(joint w/ Kari)

Wilfried, "Invt Hermitian Forms on HC modules."

$G_{\mathbb{R}}$ real reductive matrix group $\quad G$

$K_{\mathbb{R}}$ $\quad K$

$U_{\mathbb{R}} \subset G_{\mathbb{R}}$ cpt real form s.t. $U_{\mathbb{R}} \supset K_{\mathbb{R}}$.

V irred HC module, infl char χ_λ, λ real $\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$

V carries a nontrivial $U_{\mathbb{R}}$-inv hermitian form.
It is positive definite on all lowest $K_{\mathbb{R}}$-types.
If V carries a nonzero $\mathfrak{g}_{\mathbb{R}}$-invariant Hermitian form, then the two are directly related.

$(,)_{u_{\mathbb{R}}}$ $\quad U_{\mathbb{R}}$-inv. $\quad \| \quad (,) \quad G_{\mathbb{R}}$-inv.

$V^! =$ dual HC module $\quad V \xrightarrow{\sim} V'$ $K_{\mathbb{R}}$-qu

Each form determines conj. linear isom
Compose one with inverse of other, get

$$T : V \xrightarrow{\sim} V \qquad K_{\mathbb{R}}\text{-inv, linear} \quad \therefore K\text{-inv.}$$

$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$

- symmetric ops wrt $(,)_{u_{\mathbb{R}}}$
- skew w.r.t. $(,)$

$\therefore$ for $\xi \in \mathfrak{p}_{\mathbb{R}}$

$$T\xi = (-\xi)T$$
$$-\xi = \theta(\xi)$$

i.e. $T\xi = \theta\xi T \quad \forall \xi \in \mathfrak{g}$

$\therefore$ T^2 = multiple of identity.

Arrange so $T^2 = Id$.

Eigenspace decomposition: $V = V^+ \oplus V^-$

$\therefore$ $(,)_{u_R} = \pm (,)$ on $V^\pm$.

N.B. $f : V^+ \rightleftarrows V^-$ formal.

(A) Equal rank case: rank(K) = rank(G).

$H_{\mathbb{R}} \subset K_{\mathbb{R}}$ compact CSG.

$\Phi = \Phi(\mathfrak{h}, \mathfrak{g})$ $\quad \Phi^+ \subset \Phi$ $\quad \Psi$ = simple roots.

μ_0 h/w of a LKT

μ_1 some other h/w of a LKT.

$$\mu - \mu_0 = \sum_{\alpha \in \Psi}^{+} n_\alpha \alpha$$

$V(\mu)$ & $V(\mu_0)$ lie in same parity subspace

$$\Leftrightarrow \sum_{\substack{\alpha \in \Psi \\ \text{noncpt}}} n_\alpha \in 2\mathbb{Z}.$$

(B) Θ is not inner. Make it inner by extending G by θ.

$X_{\mathbb{R}}$ underlying C^∞ manifold.
X smooth quasiprojective $\mathbb{C}$-alg variety.
$\bar{X}$ opposite $\mathbb{C}$-structure. $\mathcal{O}_{\bar{X}} = \overline{\mathcal{O}_X}$.

$\mathcal{M}$ = regular holonomic sheaf of $\mathcal{D}_X$-modules.
$\mathbb{D}\mathcal{M}$ = dual module
$\bar{\mathcal{M}}$ = regular holonomic $\mathcal{D}_{\bar{X}}$-module.

Kashiwara: $\bar{\mathcal{M}} \cong \cancel{\text{sheaf of}}\ \overset{\text{sheaf}}{\mathcal{H}om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_c^{-\infty}(X_{\mathbb{R}}))$
(analytic topology) ~~conj-linear~~ conjugate isomorphism

Examples: • $X = \mathbb{C}$

(a) $\mathcal{M} = \mathcal{O}_{\mathbb{C}} \cong \mathcal{D}_{\mathbb{C}} / \mathcal{D}_{\mathbb{C}} \frac{\partial}{\partial z}$

$$\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}\partial/\partial z,\ C^{-\infty}(X_{\mathbb{R}}))$$
$$\cong \left\{ \tau \in \mathcal{C}^{-\infty}(X_{\mathbb{R}}) \;\middle|\; \frac{\partial \tau}{\partial z} = 0 \right\} = \overline{\mathcal{O}_{\mathbb{C}}}.$$

(b) $\mathcal{M} = \mathbb{C}\left[\frac{\partial}{\partial z}\right]\delta_0 \cong \mathcal{D}_{\mathbb{C}} / \mathcal{D}_{\mathbb{C}} z$

$$\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}} z,\ C^{-\infty}(X_{\mathbb{R}}))$$
$$\cong \left\{ \tau \in C^{-\infty}(X_{\mathbb{R}}) \;\middle|\; \begin{array}{l}\text{supported at origin i.e.}\\ \mathbb{C}\left[\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right]\delta_0 \text{ killed by } z\end{array} \right\} \text{ i.e.}$$
$$= \mathbb{C}\left[\frac{\partial}{\partial \bar{z}}\right]\delta_0 \ \}$$
$$\cong \bar{\mathcal{M}}.$$

Same assumption as above. Assume further that $\mathcal{M}$ is irreducible.

Defn (Saito) A polarization for $\mathcal{M}$ is an isom

$$\mathcal{M} \cong \mathbb{D}\overline{\mathcal{M}}.$$

[Saito: this notion is functorial. Needed only to understand polarization only for variations of Hodge structure on X AND when $X = \{pt\}$, i.e. Hodge structures]

Sabbah: simpler version of polarizations.
A polarization on $\mathcal{M}$ is a non-zero $(\mathcal{D}_X \times \mathcal{D}_{\overline{X}})$-invariant pairing

$$\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \longrightarrow \mathcal{C}^{-\infty}(X_{\mathbb{R}}).$$

This notion is functorial

Ex $X = \mathbb{C}$.

(a) $\mathcal{M} = \mathcal{O}_{\mathbb{C}} \cong \mathcal{D}_{\mathbb{C}} / \mathcal{D}_{\mathbb{C}} \frac{\partial}{\partial z}$

$$\mathrm{Hom}_{\mathcal{D}_{\mathbb{C}} \otimes \overline{\mathcal{D}_{\mathbb{C}}}} \left(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}},\ \mathcal{C}^{-\infty}(X_{\mathbb{R}}) \right)$$

Image of $1 \otimes 1$

$$\cong \left\{ \tau \in \mathcal{C}^{-\infty}(X_{\mathbb{R}}) \;\middle|\; \frac{\partial \tau}{\partial z} = 0 = \frac{\partial \tau}{\partial \bar{z}} \right\} \cong \mathbb{C}.$$

(b) $\mathcal{M} = \mathbb{C}\left[\frac{\partial}{\partial z}\right] \delta_0 \cong \mathcal{D}_{\mathbb{C}} / \mathcal{D}_{\mathbb{C}} \cdot z$.

example of functorial: $\mathcal{M}$ is direct image of one $\mathcal{D}$-module map pt.

As above, supported at origin ~~not~~ killed by $z, \bar{z}$, so no holomorphic nor antihol derivatives

$$\mathrm{Hom}_{\mathcal{D}_{\mathbb{C}} \otimes \overline{\mathcal{D}_{\mathbb{C}}}} \left(\mathcal{M} \otimes \overline{\mathcal{M}},\ \mathcal{C}^{-\infty}(X_{\mathbb{R}}) \right) = \mathbb{C}$$

(a) generalizes to X smooth. ??
$\mathrm{Hom}_{\mathcal{D}_X \times \mathcal{D}_{\bar{X}}}(\mathcal{O}_X \otimes \mathcal{O}_{\bar{X}}, \mathcal{C}^{\infty}(X_{\mathbb{R}}))$

section on X
$\otimes$ section on $\bar{X}$
$\rightsquigarrow$ multiply to get fn on $X_{\mathbb{R}}$ a distribution

Implications for representation theory:

$G_{\mathbb{R}}$, $U_{\mathbb{R}}$, $K_{\mathbb{R}}$ as before. X = flag variety.
$\mathfrak{h}$ = universal Cartan, $\Lambda \subset \mathfrak{h}^*$ universal weight lattice.

Fix dominant $\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$.

First case: $\lambda = \rho$. $\mathcal{D}_\rho = \mathcal{D}_X$.

Suppose $\mathcal{M}$ is an irreducible ~~HC~~ sheaf of $(\mathcal{D}_{\cancel{X}}, K)$ mod associated to K-orbit $Q \subset X$.

Polarization of $\mathcal{M}|_Q$ completely understood.
$\mathcal{D}$-module restrict $\to$ connection on Q $\quad$ ξ $\quad$ from example above.

By functorality, get polarization on M.

dm = $U_{\mathbb{R}}$-invariant measure.

$$(v, v)_{U_{\mathbb{R}}} = \int_X \mathrm{polarization}(v, \bar{v})\, dm.$$

Purely formal $U_{\mathbb{R}}$-invariant.

general "real" dominant λ.

$\hat{X}$ = enhanced flag variety. $\cong \left(G / \substack{\text{maximal} \\ \text{unipotent}} \right)$

$$\begin{array}{c} \hat{X} \\ \big\downarrow \\ X \end{array} \quad \text{fiber is } H, \text{ "abstract" CSG of } G.$$

Ex. $G_{\mathbb{R}} = SU(1,1)$ $\qquad U_{\mathbb{R}} = SU(2)$
$G = SL(2,\mathbb{C})$ $\qquad K = \mathbb{C}^*$

$\Lambda \cong \mathbb{Z}$ $\qquad$ Fix $\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}$. $\qquad \lambda$ dominant ≥ 0.

Fix $\delta \in \{0,1\}$ $\qquad 0$ = spherical, 1 = nonspherical.

irred ps: $\lambda \not\equiv \delta + 1 \pmod 2$ $\qquad$ irreducibility case.

$$\hat{X} = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \middle| \; z, w \in \mathbb{C}, \text{ not both zero} \right\}$$

$\Gamma(\mathcal{M}_{\lambda,\sigma})$ has basis $\left\{ \left(\frac{z}{w}\right)^n (zw)^{\frac{\lambda-1}{2}} \;\middle|\; n \in \mathbb{Z} + \frac{\delta}{2} \right\}$.

from $K_{\mathbb{R}}$-invariance?

Inner product $\left(\left(\frac{z}{w}\right)^n (zw)^{\frac{\lambda-1}{2}}, \left(\frac{z}{w}\right)^m (zw)^{\frac{\lambda-1}{2}} \right)_{U_{\mathbb{R}}} = \left\{ \begin{array}{l} 0 \text{ unless } m = n \\ \text{from } K_{\mathbb{R}}\text{-invariance} \end{array} \right.$

$n = m$ $\quad (\cdot \, , \cdot)_{U_{\mathbb{R}}}$

$$= \int_{U_{\mathbb{R}}/T_{\mathbb{R}} \cong X} \left| \frac{z}{w} \right|^{2n} |zw|^{\lambda-1} \, dm$$

On $U_{\mathbb{R}}$-orbit $|z|^2+|w|^2=1$ $\cong U_{\mathbb{R}}/K_{\mathbb{R}}$.

$$|zw| = \frac{\left|\frac{z}{w}\right|}{1+\left|\frac{z}{w}\right|^2}$$

$X \cong \mathbb{C}\cup\{\infty\}$,
use $z=z$, $w=1$ on ??

$$\int_{\mathbb{C}\cup\{\infty\}} |z|^{2n+\lambda-1}\left(1+|z|^2\right)^{-\lambda-1} dx\,dy.$$

The integral converges $\iff$ $|2n| < \lambda+1$

In this range, the integral is positive.

Details not finished, but in general seems like LKT integral are seen to be positive.