}(\chi)) = G^\vee/B^\vee$. Hence in our situation $\{\mathcal{F}(\lambda) \mid \lambda \in \mathcal{O}(\chi)\} = G^\vee/B^\vee$ and
$$X(\chi) = G^\vee/B^\vee \quad \dot\cup \quad G^\vee/B^\vee \times G^\vee/N$$
where $G^\vee$ acts diagonally on the second part.

3.3.1. *The subcase $F == 1$.* Let us first check the conjecture for $F = 1$. Then $S_1 = H^1(\Gamma; SL(2,\mathbb{C}))$ consists just of one element $\delta$ with $G(\mathbb{R},\delta) = SL(2,\mathbb{R})$. For any $\chi \in \mathrm{Max}Z$ let us abreviate $\mathcal{M}(SL(2,\mathbb{R}))_\chi = \mathcal{M}_\chi$. These categories break down according to $K$-types into $\mathcal{M}_\chi = \mathcal{M}_\chi^{ev} \oplus \mathcal{M}_\chi^{odd}$. Now we procede case by case.

$\chi$ **not integral.** This has already been treated in general.

$\chi$ **singular.** It is known that $\mathcal{M}_\chi^{ev}$ is isomorphic to the category of all finite dimensional continuous modules over the completion $Z_\chi^\wedge$ of $Z$ at $\chi$. Via the Harish-Chandra homomorphism we find $\mathcal{M}_\chi^{ev} \cong S^{\mathcal{W}} - Nil$, with $S = S(LieT)$ the symmetric algebra over $LieT$. But on the other hand $\mathrm{Ext}_{G^\vee}(pt) = H^\bullet(BG^\vee;\mathbb{C}) = H^\bullet(BT^\vee;\mathbb{C})^{\mathcal{W}} = S^{\mathcal{W}}$ so that indeed

$$\mathcal{M}_\chi^{ev} \cong \mathrm{Ext}_{G^\vee}(pt) - Nil.$$

The structure of $\mathcal{M}_\chi^{odd}$ is more complicated. The irreducible objects of this category are the two limits of principal series, and these extend among each other. More precisely the picture is as follows: Choose in $\mathfrak{g}$ a "standard" basis $X, H, Y$ with $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$ and such that $\mathbb{C}H \subset \mathfrak{g}$ is the complexification of the Lie algebra of the compact torus $SO(2) \subset SL(2, \mathbb{R})$. For $M \in \mathcal{M}$, $n \in \mathbb{Z}$ put $M_n = \{v \in M \mid Hv = nv\}$. Then

$$M \mapsto \left\{ \quad M_{-1} \underset{X}{\overset{Y}{\rightleftarrows}} M_{+1} \quad \right\}$$

is an equivalence of $\mathcal{M}_\chi^{odd}$ to finite dimensional representations of the quiver

$$\bullet \underset{\phi}{\overset{\psi}{\rightleftarrows}} \bullet$$

with relations $\phi\psi$ nilpotent. This is not difficult, and follows for example from the more general results in [Kho80]. I leave it to the reader to identify the quiver algebra with the twisted group ring $S^\sigma[\mathcal{W}]$, where $\sigma : \mathcal{W} \to \mathrm{Aut}_\mathbb{C} S$ is the obvious action. Putting things together, we get $\mathcal{M}_\chi^{odd} \cong S^\sigma[\mathcal{W}] - Nil$.

But on the other hand

$$\begin{aligned} \mathrm{Ext}_{G^\vee}(G^\vee/N) &= H^\bullet(BN; \mathbb{C}) \quad \text{by ??} \\ &= H^\bullet(BT^\vee; \mathbb{C})^\sigma[\mathcal{W}] \quad \text{by theorem 2.4.1 of section ??} \\ &= S^\sigma[\mathcal{W}]. \end{aligned}$$

So indeed $\mathcal{M}_\chi^{odd} \cong \mathrm{Ext}_{G^\vee}(G^\vee/N) - Nil$ and with our previous result we obtain finally $\mathcal{M}_\chi \cong \mathrm{Ext}_{G^\vee} X(\chi) - Nil$ for singular $\chi$.

$\chi$ **regular integral.** By the translation principle we may and will henceforth assume $\chi = \mathrm{Ann}_Z \mathbb{C}$ is just the annihilator of the trivial representation. Then $\mathcal{M}_\chi^{odd}$ can be identified with the category of all finite dimensional continuous modules over the completion $Z_\chi^\wedge$ of $Z$ at $\chi$, and via the Harish-Chandra homomorphism we find $\mathcal{M}_\chi^{odd} \cong S - Nil$. But $\mathrm{Ext}_{G^\vee}(G^\vee/B^\vee) = H^\bullet(BB^\vee; pt) = S$, so $\mathcal{M}_\chi^{odd} \cong \mathrm{Ext}_{G^\vee}(G^\vee/B^\vee) - Nil$.

The category $\mathcal{M}_\chi^{ev}$ is much more interesting. Using the preceding notations, we may describe it as follows: The functor

$$M \mapsto \left\{ \quad M_{-2} \underset{X}{\overset{Y}{\rightleftarrows}} M_0 \underset{X}{\overset{Y}{\rightleftarrows}} M_{+2} \quad \right\}$$

establishes an equivalence of $\mathcal{M}_\chi^{ev}$ to finite dimensional representations of the quiver

$$\begin{tikzcd} \underset{-2}{\bullet} \arrow[r, shift right, "\phi^-"'] & \underset{0}{\bullet} \arrow[l, shift right, "\psi^-"'] \arrow[r, shift right, "\phi^+"'] & \underset{+2}{\bullet} \arrow[l, shift right, "\psi^+"'] \end{tikzcd} \qquad (*)$$

with relations $\phi^-\psi^- = \psi^+\phi^+$ and $\psi^+\phi^+$ nilpotent. This is not difficult, and follows for example from the more general results in [Kho80]. Now let us attack the geometric extension algebra. Since

$$G^\vee/B^\vee \times G^\vee/N = G^\vee \times_N (G^\vee/B^\vee)$$

we have (using the induction identity) just to understand $\mathrm{Ext}_N(G^\vee/B^\vee)$. There are two $N$-orbits on $(G^\vee/B^\vee) = \mathbb{P}^1\mathbb{C} = X$, namely $Y = \{0, \infty\}$ and $U = \mathbb{C}^\times$. Here $Y$ supports just one irreducible $N$-equivariant local system $\underline{Y}$, but $U$ supports two of them, with trivial and nontrivial $N$-action, say $\underline{U}^+$ and $\underline{U}^-$. The irreducibles in $\mathcal{P}_N(X)$ are correspondingly $i_*\underline{Y}$, $\underline{X}^+[1]$ and $\underline{X}^-[1]$ where $i : Y \to X$ is the inclusion and $\underline{X}^+$ (resp. $\underline{X}^-$) denotes just the constant sheaf $\mathbb{C}$ on $X$ with trivial (resp. nontrivial) $N$-action. By definition, $\mathrm{Ext}_N X = \mathrm{End}^\bullet_N(i_*\underline{Y} \oplus \underline{X}^+[1] \oplus \underline{X}^-[1])$. But since we only need to know this ring up to Morita equivalence, let us rather compute the equivariant selfextensions of $i_*\underline{Y} \oplus i_*\underline{Y} \oplus \underline{X}^+[1] \oplus \underline{X}^-[1]$. This object has the advantage to be isomorphic to $IndRes(i_*\underline{Y} \oplus \underline{X}^+[1])$ where we consider $N$ as the semidirect product $N = T^\vee \rtimes \mathcal{W}$ and use the notations of **??**, so we may apply theorem 2.4.2 of section **??** to $\mathcal{L} = i_*\underline{Y} \oplus \underline{X}^+[1]$. The theorem tells us that we should first compute $\mathrm{End}^\bullet_{T^\vee}(Res\mathcal{L})$ with its $\mathcal{W}$-action. This is not to hard, so I just give the result.

**Proposition 3.3.1.** $\mathrm{End}^\bullet_{T^\vee}(Res\mathcal{L})$ *is the quiver algebra of the quiver*

$$\begin{tikzcd} \underset{l}{\bullet} \arrow[r, shift right, "(ml)"'] & \underset{m}{\bullet} \arrow[l, shift right, "(lm)"'] \arrow[r, shift right, "(rm)"'] & \underset{r}{\bullet} \arrow[l, shift right, "(mr)"'] \end{tikzcd} \qquad (**)$$

*with relations* $(rm)(ml) = 0 = (lm)(mr)$ *where every arrow has degree one. The reflection of the quiver along the vertical symmetry axis induces the looked-for action* $\sigma$ *of* $\mathcal{W}$ *on the quiver algebra.*

*Proof:* Left to the reader. Perhaps it would have been more suggestive to call the extremal points 0 and $\infty$ instead of $r$ and $l$.

The finite dimensional representations of $\mathrm{Ext}_N X$ can be identified by Morita equivalence with finite dimensional representations of

$$\mathrm{End}^\bullet_N(Ind\ Res\mathcal{L}) = \mathrm{End}^\bullet_{T^\vee}(Res\mathcal{L})^\sigma[\mathcal{W}],$$

and these in turn are just finite dimensional representations of the quiver with relations from the proposition equipped with an "action of $\mathcal{W}$ on the representation compatible with the above action of $\mathcal{W}$ on the quiver". Such an object consists of a diagram of finite dimensional spaces

$$\begin{tikzcd} W_l \arrow[r, shift right, "(ml)"'] & W_m \arrow[l, shift right, "(lm)"'] \arrow[r, shift right, "(rm)"'] & W_r \arrow[l, shift right, "(mr)"'] \end{tikzcd}$$

together with isomorphisms $s : W_r \to W_l$, $s : W_m \to W_m$, $s : W_l \to W_r$ such that the relations $s^2 = \mathrm{id}$, $(lm)(mr) = 0 = (rm)(ml)$, $s(mr) = (ml)s$, $s(lm) = (rm)s$ are satisfied. We got to identify the category of such objects with the category of finite dimensional representations of the related quiver $(*)$.

This can be done as follows: To a representation

$$\begin{tikzcd} M_{-2} \arrow[r, shift right, "\phi^-"'] & M_0 \arrow[l, shift right, "\psi^-"'] \arrow[r, shift right, "\phi^+"'] & M_{+2} \arrow[l, shift right, "\psi^+"'] \end{tikzcd}$$

of the related quiver $(*)$ we associate the representation

$$\begin{tikzcd} M_0 \arrow[r, shift right, "\begin{pmatrix}\phi^+\\ \phi^-\end{pmatrix}"'] & M_{+2} \oplus M_{-2} \arrow[l, shift right, "(\psi^+{,}\psi^-)"'] \arrow[r, shift right, "(\psi^+{,}-\psi^-)"'] & M_0 \arrow[l, shift right, "\begin{pmatrix}\phi^+\\ -\phi^-\end{pmatrix}"'] \end{tikzcd}$$

of the related quiver $(**)$ with $\mathcal{W}$-action given by

$$s = \mathrm{id} : M_0 \to M_0, \qquad s = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix} : M_{+2} \oplus M_{-2} \to M_{+2} \oplus M_{-2}.$$

This can be seen to be an equivalence of categories and induces an equivalence of categories $\mathcal{M}_\chi^{ev} \cong \mathrm{Ext}_N X - Nil$. So putting things together we indeed established the desired equivalence of categories $\mathcal{M}_\chi \cong \mathrm{Ext}_{G^\vee} X(\chi) - Nil$ for regular integral $\chi$.

3.3.2. *The subcase $F = Z(G)$.* Now take for $F$ the full center of $SL(2,\mathbb{C})$, i.e. $F = \{+1, -1\}$. Then $S_F$ can be seen to consist of three elements, giving rise to the three real forms $SU(2,0)$, $SU(1,1)$ and $SU(0,2)$. Certainly two of them are just the compact form $SU(2)$ and the middle one is the split form $SL(2,\mathbb{R})$. So on the representation theoretic side of our conjecture only for $\chi$ regular integral there are some changes, and we have to consider instead of $\mathcal{M}(SL(2,\mathbb{R}))_\chi = \mathcal{M}_\chi$ the category

$$\bigoplus_{i=-1,0,1} \mathcal{M}(SU(1+i, 1-i))_\chi \cong (\mathbb{C} - Nil) \oplus \mathcal{M}_\chi \oplus (\mathbb{C} - Nil).$$

On the geometric side we get $G^\vee_F = SL(2,\mathbb{C})$. By **??** homomorphisms between objects in our equivariant derived categories do not change when we pass from $G^\vee$ to $G^\vee_F$. However there might be more simple objects in $\mathcal{P}_{G^\vee_F}X(\chi)$ than in $\mathcal{P}_{G^\vee}X(\chi)$, and this happens precisely when the isotropy groups of points of $X(\chi)$ with respect to $\mathcal{P}_{G^\vee_F}$ have more components than the isotropy groups with respect to $\mathcal{P}_{G^\vee}$. In our situation, this happens only for $\chi$ regular integral and for points in the open orbit on $G^\vee/B^\vee \times G^\vee/N \subset X(\chi)$. To verify our conjecture for $F = \{+1,-1\}$ we just have to check that

$$\mathrm{Ext}_{G^\vee_F}(G^\vee/B^\vee \times G^\vee/N) \cong (\mathrm{Ext}_{G^\vee}(G^\vee/B^\vee \times G^\vee/N)) \times \mathbb{C} \times \mathbb{C}.$$

Via the "induction identity" this can be reduced to showing

$$\mathrm{Ext}_{N_F}X \cong (\mathrm{Ext}_N X) \times \mathbb{C} \times \mathbb{C}$$

where $N_F = N_{G^\vee_F}(T^\vee)$. Let $\mathcal{L} \in \mathcal{P}_N X$ be the sum of all simple objects, $\mathcal{L} = i_*\underline{Y} \oplus \underline{X}^+[1] \oplus \underline{X}^-[1]$. In $\mathcal{P}_{N_F}X$ there are an additional two simple objects. In more detail, the isotropy group in $N_F$ of a point in the open orbit $U \subset X$ is just $\mathbb{Z}/4\mathbb{Z}$, thus there are four simple objects $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ in $\mathcal{P}_{N_F}X$, corresponding to the four irreducible characters on $\mathbb{Z}/4\mathbb{Z}$. Let $j : U \to X$ be the inclusion. Then $j_{!*}\mathcal{F}_0 = \underline{X}^+[1]$, $j_{!*}\mathcal{F}_2 = \underline{X}^-[1]$ and our two new irreducible objects in $\mathcal{P}_{N_F}X$ are $j_!\mathcal{F}_1 = j_*\mathcal{F}_1$ and $j_!\mathcal{F}_3 = j_*\mathcal{F}_3$. I leave it to the reader to check that

$$\mathrm{End}^\bullet_{N_F}(\mathcal{L} \oplus j_*\mathcal{F}_1 \oplus j_*\mathcal{F}_3) \cong \mathrm{End}^\bullet_N(\mathcal{L}) \times \mathbb{C} \times \mathbb{C}.$$

This gives the conjecture for $G = SL(2,\mathbb{R})$, $F = \{+1,-1\}$.

3.4. **Complex groups.** Here comes one of the main motivations for the conjecture. Namely we check that it correctly predicts the structure of part of the representation theory of $G_c(\mathbb{C})$ for a semisimple connected complex algebraic group, where we consider $G_c(\mathbb{C})$ as a real Lie group. In fact we will only give details for $G_c$ simply connected and assume this from now on. To get in the set-up of the conjecture, choose a split antiholomorphic involution $s : G_c \to G_c$ and put $G = G_c \times G_c$ with the quasisplit antiholomorphic involution $\gamma : G \to G$, $\gamma(x,y) = (sy, sx)$. Then $G(\mathbb{R},\gamma) = G_c(\mathbb{C})$ and $1 = H^1(\Gamma; G) = S_1 = \{\gamma\}$. We prove under the above assumptions:

**Theorem 3.4.1.** *Suppose $\chi$ is a regular integral central character. Then there is an equivalence of categories*

$$\mathcal{M}(G(\mathbb{R},\gamma))_\chi \cong \mathrm{Ext}_{G^\vee}X(\chi) - Nil.$$

*Proof[Theorem]:* We begin by simplifying the right hand side. Abreviate $G^\vee_c = \mathcal{G}$ and let $\mathcal{B} \subset \mathcal{G}$ be a Borel subgroup.

**Lemma 3.4.2.** *Suppose $\chi$ is regular integral. Then $X(\chi) \cong G^\vee \times_{\mathcal{B}} (\mathcal{G}/\mathcal{B})$ for a suitable embedding $\mathcal{B} \hookrightarrow G^\vee$.*

*Proof:* Left to the reader.
Next we simplify the right hand side. Let $\mathcal{HC}h$ denote the category of "Harish-Chandra bimodules" for $\mathfrak{g}_c = LieG_c$ investigated in [Soe92]. Let $Z_c \subset U-\mathrm{mod}-U(\mathfrak{g}_c)$ be the center and $Z_c^+ = \mathrm{Ann}_{Z_c}\mathbb{C}$ the central annihilator of the trivial representation of $\mathfrak{g}_c$. Define $\mathcal{H} = \{M \in \mathcal{HC}h \mid (Z_c^+)^n M = 0 = M(Z_c^+)^n \text{ for } n \gg 0\}$ the category of Harish-Chandra bimodules with trivial central character from both sides.

**Lemma 3.4.3.** *For $\chi$ regular integral there is an equivalence of categories*
$$\mathcal{M}(G(\mathbb{R},\gamma))_\chi = \mathcal{M}(G_c(\mathbb{C}))_\chi \cong \mathcal{H}.$$

*Proof:* This is contained in the literature [BG80].
Modulo these two lemmas we see that we need to establish an equivalence of categories $\mathcal{H} \cong \mathrm{Ext}_{\mathcal{B}}(\mathcal{G}/\mathcal{B}) - Nil$. This was a conjecture in [BG86]. It was essentially solved in [Soe92], however I did not write down there the topological part. Merely I constructed in a combinatorial way a graded $\mathbb{C}$-algebra $A$ and proved the equivalence of categories $\mathcal{H} \cong A - Nil$. So we just have to establish an isomorphism $A \cong \mathrm{Ext}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})$ of graded $\mathbb{C}$-algebras.

Let $\mathcal{T} \subset \mathcal{B}$ be a maximal torus and $\mathcal{W} = N_{\mathcal{G}}\mathcal{T}/\mathcal{T}$ the Weyl group. Then the irreducible objects of $\mathcal{P}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})$ are just the intersection cohomology complexes $\mathcal{IC}_x$ of the closures of the Schubert varieties $\overline{\mathcal{B}x^{-1}\mathcal{B}/\mathcal{B}}$, for $x \in \mathcal{W}$. Now by definition $\mathrm{Ext}_{\mathcal{B}}(\mathcal{G}/\mathcal{B}) = \mathrm{End}^\bullet_{\mathcal{B}}(\oplus_{x\in\mathcal{W}}\mathcal{IC}_x)$. We got to understand this algebra better.

The $\mathcal{B}$-equivariant cohomology ring of $\mathcal{G}/\mathcal{B}$ is well known to be $\mathrm{End}^\bullet_{\mathcal{B}}(\underline{\mathcal{G}/\mathcal{B}}) = S \otimes_{S^{\mathcal{W}}} S$ where $S = S((Lie\mathcal{T})^*)$ are the regular functions on $Lie\mathcal{T}$, graded in such a way that $(Lie\mathcal{T})^*$ is the degree two part of $S$. Now equivariant hypercohomology is a functor
$$\mathbb{H}_{\mathcal{B}} = \mathrm{Hom}^\bullet_{\mathcal{B}}(\underline{\mathcal{G}/\mathcal{B}},\quad) : \mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B}) \to S \otimes_{S^{\mathcal{W}}} S - mod$$
from the equivariant derived category to the category of graded modules over the equivariant cohomology ring.

**Proposition 3.4.4.** *For all $x, y \in \mathcal{W}$, equivariant hypercohomology induces an isomorphism*
$$\mathbb{H}_{\mathcal{B}} : \mathrm{Hom}^\bullet_{\mathcal{B}}(\mathcal{IC}_x, \mathcal{IC}_y) \to \mathrm{Hom}_{S\otimes S}(\mathbb{H}_{\mathcal{B}}\mathcal{IC}_x, \mathbb{H}_{\mathcal{B}}\mathcal{IC}_y).$$

*Proof:* Postponed to **??**.
In particular we deduce an isomorphism
$$\mathrm{Ext}_{\mathcal{B}}(\mathcal{G}/\mathcal{B}) \cong \mathrm{End}_{S\otimes S}(\oplus_{x\in\mathcal{W}}\mathbb{H}_{\mathcal{B}}\mathcal{IC}_x).$$

But the algebra $A$ was defined as

$$A = \mathrm{End}_{S\otimes S}(\oplus_{x\in\mathcal{W}}\mathbb{B}_x)$$

where the $\mathbb{B}_x$ are certain graded $S\otimes S$-modules described in a combinatorial way in [Soe92]. So to prove our theorem we need just

**Lemma 3.4.5.** *For all $x \in \mathcal{W}$ there is an isomorphism of graded $S\otimes S$-modules $\mathbb{H}_{\mathcal{B}}\mathcal{IC}_x \cong \mathbb{B}_x$.*

*Proof:* The proof of this lemma is very similar to the proof of its non-equivariant analogon in [Soe90], so I will rather give a sketch of proof.

Let $(\mathcal{W}, \mathcal{S})$ be the Coxeter system associated to $\mathcal{G} \supset \mathcal{B}$ and let $\mathcal{H} = \mathcal{H}(\mathcal{W}, \mathcal{S}) = \oplus_{x\in\mathcal{W}}\mathbb{Z}[t, t^{-1}]\mathbf{T}_x$ be the corresponding Hecke algebra. For $s \in \mathcal{S}$ let $\mathcal{P}_s \supset \mathcal{B}$ be the associated parabolic. Let $\pi_s : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_s$ be the projection. One may make $\langle\mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})\rangle$ into a $\mathcal{H}$-module by the prescription $t\langle\mathcal{F}\rangle = \langle\mathcal{F}[-1]\rangle$ and $(\mathbf{T}_s+1)\langle\mathcal{F}\rangle = \langle\pi_s^*\pi_{s*}\mathcal{F}\rangle$ for all $s \in \mathcal{S}$. If $\{\mathbf{C}'_x\}_{x\in\mathcal{W}}$ is the selfdual basis of the Hecke algebra defined in [KL79], then $\mathbf{C}'_x\langle\mathcal{IC}_e\rangle = \langle\mathcal{IC}_x\rangle$ for all $x \in \mathcal{W}$.

Now let us specify that we choose the identification $H^\bullet_{\mathcal{B}}(\mathcal{G}/\mathcal{B}) = S \otimes_{S^{\mathcal{W}}} S$ such that the pullback via $\pi_s : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_s$ corresponds to the inclusion $S^s \otimes_{S^{\mathcal{W}}} S \hookrightarrow S \otimes_{S^{\mathcal{W}}} S$ on equivariant cohomology rings. Furthermore, let us identify graded $S\otimes S$-modules with graded $S$-bimodules in the obvious way. Let $S - mof - S$ denote the category of all graded $S$-bimodules which are finitely generated as left and as right modules. Then we may interpret $\mathbb{H}_{\mathcal{B}}$ as a functor

$$\mathbb{H}_{\mathcal{B}} : \mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B}) \to S - mof - S,$$

and use essentially the same arguments as in [Soe90] to prove $\mathbb{H}_{\mathcal{B}}\pi_s^*\pi_{s*}\mathcal{F} = S \otimes_{S^s} (\mathbb{H}_{\mathcal{B}}\mathcal{F})$ for $s \in \mathcal{S}$, $\mathcal{F} \in \mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})$.

Now recall from [Soe92] how we made $\langle S-mof-S\rangle$ into a $\mathcal{H}$-module. The above formula says then just that $\mathbb{H}_{\mathcal{B}}$ induces a morphism of $\mathcal{H}$-modules $\mathbb{H}_{\mathcal{B}} : \langle\mathcal{D}_{\mathcal{B}}(\mathcal{G}/\mathcal{B})\rangle \to \langle S - mof - S\rangle$. Certainly $\mathbb{H}_{\mathcal{B}}\langle\mathcal{IC}_e\rangle = \langle S\rangle$, thus $\mathbb{H}_{\mathcal{B}}\langle\mathcal{IC}_x\rangle = \mathbb{H}_{\mathcal{B}}\mathbf{C}'_x\langle\mathcal{IC}_e\rangle = \mathbf{C}'_x\mathbb{H}_{\mathcal{B}}\langle\mathcal{IC}_e\rangle = \mathbf{C}'_x\langle S\rangle = \langle\mathbb{B}_x\rangle$ by definition of $\mathbb{B}_x$. *q.e.d.[Lemma and theorem]*

We still have to prove proposition 3.4.4. Let us start with some general considerations. Let $X$ be a topological space filtered by closed subspaces $X = X_0 \supset X_1 \supset \ldots \supset X_r = \emptyset$ and let $\mathcal{F} \in D^b(ShX)$ be an object in the bounded derived category of sheaves on $X$. To calculate its hypercohomology $\mathbb{H}^\bullet\mathcal{F}$ we have to choose an injective resolution $I^\bullet$ of $\mathcal{F}$ and take cohomology of the complex of global sections $\Gamma(I^\bullet)$. Certainly this is filtered by supports, $0 \subset \Gamma_{X_r}(I^\bullet) \subset \ldots \subset \Gamma_{X_0}(I^\bullet) = \Gamma(I^\bullet)$. So its cohomology is the limit of a spectral sequence with $E_1$-term $E_1^{p,q} = \mathbb{H}^{p+q}_{X_p - X_{p+1}}(\mathcal{F})$ the local hypercohomology of $\mathcal{F}$ along $X_p -$

$X_{p+1}$. Let $j : Y \hookrightarrow X$ be the embedding of a locally closed subset. Then we can write the local hypercohomology as $\mathbb{H}^\bullet_Y(\mathcal{F}) = \mathbb{H}^\bullet v^!\mathcal{F}$, i.e. as the hypercohomology of the derived local cohomology sheaf $v^!\mathcal{F} \in D^+(ShY)$. Let $j_p : X_p - X_{p+1} \hookrightarrow X$ denote the inclusion. We may then write our $E_1$-term as $E_1^{p,q} = \mathbb{H}^{p+q} j_p^! \mathcal{F}$.

Suppose next that $X$ is a complex algebraic $\mathcal{B}$-variety and that all $X_p$ are closed $\mathcal{B}$-stable subvarieties. Applying the above considerations to $E\mathcal{B}\times_{\mathcal{B}} X$ we find that for $\mathcal{F} \in \mathcal{D}_{\mathcal{B}}(X)$ the equivariant hypercohomology $\mathbb{H}^\bullet_{\mathcal{B}}\mathcal{F}$ is the limit of a spectral sequence with $E_1$-term $E_1^{p,q} = \mathbb{H}^{p+q}_{\mathcal{B}} j_p^!\mathcal{F}$. In particular, if $\mathcal{G} \in \mathcal{D}_{\mathcal{B}}(X)$ is another object the equivariant extensions $\mathrm{Hom}^\bullet_{\mathcal{B}}(\mathcal{F},\mathcal{G}) = \mathbb{H}^\bullet_{\mathcal{B}}\mathcal{R}Hom(\mathcal{F},\mathcal{G})$ are the limit of a spectral sequence with $E_1$-term $E_1^{p,q} = \mathbb{H}^{p+q}_{\mathcal{B}} j_p^! \mathcal{R}Hom(\mathcal{F},\mathcal{G}) = \mathbb{H}^{p+q}_{\mathcal{B}}\mathcal{R}Hom(j_p^*\mathcal{F}, j_p^!\mathcal{G})$.

Suppose that $X$ is a $\mathcal{B}$-variety with a $\mathcal{B}$-stable stratification $X = \dot{\bigcup}_{w\in\mathcal{W}} X_w$ into smooth irreducible subvarieties. Put $|w| = \mathrm{dim} X_w$ and let $\mathcal{C}_w \in \mathcal{P}_{\mathcal{B}}(X_w)$ be the constant perverse sheaf $\mathcal{C}_w = \underline{X_w}[|w|]$. Let $\mathcal{IC}_w \in \mathcal{P}_{\mathcal{B}}(X)$ denote the intersection cohomology complex of the closure of $X_w$, extended by zero to $X$. Let $j_w : X_w \hookrightarrow X$ be the inclusions of the strata and suppose there exist isomorphisms

$$j_v^*\mathcal{IC}_w \cong \oplus_\nu n^\nu_{v,w}\mathcal{C}_v[\nu] \qquad (*)$$

in $\mathcal{D}_{\mathcal{B}}(X_v)$, for all $v, w \in \mathcal{W}$, with suitable integers $n^\nu_{v,w}$ defined by these isomorphisms.

**Proposition 3.4.6.** *Suppose we have in addition to the above assumptions parity vanishing, i.e. (1)* $n^\nu_{v,w} = 0$ *unless* $\nu$ *has the same parity as* $|v| + |w|$ *and (2)* $H^i_{\mathcal{B}}(X_w) = 0$ *for all* $w$ *unless* $i$ *is even. Then*

*i.)* $\mathbb{H}^\bullet_{\mathcal{B}}$ *induces an injection* $\mathrm{Hom}^\bullet_{\mathcal{B}}(\mathcal{IC}_x, \mathcal{IC}_y) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}^\bullet_{\mathcal{B}}\mathcal{IC}_x, \mathbb{H}^\bullet_{\mathcal{B}}\mathcal{IC}_y)$.

*ii.)* $\mathrm{dim}_{\mathbb{C}}\mathrm{Hom}^\nu_{\mathcal{B}}(\mathcal{IC}_x,\mathcal{IC}_y) = \sum_{a+i+j=\nu,\, z\in\mathcal{W}} n^i_{z,v} n^j_{z,w} \mathrm{dim}_{\mathbb{C}} H^a_{\mathcal{B}}(X_z)$.

*Proof:* ii.) Consider the filtration on $X$ given by $X_p = \bigcup_{|x|+p\le \mathrm{dim}X} X_x$. If we calculate $\mathrm{Hom}^\nu_{\mathcal{B}}(\mathcal{IC}_x, \mathcal{IC}_y)$ by the associated spectral sequence, we find as $E_1$-term

$$\begin{array}{rcl} E_1^{p,q} & = & \mathbb{H}^{p+q}_{\mathcal{B}}\mathcal{R}Hom(j_p^*\mathcal{IC}_x, j_p^!\mathcal{IC}_y) \\ & = & \oplus_{|z|+p=\mathrm{dim}X}\mathbb{H}^{p+q}_{\mathcal{B}}\mathcal{R}Hom(j_z^*\mathcal{IC}_x, j_z^!\mathcal{IC}_y) \\ & = & \oplus_{|z|+p=\mathrm{dim}X}\mathbb{H}^{p+q}_{\mathcal{B}}\mathcal{R}Hom(\oplus_i n^i_{z,x}\mathcal{C}_z[i], \oplus_j n^j_{z,y}\mathcal{C}_z[-j]) \\ & = & \oplus_{|z|+p=\mathrm{dim}X} \oplus_{i,j} n^i_{z,x} n^j_{z,y} H^{p+q-i-j}_{\mathcal{B}}(X_z). \end{array}$$

By our parity assumptions this vanishes unless $p+q$ has the same parity as $|x| + |y|$. In particular, our spectral sequence degenerates at this $E_1$-term and ii.) follows.

i.) Also $\mathbb{H}^i_{\mathcal{B}}\mathcal{IC}_x$ is the limit of a spectral sequence with $E_1$-term $E_1^{p,q} = \mathbb{H}^{p+q}_{\mathcal{B}} j_p^!\mathcal{IC}_x$, and again by parity vanishing it degenerates at this term. It is sufficient to show that any nonzero morphism $f : \mathcal{IC}_x \to$

$\mathcal{IC}_y[\nu]$ in $\mathcal{D}_{\mathcal{B}}(X)$ induces for some $p$ a nonzero morphism $j_p^! f : j_p^! \mathcal{IC}_x \to j_p^! \mathcal{IC}_y[\nu]$. Let $i_p : X_p \hookrightarrow X$ be the inclusion. We may decompose $X_p$ into an open and a closed subset

$$X_{p+1} \overset{i}{\hookrightarrow} X_p \overset{u}{\hookleftarrow} (X_p - X_{p+1}).$$

Then we have a distinguished triangle $(i_! i^!, \mathrm{id}, u_* u^*) i_p^! = (i_* i_{p+1}^!, i_p^!, u_* j_p^!)$ which says that $i_{p+1}^! f = 0$ and $j_p^! f = 0$ imply $i_p^! f = 0$. So if all $j_p^! f = 0$ then $i_0^! f = f = 0$. *q.e.d.*
*Proof[Proposition 3.4.4]:* It will be sufficient to (1) prove injectivity of the map and (2) check that the Dimensions coincide on both sides, where for a graded vector space $V = \oplus V^i$ we understand its Dimension to be the polynomial $\mathrm{Dim} V = \sum t^i \dim_{\mathbb{C}} V^i$. First we check that the preceding proposition applies in our situation, i.e. for the stratification of $X = \mathcal{G}/\mathcal{B}$ by the Bruhat cells $X_w = \mathcal{B} w^{-1} \mathcal{B}/\mathcal{B}$. We need

**Lemma 3.4.7.** *In $\mathcal{D}_{\mathcal{B}}(X_v)$ the $j_v^* \mathcal{IC}_w$ decompose into the direct sum of their perverse cohomology, i.e. $j_v^* \mathcal{IC}_w \cong \oplus_\nu n_{v,w}^\nu \mathcal{C}_v[\nu]$ for suitable integers $n_{v,w}^\nu$.*

*Proof:* In the nonequivariant situation this follows from pointwise purity: Namely the $j_v^* \mathcal{IC}_w$ are "pure", hence the sum of their perverse cohomology groups.

We may reduce our equivariant situation to a nonequivariant situation as follows: First choose a finite interval $I \subset \mathbb{Z}$ such that $\mathcal{IC}_w \in \mathcal{D}_{\mathcal{B}}^I(X)$. Then we have to prove the lemma for the $j_v^* \mathcal{IC}_w \in \mathcal{D}_{\mathcal{B}}^I(X_v)$. Now choose a smooth $n$-acyclic free right $\mathcal{B}$-space $M$, where $n$ is bigger then the length of $I$. Then by [BL92] we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{B}}^I(X) & \to & \mathcal{D}^I(M \times_{\mathcal{B}} X) \\ {\scriptstyle j_v^*} \downarrow & & \downarrow {\scriptstyle (\mathrm{id} \times j_v)^*} \\ \mathcal{D}_{\mathcal{B}}^I(X_v) & \to & \mathcal{D}^I(M \times_{\mathcal{B}} X_v) \end{array}$$

where the horizontal maps are fully faithful embeddings. But the nature of the singularities is the same for $M \times_{\mathcal{B}} \overline{X_w}$ and $\overline{X_w}$. Hence the intersection cohomology complex of $M \times_{\mathcal{B}} \overline{X_w}$ is also pointwise pure and the lemma follows. *q.e.d.*

Now we may in fact use proposition 3.4.6 to show that our map

$$\mathbb{H}_{\mathcal{B}} : \mathrm{Hom}_{\mathcal{B}}^\bullet(\mathcal{IC}_x, \mathcal{IC}_y) \to \mathrm{Hom}_{S \otimes S}(\mathbb{H}_{\mathcal{B}} \mathcal{IC}_x, \mathbb{H}_{\mathcal{B}} \mathcal{IC}_y).$$

is an injection. So we just have to check that the Dimensions are equal. But

$$\begin{array}{rcl} \mathrm{Dim}\mathrm{Hom}_{\mathcal{B}}^\bullet(\mathcal{IC}_x, \mathcal{IC}_y) & = & \mathrm{Dim}\mathrm{Hom}_{\mathcal{D}}^\bullet(\mathcal{IC}_x, \mathcal{IC}_y) \otimes_{\mathbb{C}} S \\ & & \text{by proposition 3.4.6} \end{array}$$

and

$$\begin{aligned}\mathrm{DimHom}_{S\otimes S}(\mathbb{H}_{\mathcal{B}}\mathcal{IC}_x, \mathbb{H}_{\mathcal{B}}\mathcal{IC}_y) &= \mathrm{DimHom}_{S\otimes S}(\mathbb{B}_x, \mathbb{B}_y) \text{ by} \\ &= \mathrm{DimHom}_{S\otimes S}(\mathbb{B}_x, \mathbb{B}_y) \otimes_S \mathbb{C} \otimes_{\mathbb{C}} S \text{ by} \\ &= \mathrm{DimHom}_S(\mathbb{B}_x \otimes_S \mathbb{C}, \mathbb{B}_y \otimes_S \mathbb{C}) \otimes_{\mathbb{C}} S \text{ by} \\ &= \mathrm{DimHom}_S(\mathbb{H}^\bullet \mathcal{IC}_x, \mathbb{H}^\bullet \mathcal{IC}_y) \otimes_{\mathbb{C}} S.\end{aligned}$$

But we know already from [Soe90] that

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(\mathcal{IC}_x, \mathcal{IC}_y) = \mathrm{Hom}_S(\mathbb{H}^\bullet \mathcal{IC}_x, \mathbb{H}^\bullet \mathcal{IC}_y).$$

*q.e.d. [Proposition 3.4.4]*

3.5. **Compact groups.** Suppose under the ABV-parametrization some geometric parameter $(Y, \tau)$ corresponds to a representation of a compact form of $G$. Our basic conjecture would say that

$$\mathrm{Hom}j_{G^\vee_F}(\mathcal{IC}(Y, \tau), \mathcal{IC}(Y', \tau')[n]) = 0$$

unless $(Y, \tau) = (Y', \tau')$ and $n = 0$. The results of [ABV92] on the duality between IC-data and KL-data tell us already that the above space of homomorphisms vanishes unless $(Y, \tau) = (Y', \tau')$. We will verify in addition that the isotropy group of points in $Y$ is finite. This implies that there are now higher selfextensions of $\mathcal{IC}(Y, \tau)$.

So let $\chi \in \mathrm{Max}Z$ be integral regular. Put $I = \{y \in {}^\Gamma G^\vee - G^\vee \mid y^2 = 1\}$. Clearly $X(\chi) = I \times G^\vee/B^\vee$ with the diagonal $G^\vee$-action. In [ABV92] it is shown that $I$ decomposes into a finite number of $G^\vee$-orbits, that these orbits are all closed and that the corresponding isotropy groups are just complexifications of maximal compact subgroups of real forms of $G^\vee$. More precisely the real forms appearing here are just all real forms of one inner class of real forms on $G^\vee$, and this inner class corresponds to the involution $(-w_\circ\gamma)$ of our based root system, where $\gamma$ is the involution of our based root system coming from the choosen inner class of real forms on $G$ and $w_\circ$ is the longest element of the Weyl group.

Hence if $G$ is endowed with the inner class of real forms containing the compact one, then $\gamma = -w_\circ$ so that the corresponding inner class of real forms on $G^\vee$ contains the split form. In particular some components of $X(\chi)$ are isomorphic to $G^\vee/K \times G^\vee/B^\vee$ where $K$ is the complexification of a maximal compact subgroup of a split form of $G^\vee$. Now the representations of compact forms of $G$ correspond just to open $G^\vee$-orbits on these $G^\vee/K \times G^\vee/B^\vee \subset X(\chi)$. And clearly the isotropy groups of points in these open orbits are finite.

3.6. **Tempered representations.** I want to suggest a link of my basic conjecture with the results of A. Guichardet [Gui92]. He considers an L-packet of irreducible tempered representations and the category $\mathcal{T}$ of all Harish-Chandra modules with composition factors only from this L-packet. He then proves that this category $\mathcal{T}$ is equivalent to a category $S^\sigma[R] - Nil$, for a suitable polynomial ring $S$ with a suitable action $\sigma : R \to \mathrm{Aut}_{\mathbb{C}} S$ of a suitable finite group $R$ on it.

In the ABV-parametrization a tempered L-packet corresponds to a collection of geometric parameters $(Y, \tau_i)$ for a fixed open orbit $Y \subset X(\chi)$. If my basic conjecture is true, there should be an equivalence of categories

$$\mathcal{T} \cong \mathrm{Ext}_{G_F^\vee} X(\chi)/I - Nil$$

where the twosided ideal $I$ is generated by all idempotents which belong to parameters other than the $(Y, \tau_i)$. Now restriction gives us a map

$$\mathrm{Ext}_{G_F^\vee} X(\chi) \to \mathrm{Ext}_{G_F^\vee} Y$$

which annihilates $I$, and by 2.4 the ring $\mathrm{Ext}_{G_F^\vee} Y$ has the form $S^\sigma[R]$. I didn't check however that these $S$, $\sigma$, $R$ are indeed those of Guichardet.

3.7. **The center and geometry.** To each additive category $\mathcal{A}$ we may associate a commutative ring $Z(\mathcal{A})$, the center of $\mathcal{A}$, as follows: $Z(\mathcal{A})$ consists just of all natural transformations of the identity functor $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ to itself, i.e. $Z(\mathcal{A}) = \mathrm{End}(\mathrm{id})$. Let $R$ be a commutative ring. By an "action of $R$ on $\mathcal{A}$" we mean a ring homomorphism $R \to Z(\mathcal{A})$.

Now consider our basic conjecture. On the representation theoretic side of the conjectured equivalence there acts certainly the completion $Z_\chi^\wedge$ at $\chi$ of the center $Z \subset U - \mathrm{mod} - U$. I want to explain what this should correspond to on the geometric side. Remark that the construction of $X(\chi)$ gives us a $G^\vee$-equivariant map $X(\chi) \to \mathcal{F}(\mathcal{O}(\chi))$. This leads to a morphism

$$H^\bullet_{G^\vee}(\mathcal{F}(\mathcal{O}(\chi))) \to \mathrm{Ext}_{G_F^\vee} X(\chi)$$

of the equivariant cohomology ring of $\mathcal{F}(\mathcal{O}(\chi))$ to our geometric extension algebra, and this morphism has central image. It should correspond to the action of $Z_\chi^\wedge$ on the representation theoretic side of the conjectured equivalence via the morphism $H^\bullet_{G^\vee}(\mathcal{F}(\mathcal{O}(\chi))) \to Z_\chi^\wedge$ defined as the composition

$$H^\bullet_{G^\vee}(\mathcal{F}(\mathcal{O}(\chi))) \overset{(1)}{=} H^\bullet_{G^\vee}(\mathcal{O}(\chi)) \overset{(2)}{=} H^\bullet(BG_\chi^\vee; \mathbb{C}) \overset{(3)}{=} S^{\mathcal{W}_\lambda} \overset{(4)}{\to} (S_0^\wedge)^{\mathcal{W}_\lambda} \overset{(5)}{\to} Z_\chi^\wedge$$

which we will now explain step by step.

1. This is clear since $\mathcal{O}(\chi) \twoheadrightarrow \mathcal{F}(\mathcal{O}(\chi))$ is a fibration with affine spaces as fibres.

2. We choose $\lambda \in \mathcal{O}(\chi) \cap LieT^\vee$ and let $G^\vee_\lambda$ be the isotropy group of $\lambda$. Then this is the induction equivalence.
3. The Weyl group of $G^\vee_\lambda$ is the stabilizer $\mathcal{W}_\lambda$ of $\lambda$ in $\mathcal{W}$. We let $S = S(LieT)$ be the regular functions on the Lie algebra $LieT^\vee$ of the maximal torus $T^\vee \subset G^\vee_\lambda$. For any linear complex algebraic group the cohomology ring of its classifying space can be identified with the regular functions on the Lie algebra of a maximal torus which are invariant under the Weyl group.
4. We let $S^\wedge_0$ be the completion of $S$ at the origin of $LieT^\vee$.
5. We will rather construct the inverse morphism. The Harish-Chandra homomorphism $\xi : Z \to S$, $\xi(Z) = S^{\mathcal{W}}$ certainly induces a morphism between the respective completions $Z^\wedge_\chi \to S^\wedge_\lambda$. If we compose with the isomorphism $S^\wedge_\lambda \to S^\wedge_0$ defined by translation, we get a morphism $Z^\wedge_\chi \to S^\wedge_0$. By some invariant theory this factorizes over an isomorphism $Z^\wedge_\chi \to (S^\wedge_0)^{\mathcal{W}_\lambda}$. The inverse of this isomorphism is the last map of our above composition.

## 4. Further conjectures

To provide additional motivation for my basic conjecture, I want to explain how it should be related to character duality [Vog82, ABV92]. Let me cite from [ABV92]. "It is natural to look for a functorial relationship between a category of representations of strong real forms of $G$, and one of our geometric categories. We do not know what form such a relationship should take, or how one may hope to establish it directly." I conjecture that the relationship in question should take the form of Koszul duality. However I have no idea how to establish such a relationship directly. I want now to give more details and start by explaining some Koszul duality.

4.1. **Generalities on Koszul duality.** Let $A = \oplus_{i\geq 0} A_i$ be a graded $\mathbb{C}$-algebra with $A_0 = \oplus_{x\in\mathcal{W}} \mathbb{C}1_x$, where $\mathcal{W}$ is a suitable finite parameter set. Let $A$-mod be the category of graded $A$-modules $M = \oplus_i M_i$. We define $M\langle 1\rangle \in A$-mod by $(M\langle 1\rangle)_i = M_{i-1}$. For $M, N \in A$-mod we let $\hom_A(M,N)$ (resp. $\mathrm{Hom}j_A(M,N)$) denote morphisms in $A$-mod (resp. $A$-Mod). So $\hom_A(M,N) = \{f \in \mathrm{Hom}_A(M,N) \mid f(M_i) \subset N_i\}$.

Let us form the bounded from above derived category $\mathcal{D} = D^-(A\text{-mod})$. It comes with three automorphisms [1] "shift of a complex", $\langle 1\rangle$ "shift of the grading on all terms of a complex" and $(1/2) = [-1]\langle -1\rangle$. The strange notation comes from the fact that later $(1/2)$ should correspond to a "root of the Tate twist". We will denote morphisms in $\mathcal{D}$ by $\hom_{\mathcal{D}}$.

We consider two t-structures on $\mathcal{D}$. The "representation theoretic" t-structure $(\mathcal{D}^{\leq 0,r}, \mathcal{D}^{\geq 0,r})$ is the obvious t-structure with heart $A$-mod.

The "geometric" t-structure $(\mathcal{D}^{\leq 0,g}, \mathcal{D}^{\geq 0,g})$ is its "Koszul dual". More precisely, $\mathcal{D}^{\leq 0,g}$ (resp. $\mathcal{D}^{\geq 0,g}$) consists of all complexes which are quasi-isomorphic to complexes $\ldots \to X^i \to X^{i+1} \to \ldots$ of projective objects $X^i$ of $A$-mod such that $X^i$ is generated by its parts of degree $\leq -i$ (resp. $\geq -i$).

The twists $\langle 1\rangle$ (resp. $(1/2)$) respect the representation theoretic (resp. geometric) t-structure. Up to twist the simple objects in the hearts of the respective t-structures are the $L_x = A_0 1_x$ (resp. the $L^x = A1_x$). Let $\mathcal{D}_r$ (resp. $\mathcal{D}_g$) be the full triangulated subcategories of $\mathcal{D}$ generated by the $L_x$ (resp. $L^x$) and their twists. The respective t-structures on $\mathcal{D}$ induce t-structures $(\mathcal{D}_r^{\leq 0}, \mathcal{D}_r^{\geq 0})$ and $(\mathcal{D}_g^{\leq 0}, \mathcal{D}_g^{\geq 0})$ on $\mathcal{D}_r$ and $\mathcal{D}_g$ respectively.

4.2. **Application to our situation.** Let us now go back to the situation of our basic conjecture. There we conjectured an equivalence of categories

$$\bigoplus_{\delta \in S_F} \mathcal{M}(G(\mathbb{R},\delta))_\chi \cong \mathrm{Ext}_{G_F^\vee}(X(\chi)) - Nil.$$

Let us abreviate the left hand side category by $\mathcal{HC}h$ and put $G_F^\vee = \mathcal{G}$, $X(\chi) = X$. Let $\{\mathcal{L}_x\}_{x\in\mathcal{W}}$ and $\{\mathcal{L}^x\}_{x\in\mathcal{W}}$ represent the simple isomorphism classes in $\mathcal{HC}h$ and $\mathcal{P}_{\mathcal{G}}(X)$ respectively, where the indices are choosen in such a way that objects with the same index correspond to each other under the ABV-parametrization $(a) \leftrightarrow (b)$ from the beginning. In particular $\mathcal{W}$ is not at all the Weyl group. Finally put $A = \mathrm{Ext}_{\mathcal{G}} X = \mathrm{End}^\bullet_{\mathcal{G}}(\oplus \mathcal{L}^x)$. This is a graded $\mathbb{C}$-algebra and $A_0 = \oplus_{x\in\mathcal{W}} \mathbb{C}1_x$ in an obvious way. The basic conjecture predicts an equivalence of categories $\mathcal{HC}h \cong A - Nil$.

Now construct to $A$ as explained before the triangulated category $\mathcal{D} = D^-(A\text{-mod})$ with its twists $[1]$, $\langle 1\rangle$, $(1/2)$ and the two t-structures giving rise to two full triangulated subcategories

$$\mathcal{D}_r \hookrightarrow \mathcal{D} \hookleftarrow \mathcal{D}_g.$$

**Conjecture 4.2.1.** *There exists a t-exact functor $v_r : \mathcal{D}_r \to D^b(\mathcal{HC}h)$ together with canonical isomorphisms $v_r(M\langle 1\rangle) = v_r M$ such that*

*i.)* $\mathrm{Hom}_{D^b(\mathcal{HC}h)}(v_r M, v_r N) = \oplus_i \mathrm{hom}_{\mathcal{D}}(M, N\langle i\rangle)$ *for all* $M, N \in \mathcal{D}_r$.

*ii.)* $v_r L_x \cong \mathcal{L}_x \quad \forall x \in \mathcal{W}$.

**Conjecture 4.2.2.** *There exists a t-exact functor $v_g : \mathcal{D}_g \to \mathcal{D}_{\mathcal{G}}(X)$ together with canonical isomorphisms $v_g(M(1/2)) = v_g M$ such that*

*i.)* $\mathrm{Hom}_{\mathcal{G}}(v_g M, v_g N) = \oplus_i \mathrm{hom}_{\mathcal{D}}(M, N(i/2))$ *for all* $M, N \in \mathcal{D}_g$.

*ii.)* $v_g L^x \cong \mathcal{L}^x \quad \forall x \in \mathcal{W}$.

The conjectures mean that the categories $\mathcal{D}_r$ and $\mathcal{D}_g$ should be "mixed" or "graded" versions of $D^b(\mathcal{HC}h)$ and $\mathcal{D}_{\mathcal{G}}(X)$ respectively, and the functors $v_r$, $v_g$ just "forget the grading". The next conjectures stipulate the existence of graded versions for "standard objects". Let me explain what I mean. Certainly any $\mathcal{IC}(Y, \tau)$ is the unique simple quotient of the zero'th perverse cohomology of $\mathcal{M}^x = i_! \tau[\dim Y]$, where $i : Y \hookrightarrow X$ is the inclusion of a $\mathcal{G}$-orbit and $\tau$ an irreducible $\mathcal{G}$-equivariant local system on $Y$, hence $\tau \in \mathcal{D}_{\mathcal{G}}(Y)$. On the other hand any $\mathcal{L}_x$ is the socle of a standard object $\mathcal{N}_x \in \mathcal{HC}h$ induced from an irreducible tempered representation.

**Conjecture 4.2.3.** *For all $x \in \mathcal{W}$ there exists $M^x \in \mathcal{D}_g$ such that $v_g M^x \cong \mathcal{M}^x$ and*
$\hom_{\mathcal{D}}(M^x, L^x) \neq 0$. *This $M^x$ is unique up to isomorphism.*

**Conjecture 4.2.4.** *For all $x \in \mathcal{W}$ there exists $N_x \in \mathcal{D}_r$ such that $v_r N_x \cong \mathcal{N}_x$ and*
$\hom_{\mathcal{D}}(N_x, L^x) \neq 0$. *This $N_x$ is unique up to isomorphism.*

The essential ingredient we would need to get the character duality of Vogan's is the following

**Conjecture 4.2.5.** *We have*

$$\hom_{\mathcal{D}}(M^x(i/2), N_y\langle n\rangle) = \begin{cases} \mathbb{C} & x = y, i = n = 0 \\ 0 & \textit{else.} \end{cases}$$

Now comes character duality. We make the Grothendieck groups $[\mathcal{D}_r]$ and $[\mathcal{D}_g]$ into $\mathbb{Z}[t, t^{-1}]$-modules by the prescription $t[M] = [M\langle 1\rangle]$ for $M \in \mathcal{D}_r$, respectively $t[M] = -[M(1/2)] = [M\langle -1\rangle]$ for $M \in \mathcal{D}_g$. We obtain a $\mathbb{Z}[t, t^{-1}]$-bilinear pairing $[\mathcal{D}_g] \times [\mathcal{D}_r] \to \mathbb{Z}[t, t^{-1}]$ between these two Grothendieck groups by the prescription

$$(M, N) \mapsto \langle M, N\rangle = \sum_{\nu, i} (-1)^{\nu} \dim_{\mathbb{C}} \hom_{\mathcal{D}}(M, N[\nu]\langle -i\rangle) t^i.$$

Certainly $[\mathcal{D}_g]$ is a free $\mathbb{Z}[t, t^{-1}]$-module with two bases $\{[L^x]\}_{x \in \mathcal{W}}$ and $\{[M^x]\}_{x \in \mathcal{W}}$, which we call the simple and the standard basis. Also $[\mathcal{D}_r]$ is a free $\mathbb{Z}[t, t^{-1}]$-module with "simple" basis $\{[L_x]\}_{x \in \mathcal{W}}$ and "standard" basis $\{[N_x]\}_{x \in \mathcal{W}}$. It is clear from the definitions that $\langle L^x, L_y\rangle = \delta_{x,y}$ and if we believe conjecture 4.2.5 we find as well $\langle M^x, N_y\rangle = \delta_{x,y}$.

Let now $P_r$ be the transition matrix from the simple basis $\{[L_x]\}$ to the standard basis $\{[N_x]\}$. So $P_r$ is a $\mathcal{W} \times \mathcal{W}$-matrix with coefficients $(P_r)_{x,y} \in \mathbb{Z}[t, t^{-1}]$ given by the equation

$$[N_y] = \sum_x (P_r)_{x,y} [L_x].$$

Similarily we define the geometric base change matrix $P_g$ by the equation

$$[M^z] = \sum_x (P_g)_{u,z}[L^u]$$

in $[\mathcal{D}_g]$. Now our formulas give us

$$\begin{aligned} \delta_{z,y} &= \langle M^z, N_y\rangle \\ &= \textstyle\sum_{u,x}(P_g)_{u,z}(P_r)_{x,y}\langle L^u, L_x\rangle \\ &= (P_g^T P_r)_{z,y} \end{aligned}$$

where the upper index $T$ transposes the matrix. In other words, we find the equation

$$P_g^T P_r = 1$$

of $\mathcal{W}\times\mathcal{W}$-matrices with coefficients in $\mathbb{Z}[t,t^{-1}]$.

This equation is essentially Vogan's character duality. To see this, write $(P_r)_{x,y} = \sum m_r(x,y;i)t^i$ so that $[N_y] = \sum_{x,i} m_r(x,y;i)[L_x\langle i\rangle]$. Applying $v_r$ to both sides, we see that if we set $t=1$ in $P_r$ then $P_r(1)$ is just the representation theoretic multiplicity matrix, $(P_r(1))_{x,y} = m_r(x,y)$ in the notation of [ABV92], (1.21). On the other hand, write $(P_g)_{u,z} = \sum m_g(u,z;j)t^j$ so that $[M^z] = \sum_{u,j} m_g(u,z;j)(-1)^j[L^u(j/2)]$. Applying $v_g$ to both sides, we see that if we set $t=-1$ in $P_g$ then $P_g(-1)$ is just the intersection cohomology matrix, $(P_g(-1))_{u,z} = m_g(u,z)$ in the notation of [ABV92], (1.22)(d).

To finish, we have to assume some parity vanishing. More precisely, for $x\in\mathcal{W}$ let $d(x)$ denote the dimension of the corresponding $\mathcal{G}$-orbit on $X$.

**Conjecture 4.2.6.** *We have $m_g(u,z;j) = 0$ unless $j$ has the same parity as $d(u)+d(z)$.*

From this we can deduce $(P_g(1))_{u,z} = (-1)^{d(u)+d(z)}(P_g(-1))_{u,z} = (-1)^{d(u)+d(z)}m_g(u,z)$ and thus specializing our equation $P_g^T P_r = 1$ to $t=1$ gives precisely Vogan's character duality

$$\sum_u (-1)^{d(u)+d(z)} m_g(u,z)m_r(u,x) = \delta_{z,x}$$

as formulated in [ABV92], Corollary 1.25. a).

4.3. **Motivation.** The first of these conjectures is almost our basic conjecture. For the other conjectures there is very little hard evidence. At least for tori everything works, and for complex groups one can find results in the spirit of the above conjectures in [BGS96].

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Universität Freiburg, Mathematisches Institut, Albertstrasse 23b, D-79104 Freiburg, Germany